

Relevant sampling applied to event-based state estimation

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Relevant Sampling applied to Event-Based State-Estimation

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Abstract—To reduce the amount of data transfer in networked control systems and wireless sensor networks, measurements are usually sampled only when an event occurs, rather than synchronous in time. Today's event sampling methodologies are triggered by the current value of the sensor. State-estimators are designed to cope with such methods. In this paper we propose a sampling method in which an event is triggered depending on the reduction of the estimator's uncertainty and estimation-error. As such, communication requirements are minimized while attaining a certain error-covariance matrix and estimation error at the state-estimator. Furthermore, it is proven that the error-covariance matrix is asymptotically bounded in case the designed sampling protocol is combined with an event-based state-estimator. An illustrative example shows that the developed protocol provides an improved state estimation, while minimizing communication between sensor and state-estimator.

Keywords—state estimation; intelligent sensors; distributed estimation

I. INTRODUCTION

State-estimation is a technique that uses measurements to estimate the state-vector of a process. The best known method for state-estimation is the Kalman filter [1]. This method assumes a linear process, Gaussian noise distributions and synchronous measurement samples. Due to its simple, yet effective set of equations, the Kalman filter forms the basis for a wide variety of state-estimators [2], [3]. In recent years the amount of applications in which communication between the sensor and the estimator is implemented via a wireless network link has been rising. Such systems are often referred to as wireless sensor networks (WSNs) [4]. In general the limiting resources in WSNs are energy and communication, which explains the need to minimize data transfer.

Event sampling has the potential to decrease data transfer, as samples are not communicated synchronously in time but only when an a-priori defined event occurs in the data monitored by sensor. Examples of event sampling can be found in [5], [6]. Therein, predefined thresholds are set in the measurement-space. In case the sensor-value crosses a threshold, a measurement sample is communicated to a state-estimator or controller. Note however, that those sampling and communication protocols are independent of both the system properties, and the estimator or controller

that uses the samples. In [7] on the other hand, an event sampling method is presented that is best suitable for a stable controller. Events are triggered depending on the properties of the controller. As such, communication can be optimized with respect to the controller's stability and/or performance.

This paper addresses a similar approach for the state-estimation case. The contribution of this paper is twofold: first, we present a sampling protocol which depends on the reduction of the error-covariance matrix of the state-estimator if the measurement sample would be sent. The measurements are sent to an event-based state-estimator which performs an *update* rather than a *prediction* event in situations that no new measurement-samples are received. Second, we show that the sampling protocol in combination with the event-based state-estimator results in a bounded error-covariance matrix of the event-based state-estimator. The importance of this property is due to the fact that an unbounded covariance matrix represents an unbounded uncertainty on the estimated value of the state-vector.

The structure of this paper is as follows : section II presents the preliminaries and Section III summarizes the event-based state-estimator. The problem is formulated in Section IV. The sampling protocol and its analysis of the covariance-matrix is presented in Section V. This is illustrated with two examples in Section VI. Conclusions are drawn in Section VII.

II. PRELIMINARIES

\mathbb{R} , \mathbb{R}_+ , \mathbb{Z} and \mathbb{Z}_+ define the set of real numbers, non-negative real numbers, integer numbers and non-negative integer numbers, respectively. Let $\mathcal{C} \subset \mathbb{R}$ be given, then $\mathbb{Z}_{\mathcal{C}} := \{c \in \mathbb{Z} | c \in \mathcal{C}\}$. A transition-matrix $A_{t_2-t_1} \in \mathbb{R}^{a \times b}$ is defined to relate the vector $u(t_1) \in \mathbb{R}^b$ to a vector $x(t_2) \in \mathbb{R}^a$ as follows: $x(t_2) = A_{t_2-t_1}u(t_1)$. The transpose, inverse and determinant of a matrix $A \in \mathbb{R}^{m \times m}$ are denoted as A^T , A^{-1} and $|A|$ respectively. The i^{th} and maximum eigenvalue of a square matrix A are denoted as $\lambda_i(A)$ and $\lambda_{\max}(A)$ respectively.

