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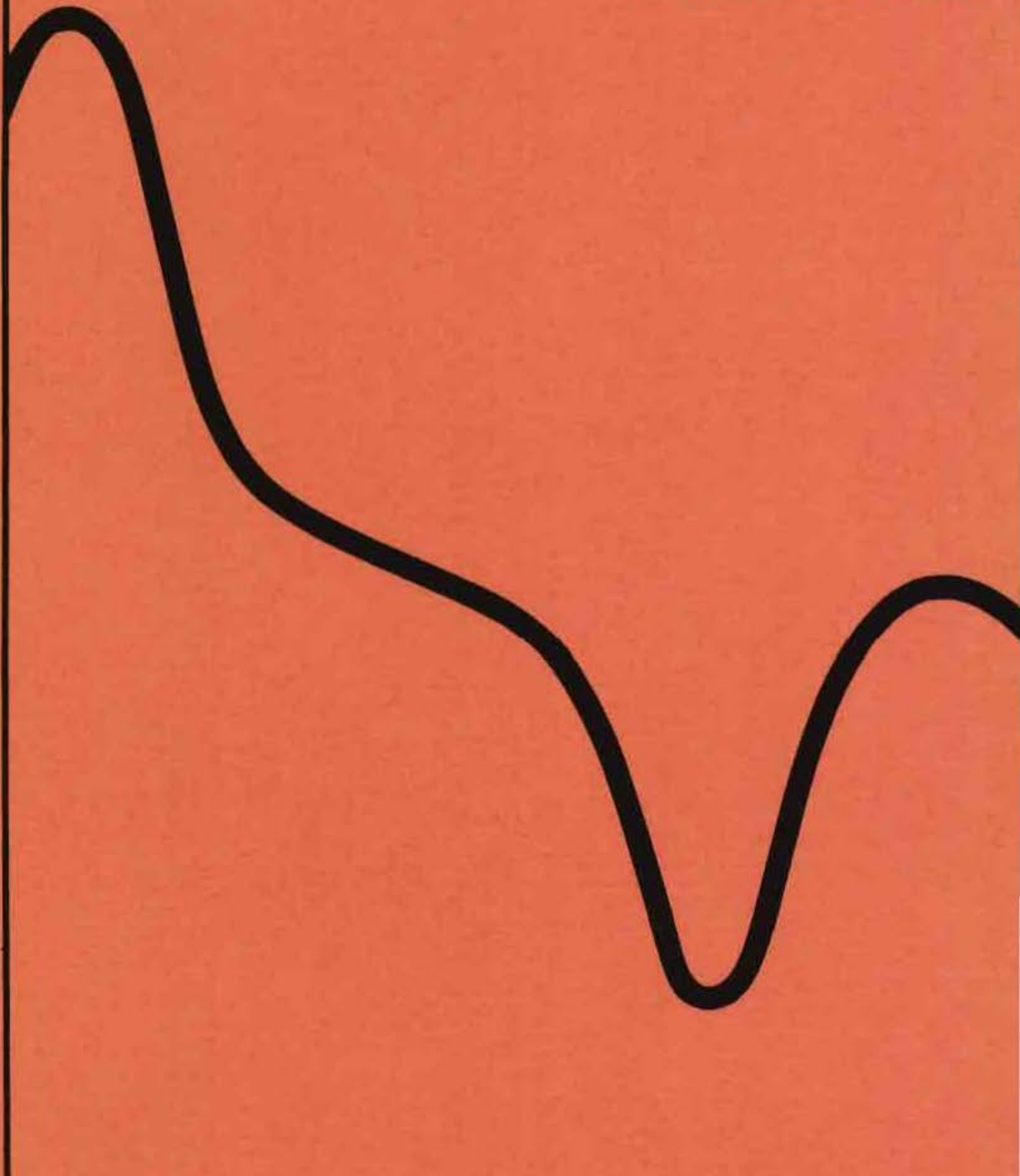

ON THE NONLINEAR THEORY OF ONE-DIMENSIONAL HOMOGENEOUS COLLISIONLESS PLASMAS

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL EINDHOVEN OP GEZAG VAN DE RECTOR MAGNIFICUS PROF.DR.IR. A.A.TH.M. VAN TRIER, HOOGLERAAR IN DE AFDELING DER ELEKTROTECHNIEK, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP DINSDAG 24 FEBRUARI 1970 DES NAMIDDAGS TE 4 UUR

DOOR

ROBERT WILLEM BOREL BEST

GEBOREN TE SOERABAJA

E R R A T A

I p. 189 eq. (27): $(4/Ak\dot{A}^2)$ should be $(4\dot{A}/Ak^2)$

II p. 240 : Read $\frac{3}{8}\,\varepsilon^2$ instead of $\frac{3}{8}\,\varepsilon_2$ in the equation.

III p. 7 eq. (12) : The first line should be

$$\left[(\underline{k}u-\omega)(\underline{k}'u-\omega')\{(\underline{k}\pm\underline{k}')u-(\omega\pm\omega')\}\right]^{-1} \equiv$$

DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR

PROF.DR. H. BREMMER

... nonlinear problems have a certain
kind of unpredictability.
Werner Heisenberg, Phys.Today 20(1967)27.

aan Rigtje

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C O N T E N T S

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INTRODUCTION

A classical physical model of the universe is a system of many particles (of infinitesimal size) interacting through electromagnetic and gravitational forces. This model is described mathematically by, and most of classical physics is thus summarized in, the equations of motion (of Newton or Einstein) together with equations for the e.m. field (Maxwell) and equations for the gravitational field. The main content of theoretical classical physics concerns a set of solutions of these equations in special configurations and limiting cases in which the equations are tractable.

Most of the universe consists of plasma, one exception being "the bubble of nearly un-ionized gas we live in"[1]). A certain condition (many particles in the local Debye sphere) is universally satisfied which justifies the so-called self-consistent field approximation[2]). Physically this means that we pass from the system of many discrete particles to a fluid: every species of particles is smeared out to a continuous distribution of mass and charge, keeping the ratio between the densities of the latter constant. Accordingly the fields become continuous in space. The system then is no longer described mathematically by the motion of each individual particle, but by distribution functions for each species of particles. These functions, continuous in 6-dimensional phase space, are governed by the continuity equation in this space, derived by Boltzmann. Since the particles do not collide directly but interact only through the field, we need the Boltzmann equation without collision term, that is the Vlasov equation. No longer N equations of motion for N particles have to be solved simultaneously, but only a few equations of motion for "test particles" of each species in the self-consistent field. (These equations constitute the characteristic equations of Vlasov's equation considered as a first-order linear partial differential equation). Through this field the latter equations are still coupled to the field equations.

In their source terms moments of the distributions occur which, in turn, are functions of the constants of motion of the test particles, according to Vlasov's equation.

In this thesis the electrostatic case is considered in which all fields and velocities are in one fixed direction (the x-axis). Only the electrons are supposed to move, the ions constituting a constant homogeneous neutralizing background. In this model the gravitational field can be incorporated in the electrostatic field, while the magnetic field has to be constant and homogeneous which makes it irrelevant for the electron motion and the electric field. Then the equations reduce to the one-dimensional Vlasov equation coupled to Poisson's equation (I eq. (48), II and III eq. (1); roman numbers refer to the subsequent papers composing this thesis). Mathematically this is the simplest set of equations which retains the essential form of the original equations: Vlasov equations for distribution functions, coupled to equations for the self-consistent field.

Physically two typical plasma properties are retained in this model. First, the electron fluid can only be described adequately by its distribution function of both position and velocity. This is due to the fact that at a certain position the electrons usually have a non-Maxwellian velocity distribution, to be determined from the equations. This is in contrast to the situation for a gas in local equilibrium, which is characterized by a few macroscopic quantities (such as the temperature). The latter, being functions of position only, depend only on corresponding moments of the velocity distribution already fixed by its Maxwellian shape apart from two parameters. Secondly, unlike particles moving in an external field so strong that the field due to the particles is negligible, here the electron fluid moves in its own field. An additional external field is not considered in this thesis. Both the distribution and the field are to be determined from the equations.

Now, imagine a continuous electron "gas" embedded in an infinitely fine ionic "grid". An equilibrium is achieved when the electron distribution is homogeneous. There is no electric field in that case. However, an inhomogeneity of the electron density disturbs the local neutrality and thus causes an electric field which accelerates the electrons so as to restore the neutrality. This restoring force, together with the electrons inertia generates oscillations at the plasma frequency (I eq. (51)), one of the most fundamental plasma properties. Apart from this collective oscillatory motion we have to do with the random thermal motion. It is a basic and only partly solved problem how the energies, associated with these two kinds of motion, are converted into each other. The plasma model studied here should include possible modes of this conversion.

It is known[3]) that in a cold plasma (no thermal motion, no velocity distribution, but a stream velocity as function of position) any collective oscillation will never convert into thermal motion, unless the oscillation amplitude exceeds a certain limit above which the electrons can overtake each other (which means that the velocity is no longer a single-valued function of position). Also in a hot plasma undamped oscillations are known to exist, the BGK waves[4]); these are travelling waves which constitute rigorous solutions of the non-linear equations. This thesis deals with some other waves and shows that these cannot be damped (or enhanced) slowly because this would violate the law of conservation of momentum or energy, or of both.

In paper I the motion of a particle in a slowly damped (or growing) travelling wave is considered first. The particle can be in either one of two states of motion: trapped between two wave crests, or free running "over hills and dales". In a damped (or growing) wave trapped particles become free (or v.v.). A trapped particle, being an (anharmonic) oscillator, is known to have an adiabatic invariant which is difficult to define precisely, but which is very nearly equal to the following quantity: the particle's velocity with respect to the wave frame (the frame of reference in which the wave has zero phase velocity), considered there as a function of its position x, and as such integrated between two turning points. With the aid of a simple canonical transformation the corresponding invariant, associated with the nearly periodic motion of a free particle, is derived: its velocity, considered as function of x, now integrated over one wavelength. Next an apparently new result is derived: for almost every particle the two invariants associated with its original and its final situation (for transition between a trapped and a free state) prove to be nearly equal[5]). Though a particle can change its invariant drastically by "sitting on a wave crest" for a long time, this is extremely unlikely to occur.

Finally, as a corollary from the constancy of the above space-averaged velocity of a free particle in a damped wave, it is pointed out that the time-averaged velocity increases (in absolute value) with respect to the wave[6]). This effect can be understood without any calculation. When averaging the velocity of a particle over a period in time, the ranges of minimum and nearly minimum absolute values of the velocity occupy a larger part of the integration range than in the case of space-averaging, so the time average will be smaller (in absolute value). But, when the wave has vanished, both averages must amount to the same value, so that the absolute time average has to increase if the space average remains constant. Returning now to the lab frame, the conclusion can be reached that a damped wave travelling faster than most particles of the electron "gas", would accellerate the latter in the opposite direction, which contradicts the conservation of momentum of the electron plasma as a whole. In fact, in this situation (assuming a positive value of the phase velocity of the wave) most particles have a negative velocity in the wave frame, the absolute value of which has to increase, thus causing an accelleration with negative sign (in both frames).

Paper II deals with a standing wave instead of a travelling one[7]). The standing wave can be considered as the result of two BGK-waves travelling in opposite directions. The non-linearity of the problem involves a coupling of these two waves (not discussed before to the author's knowledge), which gives rise to an infinite series of sum and difference "frequencies". The main problem is to show that, in spite of the infinite number of waves thus present, most electrons can be considered as free. Damping of the wave would cool the plasma for reasons, similar to those in paper I, which here lead to a violation of the conservation of energy instead of momentum.

Concerning the conversion of collective plasma oscillations into thermal motion, Landau[8]) pioneered by showing that in the linear theory waves are damped when the velocity distribution of the equilibrium state is Maxwellian. Penrose[9]) derived a criterion for instability of equilibrium velocity distributions according to which, e.g., distributions with two sufficiently separated humps are linearly unstable. These two papers are basic in an overwhelming literature on the linear theory which is now fairly well established and verified experimentally[10]). However, the linear theory is of rather limited validity. For a wave travelling a few times faster than the thermal electron velocity in a Maxwellian plasma, the linear theory is only valid either during short times or, in the case of single waves decreasing with time, for extremely small amplitudes. In the latter case the total momentum and energy can be balanced by the trapped electrons (II p. 238). Large-amplitude or growing waves quickly develop beyond the linear regime and then we arrive theoretically on flimsy grounds.

The physical concept[11]) for the non-linear regime amounts to a field that consists of a number of waves or wave packets which interact (i.e. exchange energy), both mutually and with the particles, when certain resonance conditions are fulfilled. For three waves, e.g., the latter condition may consist of the two simultaneous relations $\omega_1 = \omega_2 + \omega_3$, $k_1 = k_2 + k_3$, ω_i and k_i referring to the individual frequencies and wave numbers. A particle is supposed to move with an average velocity and a superposed oscillatory velocity. The resonance condition for wave-particle interaction expresses the coincidence of the wave's phase velocity and the particle's average velocity, cf. travelling-wave tubes. This slowly varying velocity component can only be changed by waves resonant with the particle, while the superposed oscillatory velocity is due to the non-resonant waves. Obviously, a physical model including all these resonance phenomena is not easy to catch in a rigorous mathematical formalism. Neglecting wave-wave interaction a diffusion equation for the distribution function of average-particle velocities has been derived in quasi-linear theory[12]).

Several possibilities exist for the final state of the plasma. First, the electrons might be heated so much by diffusion or other mechanisms that all oscillation energy gets exhausted. Secondly, a state of stationary turbulence (whatever this means) might develop in which wave packets are continuously destroyed and created. Thus a spontaneously growing wave packet has been found both empirically and theoretically in an apparently quiescent plasma, the MWGO echo[13]); it is like a time-reversed Landau-damped wave. Thirdly, it has been found in computer experiments[14]) for finite plasmas, admitting only a discrete wavenumber spectrum, that finally only a few waves survive whose amplitudes tend to stationary values, while the distribution of average-particle velocities tend to develop plateaux in the regions of trapped-particle velocities. This state can be described with the mathematics of paper II.

However, most plasmas are much larger than the Debye length, and then the possible wavenumber spectrum is almost continuous. This case has been considered in paper III dealing with a finite disturbance, large compared to the Debye length, in an infinite homogeneous plasma. The disturbance is assumed to be composed of a continuous spectrum of travelling waves, the width of the spectrum being small compared to the reciprocal Debye length. An individual electron, passing through the disturbance, interacts resonantly with a small part of the wave spectrum which part can be considered as a wave packet. Due to the interaction with the wave packet the electron changes slightly its mean velocity. For the electron plasma the latter changes lead to diffusion in velocity space, which involves heating of the plasma and damping of the disturbance. The mathematical formalism, which is rather similar to that in paper II, proves to admit only spatial damping, while the solution is valid in a half space The formula for the damping coefficient is very similar to the expression for spatial Landau damping in the linear theory. Spatial Landau damping of a wave spectrum then turns out to hold for much larger amplitudes than Landau damping with respect to time for a single wave, which is valid for exponentially small amplitudes only.

Summarizing, this thesis deals with a few non-linear effects in the wealth of resonance phenomena which occur in a one-dimensional homogeneous collisionless plasma, governed by the simple-looking equations of Vlasov and Poisson.

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ON THE MOTION OF CHARGED PARTICLES IN A SLIGHTLY DAMPED SINUSOIDAL POTENTIAL WAVE

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Synopsis

An adiabatic invariant is derived for the particle's nearly periodic motion. It is shown that this invariant changes little, even during transition between the free and the trapped state for nearly all particles. The time averaged energy and momentum of a particle are calculated. Free particles are accelerated in a damped wave. The so-called nonlinear Landau damping is discussed finally.

Introduction

The essential mathematical problem of this paper is the solution of the equation $d^2x/dt^2 + (1 - \varepsilon t) \sin x = 0$ for small ε. For $\varepsilon = 0$ the exact solution is known in terms of elliptic functions. For $\varepsilon \neq 0$ an approximate first integral (constant of motion) is given here.

The problem arises in plasma physics where the interaction between charged particles and waves is investigated. The wave energy can be converted slowly into kinetic energy. This process is known as collisionless damping or (nonlinear) Landau damping. It is of fundamental importance in physics as this damping process provides an example of the development of initially well organized motion (collective oscillation) into random motion (heat), without the aid of collisions but through Coulomb forces only. It is found here, however, that a simply damped travelling wave would set the plasma into motion, thus violating the law of conservation of momentum. Therefore the mechanics of nonlinear Landau damping must be different from the linear case in some important respect.

