Some methods of classical mechanics applied to continuous systems

SOME METHODS OF CLASSICAL MECHANICS APPLIED TO CONTINUOUS SYSTEMS

proefschrift

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SOME METHODS OF CLASSICAL MECHANICS

APPLIED TO CONTINUOUS SYSTEMS

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Die Mechanik ist ein Versuch alle *wahren* Sätze, die wir zur Weltbeschreibung brauchen, nach einem Plane zu konstruieren.

Mechanics is an attempt to construct according to a single plan all the true propositions that we need for the description of the world.

6.3211

Man hat ja auch davon eine Ahnung gehabt, dass es *ein* "Gesetz der kleinsten Wirkung" geben müsse, ehe man genau wusste, wie es lautete.

Indeed people even surmised that there must be *a* "law of least action" before they knew exactly how it went.

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Wir *glauben* nicht a priori an ein Erhaltungsgesetz, sondern wir *wissen* a priori die Möglichkeit einer logischen Form.

We do not have an *a priori belief* in a law of conservation, but rather *a priori knowledge* of the possibility of a logical form.

Ludwig Wittgenstein: Tractatus logico-philosophicus.

CONTENTS

0. Abstract

This work deals with several subjects taken from the field of classical mechanics (i.e. the dynamics of systems with a finite number of degrees of freedom) and their generalizations to continuum physics. A main part of the treatment is generally known and can be found in various text books [1], [2], [3], [4].

The following subjects have received special attention:

(i) Noether's theorem as a method to find conservation laws.

(ii) The concept of sesquilinear Lagrangian functions and functionals which may be used to describe first-order ordinary and partial differential equations

(iii) The concept of canonical transformations and their generating functionals in continuum physics (chapter 1,2,3,5).

A treatment on the concept of additive transferability of constants of the motion is presented in chapter 4. Here we also discuss the physical significance of the Zilch (a conserved quantity of the electromagnetic field, discovered by Lipkin in 1964).

As an application of the theory presented, the equations of motion of the inviscid compressible fluid expressed in terms of material (Lagrangian) coordinates, are discussed in both Hamiltonian and Lagrangian form. Using Noether's theorem several conservation laws are derived. The conversion from material coordinates into local (Eulerian) coordinates is described as a special case of a canonical transformation, i.c. a point transformation (chapter 6,7).

$$\frac{\delta}{\delta q_i}\mathcal{W}\{q\} = 0 \tag{3}$$

The left-hand side of (3) denotes the *functional derivative*, sometimes called *variational derivative* and implicitly defined by *)**)

$$\int_{t_1}^{t_2} \frac{\delta}{\delta q_i}\mathcal{W}\{q\} f_i(t)dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\mathcal{W}\{q+\varepsilon f\} - \mathcal{W}\{q\}\right) \tag{4}$$

where f_i is an element of a function space S_i. This function space S_i is defined by the boundary condition that at $t = t_1$ and $t = t_2$, f_i and its derivatives of order $n_i - 1$, or less, will vanish, if n_i is the order of the highest derivative of q_i occurring in $L[q]$. Furthermore the function space S_i has to yield $C_i = C_i + S_i$. Additionally certain conditions ensuring a decent behaviour should be imposed, but we will not specify and mostly not even mention these.

Equations (1) and (3) are equivalent. This can be proved with the definition (2) of $\mathcal{W}$, first-order Taylor expansion of the integrand and integration by parts, the integrated terms being zero because of the boundary condition.

In literature equation (3) is known as *Hamilton's principle*, the *variational principle* or *the action principle* and formulated as

$$\delta \int_{t_1}^{t_2} L[q]dt = 0$$

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*The left-hand side of (4) is commonly written

$$\int_{t_1}^{t_2} \frac{\delta}{\delta q_i(t)}\mathcal{W}\{q\} f_i(t)dt$$

If there is no risk of confusion we drop the explicit dependence of the arguments of functions and of the functional derivative.

**We will use Einstein's convention, throughout this work: summation over all repeated indices is implied.

Here, the symbol δ denotes the first order variation, which is essentially the same as our functional derivative. More familiar names for the action principle frequently used are the maximum principle, the minimum principle or the extremum principle. All the names are deceiving and therefore not useful. In some cases it may be proved in a rather complicated way that the action functional takes on a maximum or a minimum value for solutions of Euler's equations, but this aspect is hardly relevant to our purposes, since we dot not intend to use the variational principle as a method for approximative calculations. If one does so, both a maximum and a minimum principle are of advantage [8]. A legitimate name for the action principle is *stationary principle* or *principle of stationary action*.