The probability density function (PDF), as defined in [8], of the vector $x \in \mathbb{R}^m$ is denoted as $p(x)$. The Gaussian function (shortly noted as Gaussian) of vectors $x, u \in \mathbb{R}^n$ and matrix $P \in \mathbb{R}^{n \times n}$ is denoted as $G(x, u, P)$. If $G(x, u, P)$ is a PDF of random vector x , by definition the mean and

covariance-matrix of x are u and P , respectively. Moreover, $P \succeq 0$ is a symmetric matrix. Any Gaussian $G(x, u, P)$ can be represented by its sublevel-set $\varepsilon(\mu, P) \subset \mathbb{R}^q$, which is an ellipsoidal set defined as $\varepsilon(u, P) := \{x | (x - u)^T P^{-1} (x - u) \leq 1\}$, as is graphically depicted in Figure 1:

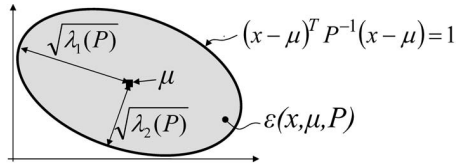


Figure 1. Graphical representation of $G(x, \mu, P)$ and its level-set $\varepsilon_{\mu, P}$. In this example the following inequality holds: $\lambda_1(P) > \lambda_2(P)$.

For a vector $x \in \mathbb{R}^m$ and a bounded Borel set [9] $Y \subset \mathbb{R}^m$, the set PDF is defined as $\Lambda_Y(x) : \mathbb{R}^n \rightarrow \{0, v\}$ with $v \in \mathbb{R}$ defined as the Lebesque measure [10] of the set Y , i.e.:

$$\Lambda_Y(x) = \begin{cases} 0 & \text{if } x \notin Y, \\ v^{-1} & \text{if } x \in Y. \end{cases} \quad (1)$$

III. AN EVENT-BASED STATE-ESTIMATOR

In this section we will summarize the result of the event-based state-estimator (EBSE), as presented in [11]. Consider a dynamical system with state vector $x \in \mathbb{R}^n$, process noise $w \in \mathbb{R}^l$, measurement vector $y \in \mathbb{R}^m$ and measurement noise $v \in \mathbb{R}^m$. This process is described by a discrete-time state-space model with $A_\tau \in \mathbb{R}^{n \times n}$ and $B_\tau \in \mathbb{R}^{n \times l}$, for all $\tau \in \mathbb{R}_+$, and $C \in \mathbb{R}^{m \times n}$, i.e.:

$$x(t) = A_\tau x(t - \tau) + B_\tau q(t - \tau), \quad (2a)$$

$$y(t) = Cx(t) + v(t). \quad (2b)$$

The above system description (2a) and (2b) can be regarded as a discrete time representation of a continuous-time plant $\dot{x}(t) = Ax(t) + Bu(t)$. In this case the matrices A_τ and B_τ would be defined with the time difference τ of two sequential sample instants, i.e.

$$A_\tau := e^{A\tau} \quad \text{and} \quad B_\tau := \int_0^\tau e^{A\eta} d\eta B.$$

However, we allow for the more general description (2a) and (2b). We assume that the process- as well as the measurement-noise are Gaussian PDFs with zero mean, for some $Q_\tau \in \mathbb{R}^{p \times p}$, $\tau \in \mathbb{R}_+$ and $R_v \in \mathbb{R}^{l \times l}$:

$$p(q(t - \tau)) := G(q(t - \tau), \underline{0}, Q_\tau), \quad p(v(t)) := G(v(t), \underline{0}, R_v).$$

The sensor uses an event sampling method which is based on y . Its sample instants are indexed by k . $y(t_k)$ denotes a measurement taken at the event instant t_k . As proposed in [11] $H_k \subset \mathbb{R}^{m+1}$ is a set in *time-measurement-space* that induces the event instants. An example of this set, in case of a two dimensional measurement-space, is graphically

depicted in Figure 2. To be precise, given that t_{k-1} was the latest event instant, the next event instant t_k is defined as:

$$t_k := \inf \left\{ t \in \mathbb{R}_+ \mid t > t_{k-1} \text{ and } \begin{pmatrix} y(t) \\ t \end{pmatrix} \notin H_k \right\}. \quad (3)$$

To illustrate the event triggering mechanism, let us present an example on how to determine set H_k . Let the events be triggered by applying the sampling method ‘‘Send-on-Delta’’. A new measurement sample $y(t_k)$ is generated when $|y(t) - y(t_{k-1})| > \Delta$. Note that this is equivalent to (3) in case

$$H_k := \left\{ \begin{pmatrix} y \\ t \end{pmatrix} \mid |y - y(t_{k-1})| \leq \Delta \right\}.$$