1. *Adiabatic invariants and rings*

1.1. Consider a one dimensional electric potential wave *

$$V(x, t) \equiv A(t)(1 - \cos kx) \tag{1}$$

* For the list of symbols, see p. 196.

and particles of charge $e > 0$ and mass m moving in it. Their equations of motion are:

$$\mathrm{d}x/\mathrm{d}t = v, \qquad m\,\mathrm{d}v/\mathrm{d}t = -e\partial V/\partial x \tag{2}$$

and their energy

$$E \equiv \tfrac{1}{2}mv^2 + eV. \tag{3}$$

We suppose that A changes slowly in time:

$$\varepsilon \equiv (|\dot{A}|/A)(m/ek^2A)^{\frac{1}{2}} \ll 1. \tag{4}$$

This means that the wave amplitude A changes little during a period of a particle oscillating deep in a wave trough.

Particles with energy $E < 2eA$ are called *trapped*, those with $E > 2eA$ we call *free*. A is supposed to be a monotonic function of time. Then the energy of a particle is also a monotonic function of time since $\mathrm{d}E/\mathrm{d}t = e\partial V/\partial t$. A particle can make at most once the transition between the free and the trapped state, in view of $|\dot{E}| \leq 2e\,|\dot{A}|$.

In phase space, the xv plane, the phase points of trapped and free particles are separated by the *separatrix* $E = 2eA$, that is

$$v = \pm 2(eA/m)^{\frac{1}{2}} \cos \tfrac{1}{2}kx. \tag{5}$$

The curves of constant energy $E = C$ constitute at any moment sets of nested closed curves inside the separatrix, and wavy curves outside, which become flatter as we go further away from the x axis. At every moment a phase point moves in the direction of the curve of constant energy through that point, since the velocity of a phase point in phase space has components $(v, \mathrm{d}v/\mathrm{d}t)$, and this vector is perpendicular to the instantaneous gradient of the energy $(\partial E/\partial x, \partial E/\partial v)$, according to eqs. (2) and (3). The curves of constant energy thus are the *streamlines* of the flow of phase points. Since the flow is not steady, however, the phase points cross the streamlines. The flux of phase points through a streamline element depends on the density of phase points (the distribution function f) and on the normal velocity of the line element, but it does not depend on the velocity of the phase points.

1.2. Now consider a *loop* of phase points, i.e. a set of phase points composing a curve in phase space at some instant. The motion of the loop is determined by the motion of its (phase) points. In this section we take a simple closed loop well inside the separatrix. Liouville's theorem tells us that its area remains constant during the motion. In general the loop will deform in a quite involved way. Outer parts of the loop, corresponding to more energetic particles, have a greater revolution time than inner parts. (In section 2 the revolution time for $A =$ const. is calculated; it is logarithmically singular on the separatrix (eq. 29)). The motion of the loop is more complicated than the evolution of a streamline $E = C$. A loop which coincides at some instant with a curve of constant energy will not continue to do so because $\mathrm{d}E/\mathrm{d}t$ depends on the phase x. There exist, however, loops, called *rings* in Kruskal's[1]) terminology, whose motion is as simple and slow as the motion of streamlines, with which they nearly (but not quite) coincide.

To show this we shall write down (following Lenard[2])) in section 2 a function $I(x, v, t)$, the so-called *adiabatic invariant*, as a series

$$I \equiv I_0 + I_1 + I_2 + \dots, \tag{6}$$

essentially an expansion in ε. The first term $I_0(x, v, t)$ is defined as half the area of the curve of constant energy through the point (x, v) at time t. I_0 is therefore a function of E and t only, A being given. Then I_1, I_2, ... can be chosen in such a way that I is a constant of the motion to all orders in ε, i.e. for all n we have

$$(\mathrm{d}/\mathrm{d}t)(I_0 + I_1 + \dots + I_n) = \mathcal{O}(\varepsilon^{n+1}), \tag{7}$$

provided that A is a sufficiently smooth function of t. The expression for I_n contains derivatives of A up to the nth order. The curves I = constant are closed curves, the rings.

We now prove that I is half the area of the curve I = constant (i.e. half the area of the curve $I = C$ is C) to all orders of ε. If A is constant, then $I = I_0$, and the statement holds in view of the definition of I_0. If $\dot{A} \neq 0$, we imagine that A is constant for $t < t_0$ and then changes slowly and smoothly until t_1, at which moment $A, \dot{A}, \ddot{A}, \dots$ have given values. (We remark that an infinitely differentiable function of a real variable such as $A(t)$ is not uniquely determined by its Taylor series around a point. Counter example: $\exp(-1/t^2)$ around $t = 0$). During this process a loop $I = I_0 = C$ at $t = t_0$ passes to a loop at $t = t_1$, keeping its area constant at the value $2C$ by Liouville's theorem. Every point of the loop keeps its value of I constant to all orders of ε, so that the equation of the loop at $t = t_1$ is given, to the same approximation, by $I = C$. This completes the proof.

The notion of rings simplifies the description of a particle's motion, essentially by distinguishing clearly between the two time scales of the motion. *A phase point travels along its ring while the latter deforms on a much larger time scale.*

1.3. We now extend this idea to the free particles. Let us map the part of phase space above the separatrix on a qp plane by the following transformation:

$$q = -2(v/k)^{\frac{1}{2}} \cos \tfrac{1}{2}kx, \qquad p = 2(v/k)^{\frac{1}{2}} \sin \tfrac{1}{2}kx \tag{8}$$

characterized by the facts that it is canonical, which implies that its Jacobian is unity, and that it maps curves in xv space, which are periodic in x with period $\lambda = 2\pi/k$ (e.g. the curves of constant energy), on closed curves in qp-space. Two periods in xv space correspond to one revolution in qp space.

The area under one period of such a curve in xv space equals half the area enclosed by the image curve in qp space: $\int_0^\lambda v\,dx = \iint dx\,dv = \iint dq\,dp$. The equations of motion (2) can be written in Hamiltonian form; since the transformation (8) is canonical, the (q, p) image points obey Hamiltonian equations of motion too. It follows that Liouville's theorem is valid for loops in qp space, and therefore the area under a "loop" in xv space is constant, if we define a loop as a piece of a curve starting at any point (x, v) and ending at $(x + \lambda, v)$. With this modification all statements about loops and rings of trapped particles, discussed in the previous section, can be transferred to free particles with positive velocity, and likewise to free particles moving in the opposite direction.

1.4. We now turn our attention to the transition of particles from free to trapped or *v.v.* Since the time of revolution of a phase point approaching the separatrix becomes larger and larger, it might be expected that the above theory breaks down for transition particles, because the theory involves different time scales for the revolution on the ring and the "drift" of the ring. Indeed, the terms I_1, I_2, ... in the series (6) for the adiabatic invariant are singular on the separatrix. There are particles which barely reach a crest of the wave and others which fall back just before the crest. By continuity a possible motion should be one that just ends on a crest, which corresponds to a phase trajectory ending in a *saddle point* $((2l + 1)\,\lambda/2, 0)$ in xv space (l integer). For these motions, ending in or skimming a saddle point, I changes considerably. However, the rest of this section is to show that these motions are very rare, and that *I changes little for most particles during transition.*

Consider an ensemble of particles with distribution function $f(x, v, t)$, where f is the number of phase points per unit area in xv space. The function f obeys the Boltzmann equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\,\frac{\partial V}{\partial x}\,\frac{\partial f}{\partial v} = 0. \tag{9}$$

Actually, in order to have a physical model which is described exactly by eq. (9) without collision term, we should take an ensemble of identical potential waves with a single particle of the same kind in each, or a fluid of an infinite number of infinitesimal particles with common e/m, such that V accounts for both the external field and the field of the fluid.

Let us assume f = constant at some moment in a neighbourhood of the separatrix. Since a constant satisfies eq. (9), f will remain constant for some time in a neighbourhood of the slowly moving separatrix. The flux of particles across one arc of the separatrix is given, according to our earlier considerations (section 1.1) about fluxes through streamlines, by the rate of change of the area under the separatrix:

$$\Phi = f(\mathrm{d}/\mathrm{d}t) \int_{-\lambda/2}^{\lambda/2} [2e(2A - V)/m]^{\frac{1}{2}} \, \mathrm{d}x = (4f\dot{A}/k)(e/mA)^{\frac{1}{2}}. \tag{10}$$

Now consider a particle with a long transition time, by which we mean that it takes a time of the order of $A/|\dot{A}|$ to move from a wave trough to the next one in time (it may be the same trough in space). Such a particle is bound to spend most of this time on the crest since the passage of the trough never takes that long. The motion near the crest is approximately described by

$$\mathrm{d}^2y/\mathrm{d}t'^2 - (a^2 + b^3t')\, y = 0, \tag{11}$$

where $y \equiv x - (2l + 1)\, \lambda/2$, with l the integer such that $|y| \ll \lambda$, represents the distance to the crest, $t' \equiv t - t_0$, t_0 being the time of reaching either the crest or the turning point just before, and

$$a^2 + b^3t' \equiv (ek^2/m)[A(t_0) + \dot{A}(t_0)\, t']. \tag{12}$$

Eq. (11) can be solved in terms of Airy functions. From further calculation postponed to section 3 it follows that a phase point has to cross the separatrix within a small distance y_1 along the x axis from the saddle point given by

$$y_1 \equiv (1/k)(2a/|b|)^{\frac{1}{2}} \exp(-a^3/2|b|^3) \tag{13}$$

in order to have a long transition time. The flux of particles with a long transition time, i.e. the part $\delta\Phi$ of Φ through the tiny little ends of the separatrix becomes, restricting the integration in eq. (10) to two small intervals of length y_1:

$$\delta\Phi = [(\mathrm{d}/\mathrm{d}t')\, f y_1^2 (a^2 + b^3t')^{\frac{1}{2}}]_{t'=0} = 4f(a^2/k^2) \exp(-a^3/|b|^3) \operatorname{sgn} b. \tag{14}$$

The ratio $\delta\Phi/\Phi$, which may be defined as the chance for a particle to have a long transition time, thus amounts to

$$\delta\Phi/\Phi = (1/\varepsilon) \exp(-1/\varepsilon). \tag{15}$$

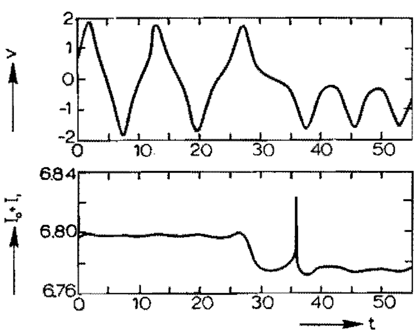

Fig. 1. Velocity and adiabatic invariant of a particle *versus* time. The curves represent a typical numerical solution of the equation $\mathrm{d}^2x/\mathrm{d}t^2 + \exp(-\varepsilon t) \sin x = 0$ with $\varepsilon = 0.01$.

Thus for most particles the wave amplitude A does not change very much during their transition. It follows that their invariant I does not change much either, since before and after the transition I equals half the area of rings which nearly coincide with the separatrix, which has not changed very much in position.

Nevertheless, it appears from numerical calculations (see fig. 1) that the invariants I of the particles of a ring change a little bit during transition, which means that the ring becomes an ordinary loop. New rings are formed at the other side of the separatrix on which the particles of the old one are distributed. The average of the distribution of the relative change of I has the sign of $\dot{A}$ and it is, together with the standard deviation, a small fraction of ε. However, there is no analysis to support these numerical results.

2. *Calculation of the adiabatic invariant*

According to the definition of I_0, just below eq. (6), we have

$$I_0(E, t) \equiv \int |v'| \, \mathrm{d}x, \tag{16}$$

where

$$v'(x, E, t) \equiv (2/m)^{\frac{1}{2}} [E - eV(x, t)]^{\frac{1}{2}} \operatorname{sgn} v \tag{17}$$

and the integration interval is one period λ for free particles ($E > 2eA$), and that part of a period where v' is real for trapped particles ($E < 2eA$). This specification applies to all integration intervals in this section, unless indicated otherwise. To simplify the notation further, all functions in this section are supposed to be written as functions of (x, E, t) instead of (x, v, t), unless indicated otherwise. The total time derivative of I_0 is:

$$\frac{\mathrm{d}I_0}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} + \frac{\mathrm{d}E}{\mathrm{d}t}\frac{\partial}{\partial E}\right) I_0 = \left(\frac{\partial}{\partial t} + e\frac{\partial V}{\partial t}\frac{\partial}{\partial E}\right) I_0 = \frac{\partial}{\partial t} I_0(x, v, t). \tag{18}$$

A useful formula, verified by straightforward calculation, is:

$$\int \frac{\mathrm{d}I_0}{\mathrm{d}t}\frac{\mathrm{d}x}{v'} = \frac{e}{m}\left(\int \frac{\mathrm{d}x}{|v'|}\right)\left(\int \frac{\partial V}{\partial t}\frac{\mathrm{d}x}{v'}\right) - \frac{e}{m}\left(\int \frac{\mathrm{d}x}{v'}\right)\left(\int \frac{\partial V}{\partial t}\frac{\mathrm{d}x}{|v'|}\right) = 0. \tag{19}$$

Note that, though I_0 depends on E and t only, $\mathrm{d}I_0/\mathrm{d}t$ is a function of x also (through V), and that $\mathrm{d}I_0/\mathrm{d}t$ in eq. (19) should not be written as a function of (x, v, t).

For the first correction on I_0 we take the following function I_1 which can be considered as an approximation to $-\int_0^t (\mathrm{d}I_0/\mathrm{d}t)\, \mathrm{d}t$, i.e. minus the true variation of I_0:

$$I_1 \equiv -\int_0^x (\mathrm{d}I_0/\mathrm{d}t)(\mathrm{d}x/v'), \tag{20}$$

where $z \equiv x - l\lambda$, with l the integer such that $|z| \leq \frac{1}{2}\lambda$ represents the distance to the nearest trough. The total time derivative of $I_0 + I_1$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_0 + I_1) = \frac{\mathrm{d}I_0}{\mathrm{d}t} + \left(\frac{\partial}{\partial t} + v'\frac{\partial}{\partial x} + e\frac{\partial V}{\partial t}\frac{\partial}{\partial E}\right)I_1 =$$
$$= \left(\frac{\partial}{\partial t} + e\frac{\partial V}{\partial t}\frac{\partial}{\partial E}\right)I_1 = \frac{\partial}{\partial t}I_1(x, v, t). \tag{21}$$

Eq. (19) implies that I_1 has the same value at both endpoints of z, i.e. $z = \pm\frac{1}{2}\lambda$ for free particles. In order to prove the same periodicity for I_2 below we note that $\int(\mathrm{d}/\mathrm{d}t)(I_0 + I_1)(\mathrm{d}x/v') = 0$ or, in view of eq. (19),

$$\int(\mathrm{d}I_1/\mathrm{d}t)(\mathrm{d}x/v') = 0, \tag{22}$$

because V is an even function of x and $(\mathrm{d}/\mathrm{d}t)(I_0 + I_1)$ is therefore odd. (So $I_1 = 0$ for $z = \pm\frac{1}{2}\lambda$ for free particles).

For the second correction I_2 we take:

$$I_2 \equiv -\int_0^z(\mathrm{d}/\mathrm{d}t)(I_0 + I_1)(\mathrm{d}x/v') + g_2(E, t) \tag{23}$$

in order to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_0 + I_1 + I_2) = \frac{\mathrm{d}}{\mathrm{d}t}(I_0 + I_1) + \left(\frac{\partial}{\partial t} + v'\frac{\partial}{\partial x} + e\frac{\partial V}{\partial t}\frac{\partial}{\partial E}\right)I_2 =$$
$$= \left(\frac{\partial}{\partial t} + e\frac{\partial V}{\partial t}\frac{\partial}{\partial E}\right)I_2 = \frac{\partial}{\partial t}I_2(x, v, t) \tag{24}$$

and also, with the aid of the additional function g_2:

$$\int(\mathrm{d}I_2/\mathrm{d}t)(\mathrm{d}x/v') = 0. \tag{25}$$

This latter relation, required for the periodicity of I_3, does not hold automatically now (i.e. with $g_2 = 0$). It can be considered as a partial differential equation for g_2. If we write g_2 as a function of (I_0, t) instead of (E, t), this equation takes the simple form $\partial g_2/\partial t$ = known function, since the term involving $\partial g_2/\partial I_0$ drops out in view of eq. (19). Thus eq. (25) determines g_2 up to an arbitrary function of I_0. Lenard[2]) shows that this function can be chosen in such a way that I_2 vanishes as soon as A becomes constant during a finite time interval.