We further note that $L[q]$ may depend on derivatives of q of higher than the first order. The methods obtained in the following will therefore also apply to dynamical systems governed by differential equations of order higher than the second. For the sake of simplicity, however, we will often restrict our attention to Lagrangians depending on t, q_i and $\dot{q}_i$ only.

Finally, we want to emphasize that so far no use has been made of the interpretation of the Lagrangian as the difference of total kinetic and potential energy. We do not require such an interpretation. In many cases it is not valid, for instance in the case of magnetic forces the Lagrangian is not such a difference, but also in other cases this need not be so, because one can always add to the Lagrangian a total time derivative of an arbitrary function $F[q]$ of the coordinates q_i, their time derivatives and the time t. This is readily seen by noting that an additional term $\frac{d}{dt} F[q]$ to $L[q]$ gives a contribution to the action functional $\mathfrak{A}$ in the form of an additional constant. Especially in one dimension ($n = 1$) one may construct very *strange* Lagrangians that give with (1) and (2) correct equations of motion [5].

The possibility of adding to the Lagrangian a term of the form $\frac{d}{dt} F[q]$ will be called a *gauge transformation*, because it is closely related to the gauge transformation in electrodynamics:

A Lagrangian for a particle, with mass m, electrical charge e and position $\underline{r}$, moving in an electromagnetic field described by the scalar potential ϕ and vector potential $\underline{A}$ is represented by:

$$L = \frac{1}{2} m\dot{\underline{r}}.\dot{\underline{r}} + e\dot{\underline{r}}.\underline{A} - e\phi$$

If we define L' by

$$L' = L + \frac{d}{dt} \chi(\underline{r},t)$$

it can be written

$$L' = \frac{1}{2} m\dot{\underline{r}}.\dot{\underline{r}} + e\dot{\underline{r}}.(\underline{A} + \frac{1}{e} \nabla\chi) - e(\phi - \frac{1}{e} \frac{\partial}{\partial t} \chi)$$

This Lagrangian describes the motion of a particle in an electromagnetic field with the scalar potential

$$\phi' = \phi - \frac{1}{e} \frac{\partial}{\partial t} \chi$$

and vector potential

$$\underline{A}' = \underline{A} + \frac{1}{e} \nabla\chi$$

It is known from electrodynamics that $\underline{A}'$ and ϕ' describe the same electromagnetic field as $\underline{A}$ and ϕ.

Although it is rather uncommon to speak about a gauge transformation when no electromagnetic field is present, this concept may be very useful here. We also note that if a gauge transformation has been applied to a Lagrangian density, the boundary conditions of the variational principle must generally be adapted.

1.2 Lagrange's function and coordinate transformations

A coordinate transformation may be useful to simplify the equations of motion. Consider such a transformation of the coordinates $q_1, q_2, \ldots q_n$ to new coordinates $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n$, where

$$q_i = f_i(\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n, t) = q_i(\bar{q}, t) \tag{1}$$

represents a one-to-one mapping, such that (1) can be uniquely inverted, which will be the case if the transformation matrix

$$\frac{\partial q_i}{\partial \bar{q}_k} \tag{2}$$

is non-singular.

We will now show, provided that the original equations of motion can be derived from a Lagrangian $L[q]$, that the transformed equations of motion can be derived from a Lagrangian which is apart from a gauge transformation equal to

$$\bar{L}[\bar{q}] = L[q(\bar{q}, t)] \tag{3}$$

Proof:

Let the action functional of the untransformed system be

$$\mathcal{W}\{q\} = \int_{t_1}^{t_2} L[q]\,dt$$

and that of the transformed system

$$\bar{\mathcal{W}}\{\bar{q}\} = \int_{t_1}^{t_2} \bar{L}[\bar{q}]\,dt$$

With (3) we obtain

$$\bar{\mathcal{W}}\{\bar{q}\} = \mathcal{W}\{q(\bar{q}, t)\}$$

and with the *chain rule* for functional differentiation (which is easily verified):

$$\frac{\delta}{\delta \bar{q}_k} \bar{\mathcal{W}} = \frac{\partial q_i}{\partial \bar{q}_k} \frac{\delta}{\delta q_i} \mathcal{W}$$