The state-estimator receives $y(t_k)$ to perform a state-update. However, the event instants t_k occur asynchronously in time while typically any functionality after the state-estimator, e.g. a controller, requires a regular and synchronous update. Hence, the state-estimator has to keep track of both event instants and synchronous instants. Let us define $\mathbb{T}_k(t)$ and $\mathbb{T}_c(t)$ as the set of time instants until time t that correspond to all event instants and synchronous instants, respectively. Therefore, if $\tau_s \in \mathbb{R}_+$ denotes the sampling time, we have that

$$\mathbb{T}_k := \{t_k \mid k \in \mathbb{Z}_+\} \quad \text{and} \quad \mathbb{T}_c := \{j\tau_s \mid j \in \mathbb{Z}_+\},$$

where the event instants t_k are generated by (3). The EBSE calculates an estimate of the state-vector and the error-covariance matrix at each sample instant $t \in \mathbb{T}$, with $\mathbb{T} := \mathbb{T}_k \cup \mathbb{T}_c$. At an event instant, i.e. $t \in \mathbb{T}_k$, the EBSE receives a new measurement $y(t_k)$ with which a state-update can be performed. At a synchronous instant, i.e. $t \in \mathbb{T}_c \setminus \mathbb{T}_k$, the EBSE does not receive a measurement. Standard estimators would perform a state-prediction using the model of the process. However, from (3) we observe that if no measurement y was received at $t > t_{k-1}$, it is still known that $(y(t)^\top, t)^\top \in H_k$. The estimator can exploit this information to perform a state-update not only at the event instants but also at all synchronous instants $t \in \mathbb{T}_c \setminus \mathbb{T}_k$. Next, we describe how this is implemented. Let us define $H_{k|t} \in \mathbb{R}^m$ as a section of H_k at the time instant $t \in \mathbb{R}_{(t_{k-1}, t_k)}$, which is graphically depicted in Figure 2, and formally defined as:

$$H_{k|t} := \left\{ y \in \mathbb{R}^l \mid \begin{pmatrix} y \\ t \end{pmatrix} \in H_k \right\}.$$

Therefore, two conditions hold for any $t \in \mathbb{T}$:

$$y(t) \in \begin{cases} \{y(t)\} & \text{if } t \in \mathbb{T}_k, \\ H_{k|t} & \text{if } t \in \mathbb{T}_c \setminus \mathbb{T}_k. \end{cases} \quad (4)$$

The estimator must first determine a PDF of the measurement $y(t)$. Therefore if $\delta(\cdot)$ denotes the Dirac-pulse and $\Lambda_Y(\cdot)$ the PDF as defined in (1), from equation (4) it follows that:

$$p(y(t)) = \begin{cases} \delta(y(t)) & \text{if } t \in \mathbb{T}_k, \\ \Lambda_{H_{k|t}}(y(t)) & \text{if } t \in \mathbb{T}_c \setminus \mathbb{T}_k. \end{cases} \quad (5)$$

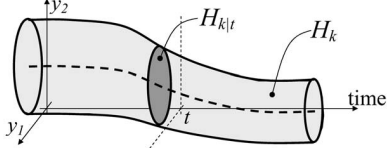


Figure 2. An example of H_k defining a set in the *time-measurement-space* and $H_{k|t}$ defining a section in the *measurement-space* at a certain time-instant $t \in \mathbb{R}_{(t_{k-1}, t_k)}$.

In [12] it was shown that any PDF can be approximated as a sum of Gaussians. Therefore, the original EBSE assumed that $p(y(t))$ could be approximated by N Gaussians. However, for the sake of clarity we will consider an EBSE in which $p(y(t))$ is approximated by a single Gaussian. Let $\hat{x}(t)$ denote the estimated state-vector and let $P(t)$ denote the error-covariance matrix at $t \in \mathbb{T}$. Furthermore, let $R(t) := R_v + R_H(t)$. The set of equations of the EBSE, in standard Kalman filter form, yields:

Step 1: prediction

$$\begin{aligned} \hat{x}^-(t) &= A_\tau \hat{x}(t - \tau), \\ P^-(t) &= A_\tau P(t - \tau) A_\tau^\top + B_\tau Q_\tau B_\tau^\top, \end{aligned} \quad (6a)$$

Step 2: measurement-update,

$$\begin{aligned} K(t) &= P^-(t) C^\top (C P^-(t) C^\top + R(t))^{-1} C, \\ P(t) &= (I - K(t) C) P^-(t), \\ \hat{x}(t) &= \hat{x}^-(t) + K(t) (\hat{y}(t) - C \hat{x}^-(t)). \end{aligned} \quad (6b)$$

In [11], [13] it was proven that all the eigenvalues of $P(t)$, i.e. $\lambda_i(P(t))$, are asymptotically bounded, if $H_{k|t}$ is a bounded set for all $t \in \mathbb{T}$.

IV. PROBLEM STATEMENT

Consider the dynamic process of (2a) and that measurements are performed according to (2b). The sensor applies an event sampling method on the sensor-data and communicates each measurement sample to the EBSE via a wireless network link. The system layout is graphically depicted in Figure 3. We assume for the sake of clarity a constant latency and no package loss between the Sensor Module and State estimator module.

