Mixed finite element for swelling of cartilaginous tissues, I

Citation for published version (APA):

Document status and date:
Published: 01/01/2007

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 04. Aug. 2019
Mixed Finite Element for Swelling of Cartilaginous Tissues, I
K. Malakpoor¹, E.F. Kaasschieter¹, J.M. Huyghe²

Introduction
The swelling and shrinking of cartilaginous tissues (CT) (like Intervertebral Disks) can be modelled by a four-component mixture theory in which a deformable and charged porous medium is saturated with a fluid with dissolved ions. This theory results in a coupled system of non-linear differential equations together with an algebraic constraint for electro-neutrality.

Four-Component Modeling of CT
=1 The two-component (Biot classical model) mixture theory is not able to describe the swelling and shrinking, that is caused by chemical and/or electrical loads.
=2 Therefore the theory is extended to four-component mixture theory.
=3 We assume incompressible and linear elastic solid saturated with incompressible and Newtonian viscous fluid. Also we neglect the inertial effects (quasi-static case) and the body forces.

Balance Equations
(degrees of freedom, material parameters)
\[-\nabla \cdot (\varphi \mu \mathbf{u}) + \lambda \nabla \cdot \mathbf{u} + \nabla p = 0,\]
\[\frac{\partial q^i}{\partial t} + \mathbf{u} \cdot \nabla q^i = 0,\]
\[\frac{\partial q^j}{\partial t} (\nabla \cdot \mathbf{u} + \varphi_0) + \nabla \cdot (q^{q_1} + q^{q_2} q^i) = 0, \varphi = +, -.

Ion balances
\[\mathbf{c} = -K (\partial^2 \mu^i + c^+ \nabla \mu^+ + c^- \nabla \mu^-), \]
\[q^{q_1} = -\frac{rT}{D_{\mu^1}} \nabla \mu^{q_1}, \beta = +, -.

Constitutive Equations
\[\varphi = \frac{1 - (\varphi_0)(1 - \nabla \cdot \mathbf{u})}{\varphi_0 (1 - \varphi_0)}, \]
\[c^{\varphi} = c^{\varphi}_0 (1 - \nabla \cdot \mathbf{u}/\varphi_0),\]
\[c^{c} = \frac{1}{2c^{\varphi}_0}, \quad c^{\varphi} = \frac{1}{2c^{\varphi}_0}, \quad c^{\varphi} = \frac{1}{2c^{\varphi}_0}, \quad c^{\varphi} = \frac{1}{2c^{\varphi}_0}, \quad c^{\varphi} = \frac{1}{2c^{\varphi}_0}.

Secondary Equations
\[\varphi = \frac{1 - (\varphi_0)(1 - \nabla \cdot \mathbf{u})}{\varphi_0 (1 - \varphi_0)}, \]
\[c^{\varphi} = c^{\varphi}_0 (1 - \nabla \cdot \mathbf{u}/\varphi_0),\]
\[p = \mu^1 - \mu^0 + RT (\Gamma^1 + \Gamma^- c^-),\]
\[\xi = \frac{1}{\varphi_0 \mu^1 - \mu^0 - RT \ln \varphi_0}, \beta = +, -.

Experiments and Boundary Conditions

Mixed Finite Element
=1 For the sake of local mass conservation, equations are discretised in space using mixed finite element.
=2 We choose \( a \in P_1(\Omega)^2, q^i, q^{q_1}_0 := q^{q_1} + c^{q_1} q^i \in \mathcal{RT}_0(\Omega) \) and \( p, \mu^1, \mu^0 \in P_0(\Omega) \).
=3 Implicit time integration results into a non-linear set of equations.
=4 Since different scales are apparent in the weak formulation, then we replace the reduced matrices by the scaled ones.
=5 Substitution in the discrete variational formulation gives:
\[\mathbf{A} \cdot \mathbf{q} = \mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} A_{ij} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_i \end{bmatrix}, \quad i = 5, j = 1,\]
\[\mathbf{A}_{ij} = \begin{cases} 0 & \text{if } i \neq 5, j \neq 1, \\
\end{cases}
=6 The upper-left block is symmetric positive definite.
=7 Given \( \varphi_0 \) and \( c^{q_1}_0 \), the solution for the linear saddle point problem exists and is unique.

Analytical Solution
=1 To verify the numerical solutions we derived a set of analytical solutions for the one-dimensional problem.
=2 The idea behind is to derive a coupled system of diffusion equations.
\[\frac{\partial}{\partial t} \begin{bmatrix} \mu^+ \\ \mu^- \end{bmatrix} = \mathbf{EP} \frac{\partial^2}{\partial x^2} \begin{bmatrix} \mu^+ \\ \mu^- \end{bmatrix},\]
=3 \( \mathbf{E} \) and \( \mathbf{P} \) are both symmetric positive definite and related to the Hessian of Helmholtz free energy and the diffusion matrix, respectively.

References

1) Eindhoven University of Technology. Dept. Mathematics and Computer Sci. P.O. Box 513. NL 5600 MB Eindhoven
2) Eindhoven University of Technology. Biomedical Engineering, Materials Technology PO Box 513 .WH 4.127 5600 MB Eindhoven. The Netherlands

/centre for analysis, scientific computing and applications