1.2
1.3.

we see because of (2)

$$\frac{\delta}{\delta \bar{q}_k}\bar{\mathcal{W}} = 0 \quad \text{if and only if} \quad \frac{\delta}{\delta q_k}\mathcal{W} = 0$$

We may now conclude: the equations of motion, which can be derived from $\mathcal{W}[q]$ and $\bar{\mathcal{W}}[\bar{q}]$ or $L[q]$ and $\bar{L}[\bar{q}]$ respectively are equivalent. As a result we obtain that the Lagrangians $L[q]$ and $\bar{L}[\bar{q}]$ give equivalent descriptions of the same physical system, or: are *equivalent*. This result may often be useful, since a transformation of the Lagrangian is usually less complicated than a transformation of the equations of motion.

I.3 Examples

I.3.1 The one-dimensional linearly-damped harmonic oscillator

The one-dimensional linearly-damped harmonic oscillator is characterized by the equation

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = 0 \tag{1}$$

Physically, this equation may represent the motion of a mass on a linear string, or a torsional pendulum, both with linear friction. We thus become involved here with friction forces. It is nevertheless possible to write down a correct Lagrangian for this problem. Although the forces do not depend explicitly on the time, the Lagrangian will do so.

Equation (1) can be transcribed into a more common form:

$$\ddot{x} + \omega^2 x = 0 \tag{2}$$

By the coordinate transformation

$$q = x \exp -\gamma t \tag{3}$$

it then becomes clear that

$$\omega^2 = \omega_0^2 - \gamma^2.$$

A Lagrangian for (2) is

$$L[x] = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 \tag{4}$$

and therefore

$$L[q] = L[x[q]] = \frac{1}{2}(\dot{q}^2 + 2\gamma q\dot{q} + (2\gamma^2 - \omega_0^2)q^2)\exp 2\gamma t \qquad (5)$$

Substitution of (4) and (5) in Euler's equations shows that they lead to the desired equations of motion (2) and (1).

We note that the Lagrangians (4) and (5) are quadratic in the coordinate q and the velocity $\dot{q}$. This is usually found to occur when dealing with second-order linear differential equations.
We further note that equation (1) is transformed by the transformation (3) into an equation for which a classical Hamiltonian exists (see section 2.1). The standard rules of quantization may be used to construct a quantum mechanical Hamilton operator. It is not obvious to us whether the eigen states of this Hamilton operator have any physical significance or not.

A different method of dealing with the damped oscillator is to transform the equation of motion (2) into a first-order differential equation.

Let $\quad y = \frac{1}{\omega}\dot{x}$

so that $\quad \ddot{x} + \omega^2 x = \omega\dot{y} + \omega^2 x = 0$

The equation of motion is then transcribed into

$$\begin{pmatrix} x \\ y \end{pmatrix}_t + \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (6)$$

In the next section we will introduce a Lagrangian for a wider class of equations of motion

$$u_t + Au = 0 \qquad (7)$$

with A a square anti-Hermitean matrix $A = -A^\dagger$ and u a n-dimensional column vector.
Obviously, equation (6) is a special form of equation (7).

1.3.2 Sesquilinear Lagrangians

As a more general, and less known, example of the use of Lagrangians, we shall now consider systems that can be described by the vector-relation

$$u_t + Au = 0 \tag{1}$$

where

$$\begin{aligned} &u = \text{column } (u_1(t), u_2(t),\ldots,u_n(t)) \\ &A = \text{constant square anti-Hermitean matrix; } A = -A^\dagger \end{aligned} \tag{2}$$

Such systems are called *linear*, because the differential equation is linear. Since matrix A does not depend on time, the system is also called *autonomous* [9], or *conservative* by some authors [10].