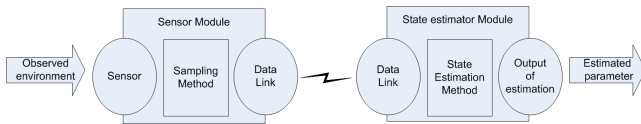


Figure 3. Schematic system representation. The sensor module contains a sensor and a sampling protocol and a datalink to send samples. The state-estimator module contains a datalink to receive samples, a state estimation method and a way to communicate the estimated state.

Most current sampling protocols, such as “Send-on-Delta” [5], [6], [14], only consider the sensor-data and not the result

of the state-estimator. Their focus is on locating the “interesting” measurement instants, i.e. events, by monitoring the data of the sensor. In case an event occurs in this sensor-data, a sample is taken. The interesting sample instants are defined as the situation at which the sensor-data crosses a predefined threshold in measurement-space. In the case of [5], [6] multiple thresholds are predefined and remain constant over time. A protocol with a time-dependent level can be found in [14], where the difference between the predicted and real measurement causes the event triggering. Note that in both examples the driving force behind sampling depends either on the measured value, or on the error between the estimated measurement and the real sensor value. However the most important parameter of any estimator is its covariance-matrix because it gives an indication of the estimation-error and therefore the uncertainty on the estimated state-vector.

Therefore the goal of this paper is to design a sampling protocol to send *relevant* measurement-samples to the state-estimator. Relevant measurements represent those measurements that if they were not sent, either the estimation-error or the covariance-matrix would become too large. Moreover, to be able to use the fact that no measurement-sample was received, the event-based state-estimator is used.

V. RELEVANT SAMPLING

In this section, we present a sampling protocol to reduce the amount of samples that are sent from the sensor to the event-based state-estimator (EBSE). In the designed sampling protocol, the sensor can decide when sending a measurement sample is relevant for the estimator. To determine what samples are relevant for a EBSE, the sensor uses the Kullback-Leibler divergence. After that, a derivation is given to determine the bounded measurement-set $H_{k|t}$ in case no measurement-sample is received at a synchronous instant. Finally, the asymptotic properties of $P(t)$ of the EBSE are analyzed for the designed sampling protocol.

A. Sampling protocol

The Kullback-Leibler divergence or relative entropy [15], as defined in (7), is a non-symmetric measure of the difference in the two PDFs $p_1(x)$ and $p_2(x)$. The PDF $p_1(x)$ is considered to be the “true” density while $p_2(x)$ the model or the approximation of density $p_1(x)$. The Kullback-Leibler divergence is sometimes also referred to as the information gain or the uncertainty reduction about x that is achieved if $p_1(x)$ can be used instead of $p_2(x)$ [15]. The definition of the Kullback-Leibler divergence of $p_1(x)$ and $p_2(x)$, yields:

$$D_{KL}(p_1(x)||p_2(x)) := \int_{-\infty}^{\infty} p_1(x) \log \frac{p_1(x)}{p_2(x)} dx \quad (7)$$

The information gain D_{KL} is used by the sensor to “measure” the relevance of sending a measurement to the estimator. A measurement is defined to be of relevance to the EBSE in case D_{KL} crosses some upper-bound $D_T \in \mathbb{R}_+$ [16]. We will

assume that both PDFs are described as a single Gaussian. To calculate D_{KL} , let us assume that sensor's last sampling instant is t_{k-1} and that at t_k is to be triggered. As such, the sensor must decide whether the current time t should be the next event instant, i.e. $t = t_k$. For that, let us define that the "true" density $p_1(x)$ is equal to the result of a standard asynchronous Kalman filter (AKF) [17] in case it is updated with the current sensor-value $y(t)$. The approximated density $p_2(x)$ is defined as the result of the AKF in case it would predict its state-estimates from time t_{k-1} to current time t . As $x(t)$ depends on time, the same should hold for $D_{KL}(t)$. As such, the next sample instant t_k , for some threshold D_T and $t > t_{k-1}$, is defined as:

$$t_k := \inf \{t \in \mathbb{R}_+ \mid t > t_{k-1} \text{ and } D_{KL}(t) > D_T\}, \quad (8a)$$

$$\begin{aligned} \text{where } p_1(x) &:= p(x(t)|y(t_0), \dots, y(t_{k-1}), y(t)) \\ \text{and } p_2(x) &:= p(x(t)|y(t_0), \dots, y(t_{k-1})). \end{aligned} \quad (8b)$$

The next step is to determine how $p_1(x)$ and $p_2(x)$ are calculated. Let us assume that the EBSE provides the sensor with its updated state-vector $\hat{x}(t_{k-1})$ and error-covariance matrix $P(t_{k-1})$. As such, the sensor knows that the current PDF of the EBSE, i.e.,

$$p(x(t_{k-1})|y(t_0), \dots, y(t_{k-1})) = G(x(t_{k-1}), \hat{x}(t_{k-1}), P(t_{k-1})).$$