The nth correction is determined quite analogous to I_2. It is plausible that the series generated in this way satisfies eq. (7) if $A(t)$ is such that every differentiation with respect to t lowers the order. For a rigorous treatment the reader is referred to Lenard's beautiful article.

I_0 and I_1 can be expressed in terms of elliptic integrals

$$\mathsf{E}(x, r) \equiv \int_0^x(1 - r\sin^2 u)^{\frac{1}{2}}\,\mathrm{d}u, \qquad \mathsf{F}(x, r) \equiv \int_0^x(1 - r\sin^2 u)^{-\frac{1}{2}}\,\mathrm{d}u$$

as follows[3]):

$$
\begin{aligned}
I_0 &= (4/k)(2E/m)^{\frac12}\,\mathsf{E}(\tfrac12\pi,\,1/r) \qquad r > 1 \\
&= (8/k)(eA/m)^{\frac12}\,[\mathsf{E}(\tfrac12\pi,\,r) - (1-r)\,\mathsf{F}(\tfrac12\pi,\,r)] \qquad r < 1;
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
I_1 &= (4\dot{A}/Ak^2)[\mathsf{F}(\tfrac12\pi,\,1/r)\,\mathsf{E}(\tfrac12 kx,\,1/r) - \mathsf{E}(\tfrac12\pi,\,1/r)\,\mathsf{F}(\tfrac12 kx,\,1/r)]\operatorname{sgn} v \\
&\qquad\qquad r > 1 \\
&= (4/Ak\dot{A}^2)[\mathsf{F}(\tfrac12\pi,\,r)\,\mathsf{E}(w,\,r) - \mathsf{E}(\tfrac12\pi,\,r)\,\mathsf{F}(w,\,r)]\operatorname{sgn} v \qquad r < 1
\end{aligned} \tag{27}
$$

with $r = E/2eA$ and $w = \arcsin\,(r^{-\frac12}\sin\tfrac12 kz)$. Note that I_1/I_0 is of order ε.

From eq. (26) we derive the time period T_0 for a particle in a constant wave ($\dot{A} = 0$):

$$
\begin{aligned}
T_0 &\equiv \int (\mathrm{d}x/|v'|) = m\partial I_0/\partial E = \\
&= (2/k)(2m/E)^{\frac12}\,\mathsf{F}(\tfrac12\pi,\,1/r) \qquad r > 1 \\
&= (2/k)(m/eA)^{\frac12}\,\mathsf{F}(\tfrac12\pi,\,r) \qquad r < 1.
\end{aligned} \tag{28}
$$

Near the separatrix we have

$$T_0 = -(1/k)(m/eA)^{\frac12}\ln|r-1| \qquad r \approx 1. \tag{29}$$

3. *Calculation of the motion near a saddle point*

With the new independent variable τ, defined by $b^2\tau \equiv a^2 + b^3t'$, eq. (11) becomes

$$\mathrm{d}^2y/\mathrm{d}\tau^2 - \tau y = 0, \tag{30}$$

which[4]) has the basic solutions $\mathrm{Ai}(\tau)$, $\mathrm{Bi}(\tau)$, whose Wronskian equals $1/\pi$. Thus we find for a particle with velocity $v = v_0$ at $y = 0$, $t = t_0$ ($t' = 0$, $\tau = a^2/b^2 \equiv \tau_0$):

$$
\begin{aligned}
y &= (\pi v_0/b)[\mathrm{Ai}(\tau_0)\,\mathrm{Bi}(\tau) - \mathrm{Bi}(\tau_0)\,\mathrm{Ai}(\tau)], \\
v &= \pi v_0[\mathrm{Ai}(\tau_0)\,\mathrm{Bi}'(\tau) - \mathrm{Bi}(\tau_0)\,\mathrm{Ai}'(\tau)],
\end{aligned} \tag{31}
$$

and for a particle with position $y = y_0$ when $v = 0$, $t = t_0$:

$$
\begin{aligned}
y &= \pi y_0[\mathrm{Bi}'(\tau_0)\,\mathrm{Ai}(\tau) - \mathrm{Ai}'(\tau_0)\,\mathrm{Bi}(\tau)], \\
v &= \pi b y_0[\mathrm{Bi}'(\tau_0)\,\mathrm{Ai}'(\tau) - \mathrm{Ai}'(\tau_0)\,\mathrm{Bi}'(\tau)].
\end{aligned} \tag{32}
$$

The first particle crosses the separatrix (line of constant energy $\tfrac12(\mathrm{d}y/\mathrm{d}\tau)^2 - \tfrac12\tau y^2$ through ($y = 0$, $v = 0$)) given by

$$v = b\tau^{\frac12}y \tag{33}$$

at a time τ which is the root of an equation derived from eqs. (31) and (33) after elimination of y and v:

$$\mathrm{Ai}(\tau_0)[\tau^{\frac12}\mathrm{Bi}(\tau) - \mathrm{Bi}'(\tau)] = \mathrm{Bi}(\tau_0)[\tau^{\frac12}\mathrm{Ai}(\tau) - \mathrm{Ai}'(\tau)]. \tag{34}$$

The particle to which eq. (32) refers crosses another branch of the separatrix, given by

$$v = -b\tau^{\frac{1}{2}}y, \tag{35}$$

at a time τ determined by eq. (36), derived from eqs. (32) and (35):

$$\mathrm{Ai}'(\tau_0)[\tau^{\frac{1}{2}}\mathrm{Bi}(\tau) + \mathrm{Bi}'(\tau)] = \mathrm{Bi}'(\tau_0)[\tau^{\frac{1}{2}}\mathrm{Ai}(\tau) + \mathrm{Ai}'(\tau)]. \tag{36}$$

Since $\tau_0 = \varepsilon^{-\frac{2}{3}} \gg 1$ we use the asymptotic expansions:

$$\begin{aligned}
\mathrm{Ai}(\tau) &= \tfrac{1}{2}\pi^{-\frac{1}{2}}\,\tau^{-\frac{1}{4}}\,\mathrm{e}^{-\xi}(1 - 5/72\xi + \ldots),\\
\mathrm{Ai}'(\tau) &= -\tfrac{1}{2}\pi^{-\frac{1}{2}}\,\tau^{\frac{1}{4}}\,\mathrm{e}^{-\xi}(1 + 7/72\xi - \ldots),\\
\mathrm{Bi}(\tau) &= \pi^{-\frac{1}{2}}\,\tau^{-\frac{1}{4}}\,\mathrm{e}^{\xi}(1 + 5/72\xi + \ldots),\\
\mathrm{Bi}'(\tau) &= \pi^{-\frac{1}{2}}\,\tau^{\frac{1}{4}}\,\mathrm{e}^{\xi}(1 - 7/72\xi - \ldots),
\end{aligned} \tag{37}$$

in which $\xi \equiv \frac{2}{3}\tau^{-\frac{3}{2}}$. Substituting these expressions in eqs. (34) and (36) we find the equations:

$$\exp(2\xi - 2\xi_0) = 12\xi \qquad \text{and} \qquad \exp(2\xi_0 - 2\xi) = 12\xi \tag{38}$$

respectively in lowest approximation, where $\xi_0 \equiv \frac{2}{3}\tau_0^{\frac{3}{2}}$. The solutions are, again to zero order:

$$\xi - \xi_0 = \tfrac{1}{2}\ln 12\xi_0 \qquad \text{and} \qquad \xi - \xi_0 = -\tfrac{1}{2}\ln 12\xi_0 \tag{39}$$

respectively. Substituting this into eqs. (31) and (32) respectively, after expansion, we find for the abscissa of the point of crossing of the separatrix:

$$y = 2^{\frac{1}{2}}v_0a^{\frac{1}{2}}\,|b|^{-\frac{3}{2}}\,\mathrm{sgn}\,b \qquad \text{and} \qquad y = 2^{\frac{1}{2}}y_0a^{\frac{3}{2}}\,|b|^{-\frac{3}{2}} \tag{40}$$

respectively.

Now we have to find values for v_0 and y_0 corresponding to particles with a long transition time. This means that $|\xi - \xi_0| \approx |at'| \gg 1$, but still $|b^3t'| \ll a^2$, so that eqs. (31) and (32) become

$$y = \tfrac{1}{2}(v_0/a)\exp(a\,|t'|)\,\mathrm{sgn}\,t' \qquad \text{and} \qquad y = \tfrac{1}{2}y_0\exp(a\,|t'|) \tag{41}$$

respectively. Putting $y = \pm 1/k$ and $2t' = \pm A/\dot{A} = \pm a^2/b^3$ we find

$$v_0 = \pm\left(\frac{2a}{k}\right)\exp\left(\frac{-a^3}{2\,|b|^3}\right) \qquad \text{and} \qquad y_0 = \pm\left(\frac{2}{k}\right)\exp\left(\frac{-a^3}{2\,|b|^3}\right) \tag{42}$$

respectively for the required limit values. Substitution of these values into eq. (40) yields $|y| = y_1$, where y_1 is given by eq. (13).

4. *Momentum and kinetic energy*

As in section 1.4 we consider an ensemble of particles, with distribution function f, moving in a potential V. Let us assume that at some instant f

is a function of the adiabatic invariant I only:

$$f(x, v, t) = F(u), \qquad u \equiv I/\lambda. \tag{43}$$

To the approximation that I is a constant of the motion this function $F(u)$ satisfies the Boltzmann equation (9), i.e. $\mathrm{d}F/\mathrm{d}t = 0$. After some time there may be a small deviation from eq. (43) only in the part of the phase plane swept out by the separatrix during this time.

The quantity u has a simple physical meaning for a free particle. It is the modulus of the space averaged velocity of its ring. We do not want f to be necessarily equal on free rings with common value of u but at opposite sides of the x axis. So instead of $F(u)$ we should write $F(u, \operatorname{sgn} v)$ for the free particles in eq. (43) but the second "argument" is suppressed in what follows. In integrations with respect to u it will be understood that the integral denotes a sum of three integrals, one for each direction of the velocity for the free particles, and one for the trapped particles.

The space-averaged number density

$$N \equiv \lambda^{-1} \iint f \,\mathrm{d}x\, \mathrm{d}v = \lambda^{-1} \int F(u)\, \mathrm{d}I = \int F(u)\, \mathrm{d}u \tag{44}$$

is conserved; this also follows from eq. (9) after an integration over x and v. In this section, unless stated otherwise, integrations run over the full range of the variable, except for x which occurs in double integrals only; their region of integration is one x-period of phase space. The middle equality in eq. (44) follows from the significance of I as an area.

The space averaged momentum density is

$$P \equiv \lambda^{-1} \iint mvf \,\mathrm{d}x\, \mathrm{d}v = \lambda^{-1} \iint F(u)\, \mathrm{d}x\, \mathrm{d}E. \tag{45}$$

We put $u = u_0 + u_1 + \ldots$ with $u_n \equiv I_n/\lambda$ and expand around $u = u_0$:

$$P = \lambda^{-1} \iint F(u_0)\, \mathrm{d}x\, \mathrm{d}E + \lambda^{-1} \iint F'(u_0)\, u_1\, \mathrm{d}x\, \mathrm{d}E + \ldots.$$

Taking into account that u_0 is a function of E (and t) only and that the integration with respect to E involves two integrations (for the two directions of v) which cancel for the trapped particles in the first term, we find that

$$P = \int F(u_0)\, \mathrm{d}E + \ldots = \int F(u_0)(m\lambda/T_0)\, \mathrm{d}u_0 + \ldots. \tag{46}$$

For the last equality eq. (28) has been used. The remaining integration, covering the free particles only, has a lower bound $E = 2eA$ or $u_0 = (4/\pi)(eA/m)^{\frac{1}{2}}$ according to eq. (26). P is the difference of the contributions from either direction of velocity of the free particles.

The space averaged kinetic energy density is

$$K \equiv \lambda^{-1} \iint \tfrac{1}{2}mv^2 f \,\mathrm{d}x\, \mathrm{d}v = \lambda^{-1} \iint \tfrac{1}{2}\, |v'|\, F(u)\, \mathrm{d}x\, \mathrm{d}E =$$
$$= \int \tfrac{1}{2}u_0 F(u_0)\, \mathrm{d}E + \ldots = \int F(u_0)\, (\tfrac{1}{2}mu_0\lambda/T_0)\, \mathrm{d}u_0 + \ldots, \tag{47}$$

where again u is approximated by u_0 to find an adiabatic approximation ($\varepsilon \to 0$), and definition (16) is used.

When we think every particle of an arbitrary ensemble (where f is a function of u *and* the phase along the ring labelled by u) smeared out over its ring, leaving us with an ensemble of rings, then $F(u)$ is the *ring density* (the number of rings per unit of u and per wavelength), $\frac{1}{2}m\lambda u_0/T_0 = \frac{1}{2}mI_0/T_0$ is the kinetic energy of a ring in the adiabatic limit, while eq. (46) indicates that the momentum equals $m\lambda/T_0$ for a free ring and vanishes for a trapped ring in the same limit. The kinetic energy and momentum of actual particles oscillate around these ring averages.

5. *On the Landau problem in plasma physics*

5.1. Up to now we considered the motion of particles in a given field. This section deals with a system of particles moving in their own field. Therefore we look at the Vlasov equation (9) together with Poisson's equation in one dimension (MKS system):

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial V}{\partial x}\frac{\partial f}{\partial v} = 0, \qquad \frac{\partial^2 V}{\partial x^2} = \frac{e}{\varepsilon_0}\left(N - \int f\,\mathrm{d}v\right). \tag{48}$$

The same convention about unindicated integration ranges is adopted as in the preceding section (under eq. 44). The set of eqs. (48) describes the one-dimensional motion of charged particles against a neutralizing (smoothed) background of fixed particles with charge density $-Ne$. We take N constant.

In 1946 Landau[5]) solved this set of equations in the linearized approximation

$$f = f_0(v) + f_1(x, v, t), \qquad |f_1| \ll f_0, \qquad \int f_0\,\mathrm{d}v = N, \tag{49}$$

which reduces the set to

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m}\frac{\partial V}{\partial x}\frac{\mathrm{d}f_0}{\mathrm{d}v} = 0, \qquad \frac{\partial^2 V}{\partial x^2} = -\frac{e}{\varepsilon_0}\int f_1\mathrm{d}v. \tag{50}$$

He found for a Maxwellian f_0 that a smooth initial disturbance $f_1(x, v, 0)$ causes a field V whose spatial Fourier components damp out rapidly (in a time of order $1/kv_{\mathrm{th}}$) except for the components with wavelength large compared to the Debye length ($k \ll \omega_{\mathrm{p}}/v_{\mathrm{th}}$). The latter components set up travelling waves with frequency near to the plasma frequency ω_{p} which are damped as $\exp(-\gamma t)$ with very small but finite γ. Here

$$\omega_{\mathrm{p}}^2 = Ne^2/\varepsilon_0 m, \qquad v_{\mathrm{th}}^2 = N^{-1}\int v^2 f_0\,\mathrm{d}v, \qquad \gamma = \tfrac{1}{2}\pi(\omega_{\mathrm{p}}^3/k^2N)\,f_0'(-\omega_{\mathrm{p}}/k). \tag{51}$$

The requirement that f_1 is a smooth function of v is crucial for the damping since the equations permit an arbitrary V, i.e. eqs. (50) determine a relation only between $V(x, t)$ and $f_1(x, v, 0)$. However, functions V not showing the Landau damping correspond to sharply peaked functions $f(x, v, 0)$, to very special initial conditions[6]).

The not linearized eqs. (48) have not yet been solved. Again, the general solution f involves an arbitrary function of two variables, $f(x, v, 0)$ or $V(x, t)$, and it is the aim to find the natural asymptotic behaviour of V which presumably corresponds to smooth, unspecific initial conditions $f(x, v, 0)$.

The set (48) was solved by Bernstein, Greene and Kruskal[7]) for the special case that the initial distribution $f(x, v, 0)$ is a function of the particle energy E only (apart from sgn v for free particles; cf. the discussion of $f = F(u, \operatorname{sgn} v)$ in the preceding section) in some inertial frame of reference. Eqs. (48) are invariant with respect to Gallilean transformations. In the frame of reference mentioned the solution is time independent then. Indeed, $f = f(E)$ is the solution of Vlasov's equation (9) for $\partial f/\partial t = 0$.