It is not possible to introduce in the normal way a Lagrangian depending on the coordinates $q_i = u_i$ and velocities $\dot{q}_i = \dot{u}_i$ describing equation (1). Therefore, we introduce the rather unfamiliar concept of the *sesquilinear Lagrangian*.

$$\text{Let } L[v,u] = v^\dagger u_t + v^\dagger Au \tag{3}$$

where

$$v = \text{column } (v_1(t), v_2(t),\ldots,v_n(t))$$

and $\dagger$ denotes Hermitean conjugation

It is easily verified that for any complex numbers λ_1 and λ_2

$$\begin{aligned} L[v,\lambda_1 u_1 + \lambda_2 u_2] &= \lambda_1 L[v,u_1] + \lambda_2 L[v,u_2] \\ L[\lambda_1 v_1 + \lambda_2 v_2,u] &= \lambda_1^* L[v_1,u] + \lambda_2^* L[v_2,u] \end{aligned} \tag{4}$$

where $*$ denotes complex conjugation.
According to the terminology Jauch [10] uses for functionals, L is called *linear* in u, and *conjugate-linear* in v and therefore *sesquilinear**).
We may consider (3) as a Lagrangian for the coordinates u_i and v_i. With equation (1.1.1) we can determine Euler's equations associated with (3). However, we shall derive them directly and into more detail for further reference.

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*) *sesqui* literally means: *and a half*

Let $\delta u = \varepsilon f$ and $\delta v = \varepsilon g$ be infinitesimal variations of the vectors u and v, and define the *first-order variation* δL of L by

$$\delta L = \varepsilon \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(L[u + \delta u, v + \delta v] - L[u,v] \right) \tag{5}$$

so that

$$\begin{aligned} \delta L &= \delta v^\dagger u_t + v^\dagger \delta u_t + \delta v^\dagger A u + v^\dagger A \delta u \\ &= \frac{d}{dt}(v^\dagger \delta u) + \delta v^\dagger u_t - v_t^\dagger \delta u + \delta v^\dagger A u + v^\dagger A \delta u \\ &= \frac{d}{dt}(v^\dagger \delta u) + \delta v^\dagger (u_t + A u) - (v_t + A v)^\dagger \delta u \end{aligned} \tag{6}$$

with Hamilton's principle this leads to following Euler's equations:

$$u_t + A u = 0 \tag{7}$$

and

$$v_t + A v = 0 \tag{8}$$

We see that (7) is in agreement with (1). The Lagrangian (3) can properly describe the system (1).

The choice of v is still arbitrary, apart from the fact that it must satisfy equation (8). This implies that v may be chosen identical with u, but this is not necessary. Since any solution of (8) is correct, we may take $v = Mu$, where M is a constant $n \times n$ matrix commuting with A, $AM - MA = 0$. We will use this later. However, it is essential that u and v are considered independent during the process of variation.
If for instance v is substituted by u in equation (3), we obtain with (2)

$$L = u^\dagger u_t = \frac{d}{dt} \left(\frac{1}{2} u^\dagger u \right)$$

and this certainly does not describe the equation of motion (1).

As the equations (7) and (8) are equivalent it might seem remarkable that u and v appear in the expression for L in obviously different ways. However, it is possible to construct an expression without this dissimilarity:
(7) and (8) will also follow from the Lagrangian

$$L'[v,u] = \frac{1}{2} v^\dagger u_t - \frac{1}{2} v_t^\dagger u + v^\dagger A u$$

The difference between the two Lagrangians is

$$L[v,u] - L'[v,u] = \frac{d}{dt} \left(\frac{1}{2} v^\dagger u \right)$$

i.e. a total derivative. So here we have an example of the gauge transformation discussed previously (section 1.1).
Multiplying L'[v,u] by a purely imaginary constant we may even construct a *symmetric* Lagrangian L", in the sense that

$$L"[v,u] = (L"[u,v])^*$$

which is equivalent to L and to L'.

1.4 Noether's theorem

In this section we shall investigate the relation between certain coordinate transformations and the existence of conservation laws. The results can be considered as a restriction of Noether's famous theorem to mechanical systems [11]. Many text books do so in a way which is in our opinion rather unreadable. To our knowledge, the best articles are those of Noether herself [11] and Hill [12]. More recent papers on the subject are those of Broer-Kobussen [13] and Rosen [14].