The sensor determines the next sample instant, i.e. $t_k = t$, by calculating $D_{KL}(t)$. Note that to calculate $p_1(x)$ and $p_2(x)$ of (8b), the sensor performs an AKF rather than the EBSE. The main reason for this choice is to limit the processing demand at the sensor. As a consequence, the sensor's state-estimates are slightly different than the ones of the EBSE. Therefore, let us use the subscript s to emphasize that the estimates are calculated at the sensor rather than at the state-estimator. Let the predicted PDF $p_2(x)$ and the updated PDF $p_1(x)$ at current time t be described with the following Gaussians:

$$\begin{aligned} p_2(x) &= G(x(t), \hat{x}_s^-(t), P_s^-(t)) \\ p_1(x) &= G(x(t), \hat{x}_s(t), P_s(t)), \end{aligned}$$

for some $\hat{x}_s^-(t), \hat{x}_s(t) \in \mathbb{R}^n$ and $P_s^-(t), P_s(t) \in \mathbb{R}^{n \times n}$. If $\tau := t - t_{k-1}$, the values of these parameters are found by applying the equations of the AKF on their definitions as shown in (8b), i.e.,

$$\begin{aligned} P_s^-(t) &= A_\tau P(t_{k-1}) A_\tau^T + B_\tau Q_\tau B_\tau^T, \\ \hat{x}_s^-(t) &= A_\tau \hat{x}(t_{k-1}), \\ P_s(t) &= \left((P_s^-(t))^{-1} + C^T R_v^{-1} C \right)^{-1}, \\ \hat{x}_s(t) &= P_s(t) \left((P_s^-(t))^{-1} \hat{x}_s^-(t) + C^T R_v^{-1} y(t) \right). \end{aligned} \quad (9)$$

Substituting the result of (9) in $p_1(x)$ and $p_2(x)$ of (7), gives that the $D_{KL}(t)$ is constructed from a *dispersion*-term, denoted with α , and *signal*-term [18]:

$$\begin{aligned} D_{KL}(t) &= \alpha(t) \\ &+ \frac{1}{2} (x_s(t) - x_s^-(t))^T (P_s^-(t))^{-1} (x_s(t) - x_s^-(t)), \end{aligned} \quad (10)$$

where,

$$\alpha(t) := \frac{1}{2} \left(\log |P_s^-(t)| |P_s(t)|^{-1} + \text{tr} \left((P_s^-(t))^{-1} P_s(t) \right) - n \right).$$

The *dispersion*-term α depends on the error-covariance matrices of the update P_s and of the prediction P_s^- . The more a possible event sample reduces P_s compared to P_s^- the more relevant a measurement-sample becomes for the estimator. The same holds for the *signal*-term depending on the difference in means, i.e. x_s and x_s^- . The bigger the difference of the predicted and the updated state, the more relevant a new event sample instant becomes.

Note that Relevant Sampling is defined with a corresponding divergence threshold D_T . This threshold is used to design the bounded measurement-set $H_{k|t}$ that is required for a state-update at the EBSE at synchronous instants. The derivation from D_T to $H_{k|t}$ is presented next.

B. Measurement set

To use the Relevant Sampling efficiently in the EBSE, the set $H_{k|t}$ in *measurement*-space must be determined at the synchronous instants $t \in \mathbb{T}_c$. In this section, a method to design $H_{k|t}$ at a single sample instants t is presented. Therefore, we will omit the time representation of time-dependent variables in the section, with the exception of $H_{k|t}$. Let us assume that the EBSE can calculate the same predicted and updated state-estimates of the sensor, as shown in (9). This is not a limiting requirement as the processor that performs the EBSE has a high processing capacity. As such, also D_{KL} and α of (10) can be calculated at the EBSE.

Derivation of the set $H_{k|t}$ starts by rewriting $x_s - x_s^-$:

$$\begin{aligned} x_s - x_s^- &= P_s \left((P_s^-)^{-1} x_s^- + C^T R_v^{-1} y \right) - x_s^-, \\ &= P_s \left((P_s^-)^{-1} x_s^- + C^T R_v^{-1} y \right) \\ &\quad - P_s \left((P_s^-)^{-1} + C^T R_v^{-1} C \right) x_s^-, \\ &= P_s C^T R_v^{-1} (y - C x_s^-). \end{aligned} \quad (11)$$