5.2. It is tempting now to look for slightly damped finite amplitude waves as solutions of eqs. (48) in view of the theory in the preceding sections, in order to extend Landau damping to the nonlinear regime, which has the attention of many authors nowadays[8]). However, it will be shown now that such a configuration is impossible because of violation of the conservation of momentum. *A slightly damped travelling wave* (with constant wave length and phase velocity) *cannot correspond to a simple initial distribution function similar to that of the linearized case.* This result is compatible with the linearized case since the change in momentum turns out to be of second order in the wave amplitude (eq. 55).

We look for solutions periodic in space, with wavelength $\lambda = 2\pi/k$. Then eqs. (48) imply conservation of momentum:

$$\lambda \frac{\mathrm{d}P}{\mathrm{d}t} = \iint mv \frac{\partial f}{\partial t} \,\mathrm{d}x\,\mathrm{d}v = \iint ve \frac{\partial V}{\partial x} \frac{\partial f}{\partial v} \,\mathrm{d}x\,\mathrm{d}v = -\iint e \frac{\partial V}{\partial x} f \,\mathrm{d}x\,\mathrm{d}v =$$

$$= \int \frac{\partial V}{\partial x} \left\{ \varepsilon_0 \frac{\partial^2 V}{\partial x^2} - eN \right\} \mathrm{d}x = \int \frac{\partial}{\partial x} \left[\tfrac{1}{2}\varepsilon_0 \left(\frac{\partial V}{\partial x} \right)^2 - eNV \right] \mathrm{d}x = 0. \quad (52)$$

In accordance with Landau's long wavelength case we consider a nearly Maxwellian distribution of particles (in the fixed neutralizing background) and a potential wave travelling with phase velocity v_{ph} much larger than the thermal velocity v_{th}. We suppose that there exists an inertial frame, the wave frame, in which the wave is static apart from a slow change in amplitude and (possibly) wave form. It will be shown in section 5.3 that it is consistent to take the wave form nearly sinusoidal. Then all of the theory in the preceding sections is valid, apart from correction factors, near to unity, due to the deviation from a sinusoid. The only qualitative change is the need for an additive function $g_1(E, t)$ in eq. (20) to satisfy eq. (22) if V is not exactly even any more; cf. eq. (23) and its discussion.

In the wave frame moving with $v_{\mathrm{ph}} \gg v_{\mathrm{th}}$ all but an exponentially small part of the particles are free. Therefore the ring density is a nearly Maxwellian

$F(u)$ with the mean shifted to $u = v_{\mathrm{ph}}$. The coarse time behaviour of the momentum P is already determined by the ring density; the phase dependence of f along the rings can only give an extra ripple on $P(t)$. We calculate P according to eq. (46) in the further approximation that $E \gg eV$, valid for most particles. Expanding the square root in eq. (17) for small eV/E and substituting into eq. (16) we find

$$u_0 = (2E/m)^{\frac{1}{2}} (1 - e^2\overline{V^2}/2E^2 \ldots), \tag{53}$$

where the bar denotes averaging over λ and the zero point of V is chosen so as to have $\overline{V} = 0$ (deviating from eq. (1)). Inverting the series (53) yields

$$E = \tfrac{1}{2}mu_0^2 + e^2\overline{V^2}/2mu_0^2 + \ldots. \tag{54}$$

Thus

$$P \approx \int F(u_0)\,\mathrm{d}E = m \int uF(u)\,\mathrm{d}u - (e^2/m)\,\overline{V^2} \int u^{-3}F(u)\,\mathrm{d}u \ldots \tag{55}$$

clearly increases as $\overline{V^2}$ decreases. The last integral requires a positive lower bound on which it is only weakly dependent in a wide range of values. Thus the bulk of the particles accelerates in a damped wave. This is also easily derived from a differentiation of eq. (54) with respect to u_0 to find T_0 according to eq. (28):

$$T_0 = m\lambda\partial u_0/\partial E = \lambda/u_0 + \lambda e^2\overline{V^2}/m^2u_0^5 + \ldots. \tag{56}$$

The momentum gain of the bulk cannot be compensated by the trapped or nearly trapped particles, which are exponentially few in number in a Maxwellian plasma for $v_{\mathrm{ph}} \gg v_{\mathrm{th}}$.

5.3. Let us insert $f = F(u_0)$ into Poisson's equation (48). This covers also a BGK solution[7]) since u_0 is a function of E only for $\partial/\partial t = 0$. We expand again with respect to eV/E, which is small for most particles.

$$\partial^2V/\partial x^2 = (e/\varepsilon_0)[N - \int F(u_0)(mv')^{-1}\,\mathrm{d}E] =$$

$$= (e/\varepsilon_0)[N - \int F(u_0)(2m)^{-\frac{1}{2}}(E^{-\frac{1}{2}} + \tfrac{1}{2}eVE^{-\frac{3}{2}} + \ldots)\,\mathrm{d}E].$$

Systematical use of eq. (54) to change to u_0 as integration variable yields after some calculation

$$\partial^2V/\partial x^2 + (e^2/\varepsilon_0 m)\,V \int u^{-2}\,F(u)\,\mathrm{d}u + \ldots = 0. \tag{57}$$

Here we have used the neutrality condition $\int F(u)\,\mathrm{d}u = N$; cf. eq. (44). The integral $\int u^{-2}\,F(u)\,\mathrm{d}u$ needs a lower bound as in eq. (55); it is approximately equal to N/v_{ph}^2. Thus eq. (57) shows that Poisson's equation admits a nearly sinusoidal wave form with a wavelength given by $k = \omega_{\mathrm{p}}/v_{\mathrm{ph}}$.

Finally we show that the decrease of the potential energy of the wave is just equal to the increase of kinetic energy of the bulk of the particles in first approximation. The wave energy density becomes, applying eq. (57),

$$W \equiv \frac{\varepsilon_0}{2\lambda}\int\left(\frac{\partial V}{\partial x}\right)^2 \mathrm{d}x = -\frac{\varepsilon_0}{2\lambda}\int V\frac{\partial^2 V}{\partial x^2}\,\mathrm{d}x = \frac{e^2}{2m}\,\overline{V^2}\int\frac{F(u)}{u^2}\,\mathrm{d}u + \dots \tag{58}$$

and the kinetic energy density is, according to eq. (47) and (54),

$$K \approx \int \tfrac{1}{2}u_0 F(u_0)\,\mathrm{d}E = \tfrac{1}{2}m\int u^2 F(u)\,\mathrm{d}u - (e^2/2m)\,\overline{V^2}\int u^{-2}F(u)\,\mathrm{d}u \dots . \tag{59}$$

The total energy $W + K$ is conserved in this approximation.

The conclusion is that a slightly damped, fast travelling wave could exist in a Maxwellian plasma as far as the wave form and the energy balance are concerned, but non-conservation of momentum renders this configuration impossible. Numerical work[8]) on the nonlinear Vlasov equation is concerned with a standing wave rather than a travelling one, and then momentum can be conserved by symmetry. Another configuration in which momentum might be conserved is an accelerated travelling wave. However, no inertial frame of reference exists in which one of the latter waves is almost static, a prerequisite in the theory of adiabatic invariants.

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LIST OF SYMBOLS

The numbers between parentheses refer to the equation in or near to which the symbol is introduced.

a (12)	n positive integer	v' (17)
b (12)	p, q (8)	v_0 (31)
e (2)	r (26)	v_{ph} (52)
f (9)	t (1)	v_{th} (51)
f_0, f_1 (49)	t' (11)	w (27)
g_1 (52)	t_0 (12)	x (1)
g_2 (23)	u (43)	y (11)
k (1)	$u_0, u_1 \ldots$ (45)	y_0 (32)
l integer	v (2)	y_1 (13)
m (2)		z (20)

A (1)	K (47)	γ (51)	τ (30)
E (3)	N (44)	ε (4)	τ_0 (31)
E, F (26)	P (45)	λ (8)	ω_p (51)
F (43)	T_0 (28)	ξ (37)	Φ (10)
$I, I_0, I_1, \ldots$ (6)	V (1)	ξ_0 (38)	$\delta\Phi$ (14)
	W (58)		

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ON THE MOTION OF CHARGED PARTICLES IN A SELF-CONSISTENT STANDING POTENTIAL WAVE

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Synopsis

This paper deals with an asymptotic solution of the equations of Vlasov and Poisson describing one-dimensional plasma motion. The self-consistent, non-external, potential is in lowest order a standing wave. The solution involves expansions in the wave amplitude. Higher order coefficients of the potential series are expressed in terms of the distribution function, which is a function of one constant of motion only, obtained by integrating once the equation of motion. The omission of a second integration, and the associated constant of motion, amounts to an assumption of random phases. The wave is shown to be stable.

1. *Basic equations.* We consider the one-dimensional motion of electrons (charge e, mass m) in a neutralizing background of fixed ions with constant charge density $-Ne$. The electron distribution function $f(x, v, t)$ is governed by the equations of Vlasov and Poisson

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial V}{\partial x}\frac{\partial f}{\partial v} = 0, \qquad \frac{\partial^2 V}{\partial x^2} = \frac{e}{\varepsilon_0}(N - \int f\,\mathrm{d}v), \tag{1}$$

in which we assume f and the potential $V(x, t)$ with period $\lambda = 2\pi/k$ in x. Further we assume, averaged over a wavelength, charge neutrality and no current:

$$\int_0^\lambda \mathrm{d}x \int \mathrm{d}v\, f = N\lambda, \qquad \int_0^\lambda \mathrm{d}x \int \mathrm{d}v\, vf = 0. \tag{2}$$

Eqs. (1) have been solved exactly in a number of limit cases.

a) $\partial/\partial t = 0$. Then $f = F(\frac{1}{2}mv^2 + eV)$ and Poisson's equation amounts to an integral relation between V and F, which has been solved by Bernstein, Greene and Kruskal (BGK wave)[1]).

b) $\partial/\partial x = 0$, $k = 0$. Before going to this limit it is appropriate to replace Poisson's equation by Maxwell's equation

$$\partial^2 V/\partial x \partial t = (e/\varepsilon_0) \int vf\,\mathrm{d}v \tag{3}$$

(which can be derived from eqs. (1) and (2)) and to put $\partial V/\partial x = -E$. Then we have

$$\partial f/\partial t + (e/m)\, E\, \partial f/\partial v = 0, \qquad \mathrm{d}E/\mathrm{d}t + (e/\varepsilon_0) \int vf\, \mathrm{d}v = 0 \tag{4}$$

and the solution

$$E = E_0 \sin(\omega_\mathrm{p} t + \varphi), \qquad f = F\{v + (eE_0/m\omega_\mathrm{p}) \cos(\omega_\mathrm{p} t + \varphi)\} \tag{5}$$

with

$$\int F(w)\, \mathrm{d}w = N, \qquad \int wF(w)\, \mathrm{d}w = 0, \qquad \omega_\mathrm{p}^2 \equiv Ne^2/\varepsilon_0 m. \tag{6}$$

c) $f = n(x, t)\, \delta(v - \bar{v}(x, t))$, the cold-plasma limit. Substitution in (1) yields the set of eqs.

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\bar{v}) = 0, \quad \frac{\partial \bar{v}}{\partial t} + \bar{v}\frac{\partial \bar{v}}{\partial x} + \frac{e}{m}\frac{\partial V}{\partial x} = 0, \quad \frac{\partial^2 V}{\partial x^2} = \frac{e}{\varepsilon_0}(N - n), \tag{7}$$

solved by Kalman[2]) who introduced a Lagrangian variable.

d) $V = 0$, free streaming: $f = F(x - vt, v)$, $\int F\, \mathrm{d}v = N$. For very small but finite V the equations can be linearized according to $f = F(v) + f_1$. The resulting set of equations

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} - \frac{e}{m}\frac{\partial V}{\partial x} F'(v) = 0, \qquad \frac{\partial^2 V}{\partial x^2} + \frac{e}{\varepsilon_0}\int f_1\, \mathrm{d}v = 0 \tag{8}$$

was solved by Landau[3]) who applied Fourier and Laplace transforms with respect to x and t respectively. This solution breaks down, due to the linearization, for $t^2 = \mathcal{O}(m/ek^2V)$. See, *e.g.*, O'Neil[4]).

The general solution of eqs. (1) contains an arbitrary function of two variables, *e.g.*, the initial condition $f(x, v, 0)$. Alternatively, the potential $V(x, t)$ could be prescribed. In a given field the electron position and velocity $x(t; y, u)$, $v(t; y, u)$ are determined where y and u are two integration constants of the equation of motion

$$m\ddot{x} + e\partial V/\partial x = 0. \tag{9}$$

Then the solution of Vlasov's equation is $f = F(y, u)$, a positive function of y and u, expressed as functions of x, v, t. Finally F is determined by Poisson's equation for given V. If the constants of motion are the initial values $y = x(t = 0)$, $u = v(t = 0)$, then we have $f(x, v, 0) = F(x, v)$.

Though the general solution of eqs. (1) is unknown, some properties of it have become clear from physical considerations[5]).

1) Density fluctuations on a scale smaller than the Debye length

$$\lambda_\mathrm{D} \equiv v_\mathrm{th}/\omega_\mathrm{p}, \qquad v_\mathrm{th}^2 \equiv N^{-1} \int v^2 f\, \mathrm{d}v \tag{10}$$

die out rapidly, in a time of the order of ω_p^{-1}.

2) Density disturbances on a larger scale oscillate with nearly the plasma frequency ω_p.

3) Energy exchange of this oscillation and the heat motion occurs mainly for resonant electrons (*i.e.* those of which the thermal speed is near to the phase velocity of a wave which is a component of the total V). The exchange time is of the order of $|m/ek^2A|^{\frac{1}{2}}$, the oscillation period of an electron in a trough of the potential wave with amplitude A.

It appears that not all functions $V(x, t)$, mathematically permitted, are physically significant. This is (in the linearized case d) and presumably in general) because $V(x, t)$ which are not associated with the properties 1), 2), 3) correspond to impossible or improbable initial conditions $f(x, v, 0)$.

The solution presented in the next sections involves an expansion with respect to the wave amplitude A. It is plausible that the expansion can be continued to all orders in principle. The solution is physically significant for values of A not so small that the linearized theory d) would hold, and not so large that most electrons would be trapped in wave troughs. It is a stationary asymptotic solution, conceivable as a superposition of two similar BGK waves moving in opposite directions and giving rise, as a consequence of the nonlinearity, to a series of higher order waves.

2. *Method of solution.* Our method to find a solution of eqs. (1) is to take for V the series *

$$\begin{aligned} -V = {} & A_1 \cos(kx - \omega t) + A_1 \cos(kx + \omega t) + \\ & + A_2 \cos(2kx - 2\omega t) + A_2 \cos(2kx + 2\omega t) + C_2 \cos 2kx \ldots, \end{aligned} \tag{11}$$

whose leading terms constitute a standing wave $2A_1 \cos kx \cos \omega t$, $A_1 > 0$, and for v the series

$$\begin{aligned} v = {} & u + a_1 \cos(kx - \omega t) + b_1 \cos(kx + \omega t) + \\ & + a_2 \cos(2kx - 2\omega t) + b_2 \cos(2kx + 2\omega t) + c_2 \cos 2kx + \ldots \end{aligned} \tag{12}$$

to satisfy the equation of motion (9). The smallness parameter in these series is A_1 and the index of the coefficients A_2, a_1, etc. indicates the order. The n-th order terms contain the n-th order sum or difference "frequencies"

$$\cos\{p(kx - \omega t) \pm q(kx + \omega t)\} \quad \text{with} \quad p + q = n, \quad p = 0, 1, \ldots, n. \tag{13}$$

Terms independent of x are omitted as only $\partial V/\partial x$ is used. For additional remarks see section 3.