Consider a dynamical system described by the coordinates $q_1,\ldots,q_n$ and the Lagrangian

$$L[q] = L(q_1,q_2,\ldots,q_n,\ \dot{q}_1,\ldots,\dot{q}_n,\ \ddot{q}_1,\ldots,\ddot{q}_n,\ldots,t)$$

The equations of motion are then given by

$$\frac{\delta}{\delta q_i}\mathcal{W}\{q\} = 0 \tag{1}$$

with

$$\mathcal{W}\{q\} = \int_{t_1}^{t_2} L[q]dt \tag{2}$$

We will investigate the behaviour of $\mathcal{W}\{q\}$ for infinitesimally small transformations (*variations*) of the coordinates q_i:

$$\bar{q}_i(t) = q_i(t) + \varepsilon g_i(q,\dot{q},\ddot{q},\ldots,t) \tag{3}$$

where ε is a vanishingly small parameter, and $g_i(\alpha,\beta,\ldots,t)$ an arbitrary function of α, β,... and t.
We write

$$\varepsilon\ g_i(q,\dot{q},\ddot{q},\ldots,t) = \varepsilon\ g_i[q] = \delta q_i \tag{4}$$

Note that we did not require q_i to satisfy the equations of motion.

If we consider the *first-order variation* $\delta\mathcal{W}$ of $\mathcal{W}$ defined by

$$\begin{aligned}\delta\mathcal{W} &= \varepsilon \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\mathcal{W}\{q + \varepsilon g\} - \mathcal{W}\{q\}\right) \\ &= \varepsilon \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\mathcal{W}\{q + \delta q\} - \mathcal{W}\{q\}\right)\end{aligned} \tag{5}$$

and compare this relation with the definition of the functional derivative, we obtain

$$\delta\mathcal{W} = \int_{t_1}^{t_2} \frac{\delta\mathcal{W}}{\delta q_i}\, \delta q_i \, dt + \text{additional terms}$$

The additional terms would vanish if the δq_i's would satisfy the boundary conditions required in the definition of the functional derivative.
We therefore write

$$\delta\mathcal{W} = \int_{t_1}^{t_2} \left(\frac{\delta\mathcal{W}}{\delta q_i}\, \delta q_i + \frac{d}{dt}\, \chi[\delta q, q]\right) dt, \tag{6}$$

where $\chi[\delta q, q]$ represents a function of q_i, δq_i and their time derivatives, comprising the stock terms of the integration by parts mentioned in section (1.1). The function χ is therefore linear in δq and its derivatives and is independent of the integration boundary t_1, t_2. Furthermore χ is known if L is known.

Equation (6) being satisfied for arbitrary t_1 and t_2, the assumption

$$\delta\mathcal{W} = 0$$

yields

$$\frac{d}{dt}\, \chi[\delta q, q] = -\frac{\delta\mathcal{W}}{\delta q_i}\, \delta q_i \tag{7}$$

This leads to the following theorem:

Theorem:

If there exists a variation of the coordinates

$$\bar{q}_i = q_i + \delta q_i = q_i + \varepsilon\, g_i[q]$$

such that it leaves the action functional $\mathcal{W}$ for arbitrary t_1 and t_2

invariant in first order of ε, there exists a function χ satisfying the relation

$$\frac{d}{dt}\chi = \frac{\delta \mathfrak{W}}{\delta q_i}\delta q_i \tag{8}$$

This theorem is the same as Noether's theorem with the restriction that only mechanical systems are concerned and the non-essential restriction that transformations of the independent variable are not considered. There also exists an extended version of Noether's theorem, also published by Noether. The restricted (in the sense of this paragraph) version of this extended theorem is also evident now.

Theorem:

If there exists a variation of the coordinates

$$\bar{q}_i = q_i + \delta q_i = q_i + \varepsilon\, g_i[q]$$

such that it leaves the action functional $\mathfrak{W}$ in first order of ε invariant up to a term

$$\int_{t_1}^{t_2} \frac{d}{dt}\psi[q]dt,$$

there exists a function χ satisfying the relation

$$\frac{d}{dt}(\chi - \psi) = \frac{\delta \mathfrak{W}}{\delta q_i}\delta q_i \tag{9}$$

The variations mentioned in the theorems will be called *invariant transformations* or *invariant variations*.