With this result, the EBSE can derive the sensor's D_{KL} by substituting the result of (11) into the *signal*-term, i.e.:

$$\begin{aligned} D_{KL} &= \alpha + \frac{1}{2} (y - C x_s^-)^T W^{-1} (y - C x_s^-), \\ \text{where } W &:= \left(R_v^{-1} C P_s (P_s^-)^{-1} P_s C^T R_v^{-1} \right)^{-1}. \end{aligned} \quad (12)$$

As no measurement was received at synchronous instants, it follows that $D_{KL} \leq D_T$, for all $t \in \mathbb{T}_c$, and thus:

$$2(D_T - \alpha) \leq (y - C x_s^-)^T W^{-1} (y - C x_s^-). \quad (13)$$

Note that (13) equals the definition of a sublevel-set, as it was graphically depicted in Figure 1:

$$(y - C x_s^-)^T \left(\frac{1}{2(D_T - \alpha)} W^{-1} \right) (y - C x_s^-) \leq 1. \quad (14)$$

Therefore if the EBSE did not receive a new measurement sample at the sample instant $t \in \mathbb{T}_c$, the following condition holds for the measured value y at the sensor:

$$y \in \varepsilon(Cx_s^-, 2(D_T - \alpha)W).$$

Therefore, when re-introducing the time-dependencies, the update in the EBSE at the sample instants synchronously in time can be performed as follows:

$$H_{k|t} := \varepsilon(Cx_s^-(t), 2(D_T - \alpha(t))W(t)), \forall t \in \mathbb{T}_c. \quad (15)$$

C. Asymptotic analysis

An important property of the EBSE combined with Relevant Sampling is the stability of its estimation. Which means that all eigenvalues of $P(t)$ are asymptotically bounded in time. Note that this property was proven in [11], [13] under the condition that $H_{k|t}$ is a bounded set for all $t \in \mathbb{T}$. The definition of $H_{k|t}$ is shown in (15). From that we can derive that $H_{k|t}$ is bounded for all $t \in \mathbb{T}_c$ if $2(D_T - \alpha(t))W(t)$ is a bounded matrix, i.e. $\lambda_i(2(D_T - \alpha(t))W(t)) < \infty$ for all i . As $D_T - \alpha(t)$, these eigenvalues can be rewritten into:

$$\lambda_i(2(D_T - \alpha(t))W(t)) = 2(D_T - \alpha(t))\lambda_i(W(t)).$$

Therefore, stability of the EBSE is proven if $D_T - \alpha(t)$ is a bounded scalar and $\lambda_i(W(t)) < \infty$ for all i . Let us start with $D_T - \alpha(t)$ after which we will continue with $\lambda_i(W(t)) < \infty$.

Lemma V.1 *Let $P(t_{k-1}), R_v \succ 0$ be given, let both $P_s^-(t)$ and $P_s(t)$ be defined as (9) and let $\alpha(t)$ be defined as (10). If $t_{k-1} < t < t_k$, then it holds that $\alpha(t) \geq 0.5n$, for all $t \in \mathbb{T}_c$, and thus $D_T - \alpha(t) \leq D_T + 0.5n$.*

Proof: For brevity, let us omit the time-index t and recall that $2\alpha + n = \log|P_s^-||P_s|^{-1} + \text{tr}((P_s^-)^{-1}P_s)$. Then the Lemma is proven if $\log|P_s^-||P_s|^{-1} + \text{tr}((P_s^-)^{-1}P_s) \geq 0$.

Let us start by proving that $\text{tr}((P_s^-)^{-1}P_s) > 0$. Note that $P_s, P_s^- \succ 0$. Hence, from Proposition 8.6.5 of [19], i.e. for any $A \succ 0$ it also holds that $A^{-1} \succ 0$, it follows that $(P_s^-)^{-1} \succ 0$. Applying Lemma 2.2 of [20], i.e. for any $A, B \succ 0$ it holds that $\text{tr}(AB) > 0$, on $\text{tr}((P_s^-)^{-1}P_s)$ completes the first part of the proof.

The last step is to prove that $\log|P_s^-||P_s|^{-1} \geq 0$. From (9) it follows that $P_s^{-1} = (P_s^-)^{-1} + C^T R_v^{-1} C$. Starting from $R_v \succ 0$ Proposition 8.1.2 of [19], i.e. if $A \succ 0$ it holds that $SAS^T \succeq 0$ for any suitable S , gives that $C^T R_v^{-1} C \succeq 0$ and thus $P_s^{-1} \succeq (P_s^-)^{-1}$. Applying Corollary 8.4.10 of [19] on this inequality, i.e. for all $A \succeq B$ it holds that $|A| \geq |B|$, results in $|P_s|^{-1} \succeq |P_s^-|^{-1}$. Hence, $|P_s^-||P_s|^{-1} \geq 1$ and thus also $\log|P_s^-||P_s|^{-1} \geq 0$, which completes this proof. ■

The last property to prove, in order for $H_{k|t}$ to be bounded, is that $W(t)$ is a bounded matrix.

