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Mixed Finite Element for Swelling of Cartilaginous Tissues, I
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Introduction

The swelling and shrinking of cartilaginous tissues (CT) (like Intervertebral Disks) can be modelled by a four-component mixture theory in which a deformable and charged porous medium is saturated with a fluid with dissolved ions. This theory results in a coupled system of non-linear differential equations together with an algebraic constraint for electro-neutrality.

Four-Component Modeling of CT

We assume incompressible and linear elastic solid saturated with incompressible and Newtonian viscous fluid. Also we neglect the inertial effects (quasistatic case) and the body forces.

Balance Equations

(degrees of freedom, material parameters)

\[ -\nabla \cdot (2\mu u) + \lambda \nabla \cdot u + \nabla p = 0, \quad \text{force balance} \]
\[ \partial_t (\nabla \cdot u + \rho \nabla \cdot q^i) = 0, \quad \text{volume balance} \]
\[ \partial_t (\nabla \cdot (u + \rho \phi)) + \nabla \cdot (q^i + c^i \mu^-) = 0, \quad \beta = +, - , \quad \text{ion balances} \]

Constitutive Equations

\[
\begin{align*}
 q^i &= -K(\nabla \cdot \rho^i + c^i \nabla \mu^i + c^- \nabla \mu^-), \quad \text{Darcy’s law} \\
 q^i &= -\frac{RT}{\phi} \nabla \mu^i, \quad \beta = +, - \\
\end{align*}
\]

Fick’s law

Secondary Equations

\[
\begin{align*}
 \varphi &= 1 - (1 - \varphi_0)(1 - \nabla \cdot u) = c_{\varphi} = c_{\varphi}^0 (1 - \nabla \cdot u/\varphi_0) \\
 c^i &= \frac{1}{2} \alpha^i \epsilon^{i} + \frac{1}{2} (c^i)^2 + 4k^2 \exp \left( \frac{\mu^i - \mu^0_i + \mu^- - \mu^-_0}{RT} \right) \\
 p &= \mu^i - \mu^0_i + RT(\Gamma^i c^i + \Gamma^- c^-), \\
 \xi &= \frac{1}{\Sigma^i F} (\mu^i - \mu^0_i - RT \ln \frac{e^i}{e}), \quad \beta = +, - .
\end{align*}
\]

Experiments and Boundary Conditions

Mixed Finite Element

For the sake of local mass conservation, equations are discretised in space using mixed finite element.

We choose \( \Omega \in \mathbb{P}_1(\Omega), q_i, q_{tot} := q^i + c^i \mu^- \in \mathbb{R}T_0(\Omega) \) and \( \mu^i, \mu^- \in \mathbb{P}_0(\Omega) \).

Implicit time integration results into a non-linear set of equations.

Since different scales are apparent in the weak formulation, then we replace the reduced matrices by the scaled ones.

Substitution in the discrete variational formulation gives:

\[
\begin{align*}
 \mathbf{a}(\varphi, e^+ - e^-) \frac{\partial y}{\partial t} + \mathbf{b}(\varphi, e^+ - e^-) y &= \mathbf{g}(\varphi, c^+, c^-, c^+), \\
 \mathbf{a}_j &= \begin{bmatrix} \mathbf{B}^T, i = 5, j = 1, 0, \eta \neq 5, j \neq 1, \end{bmatrix}, \\
 \mathbf{b} &= \begin{bmatrix} A_0 0 0 B 0 0 0 0 C 0 0 0 0 D 0 0 0 0 E \end{bmatrix}.
\end{align*}
\]

The upper-left block is symmetric positive definite.

Given \( \varphi_0 \) and \( c^0_\phi \), the solution for the linear saddle point problem exists and is unique.

Analytical Solution

To verify the numerical solutions we derived a set of analytical solutions for the one-dimensional problem.

The idea behind is to derive a coupled system of diffusion equations.

\[
\begin{align*}
 C_{\mu^i} + C_{\mu^-} &= 0, \quad \mu^i = \mu^- = 0, \\
 C_{\mu^i} + C_{\mu^-} &= 0, \quad \mu^i = \mu^- = 0.
\end{align*}
\]

\( \mathbf{E} \) and \( \mathbf{P} \) are both symmetric positive definite and related to the Hessian of Helmholtz free energy and the diffusion matrix, respectively.

References


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