It is easily seen that both theorems can be inverted, e.g. for the second theorem:

Inverted theorem:

If there exist a function $\phi[q]$ and a set of functions $f_i[q]$, $i = 1,2,\ldots,n$, mutually satisfying the relation

$$\frac{d}{dt}\phi[q] = \frac{\delta \mathfrak{W}}{\delta q_i} f_i[q]$$

the variation of the coordinates

$$\bar{q}_i = q_i + \varepsilon f_i$$

is an invariant variation and yields the invariance of the action functional $\mathfrak{W}$ in first order of ε up to a term

$$\int_{t_1}^{t_2} \frac{d}{dt} (\chi[\varepsilon f, q] - \phi[q])dt$$

The advantages of Noether's theorems become apparent when they are applied to functions q_i which satisfy the equations of motion (1). In that case the right-hand side of equations (8) en (9) will vanish, which yields us a method to construct conservation laws. In the following section we will give some examples of this. An old interesting application of Noether's theorem is found in the paper of Bessel-Hagen [15].

An interesting question is whether Noether's theorem applied to solutions of the equations of motion, in the form where the right-hand side of (8) and (9) is replaced by zero, can also be inverted. Or: does an invariant transformation exist for any constant of the motion?
On the basis of the previous remarks it is clear that this is true for conserved quantities Ω the time derivative of which can be written identically (i.e.: independent of the question whether the q_i's satisfy the equations of motion or not) as

$$g_i \frac{\delta}{\delta q_i} \mathfrak{W}$$

where g_i is finite for solutions of the equations of motion.
Fletcher [16] and Dass [17] proved for any constant of the motion $\Omega[q]$ that another constant of the motion $\Omega'[q]$ exists with the following properties:

$$\Omega[q] = \Omega'[q] \text{ if } q_i \text{ satisfies } \frac{\delta \mathfrak{W}}{\delta q_i} = 0$$

$$\frac{d}{dt} \Omega'[q] = \frac{\delta \mathfrak{W}}{\delta q_i} g_i[q] \text{ for all } q_i$$

Moreover, we observe that in the foregoing version of Noether's theorem there is no need, to introduce explicit transformation of the independent variable t. In this aspect our treatment differs from those of Noether [11], Hill [12], Courant & Hilbert [18] and many others. They consider infinitesimally small

transformations of both the independent variables (in this section only t) and the dependent variables - the coordinates q_i. It is obvious that we did not need such a transformation of the time t. Steudel [19] proved in an elegant way that any infinitesimal transformation involving both the q_i's and t can be replaced by another transformation involving only the q_i's and giving the same conservation law. For instance, equivalent with the translation of the time $\bar{t} = t+\varepsilon$ is the transformation $\bar{q}_i(t) = q_i(t) + \varepsilon\dot{q}_i(t)$

Noether's theorem, is often easier to handle in combination with the Lagrangian than with the action functional. Analogous to (5) the first-order variation δL of $L[q]$ is defined by

$$\delta L = \varepsilon \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} (L[q + \delta q] - L[q]) \tag{10}$$

The equations of motion are written as

$$\Lambda_i L = \frac{\delta \mathfrak{A}}{\delta q_i} = 0 \tag{11}$$

Analogous to (6):

$$\delta L = \frac{d}{dt} \chi[\delta q, q] + \Lambda_i L \, \delta q_i \tag{12}$$

or for solutions of the equations of motion, equation (12) takes the form

$$\delta L \doteq \frac{d}{dt} \chi[\delta q, q] \tag{13}$$

The symbol $\doteq$ denotes an equality for solutions of Euler's equations and is used instead of = when confusion might arise.
If now certain fixed δq_i's yield

$$\delta L = \frac{d}{dt} \psi[q] \tag{14}$$

then by (13) and (14)

$$\frac{d}{dt} (\chi[\delta q, q] - \psi[q]) \doteq 0 \tag{15}$$

Finally, we want to stress that because an infinitesimal invariant transformation leaves the Lagrangian invariant up to a gauge transformation, it does not affect the equations of motion (in first order).

1.5 Applications of Noether's theorem

1.5.1 Sesquilinear Lagrangians

Noether's theorem can be applied to the sesquilinear Lagrangian of section 1.3.2. The results obtained in this way will be demonstrated in a discussion of the two-dimensional harmonic oscillator (section 1.5.2) and of the *one*-dimensional linearly-damped harmonic oscillator (section 1.5.3).