Lemma V.2 *Let $P(t_{k-1}), R_v \succ 0$ be given, let both $P_s^-(t)$ and $P_s(t)$ be defined as (9) and let $C \in \mathbb{R}^{m \times n}$ be such that*

$\text{rank}(C) = n$. If $W(t)$ is defined according to (12), it holds that $\lambda_i(W(t)) < \infty$, for all $i \in \mathbb{Z}_{[1,m]}$ and for all $t \in \mathbb{T}_c$.

Proof: For brevity, let us omit the time-index t . First note that from $P_s^- \succ 0$ it follows that $(P_s^-)^{-1} \succ 0$. Applying Proposition 8.1.2 of [19], i.e. for any $B \succ 0$ and $A \in \mathbb{R}^{m \times n}$ it holds that $ABA^T \succ 0$ if $\text{rank } A = m$, we have that $P_s(P_s^-)^{-1}P_s \succ 0$, $CP_s(P_s^-)^{-1}P_sC^T \succ 0$ and also $R_v^{-1}CP_s(P_s^-)^{-1}P_sC^TR_v \succ 0$. As such, it holds that $W^{-1} \succ 0$ which inheritably results in $\lambda_i(W^{-1}) > 0$ and thus $\lambda_i(W) < \infty$, for all $i \in \mathbb{Z}_{[1,m]}$, which completes this proof. ■

Remark V.3 Lemma V.2 required that the measurement-matrix $C \in \mathbb{R}^{m \times n}$ is such that $\text{rank}(C) = m$. The only systems that do not apply to this conditions are those using multiple sensors to measure the same state-element (mixture of elements). In this case multiple rows within C are equal due to which $\text{rank}(C) < m$. To circumvent this issue one should first fuse the uncorrelated measurements, which have an equal representation, into one fused measurement. This can be done via standard probability theory. The new matrix C will not have equal rows anymore and therefore have the correct rank.

Note that the Lemmas V.1 and V.2 proof that $H_{k|t}$ is a bounded set for all $t \in \mathbb{T}_c$. Therefore, it also holds that all eigenvalues of $P(t)$ are asymptotically bounded and that the EBSE, combined with Relevant Sampling, results in a stable estimator. The next section presents a small application example.

VI. ILLUSTRATIVE EXAMPLE

In this section we illustrate the effectiveness of the developed embedded protocol. The first case study is a virtual 1D object-tracking system. The second case is an actual 3D airplane tracking system.

In both cases the states $x(t)$ of the object are position and speed while the measurement vector $y(t)$ is position. The process-noise $w(t)$ represents the object's acceleration. We will use the EBSE as standard state-estimator and compare two different sampling methodologies in the two cases. The first one is Relevant Sampling (RS). The determination of $H_{k|t}$ in the case that no measurement is received is implemented as proposed in Section V. The second sampling method is Send-on-Delta (SoD), for which it follows that $H_{k|t} := \{y \in \mathbb{R} \mid |y(t) - y_{k-1}| \leq \Delta\}$. In both cases we must approximate the PDF $\Lambda_{H_{k|t}}(y(t))$ into the Gaussian $G(y(t), \hat{y}(t), R_H(t))$, for all $t \in \mathbb{T}_c \setminus \mathbb{T}_k$, to be able to update the estimated state of the EBSE. This is done as follows:

$$\text{RS: } \hat{y}(t) = Cx_s^-(t), \quad R_H = 2(D_T - \alpha(t))W(t)/4.$$

$$\text{SoD: } \hat{y}(t) = y_{k-1}, \quad R_H = \Delta^2/4.$$

In the first case we simulate a moving object. The object's position, speed and acceleration that are used in this simulation-example are presented in Figure 4.

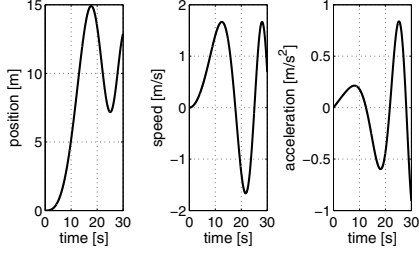


Figure 4. The position, speed and acceleration of the object.

As the covariance of the acceleration is 0.5, let us define that $Q = 0.5$. The process-model yields $A = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} \frac{\tau^2}{2} & \tau \end{pmatrix}^T$, $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $D = 0$, which is in fact a discrete-time double integrator. The sampling time is $\Delta_s = 0.1[s]$ and the measurement-noise covariance is $R_v = 0.2 \cdot 10^{-3}$.