Differentiation of v yields

$$\begin{aligned} \ddot{x} = \dot{v} = {} & \partial v/\partial t + v \partial v/\partial x = \\ = {} & -a_1(ku - \omega) \sin(kx - \omega t) - b_1(ku + \omega) \sin(kx + \omega t) - \\ & - \{\tfrac{1}{2}ka_1^2 + 2a_2(ku - \omega)\} \sin(2kx - 2\omega t) - \\ & - \{\tfrac{1}{2}kb_1^2 + 2b_2(ku + \omega)\} \sin(2kx + 2\omega t) - \\ & - (ka_1b_1 + 2c_2ku) \sin 2kx - \ldots, \end{aligned} \tag{14}$$

* For the list of symbols see p. 240.

substitution of which into the equation of motion (9) gives the relations between the coefficients up to second order, correct in the same order:

$$a_1 = \frac{ekA_1}{m(ku-\omega)}, \qquad b_1 = \frac{ekA_1}{m(ku+\omega)};$$

$$a_2 = \frac{ekA_2}{m(ku-\omega)} - \frac{e^2k^3A_1^2}{4m^2(ku-\omega)^3}, \quad b_2 = \frac{ekA_2}{m(ku+\omega)} - \frac{e^2k^3A_1^2}{4m^2(ku+\omega)^3},$$

$$c_2 = \frac{eC_2}{mu} - \frac{e^2k^3A_1^2}{4m^2\omega^2}\left(\frac{-2}{ku} + \frac{1}{ku-\omega} + \frac{1}{ku+\omega}\right). \tag{15}$$

It appears that the expansion for v diverges in a series of "resonant regions" at $ku/\omega = \pm 1, 0, \ldots$, to be discussed in the next sections. In section 4 the function $v(x, t; u)$ is found to be continuous in u.

To round off the calculation we assume that the non-resonant electrons, characterized by a constant of motion u for which (12) converges, have a distribution f which is a function of u only, $u(x, v, t)$ being given by eq. (12). The omission of the other constant of motion implies a random phase (see section 3). Further, we assume (see section 4) that the distribution function is constant in the vicinity of each individual resonance. In any case f defined by

$$f = F(u), \qquad \text{constant near} \qquad u = \pm\omega/k,\, 0, \ldots, \tag{16}$$

satisfies Vlasov's equation.

Substitution of $F(u)$ into Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{e}{\varepsilon_0}\left\{N - \int F(u)\,\mathrm{d}v\right\} = \frac{e}{\varepsilon_0}\left\{N + \int v\,\mathrm{d}F(u)\right\}. \tag{17}$$

provides us, using eqs. (11), (12) and (15), with *the conditions for self-consistency of the solution*:

$$N = \int F\,\mathrm{d}u, \quad k = \frac{e^2}{\varepsilon_0 m}\int \frac{\mathrm{d}F}{ku \pm \omega}, \qquad A_2 = -\frac{e^3A_1^2k}{12\varepsilon_0 m^2}\int \frac{\mathrm{d}F}{(ku \pm \omega)^3},$$

$$C_2 = -\frac{ek^2A_1^2}{2m\omega^2}\left(k^2 - \frac{e^2}{\varepsilon_0 m}\int \frac{\mathrm{d}F}{u}\right)\left(4k^2 - \frac{e^2}{\varepsilon_0 m}\int\frac{\mathrm{d}F}{u}\right)^{-1}. \tag{18}$$

The integrals are well defined since $\mathrm{d}F$ vanishes in the resonant points. To evaluate them we take F Maxwellian, apart from a flattening in the resonant regions, with r.m.s. value of u well below ω/k:

$$\eta \equiv \frac{k^2}{N\omega^2}\int u^2F\,\mathrm{d}u \ll 1, \qquad F(u) \approx \frac{kN}{\omega(2\pi\eta)^{\frac{1}{2}}}\exp\frac{-k^2u^2}{2\eta\omega^2}. \tag{19}$$

Then the second relation of (18), the well known dispersion relation for linear waves (section 1, case d)), reads (expanding the denominator for small ku/ω; for the ε^2 term see the appendix)

$$\omega^2 = \omega_p^2\{1 + 3\eta(1 + \tfrac{57}{8}\varepsilon^2)\} \ldots \tag{20}$$

For A_2 and C_2 we find

$$A_2 = \tfrac{1}{4}\varepsilon A_1(1 + 10\eta) \ldots, \qquad C_2 = \tfrac{1}{2}\varepsilon A_1(1 + \eta)\,(1 + 4\eta)^{-1} \ldots, \tag{21}$$

where

$$\varepsilon \equiv |e|\, k^2 A_1/m\omega^2 \ll 1 \tag{22}$$

is clearly the dimensionless parameter which has to be small in order that the series (11) and the whole method of solution make sense.

3. *Additional remarks.* Looking back upon the calculation in the previous section we see that (for our particular choice of the leading terms for V, eq. (11)) only cosine terms were needed in the series (11) and (12) since in the calculation of $\ddot{x}$ we have only sines, and products of cosines and sines which yield sines again. No term $\sin(pkx + q\omega t)$ with $p = 0$ in this series for $\ddot{x}$ will remain, as a consequence of the following simple lemma:

If $v(x, t)$ has a period λ in x while its aperiodic part is independent of t, that is, if $\int_a^{a+\lambda} v(x, t)\, dx = u$ independent of a and t, then we have

$$\int_a^{a+\lambda} \dot{v}\, dx = \int_a^{a+\lambda} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}\right) dx = \frac{d}{dt} \int_a^{a+\lambda} v\, dx + \tfrac{1}{2} \int_a^{a+\lambda} \frac{\partial v^2}{\partial x}\, dx = 0. \tag{23}$$

Thus $\ddot{x}$ cannot have an aperiodic part in x, as it should be because neither has $\partial V/\partial x$.

A second remark is that we have integrated the equation of motion only once to find $v(x, t; u)$; a second integration should give $x(t; y, u)$. However, taking into account also a second integration constant y is probably not relevant, on the analogy of the case of the travelling wave dealt with in a previous paper[6]). The present theory reduces to the latter case when we look only at terms containing $p(kx - \omega t)$ in eqs. (11) and (12). Thus $\lambda u = \int_0^\lambda v\, dx$ resembles an action variable (which, however, is defined only for time-periodic motions, as occurs in a travelling wave but not in a standing wave). In fact $|u|$ equals the quantity u for the free particles in the article mentioned. There it was made clear (at the end of section 4, and near eq. (53)) that carrying along the appropriate angle variable which describes the position of particles along the lines u = const., would only account for oscillations around a coarse time behaviour described by the u distribution. Moreover, by phase-mixing these oscillations are likely to die out because the oscillation periods differ for various particles. This makes it plausible that the omission of the second integration of the equations of motion amounts to disregarding some transient phenomena and selecting a stationary asymptotic solution of eqs. (1).

A third remark concerns the infinite series of resonant regions. In n-th order we find resonances at (see eq. (13))

$$\frac{ku}{\omega} = \pm 1, \pm\left(\frac{n-2}{n}\right)^{\pm 1}, \pm\left(\frac{n-4}{n}\right)^{\pm 1}, \pm\left(\frac{n-6}{n}\right)^{\pm 1} \dots \tag{24}$$

This row ends at $ku/\omega = \pm n^{\pm 1}$ for n odd and at $u = 0$ for n even. Some of these $2n$ or $2n - 1$ values coincide with lower order resonant points; at most $2n - 2$ are new (this maximum occurs for n prime). Every rational value for ku/ω represents a resonance. At the end of this section it will appear (eq. (30)) that the width of an n-th order resonant region is $\mathcal{O}(\varepsilon^{\frac{1}{2}n}\omega/k)$. Since $\sum_1^\infty n\varepsilon^{\frac{1}{2}n} = \varepsilon^{\frac{1}{2}}(1 - \varepsilon^{\frac{1}{2}})^{-2}$, the total width of all resonant regions is $\mathcal{O}(\varepsilon^{\frac{1}{2}}\omega/k)$. The set of resonant points u (for which (12) diverges) is dense on the u axis, but its measure is $\mathcal{O}(\varepsilon^{\frac{1}{2}}\omega/k)$. The graph of $F(u)$ really consists of a denumerable infinite set of horizontal line pieces at the rational values of ku/ω, together with a non-denumerable infinite set of points at irrational values of ku/ω, plotted in such a way that the complete graph approximates a Maxwellian as much as possible.

We close this section with an investigation of the convergence of (12). Therefore, we derive a recurrence relation for the coefficients (15). The series for V has the form

$$V = -\sum_{p=1}^{\infty} \sum_{q=-\infty}^{\infty} A_{pq} \cos(pkx + q\omega t) \tag{25}$$

and for v:

$$v = u + \sum_{p=1}^{\infty} \sum_{q=-\infty}^{\infty} a_{pq} \cos(pkx + q\omega t). \tag{26}$$

The order of A_{pq} and a_{pq} equals $\max(p, |q|)$ if p and q are both even or both odd, else the order is twice this maximum (see eq. (24)). Differentiating v we find (q and q' run from $-\infty$ to ∞)

$$\begin{aligned}\dot{v} = &-\sum_{p=1}^{\infty} \sum_{q} [a_{pq}(pku + q\omega) + \sum_{p'=1}^{\infty} \sum_{q'} kpa_{pq}a_{p'q'} \cos(p'kx + q'\omega t)] \cdot \\ &\cdot \sin(pkx + q\omega t) = -\sum_{p=1}^{\infty} \sum_{q} \{a_{pq}(pku + q\omega) + \\ &+ \tfrac{1}{2} \sum_{p'=1}^{p-1} \sum_{q'} kp'a_{p'q'}a_{p-p',\,q-q'} + \\ &+ \tfrac{1}{2} \sum_{p'=p+1}^{\infty} \sum_{q'} kp'a_{p'q'}a_{p'-p,\,q'-q} - \\ &- \tfrac{1}{2} \sum_{p'=1}^{\infty} \sum_{q'} kp'a_{p'q'}a_{p'+p,\,q'+q}\} \sin(pkx + q\omega t).\end{aligned} \tag{27}$$

Substitution into the equation of motion yields the recurrence relation

$$\{...\} = epkA_{pq}/m \tag{28}$$

with, between the curly brackets, the same expression as before (eq. 27).

Now consider a particular singularity $u = u_n = -q\omega/pk$, which occurs in (12) for the first time in n-th order in the coefficient a_n, which has the form, correct to the same order (α_p is a number)

$$a_n = (u - u_n)^{-1} (eA_n/m + \sum \sum \alpha_p a_p a_{n-p}). \tag{29}$$

The coefficients a_p and a_{n-p} are of lower order and do not contain the singularity $(u - u_n)^{-1}$ in n-th order. Higher order a_q, $q > n$, do contain this singularity if a_n plays the role of an a_p in the expression for a_q. Not until the $2n$-th order there are coefficients (for $n > 1$) which contain the singularity $(u - u_n)^{-2}$, and a more important one with $(u - u_n)^{-3}$ in view of the term with $p = n$ in the series (29) for a_{2n}; a singularity with $(u - u_n)^{-5}$ occurs for the first time in $3n$-th order, etc. The series (12) can converge, as far as this singularity is concerned, outside a δu neighbourhood of $u = u_n$ determined by the ratio of the successive terms mentioned, $|e| A_n/m(\delta u)^2 < 1$. Since $A_n = \mathcal{O}(A_1 \varepsilon^{n-1})$, we have

$$\delta u = \mathcal{O}(\varepsilon^{\frac{1}{2}n} \omega/k), \tag{30}$$

the expression quoted above.

4. *Resonant electrons.* It appeared in section 2 that the Ansatz (12) fails for electrons whose "average" velocity u approaches the phase velocity of one of the travelling waves composing the potential V. These electrons perform a slow motion $x_s(t)$ governed mainly by the resonant wave, and at the same time a rapid oscillation $x_r(t)$ caused by the other waves. Accordingly we substitute

$$x = x_s + x_r, \qquad kx_r = \mathcal{O}(\varepsilon) \tag{31}$$

into the equation of motion and expand with respect to the quantity kx_r presumed as small:

$$\begin{aligned} m\ddot{x}_s + m\ddot{x}_r &+ ekA_1 \sin(kx_s - \omega t) + ekA_1 \sin(kx_s + \omega t) + \\ &+ ek^2 A_1 x_r \cos(kx_s - \omega t) + ek^2 A_1 x_r \cos(kx_s + \omega t) + \\ &+ 2ekA_2 \sin(2kx_s - 2\omega t) + 2ekA_2 \sin(2kx_s + 2\omega t) + \\ &+ 2ekC_2 \sin 2kx_s + \ldots = 0. \end{aligned} \tag{32}$$

This equation has to be split into a sum of rapid terms and a sum of slow terms (marked by indices r and s), which vanish separately. The splitting depends on the resonant region considered.

Case 1. $v_s \equiv \dot{x}_s \approx \omega/k$.

Then we have the two equations

$$m\ddot{x}_r + ekA_1 \sin(kx_s + \omega t) + ek^2 A_1 x_r \cos(kx_s - \omega t) + \\ + ek^2 A_1 \{x_r \cos(kx_s + \omega t)\}_r + \\ + 2ekA_2 \sin(2kx_s + 2\omega t) + 2ekC_2 \sin 2kx_s + \ldots = 0,$$

$$m\ddot{x}_s + ekA_1 \sin(kx_s - \omega t) + ek^2 A_1 \{x_r \cos(kx_s + \omega t)\}_s + \\ + 2ekA_2 \sin(2kx_s - 2\omega t) + \ldots = 0. \tag{33}$$

The first equation is solved by iteration. To first order we find, choosing the integration constants in accordance with the condition $kx_r \ll 1$,

$$v_r = \frac{ekA_1}{m(kv_s + \omega)} \cos(kx_s + \omega t), \qquad x_r = \frac{ekA_1}{m(kv_s + \omega)^2} \sin(kx_s + \omega t), \tag{34}$$

which yields upon substitution in the two above eqs. (33)

$$m\ddot{x}_r + ekA_1 \sin(kx_s + \omega t) + \frac{e^2k^3A_1^2}{m(kv_s + \omega)^2} \cos(kx_s - \omega t) \sin(kx_s + \omega t) + \\ + \left(2ekA_2 + \frac{e^2k^3A_1^2}{2m(kv_s + \omega)^2}\right) \sin(2kx_s + 2\omega t) + \\ + 2ekC_2 \sin 2kx_s + \ldots = 0,$$

$$m\ddot{x}_s + ekA_1 \sin(kx_s - \omega t) + 2ekA_2 \sin(2kx_s - 2\omega t) + \ldots = 0. \tag{35}$$

One integration gives for the rapid motion

$$v_r = \frac{ekA_1}{m(kv_s + \omega)} \cos(kx_s + \omega t) + \\ + \left(\frac{ekA_2}{m(kv_s + \omega)} + \frac{e^2k^3A_1^2}{4m^2(kv_s + \omega)^3}\right) \cos(2kx_s + 2\omega t) + \\ + \left(\frac{eC_2}{mv_s} + \frac{e^2k^2A_1^2}{2m^2v_s(kv_s + \omega)^2}\right) \cos 2kx_s + \ldots \tag{36}$$

and a constant of the motion (energy integral)

$$E_s = \tfrac{1}{2}m(v_s - \omega/k)^2 - eA_1 \cos(kx_s - \omega t) - eA_2 \cos(2kx_s - 2\omega t) - \ldots \tag{37}$$

Clearly electrons which are trapped in a trough of the potential wave $A_1 \cos(kx_s - \omega t) + \ldots$, *i.e.* electrons with energy $E_s < eA_1$, move in a way which cannot be represented by eq. (12), but for a free electron ($E_s > eA_1$)

the velocity

$$v = v_s + v_r = \frac{\omega}{k} +$$
$$+\left(\frac{2}{m} E_s + 2\frac{e}{m} A_1 \cos(kx_s - \omega t) + 2\frac{e}{m} A_2 \cos(2kx_s - 2\omega t)\right)^{\frac{1}{2}} + v_r \tag{38}$$

must be equal to the series (12) for a certain relation between E_s and u. Indeed, if we start from $E_s = \frac{1}{2}m(u - \omega/k)^2$ as a first approximation, expand the square root for $eA_1 < E_s$, substitute $x_s = x - x_r$ and expand for small x_r, then we recover the expression (12) for v with coefficients (15) in all details, apart from some x and t independent terms which, however, can be removed by an improved choice for E_s:

$$E_s = \tfrac{1}{2}m\left(u - \frac{\omega}{k}\right)^2 + (ku - \omega)\frac{e^2k^2A_1^2}{4m}\left(\frac{1}{(ku-\omega)^3} - \frac{2}{(ku+\omega)^3}\right) + \ldots \tag{39}$$

Obviously, the case $v_s \approx -\omega/k$ is derived from the above one after inversion of the sign of ω.

Case 2. $v_s \approx 0$.