Let $$L[v,u] = v^\dagger u_t + v^\dagger A u, \tag{1}$$

where $$A = -A^\dagger \tag{2}$$

is a matrix independent of time. As discussed in section (1.3.2) the equations of motion are

$$\begin{aligned} u_t + Au &= 0 \\ v_t + Av &= 0 \end{aligned} \tag{3}$$

Consider infinitesimally small transformations (variations) of the *coordinate vectors* u and v:

$$\begin{aligned} \bar{u} &= u + \varepsilon g = u + \delta u, \\ \bar{v} &= v + \varepsilon h = v + \delta v, \end{aligned}$$

g and h being column vectors depending on the time t.

According to (1.3.2.6) the first order variation of L is

$$\delta L = \delta v^\dagger (u_t + Au) - (v_t + Av)^\dagger \delta u + \frac{d}{dt}(v^\dagger \delta u) \tag{4}$$

or

$$\delta L \doteq \frac{d}{dt}(v^\dagger \delta u) \tag{5}$$

We will investigate if there are invariant variations of the form

$$\begin{aligned} \delta u &= \varepsilon\, Mu \\ \delta v &= \varepsilon\, N^\dagger v, \end{aligned} \tag{6}$$

where M and N are two constant square matrices. Direct application of the definition of δL (1.4.10) or (1.3.2.5) to (1) gives

$$\frac{1}{\epsilon}\,\delta L = v^{\dagger}(N+M)u_t + v^{\dagger}(NA+AM)u \tag{7}$$

This should now be the time derivative of some function of u and v, which is only possible if we require

$$N+M = 0;\ NA+AM = 0$$

or

$$N = -M\ ;\ MA-AM = 0 \tag{8}$$

Then

$$\delta L = 0 \tag{9}$$

Hence by (5), (6), (8) and (9)

$$\frac{d}{dt}\,(v^{\dagger}Mu) = 0;\quad MA-AM = 0 \tag{10}$$

We derived this equation for any solution of (3). We can write therefore

$$\frac{d}{dt}\,(v^{\dagger}Mw) = 0;\ MA = AM \tag{11}$$

where v and w have to satisfy (3).
Relations of this kind are known as *reciprocity laws* [20].

As the identity matrix I commutes with any square matrix, we can get from (11)

$$\frac{d}{dt}\,(v^{\dagger}w) = 0 \tag{12}$$

If u is a solution of $u_t + Au = 0$, Mu is a solution too if AM = MA. We can also take

$$v = u \text{ and } w = Mu$$

so that

$$\frac{d}{dt}\,(u^{\dagger}Mu) = 0 \text{ and } AM-MA = 0 \tag{13}$$

which represents a conservation law for any M.
Clearly we obtain the same relation as we would have found after formally identifying u and v in equation (10).

In a similar way, new conservation laws are found by taking

$$v = u;\ w = Nu_t;\ AN-NA = 0$$

Substitution in (12) gives

$$\frac{d}{dt}\,(u^{\dagger}Nu_t) \triangleq 0;\ AN\text{-}NA = 0 \qquad (14)$$

This obviously also represents a conservation law for any N.
This conservation law can also be found in a way analogous to (10), beginning with

$$\delta u = \varepsilon\, Mu_t, \quad \delta v = -\varepsilon\, N^{\dagger}v_t$$

and identifying u and v afterwards.

According to the classification of Steudel [21] the set of conservation laws (13) is *trivially equivalent* to the set given by (14). The conserved quantity $u^{\dagger}Nu_t$ of (14) is numerically equal to $u^{\dagger}NAu$ for any solution of the equation of motion (3). This quantity represents the set of conserved quantities given by (13), because, if AN = NA we also have AM = MA with M = NA and vice versa.
For fixed M = N equation (14) can be derived from (13) by differentiation with respect to t. Therefore, we call the conservation laws *dynamically-equivalent* (see also section 3.4).

We can extend the given theory by omitting the restriction to constant matrices M and N in (6). If M and N are allowed to depend on time equation (7) becomes

$$\frac{1}{\varepsilon}\,\delta L = v^{\dagger}(N+M)u_t + v^{\dagger}(M_t + NA + AM)u$$

where M_t consists of the matrix elements, which are the time derivatives of the matrix elements of M. The assumption $\delta L = 0$ now requires

$$N = -M;\ M_t = MA\text{-}AM$$

which results in the conservation law

$$\frac{d}{dt}\,(u^{\dagger}Mu) \triangleq 0;\quad M_t = MA\text{-}AM \qquad (15)$$