In this simulation we used for the RS protocol the threshold $D_T = 1.5$ and for the SoD protocol the threshold of $\delta = .5$. These thresholds were chosen so both protocols communicate roughly the same amount of samples. The results of the simulation with respect to communication are shown in Figure 5.

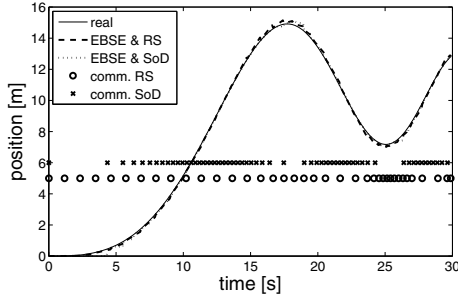


Figure 5. The communication results of the Relevant Sampling versus Send-on-Delta.

Note that the time when communication takes place differs significantly. In the case of Send-on-Delta most communication takes place when the object has a high speed. In the case of Relevant Sampling it is just the opposite. Here most communication takes place when the *change* in speed is large. In total, the amount of samples that were sent by Send-on-Delta is 54 with an mean squared error (MSE) of $0.75[m]$. Relevant Sampling uses 51 communications and results in a MSE of $0.42[m]$. The next figure shows that results of the modeled estimation-error, i.e. $\text{tr}(P(t))$, and the true estimation-error, i.e. $(x(t) - \hat{x}(t))^T(x(t) - \hat{x}(t))$, for both types of sampling.

The next case is a recorded 2.5 hour track of an airplane used for Synthetic Aperture Radar research. It is a highly accurate track generated from differential GPS and a special developed IMU. The track is illustrated in Figure 7. In this case we use this 3D track as the input signal for the RS and SoD protocols.

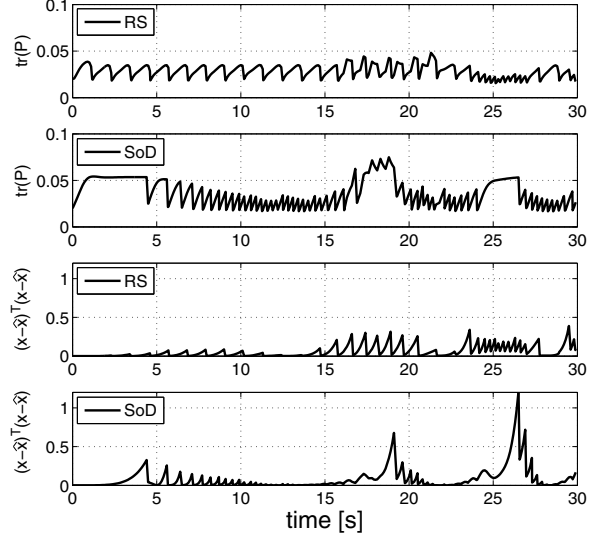


Figure 6. The estimation results of the Relevant Sampling versus Send-on-Delta.

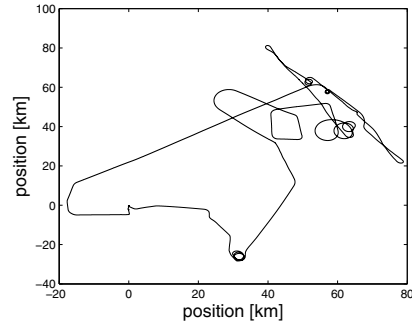


Figure 7. The airplane trajectory.

The process-model in this case uses the in 3D extended A , B , C and D matrices as defined in the case above. The sampling time is $\Delta_s = 0.1[s]$. The measurement-noise covariance R_v is a 3×3 matrix and has its diagonal filled with $0.2 \cdot 10^{-3}$ and the process-noise Q is a three by three matrix and has its diagonal filled with 2.

In this simulation we used for the RS protocol the threshold $D_T = 15$ and for the SOD protocol the threshold of $\delta = .4[km]$ to ensure both methods communicate the same amount of samples. In total, the amount of samples that were sent by SoD is 1824 with an mean squared error of $6.3 \cdot 10^{-4}[km]$. RS uses 1797 communications and results in a MSE of $3.4 \cdot 10^{-4}[km]$.

In both cases the RS protocol outperformed the SoD protocol in accuracy.

VII. CONCLUSIONS

This paper proposes the design of a measurement-sampling protocol that is used in combination with an event-based state-estimator. The protocol minimizes communica-

tion and the state estimation is accurate and stable even when no samples are sent.

Analysis of the RS protocol in a simulation and a actual example showed the positive result on the estimation error while minimizing communication.

Choosing threshold D_T influences the amount samples and so the error of the state estimator, the covariance of the state estimator and the average time interval between samples. This relationship will be explored in future work.

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