Then eq. (32) splits into

$$m\ddot{x}_r + ekA_1 \sin(kx_s - \omega t) + ekA_1 \sin(kx_s + \omega t) +$$
$$+ ek^2A_1\{x_r \cos(kx_s - \omega t)\}_r + ek^2A_1\{x_r \cos(kx_s + \omega t)\}_r +$$
$$+ 2ekA_2 \sin(2kx_s - 2\omega t) + 2ekA_2 \sin(2kx_s + 2\omega t) + \ldots = 0,$$
$$m\ddot{x}_s + ek^2A_1\{x_r \cos(kx_s - \omega t)\}_s + ek^2A_1\{x_r \cos(kx_s + \omega t)\}_s +$$
$$+ 2ekC_2 \sin 2kx_s + \ldots = 0. \tag{40}$$

First approximations for v_r and x_r are

$$v_r = \frac{ekA_1}{m(kv_s - \omega)} \cos(kx_s - \omega t) + \frac{ekA_1}{m(kv_s + \omega)} \cos(kx_s + \omega t),$$
$$x_r = \frac{ekA_1}{m(kv_s - \omega)^2} \sin(kx_s - \omega t) + \frac{ekA_1}{m(kv_s + \omega)^2} \sin(kx_s + \omega t). \tag{41}$$

Substitution of x_r gives

$$m\ddot{x}_r + ekA_1 \sin(kx_s - \omega t) + ekA_1 \sin(kx_s + \omega t) +$$
$$+\left(2ekA_2 + \frac{e^2k^3A_1^2}{2m(kv_s - \omega)^2}\right)\sin(2kx_s - 2\omega t) + (\omega \rightarrow -\omega) +$$
$$+\frac{e^2k^3A_1^2}{2m}\left(\frac{1}{(kv_s+\omega)^2} - \frac{1}{(kv_s-\omega)^2}\right)\sin 2\omega t + \ldots = 0,$$

$$m\ddot{x}_s + \left\{2ekC_2 + \frac{e^2k^3A_1^2}{2m}\left(\frac{1}{(kv_s+\omega)^2} + \frac{1}{(kv_s-\omega)^2}\right)\right\}\cdot$$
$$\cdot\sin 2kx_s + \ldots = 0, \tag{42}$$

where $(\omega \to -\omega)$ stands for the preceding term with ω replaced by $-\omega$. Integration yields

$$v_r = \frac{ekA_1}{m(kv_s-\omega)}\cos(kx_s-\omega t) + \frac{ekA_1}{m(kv_s+\omega)}\cos(kx_s+\omega t) +$$
$$+\left(\frac{ekA_2}{m(kv_s-\omega)} + \frac{e^2k^3A_1^2}{4m^2(kv_s-\omega)^3}\right)\cos(2kx_s-2\omega t) + (\omega \to -\omega)$$
$$+\frac{e^2k^3A_1^2}{4m^2\omega}\left(\frac{1}{(kv_s+\omega)^2} - \frac{1}{(kv_s-\omega)^2}\right)\cos 2\omega t + \ldots,$$
$$E_s = \tfrac{1}{2}mv_s^2 - e\left\{C_2 + \frac{ek^2A_1^2}{4m}\left(\frac{1}{(kv_s+\omega)^2} + \frac{1}{(kv_s-\omega)^2}\right)\right\}\cos 2kx_s - \ldots \tag{43}$$

For $E_s > eC_2$ again eqs. (12) and (15) can be recovered if we take

$$E_s = \tfrac{1}{2}mu^2 - u\,\frac{e^2k^3A_1^2}{2m}\left(\frac{1}{(ku-\omega)^3} + \frac{1}{(ku+\omega)^3}\right) - \ldots \tag{44}$$

It appears (eq. (43)) that the slow motion is governed not only by the C_2 term of V but also by a (fictive) potential $-(ek^2A_1^2/2m\omega^2)\cos 2kx_s$, neglecting kv_s/ω. The latter potential equals the x_s-dependent part of $\frac{1}{2}m(v_r^2)_s/e$ in the same approximation, a well known property of motion in rapidly oscillating fields [7]).

Now we assume that the resonant electron distribution is a function of E_s only, E_s being given as a function of x, v, t, differently for the various resonant regions. This assumption implies again a random phase, namely the phase of the slow oscillation in a wave trough. The distribution $f = G(E_s)$ cannot depend on the sign of v_s as far as the trapped electrons are concerned, because their v_s changes sign twice per oscillation period. For the free electrons $f = G(E_s, \operatorname{sgn} v_s) = F(u)$. As $F(u)$ thus has to have the same value at both edges of any resonant region, which is small in addition, we assume for simplicity that $f =$ constant throughout any resonant region.

It is difficult to find out what the rather involved analysis of the motion of resonant electrons looks like in higher order. Probably the slow motion is governed by an equation of the form

$$m\ddot{x}_s + ekA_n\sin(pkx_s + q\omega t) + \ldots = 0, \tag{45}$$

where A_n is determined by the resonant wave, apart from the contribution of a fictive potential assumed to be of at most the same order. Then the width of the resonant region is immediately found to be given by eq. (30).

5. *Stability.* One can ask whether the standing wave found in the preceding sections, not necessarily being an exact solution for all t, is perhaps in fact not stationary but slowly damped or enhanced. In this section we show, analogous to the case of the travelling wave[6]), that such a slow variation in amplitude is impossible because it would violate conservation of energy.

The first step is to show that a function $u(x, v, t)$, which for $\dot{A}_1 = 0$, $\dot{A}_2 = 0, \ldots$ is given by eq. (12), is adiabatically invariant for an electron moving in the standing wave (11) when $A_1, A_2 \ldots$ vary slowly. Therefore we verify that

$$\begin{aligned} v = u &+ a_1 \cos(kx - \omega t) + b_1 \cos(kx + \omega t) + \\ &+ \alpha_1 \sin(kx - \omega t) + \beta_1 \sin(kx + \omega t) + \\ &+ a_2 \cos(2kx - 2\omega t) + b_2 \cos(2kx + 2\omega t) + c_2 \cos 2kx + \\ &+ \alpha_2 \sin(2kx - 2\omega t) + \beta_2 \sin(2kx + 2\omega t) + \gamma_2 \sin 2kx + \ldots \end{aligned} \tag{46}$$

can satisfy the equation of motion to all orders if $A_1, A_2, \ldots$ are slowly varying functions of time, $a_1, b_1, \alpha_1, \beta_1, \ldots$ are properly chosen as functions of u, t and u is *constant.* It is enough to observe that substitution of v into the equation of motion and taking together the coefficients of equal trigonometric functions, yields as many equations as unknowns $a_1, b_1, \ldots$. The first few equations read (to first order in A_1)

$$\begin{aligned} &ma_1(ku - \omega) - m\dot{\alpha}_1 = ekA_1, \qquad \alpha_1(ku - \omega) + \dot{a}_1 = 0, \\ &mb_1(ku + \omega) - m\dot{\beta}_1 = ekA_1, \qquad \beta_1(ku + \omega) + \dot{b}_1 = 0, \end{aligned} \tag{47}$$

solvable by iteration for small dotted terms. Moreover, the lemma (23) guarantees that no x-independent terms occur in $\dot{v}$.

The next step is to derive from the constancy of u that the time averaged velocity magnitude of a free electron, if smaller than ω/k, decreases in a damped wave and that the total time averaged kinetic energy of the free electrons decreases accordingly. Writing

$$v = u + a_1 \cos(kx - \omega t) + b_1 \cos(kx + \omega t) + \ldots \tag{48}$$

in the form $v = v_s + v_r$, with $x = x_s + x_r$, where v_r and x_r are given in first approximation by eqs. (41), we find for v_s

$$v_s = u - \frac{e^2k^3A_1^2}{2m^2}\left(\frac{1}{(ku - \omega)^3} + \frac{1}{(ku + \omega)^3}\right)\ldots \tag{49}$$

Thus $|v_s|$ decreases for decreasing A_1 if $|u| < \omega/k$. The instantaneous kinetic energy density of the free electrons is (the index nr means that the integration runs over the non-resonant region only)

$$\lambda^{-1}\iint \tfrac{1}{2}mv^2F_{\mathrm{nr}}(u)\,\mathrm{d}x\,\mathrm{d}v = \lambda^{-1}\iint \tfrac{1}{6}m(\partial v^3/\partial u)\,F_{\mathrm{nr}}(u)\,\mathrm{d}x\,\mathrm{d}u =$$

$$= \frac{1}{\lambda}\iint \tfrac{1}{6}mF_{\mathrm{nr}}(u)\,\frac{\partial}{\partial u}\left(u - 2\frac{ekA_1}{m\omega}\sin kx \sin \omega t\right)^3 \mathrm{d}x\,\mathrm{d}u =$$

$$= \int \tfrac{1}{2}mu^2F_{\mathrm{nr}}(u)\,\mathrm{d}u + \frac{e^2k^2A_1^2}{m\omega^2}\int F_{\mathrm{nr}}(u)\,\mathrm{d}u\,\sin^2\omega t\,\ldots \tag{50}$$

The rapid time dependence of this expression disappears when it is added to the instantaneous wave energy density

$$(\varepsilon_0/2\lambda)\int(\partial V/\partial x)^2\,\mathrm{d}x = \varepsilon_0k^2A_1^2\cos^2\omega t + \ldots \tag{51}$$

The sum is constant if A_1 is constant ($\int F_{\mathrm{nr}}\,\mathrm{d}u \approx N$, $\omega \approx \omega_{\mathrm{p}}$), but decreases with decreasing A_1.

The energy loss $\varepsilon_0k^2A_1^2$ must be absorbed by the resonant electrons if the wave damps slowly. This occurs in the case of Landau damping, where the velocity distribution at $v = \pm\omega/k$ becomes flattened over a region δv with $\frac{1}{2}m(\delta v)^2 \approx |e|\,A_1$, in a time of about $|m/ek^2A_1|^{\frac{1}{2}}$, the oscillation period of resonant electrons. A comparison of their gain in energy,

$$|F'(\omega/k)|\,(\delta v)^2\,\delta(\tfrac{1}{2}mv^2) = |F'(\omega/k)|\,m(\omega/k)(\delta v)^3,$$

with the quantity $\varepsilon_0k^2A_1^2$, shows that Landau damping can occur only for exponentially small A_1:

$$\varepsilon = |e|\,A_1k^2/m\omega^2 = \mathcal{O}\{\omega^4F'^2(\omega/k)/k^4N^2\} = \mathcal{O}(\eta^{-3}\,\mathrm{e}^{-1/\eta}), \tag{52}$$

in a time $|m/ek^2A_1|^{\frac{1}{2}} \approx k^2N/\omega^3|\,F'(\omega/k)|$, just what Landau found.

For larger A_1 than according to eq. (52) the resonant electrons cannot absorb all of $\varepsilon_0k^2A_1^2$ and the wave stabilizes after flattening of the distribution in the resonant regions. This process has been studied by O'Neil[4]) and in the so-called quasi-linear theory initiated by Drummond and Pines[8]) and Vedenov, Velikhov, and Sagdeev[9]). The latter theory deals also (and mainly) with a growing wave. Then the energy $\varepsilon_0k^2A_1^2$ increases, counterbalanced by a transport of electrons from initially present humps in the tails towards the main part of the distribution, until the humps are flattened out.

For A_1 so large that $\varepsilon = \mathcal{O}(1)$ and most electrons are resonant, there are only numerical treatments, *e.g.* Armstrong[10]) and also Dawson and Shanny[11]), who found damping in a few plasma periods ω_{p}^{-1} (wave breaking).

We may conclude that a homogeneous plasma permits a stable wave given by eqs. (11) and (18), provided that the inequalities (19) and (22) are satisfied and the distribution $F(u)$ is even, non-increasing for $u > 0$, and flat in the regions given by eqs. (24) and (30). Landau damping and quasi-linear growth can be considered as transient phenomena to reach this stable wave. The theory might be generalized to include more than two lowest order travelling waves with unequal A_1, ω and k.

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APPENDIX

Amplitude correction on the frequency. Eq. (20) contains an ε^2 term which is the more interesting result of a third order calculation. The latter starts by including terms with $\cos(3kx \pm 3\omega t)$, $\cos(3kx \pm \omega t)$, $\cos(kx \pm 3\omega t)$ in the series (11) and (12), and by substituting these series into the equation of motion (9). Thus we find, apart from ten other relations, the following one:

$$a_1(ku - \omega) + \tfrac{1}{2}ka_1a_2 + \tfrac{1}{2}kb_1c_2 = ekA_1/m. \tag{53}$$

Together with the approximation already fixed by (15) we get the improved expression for a_1:

$$\begin{aligned} a_1 = {} & \frac{ekA_1}{m(ku-\omega)} - \frac{e^2k^3A_1A_2}{2m^2(ku-\omega)^3} + \frac{e^3k^5A_1^3}{8m^3(ku-\omega)^5} - \\ & - \frac{e^2k^3A_1}{4m^2\omega^2}\left(C_2 + \frac{ek^2A_1^2}{2m\omega^2}\right)\cdot \\ & \cdot\left(\frac{-2}{ku} + \frac{1}{ku-\omega} + \frac{1}{ku+\omega}\right) + \\ & + \frac{e^3k^5A_1^3}{16m^3\omega^3}\left(\frac{1}{(ku-\omega)^2} - \frac{1}{(ku+\omega)^2}\right), \end{aligned} \tag{54}$$

which is to be substituted into the $\cos(kx - \omega t)$ component of Poisson's equation:

$$k^2A_1 = (e/\varepsilon_0)\int a_1\, \mathrm{d}F. \tag{55}$$

This equation corresponds to the second relation of (18). Now, using the relations (19), (21) and (22) and expanding the denominators for small ku/ω

we find the following contributions to eq. (20), corresponding to the successive terms of eq. (54):

$$\omega^2 = \omega_p^2\{(1 + 3\eta) + \tfrac{3}{8}\varepsilon_2\,(1 + 20\eta) + \tfrac{5}{8}\varepsilon^2\,(1 + 21\eta) - \tfrac{3}{4}\varepsilon^2(1 - 3\eta) - \\ - \tfrac{1}{4}\varepsilon^2(1 + 6\eta)\}.$$

This is identical to (20). For $\eta \to 0$ we have $\omega = \omega_p$, correct in a cold plasma.

LIST OF SYMBOLS

Symbols used in sections 2 to 5 are listed with the number of the equation in or near to which the symbol is introduced.

a_1, a_2 (12)
a_{pq} (26), a_n (29)
b_1, b_2 (12)
c_2 (12)
e(1)
f (1)
k (1)
m (1)
n pos. integer
p, q integer
r, s index (31)
t (1)
u (12), u_n (29)
v (1)
x (1)
y (23)

A_1, A_2 (11)
A_{pq} (25), A_n (29)
C_2 (11)
E_s (37)
F (16), F_{nr} (50)
G (45)

α_1, α_2 (46)
α_p (29)
β_1, β_2 (46)
γ_2 (46)
ε (22), ε_0 (1)
η (19)
λ (1)
ω (11), ω_p (6)

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PREPRINT

ON THE MOTION OF CHARGED PARTICLES IN A SELF-CONSISTENT CONTINUOUS SPECTRUM OF WAVES

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Synopsis

This paper deals with a solution of the equations of Vlasov and Poisson describing one-dimensional plasma motion. The self-consistent potential assumed consists of a broad spectrum of travelling waves. The solution to be derived first involves expansions in the spectral amplitude. At first order the linear dispersion relation, including spatial Landau damping, is recovered. The second order, which shows no secularity, yields coefficients of the series for the potential. They are expressed in terms of the distribution function, which is assumed as a function of one constant of motion only, obtained by integrating once the equation of motion. The solution holds valid for a half-space.

Combining two of the above solutions, the velocity of a particle through the entire space is derived when it passes through a local disturbance described by a broad spectrum of undamped travelling waves. The mean velocity here changes slightly by resonant interaction, which provides a mechanism for the above-mentioned Landau damping.

1. *Introduction.* We consider the one-dimensional motion of electrons (charge e, mass m) in a neutralizing homogeneous background of fixed ions with constant charge density $-Ne$. The electron distribution function $f(x,v,t)$ is governed by the equations of Vlasov and Poisson

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial V}{\partial x}\frac{\partial f}{\partial v} = 0, \quad \frac{\partial^2 V}{\partial x^2} = \frac{e}{\varepsilon_o}\left(N - \int f dv\right), \qquad (1)$$

in which $V(x,t)$ is the potential. Further we assume, at infinity, charge neutrality, absence of currents, and a vanishing potential:

$$\int f dv = N, \quad \int v f dv = 0, \quad V = 0 \quad \text{for } x \to \infty. \qquad (2)$$

This configuration differs from the one examined in a previous paper[1]) by the fact that it is not periodic in x, yet it will be treated with similar mathematics.

Recently, Armstrong and Montgomery[2]) performed a huge computer experiment showing a typical oscillation in a finite plasma, treated mathematically by Fourier series as a space-periodic infinite plasma. The initial stage of exponentially growing waves turned out to be in accordance with the linear theory. The next stage is difficult to describe analytically, but its essential feature, a damping coefficient (initially negative) which oscillates, slowly, to zero, due to phase mixing of the trapped particles, has been treated by O'Neil[3]). The stationary wave which appears to be the final stage of the plasma oscillation resembles the standing wave found in ref.1. Thus the mathematics of the latter paper (see also ref. 4) fit the asymptotic behaviour of plasma oscillations, provided that only a few waves survive (essentially a single standing wave in the above-mentioned paper[2])). This proviso is likely to be satisfied, in the long run, in any periodic or finite plasma not too large, since the latter restricts the possible wavenumbers below the Debye wavenumber k_D to only a few. (Wavenumbers above k_D are generally associated with strong damping anyway). The numerical experiment shows that the proviso can even be satisfied in a plasma so large that the ground-wave-number is 0.15 k_D.

However, most plasmas are much larger. Then the wavenumber spectrum is essentially continuous. This case will be considered in the present paper, where we assume a potential which reads to first order:

$$V = - \int A(k) \cos\{kx - \omega(k)t\} dk \ . \tag{3}$$

The "dispersion function" $\omega(k)$ will be chosen properly later on, and the continuous spectral function $A(k)$ is assumed to be concentrated near $k = 0$ with a width small compared to k_D. Hence the potential V differs appreciably from zero in a finite x-region only, large compared to the Debye-length. Since such a confined disturbance will tend to spread out in a hot plasma, no stationary free oscillation is to be expected.

Moreover, for the stationary wave in ref. 1 it turned out to be necessary that only a small range of average velocities u is associated with resonant electrons. This condition could be satisfied, in spite of the fact that (as a result of an unlimited series of sum and difference "frequencies" of the two ground waves) every rational value of ku/ω leads to a resonant-electron region for u. In the present case of a continuous spectrum of ground waves, the range of phase velocities of these waves and their sum and difference "frequencies" are likely to cover all values of u. That is, all electrons are resonant with waves, contrary to the situation in ref. 1 where most electrons are non-resonant.

In section 2 we present a rough picture of the motion of an electron in the potential (3). The result will be confirmed in section 5 in which the equation of motion with associated potential (3) will be integrated once to give the velocity as a function of x and t in the form of a series, A playing the role of the expansion parameter. It seems to be impossible to make this potential self-consistent, i.e. to satisfy also Poisson's equation with the formalism to be presented. However, in section 3 a solution of both equations (1) will be derived, valid in a half-space $x > 0$ or $x < 0$. Two of such solutions are combined in section 5 which is only possible at the expense of the self-consistency.

2. *Resonant wave packet.* In this section the possible interaction of an electron with resonant waves contained in a continuous spectrum is made understandable. Since the potential (3) vanishes at infinity, $x \to \infty$, the velocity there is constant, $v = u$. Approaching the disturbance around $x = 0$ the electron interacts resonantly with waves having wavenumbers around $k = k_u$, fixed by the resonance condition $ku = \omega(k)$. This interaction presumably changes the average velocity u of the electron to a neighbouring value $u + \delta u$, corresponding to a neighbouring wavenumber $k = k_u + \delta k$ fixed by $k(u + \delta u) = \omega(k)$. We now want to estimate δk.

To this end we take the degenerated dispersion relation $\omega = \omega_p \equiv (Ne^2/\varepsilon_o m)^{\frac{1}{2}}$, which is justified by assuming k_u small, $k_u \ll k_D$. We then find that the relation between δu and δk is expressed by $\delta u = -(\omega_p/k_u^2)\delta k$. A second relation between these two quantities can be derived by assuming that the velocity change results from interaction of the electron with the wave packet

$$\int_{k_u}^{k_u+\delta k} A(k)\cos(kx - \omega t)dk \approx A(k_u)\int_{k_u}^{k_u+\delta k} \cos(kx - \omega_p t)dk =$$

$$= A(k_u)\,\frac{\sin x\,\delta k/2}{x/2}\cos\left\{\left[k_u + \frac{\delta k}{2}\right]x - \omega_p t\right\}\,. \tag{4}$$

The maximum amplitude of this potential is $A(k_u)\delta k$, from which we may expect a relation $m(\delta u)^2 \approx eA(k_u)\delta k$ between the orders of magnitude of δk and δu. From their two above relations, together with the resonance condition, we find

$$|\delta u/u| \sim |\delta k/k| \sim eA(k_u)k_u^3/m\omega_p^2 \ . \tag{5}$$

This quantity should be small in view of the used approximations.

Thus we expect an electron to leave the disturbance with a slight change in velocity having an order of magnitude indicated in (5). There is little chance that an electron interacts "adiabatically" with the wave packet[6]), leaving it with unchanged velocity, because the oscillation period of an electron in a wave trough with amplitude $A(k_u)\delta k$ has an order of magnitude $\{ek_u^2A(k_u)\delta k/m\}^{-\frac{1}{2}} \approx m\omega_p/ek_u^3A(k_u)$, which is just about the time of flight $1/u\delta k$ through the wave packet. For adiabatic interaction the oscillation period should be much smaller.

3. *Method of solving equations* (1). Analogous to section 2 of ref. 1 we take for the potential and the velocity as an Ansatz the following series, each term of which now consists of a continuous spectrum ($^\times$ marks complex conjugate):

$$-V = \int \underline{A}_1 \exp i(\underline{k}x-\omega t)dk + \iint \underline{A}_2 \exp i\{(\underline{k}+\underline{k}')x - (\omega+\omega')t\}dkdk' +$$

$$+ \iint \underline{B}_2 \exp i\{(\underline{k}-\underline{k}'^\times)x - (\omega-\omega')t\}dkdk' + \dots + \text{c.c.},$$

$$v - u = \int \underline{a}_1 \exp i(kx-\omega t)dk + \iint \underline{a}_2 \exp i\{(\underline{k}+\underline{k}')x - (\omega+\omega')t\}dkdk' +$$

$$+ \iint \underline{b}_2 \exp i\{(\underline{k}-\underline{k}'^\times)x - (\omega-\omega')t\}dkdk' + \dots + \text{c.c.} \tag{6}$$

c.c. is used as abbreviation for the complex conjugate of all preceding terms, while underlining marks complex quantities. We assume the following functional dependences:

$$\underline{k} \equiv k + i\kappa(k);\ \underline{A}_1 = \underline{A}_1(k);\ \omega = \omega(k) > 0;\ \underline{A}_2 = \underline{A}_2(k,k');\ \underline{B}_2 = \underline{B}_2(k,k');$$

$$\omega' \equiv \omega(k');\ \underline{a}_1 = \underline{a}_1(k,u);\ \underline{a}_2 = \underline{a}_2(k,k',u);\ \underline{b}_2 = \underline{b}_2(k,k',u). \tag{7}$$

All these functions are to be determined for given $\underline{A}_1(k)$. All integration variables k,k', the real parts of the complex wavenumbers $\underline{k},\underline{k}'$ occurring in the exponentials, run from $-\infty$ to $+\infty$.

The Ansatz (6) results from a trial-and-error method, starting from a potential (3), to satisfy the equation of motion

$$m\ddot{x} + e\partial V/\partial x = 0 \; . \tag{8}$$

The quantity $|\underline{A}_1|$ should be the smallness parameter in the series (6); the indices of $\underline{A}_2, \underline{a}_2$, etc. then indicate the orders An example of a higher order term in these series contains an exponential

$$\exp \mathrm{i}\{(\underline{k}+\underline{k}' - k''^{\times} + \underline{k}''' \ldots)x - (\omega+\omega' - \omega'' + \omega''' \ldots)t\} \; . \tag{9}$$

Differentiation of v yields

$$\ddot{x} = \dot{v} = \partial v/\partial t + v\partial v/\partial x = \int \underline{a}_1 \mathrm{i}(\underline{k}u-\omega) \exp\mathrm{i}(\underline{k}x-\omega t)\mathrm{d}k +$$

$$+ \iint \left[\underline{a}_2 \mathrm{i}\{(\underline{k}+\underline{k}')u-(\omega+\omega')\}+\mathrm{i}\underline{k}\underline{a}_1\underline{a}_1'\right] \exp\mathrm{i}\{(\underline{k}+\underline{k}')x-(\omega+\omega')t\}\mathrm{d}k\mathrm{d}k' +$$

$$+ \iint \left[\underline{b}_2 \mathrm{i}\{(\underline{k}-\underline{k}'^{\times})u-(\omega-\omega')\}+\mathrm{i}\underline{k}\underline{a}_1 a_1'^{\times}\right] \exp\mathrm{i}\{(\underline{k}-\underline{k}'^{\times})x-(\omega-\omega')\}\mathrm{d}k\mathrm{d}k' +$$

$$+ \ldots + \mathrm{c.c.} \tag{10}$$

Substitution of (10) into the equation of motion (8) yields the following relations between the coefficients up to second order, correct in the same order:

$$(\underline{k}u - \omega)\underline{a}_1 = e\underline{k}\underline{A}_1/m \; ,$$

$$\{(\underline{k}+\underline{k}')u - (\omega+\omega')\}\underline{a}_2 + \underline{k}\underline{a}_1\underline{a}_1' = e(\underline{k}+\underline{k}')\underline{A}_2/m \; ,$$

$$\{(\underline{k}-\underline{k}'^{\times})u - (\omega-\omega')\}\underline{b}_2 + \underline{k}\underline{a}_1\underline{a}_1'^{\times} = e(\underline{k}-\underline{k}'^{\times})\underline{B}_2/m \; . \tag{11}$$

Using the algebraic identities

$$')\{(\underline{k}\pm\underline{k}')u-(\omega\pm\omega')\}]^{-1} \equiv$$

$$\left[\frac{\underline{k}^2}{\underline{k}u-\omega} \pm \frac{\underline{k}'^2}{\underline{k}'u-\omega'} - \frac{(\underline{k}\pm\underline{k}')^2}{(\underline{k}\pm\underline{k}')u-(\omega\pm\omega')}\right] , \tag{12}$$

where $\underline{k}'$ must be replaced by $\underline{k}'^{\times}$ when reading this identity with the lower signs of $\pm$ and $\mp$, the solutions for the coefficients $\underline{a}_1, \underline{a}_2$, and $\underline{b}_2$ can be represented as follows:

$$\underline{a}_1 = e\underline{k}\underline{A}_1/m(\underline{k}u-\omega) ,$$

$$\underline{a}_2 = \frac{e(\underline{k}+\underline{k}')\underline{A}_2}{m\{(\underline{k}+\underline{k}')u-(\omega+\omega')\}} - \frac{e^2\underline{k}^2\underline{k}'\underline{A}_1\underline{A}_1'}{m^2(\underline{k}\omega'-\underline{k}'\omega)^2}\left[\frac{\underline{k}^2}{\underline{k}u-\omega} + \frac{\underline{k}'^2}{\underline{k}'u-\omega'} - \frac{(\underline{k}+\underline{k}')^2}{(\underline{k}+\underline{k}')u-(\omega+\omega')}\right],$$

$$\underline{b}_2 = \frac{e(\underline{k}-\underline{k}'^{\times})\underline{B}_2}{m\{(\underline{k}-\underline{k}'^{\times})u-(\omega-\omega')\}} + \frac{e^2\underline{k}^2\underline{k}'^{\times}\underline{A}_1\underline{A}_1'^{\times}}{m^2(\underline{k}\omega'-\underline{k}'^{\times}\omega)^2}\left[\frac{\underline{k}^2}{\underline{k}u-\omega} - \frac{\underline{k}'^{\times 2}}{\underline{k}'^{\times}u-\omega'} - \frac{(\underline{k}-\underline{k}'^{\times})^2}{(\underline{k}-\underline{k}'^{\times})u-(\omega-\omega')}\right]. \tag{13}$$

It will appear later from eq. (20) that ω and $\kappa = \mathrm{im}\,\underline{k}$ are non-vanishing, even functions of k. Thus $\underline{a}_1, \underline{a}_2$, and $\underline{b}_2$ have no singularities when considered as functions of u. Moreover the denominator $(\underline{k}\omega'-\underline{k}'\omega)^2$ in the expression for $\underline{a}_2$, though vanishing for $k = k'$, yet does not give rise to a singularity, the l.h.s. of eq. (12) being regular there.

Next we assume, still following ref. 1, that the distribution of the electrons is a function of the constant of motion u only, $u(x,v,t)$ being given implicitly by eq. (6), second relation. Hence we take

$$f = F(u) . \tag{14}$$

Substitution of $F(u)$ into Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{e}{\varepsilon_0}\left\{N - \int F(u)\,\mathrm{d}v\right\} = \frac{e}{\varepsilon_0}\left\{N + \int v\,\mathrm{d}F(u)\right\} \tag{15}$$

provides us, using eqs. (6) and (13), with the *conditions of self-consistency of the solution:*

$$N = \int F\mathrm{d}u \,, \qquad \underline{k} = \frac{e^2}{\varepsilon_o m}\int \frac{\mathrm{d}F}{\underline{k}u-\omega}\,,$$

$$\underline{A}_2 = -\frac{e\underline{k}^2\underline{k}'\underline{A}_1\underline{A}_1'}{m(\underline{k}\omega'-\underline{k}'\omega)^2}\,\frac{\underline{k}^2-\underline{k}\underline{k}'+\underline{k}'^2-(\underline{k}+\underline{k}')\underline{K}(\underline{k}',\omega')}{\underline{k}+\underline{k}'-\underline{K}(\underline{k}',\omega')}\,,$$

$$\underline{B}_2 = \frac{e\underline{k}^2\underline{k}'^{\times}\underline{A}_1\underline{A}_1'^{\times}}{m(\underline{k}\omega'-\underline{k}'^{\times}\omega)^2}\,\frac{\underline{k}^2+\underline{k}\underline{k}'^{\times}+\underline{k}'^{\times 2}-(\underline{k}-\underline{k}'^{\times})\underline{K}(-\underline{k}'^{\times},-\omega')}{\underline{k}-\underline{k}'^{\times}-\underline{K}(-\underline{k}'^{\times},-\omega')}\,, \tag{16}$$

where

$$\underline{K}(\underline{k}',\omega') \equiv \frac{e^2}{\varepsilon_o m}\int \frac{\mathrm{d}F}{(\underline{k}+\underline{k}')u-(\omega+\omega')}\,.$$

To evaluate the integrals we assume F even and single-humped, like for a Maxwellian distribution. The evenness is in agreement with the second relation of (2), the absence of any current at infinity. Further we had assumed (below eq. (3)) the functions F and $\underline{A}_1$ to be such that

$$\overline{u^2} \equiv N^{-1}\int u^2 F\mathrm{d}u << \omega_p^2/k^2 \tag{17}$$

for all k for which $\underline{A}_1$ is appreciably different from zero. Then we may assume

$$|\kappa/k| << 1\,, \tag{18}$$

an assumption to be verified by the result (eq. (20), second relation). The second relation of (16) can now be written as follows, in the limit $\kappa \to 0$:

$$k+\mathrm{i}\kappa = \frac{e^2}{\varepsilon_o m}\int \frac{ku-\omega-\mathrm{i}\kappa u}{(ku-\omega)^2+\kappa^2u^2}\,\mathrm{d}F \approx$$

$$\approx \frac{e^2}{\varepsilon_o m}\left\{\mathrm{P}\int\frac{\mathrm{d}F}{ku-\omega}-\mathrm{i}\int \pi\delta(ku-\omega)\mathrm{d}F\ \mathrm{sgn}\frac{\kappa\omega}{k}\right\}\,. \tag{19}$$

Expanding, as usual, the integrand of the principal-value integr for small ku/ω, we find

$$\omega^2 = \omega_p^2 + 3k^2\overline{u^2} \ldots, \qquad |\kappa| = -\pi\omega_p^2 F'(\omega/k)/Nk \; . \tag{20}$$

Note that ω was assumed positive (eq. (7)); that κ is very small in view of (17), justifying the assumption (18); however, the sign of κ can still be chosen arbitrarily. It becomes evident at this point why in the Ansatz (6) $\underline{k}$ was taken complex and ω real: for real k the dispersion relation (16), second relation, has no ω-roots, a result well-known in the linear theory[5]).

To get an idea what $\underline{A}_2$ and $\underline{B}_2$ look like (eq. (16), 3rd and 4th relation), we substitute in these complicated expressions $\underline{k} = k$ and $\omega = \omega' = \omega_p$. The expansion of the integrand of $K(k',\omega')$ for small $(k+k')u/2\omega_p$ then yields as a first approximation for this integral $\frac{1}{4}(k+k')$. For a Maxwellian F we have:

$$K(-k',-\omega') = \frac{e^2}{\varepsilon_o m}\,\frac{1}{k-k'}\int\frac{dF}{u} = -\frac{1}{k-k'}\,\frac{\omega_p^2}{\overline{u^2}} \; . \tag{21}$$

We then find, also using the inequality (17) for the second relation:

$$\underline{A}_2 \approx -\frac{ek^2k'\underline{A}_1\underline{A}_1'}{m\omega_p^2(k+k')}\,, \qquad \underline{B}_2 \approx \frac{ek^2k'\underline{A}_1\underline{A}_1'^{\times}}{m\omega_p^2(k-k')} \; . \tag{22}$$

The points $k' = \pm k$ are not really singular; this will be investigated more carefully in the next section.

4. *Asymptotic behaviour*. The behaviour of V and v for $x \to \infty$ is the subject of this section. It will be shown that

$$V \to 0 \quad \text{and} \quad v \to u \quad \text{if } x \to \infty\,\mathrm{sgn}\,\kappa \; . \tag{23}$$

On the other hand, for $\kappa x < 0$, the formalism of section 3 leads to physically unrealistic results and therefore must be discarded in this half-space.

The first term of the series for the potential (6) vanishes for $x \to \infty\,\mathrm{sgn}\,\kappa$. (It even vanishes for $x \to -\infty\,\mathrm{sgn}\,\kappa$ if $\underline{A}_1(k)$ is sufficiently smooth, but the proof will be omitted as being irrelevant).

In the second and third term we have to investigate the "singularities" $k' = \pm k$ (eq. (22)) since the singularities of a function determine the asymptotic behaviour of its Fourier transform.

Considering first $\underline{A}_2$, we substitute there $k' = -k$ everywhere except in the combination $\underline{k}+\underline{k}'$. Also we expand the integrand in $\underline{K}(\underline{k}',\omega')$ with respect to the very small quantity $(\underline{k}+\underline{k}')u/2\omega$; this is justified by the fact that $F\{2\omega/(k+k')\}$ is completely negligible. This yields that $\underline{K}(k',\omega')$ tends to $(\omega_p^2/4\omega^2)(k+k') \approx \frac{1}{4}(k+k')$ in the neighbourhood of $k' = -k$ when $\omega \approx \omega_p$. The integral with $\underline{A}_2$ in eq. (6) then proves to become for large x:

$$\iint \frac{e\underline{k}^3\underline{A}_1(k)\underline{A}_1(-k)}{4m\underline{k}^2\omega_p^2}\,\frac{3\underline{k}^2}{\frac{3}{4}(\underline{k}+\underline{k}')}\exp i\{(\underline{k}+\underline{k}')x-2\omega_p t\}\,dk\,dk' =$$

$$= \int \frac{e\,\underline{k}^3\underline{A}_1(k)\underline{A}_1(-k)}{m\omega_p^2}\,dk\,e^{-2i\omega_p t}\int \frac{e^{i(\underline{k}+\underline{k}')x}}{\underline{k}+\underline{k}'}\,d(k+k') \ . \tag{24}$$

The last integral equals

$$\int \frac{e^{i\underline{k}x}}{\underline{k}}\,dk = -2\pi i\,H(-\kappa x)\,\mathrm{sgn}\,\kappa = 2\pi i\,H(-\kappa x)\,\mathrm{sgn}\,x\,, \tag{25}$$

H being the Heaviside unit-step function.

Passing next to $\underline{B}_2$, we put $k' = k$ everywhere except in the combination $\underline{k}-\underline{k}'^{\times}$. Also we note that, in view of the approximation $\omega = \omega_p + \frac{3}{2}k^2\overline{u^2}/\omega_p$ (eq. (20), first relation), the quantity

$$\underline{\varepsilon} \equiv \frac{\omega-\omega'}{\underline{k}-\underline{k}'^{\times}} \approx \frac{3}{2}\,\frac{\overline{u^2}}{\omega_p}\,\frac{k^2-k'^2}{k-k'+2i\kappa}\ , \text{ hence } |\underline{\varepsilon}|^2 \ll \overline{u^2}\ .$$

Therefore, $\int dF/(u-\underline{\varepsilon})$ is nearly constant for $k' \approx k$. (Integrals of this kind have been extensively studied in the linear theory[5])) This gives that $\underline{K}(-k'^{\times},-\omega')$ behaves like $-\omega_p^2/\overline{u^2}(\underline{k}-\underline{k}'^{\times})$ in the neighbourhood of $k' = k$. For large x the integral with $\underline{B}_2$ in eq. (6) thus amounts to

$$\int \frac{e\underline{k}^2\underline{k}^{\times}\underline{A}_1\underline{A}_1^{\times}}{m\omega_p^2}\,dk\int \frac{e^{i(\underline{k}-\underline{k}'^{\times})x}}{\underline{k}-\underline{k}'^{\times}}\,d(k-k') \ . \tag{26}$$

The last integral again equals (25).

The asymptotic behaviour of the velocity can be obtained in the same way. The $\underline{a}_1$-term in (6), with a nearly singular point at $k = \omega/u$, behaves for large x like

$$\frac{e\underline{k}\underline{A}_1}{mu} \exp i\omega\left(\frac{x}{u} - t\right) \cdot \int \frac{e^{i(\underline{k}-\omega/u)x}}{\underline{k}-\omega/u}\, dk \;, \tag{27}$$

where $\underline{k}$ and ω, outside the integral, are to be taken at the k-root of the equation $ku - \omega(k) = 0$. The last integral again equals (25). The $\underline{a}_2$ and $\underline{b}_2$ term of (6) also yield in a straightforward manner terms including the Heaviside function $H(-\kappa x)$ as a factor. Thus the statement (23) is verified.

On the other hand, for $\kappa x < 0$ the asymptotic expressions do not make much sense. Instead of the expected small change in velocity δu (section 2) we found the expression (27).

5. *Solution of* (8) *for the entire space. Conclusion.* The formulae (6) up to (13) inclusive contain a special solution of the equation of motion (8) with the potential according to eq. (3). This solution results by substituting:

$$\begin{aligned} &\underline{A}_1 = \tfrac{1}{2}A \;, \quad A_n = B_n = \ldots = 0 \text{ for } n > 1 \;, \\ &\kappa \to +0 \;, \quad u = u_+ \quad \text{for } x > 0 \;, \\ &\kappa \to -0 \;, \quad u = u_- \quad \text{for } x < 0 \;. \end{aligned} \tag{28}$$

Thus we find two solutions, each of which refers to a single half-space. In particular, the velocity reads to first order, for $\operatorname{sgn} x = \pm 1$:

$$v = u_\pm + \frac{e}{m}\, \mathrm{P} \int \frac{kA}{ku_\pm - \omega} \cos(kx - \omega t)\, dk \pm \frac{\pi e\omega A}{mu_\pm^2} \sin\omega\left(\frac{x}{u_\pm} - t\right) \ldots \;, \tag{29}$$

where ω and A in the last term have to be taken at the k-root of $ku_\pm - \omega = 0$. To get the solution for the entire space, the two above solutions must be matched to each other at $x = 0$. Therefore, equating the two expressions for v at $x = 0$, at the time t_o at which the particle passes there, we obtain the relation between u_+ and u_-. As a first approximation we find

$$\delta u = u_+ - u_- = -\frac{2\pi e\omega A}{mu^2} \sin \omega t_o \ldots , \tag{30}$$

where again ω and A are to be taken at k satisfying $ku-\omega = 0$. Note that δu thus obtained, agrees with the order of magnitude derived in eq. (5). Not only v itself but also its derivative $\dot{v}$ proves to be continuous at $x = 0$, $t = t_o$, since v satisfies the equation of motion (8) while the potential is continuously differentiable at this point. However, starting from a potential (6) with $\kappa \neq 0$, the matching would be impossible (see section 1, last sentence) in view of discontinuities at $x = 0$.

Summarizing, we have shown in section 3 and 4 that the distribution function (14), $u(x,v,t)$ being implicitly defined by the second relation of (6), together with the potential given in the first relation of (6), constitute a solution of the equations of Vlasov and Poisson (1), either for $x > 0$ if $\kappa > 0$ or for $x < 0$ if $\kappa < 0$. The potential vanishes at ∞ sgnκ while the electron velocity becomes constant there. Concerning the convergence of the series (6) we can say that, though a proof of convergence is obviously quite beyond the present state of the theory, there is no secularity, and the terms, being ordered according to powers of $\underline{A}_1$, will decrease when $|\underline{A}_1|$ is sufficiently small. The required smallness can be inferred from the approximate expressions (22) for $\underline{A}_2$ and $\underline{B}_2$, compared with $\underline{A}_1$, while accounting for the difference in dimension of $\underline{A}_1$ and $\underline{A}_2$. It is then found that the quantity indicated in (5) is indeed the dimensionless parameter in the problem which has to be small compared to unity in order that the series (6) can converge. In (5) we have to substitute for k_u a characteristic value, e.g., the width of $|\underline{A}_1(k)|$.

The derived solution involves spatial Landau damping according to the second relation of eq. (20). This leads to the conclusion that Landau damping occurs in this case of a spatially damped broad spectrum of waves for much larger amplitudes than in the previously considered[1]) case of a single travelling or standing wave damped with respect to time. In fact, in the latter case Landau damping proved to hold only for exponentially small amplitudes.

Next we have found in section 2 and 5 that an electron changes slightly its mean velocity when passing through a disturbance composed of a broad spectrum of travelling waves, due to resonant interaction with some of them. The velocity change δu (eq. (30)) suggests a physical mechanism for the above spatial Landau damping. The "random" increase or decrease of the mean velocity, depending on the phase ωt_0, leads for a set of particles to a diffusion in velocity space, which involves heating of the electron plasma. This mechanism is rather different from the well-known physical picture that serves to explain Landau damping with respect to time of a single sinusoidal wave[5]).

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2) Armstrong, T.P. and Montgomery, D., Phys. Fluids 12 (1969) 2094.
3) Sagdeev, R.Z. and Galeev, A.A., Nonlinear plasma theory, Benjamin, Amsterdam (1969) p. 42.
4) Lewak, G.J., J. Plasma Physics 3 (1969) 243. Discusses independently the same mathematical set up as a method to avoid secular terms.
5) Bernstein, I.B., Trehan, S.K. and Weenink, M.P.H., Nuclear Fusion 4 (1964) 61.
6) Knorr, G., Plasma Physics 11 (1969) 917.

LIST OF SYMBOLS

The numbers between parentheses refer to the equation in or near to which the symbol is introduced.

$\underline{a}_1, \underline{a}_2$ (6)	A(3)	$\delta k, \delta u$(4)
$\underline{b}_2$(6)	$\underline{A}_1, \underline{A}_2$(6)	ε_o(1), $\underline{\varepsilon}$(24)
c.c.(6)	$\underline{B}_2$(6)	κ(7)
e, f(1)	F(14)	ω(3), ω_p(4)
k, k_D(3)	$\underline{K}$(16)	ω'(6)
k_u(4)	N, V(1)	
$\underline{k}, \underline{k}', k'$(6)		
m, t(1), t_o(30)		
u(4), $u_{\pm}$(28)		
v, x(1)		

Na het eindexamen gymnasium in 1954 heb ik elektrotechniek gestudeerd in Delft. Het afstudeerwerk (1961) o.l.v. prof. dr.ir. F.A. Heyn betrof ionenbundels. Van invloed op mijn wetenschappelijke opleiding waren voorts een jaar (1960) besteed in de werkgroep te Amsterdam van de Werkgemeenschap "Onderzoek thermonucleaire reacties" van de Stichting voor Fundamenteel Onderzoek der Materie, en vooral studie van de "Course in Theoretical Physics" van L.D. Landau en E.M. Lifshitz, tijdens militaire dienst in 1962-'63. Teruggekeerd in dezelfde FOM-Werkgemeenschap, nu in de theoretische groep te Jutphaas, werd ik door dr. M.P.H. Weenink op het spoor gezet dat voerde naar dit proefschrift, sinds 1966 onder de stimulerende leiding van prof.dr. H. Bremmer.

STELLINGEN

1. De oude ergodenhypothese van Maxwell en Boltzmann (1871) is in directe tegenspraak met de stelling van Brouwer over de invariantie van de dimensie (1913) en de somstelling uit de dimensie-analyse.

   Rosenthal, A., Ann. Physik 42 (1913) 796.

2. De vergelijkingen voor de reflectie- en transmissiecoëfficiënten van een inhomogene laag, zoals die bijv. door De Jager en Levine worden gegeven, zijn op eenvoudige wijze af te leiden uit de recurrente betrekkingen voor de termen van de Bremmerreeks.

   De Jager, E.M., Levine, H., Appl. scient. Res. 21 (1969) 87.

   Bremmer, H., Comm. pure appl. Math. 4 (1951) 105.

3. Uit een serie metingen van dezelfde grootheid met twee gelijktijdig en onafhankelijk werkende meetinstrumenten, zijn zuivere schattingen te verkrijgen van de varianties van zowel de gemeten grootheid als de fouten van de beide instrumenten.

   Best, R.W.B., Rapport RvR 157, 's Gravenhage 1963, Min.v.Def., K.L., I.T.D.

4. Bij een Landau-gedempte lineaire plasmagolf is de ontwikkeling in de tijd van de storing $f(v,t)e^{ikx}$ op de elektronenverdelingsfunctie het eenvoudigste te beschrijven met de naar v Fourier-getransformeerde $\hat{f}(q,t)$. De beginstoring $\hat{f}(q,0)$ splitst zich in een gedempte vrije trilling $\hat{f}_1(q)e^{-i\omega t}$, en een vrije stroming $\hat{f}_2(q,t) = \hat{f}_2(q+kt,0)$. Ten onrechte stelt Simon dat $\hat{f}(q,0) = \hat{f}_2(q,0)$.

   Best, R.W.B., Euratom symp. on theor. plasma phys., Varenna 1966, pt. 1, p. 39.

   Simon, A., IAEA seminar on plasmaphys., Trieste 1964, p. 182.

5. De oplossingen van de vergelijkingen van Vlasov en Poisson in dit proefschrift zijn verkregen door de deeltjessnelheid v te schrijven als functie van x,t en één integratieconstante u, en door te veronderstellen dat de verdeling f een functie is van u alleen. De functie v voldoet aan:

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+v\,\frac{\partial v}{\partial x}\right]+\frac{\omega_p^2}{N}\int(v-u)\,df=0.$$

6. Het vergt vaak te veel moeite van de lezer vertrouwd te raken met de termen en symbolen in een lang natuurkundig artikel. Het is nuttig een lijst op te nemen van de introductieplaatsen van de symbolen, althans van die welke afwijken van de aanbeveling van de IUPAP, en van die termen die niet voorkomen in de gangbare handboeken.
Een lijst van definities en eenheden zou veel ruimte in beslag nemen.

7. In een wetenschappelijk artikel dient de schijn van onaantastbaarheid vermeden te worden, die gewekt wordt door ingewikkelde, quasi-geleerde taal en quasi-exacte afleidingen.

Woodford, F.P., Science 156 (1967) 743.
Von Mises, R., Positivism, p. 113, Braziller, N.Y. 1956.

8. Eerst een studie van de factoren, die de variatie in de morfologische kenmerken teweegbrengen, maakt van de taxonomie der koralen een inductieve wetenschap.

Best, Maya R.B., Bijdragen tot de dierkunde 38 (1968) 17.

9. Verkeersregels zijn ontworpen voor een eenvoudig verkeersmodel, dat gebrekkiger vastgelegd is dan de regels. Deze kunnen onbillijk zijn in uitzonderingssituaties. Dan dient een eenvoudige mogelijkheid tot vrijspraak na overtreding te bestaan.

10. Waardevrije wetenschap, vermenging van wetenschap en politiek, het zijn strijdpunten passend in de historische reeks van conflicten tussen hen die hun ideeën trachten aan te passen aan de feiten (Ernst Mach), en hen die de wereld proberen te hervormen naar hun ideeën.

Robert W.B. Best
24 februari 1970