To apply the present theory, we must first find solutions of the matrix equations

$$MA\text{-}AM = 0 \quad \text{or} \quad M_t = MA\text{-}AM$$

Solving the first one is equivalent to finding the commutator ring of matrix A. This problem has been thoroughly discussed by Gantmacher [22]. The solutions M(t) of the second equations can, as is easily verified, be

written in the form

$$M(t) = \exp -tA.\ M_0.\exp tA \tag{16}$$

where M_0 is a constant arbitrary square matrix.
A direct method to obtain this result is to start with the solution of the initial value problem of $u_t + Au = 0$ which is given by

$$u = \exp -tA.u_0;\ u_0 = u(0)$$

Hence by (16) if M_0 is constant and arbitrary

$$u_0^\dagger M_0 u_0 = u(t)^\dagger M(t) u(t) \doteq \text{constant}$$

or

$$\frac{d}{dt}\,(u^\dagger M(t) u) \doteq 0$$

As to the case investigated, that of a first order differential equation, we draw attention to the fact that a sesquilinear Lagrangian had to be used. In problems governed by second-order linear differential equations, such as

$$u_{tt} + Au = 0;\quad A = A^\dagger \tag{17}$$

one normally uses Lagrangians, quadratic in the coordinates and velocities. A Lagrangian describing (17) is for instance

$$L[u] = u_t^\dagger u_t - u^\dagger A u$$

With Noether's theorem a number of conservation laws can then be derived such as

$$\frac{d}{dt}\,(u_t^\dagger M u_t + u^\dagger M A u) \doteq 0;\ MA-AM = 0;\ M = M^\dagger$$

and

$$\frac{d}{dt}\,(-u^\dagger M u_t + u_t^\dagger M u) \doteq 0;\ MA-AM = 0;\ M = -M^\dagger$$

But also in this case it is possible to use a sesquilinear Lagrangian. For instance

$$L[v,u] = v_t^\dagger u_t - v^\dagger A u$$

describes (17) as well as

$$v_{tt} + Av = 0$$

Noether's theorem gives then reciprocity laws such as

$$\frac{d}{dt}\,(v_t^\dagger M u_t + v^\dagger M A u) \doteq 0;\ MA-AM = 0;\ M = M^\dagger$$

and

$$\frac{d}{dt}\,(-v^\dagger M u_t + v_t^\dagger M u) \doteq 0;\ MA-AM = 0;\ M = -M^\dagger$$

from which previous conservation laws can be derived by identifying u and v.

1.5.2 The two-dimensional harmonic oscillator

The two-dimensional harmonic oscillator can be represented by the equations of motion

$$x_{tt} + \omega_1^2 x = 0$$

$$y_{tt} + \omega_2^2 y = 0$$

This set of two second-order differential equations can be transformed into four first-order differential equations written as

$$u_t + Au = 0$$

where

$$u = \text{column } (u_1, u_2, u_3, u_4)$$

$$u_1 = x;\ u_2 = y,\ u_3 = \frac{1}{\omega_1}\,\dot{x},\ u_4 = \frac{1}{\omega_2}\,\dot{y}$$

A = square real skew-symmetric matrix

$$A = \begin{pmatrix} & & -\omega_1 & 0 \\ & & 0 & -\omega_2 \\ \omega_1 & 0 & & \\ 0 & \omega_2 & & \end{pmatrix}$$

(we will omit zeroes in matrices unless confusion might arise)

To find the time-independent conserved quantities belonging to (1.5.1.13) we have to find the commutator ring of matrix A. It is easily verified that all commutators of A are given by the linear combinations of M_1 through M_4 if $\omega_1 \neq \omega_2$ and M_1 through M_8 if $\omega_1 = \omega_2$

$$M_1 = \text{diagonal } (1,0,1,0)$$

$$M_2 = \text{diagonal } (0,1,0,1)$$

$$M_3 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & \end{pmatrix} \qquad M_4 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix} \qquad M_6 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

$$M_7 = \begin{pmatrix} & & 0 & 1 \\ & & -1 & 0 \\ 0 & -1 & & \\ 1 & 0 & & \end{pmatrix} \qquad M_8 = \begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ 0 & -1 & & \\ -1 & 0 & & \end{pmatrix}$$

Only the matrices M_1, M_2, M_5 and M_7 are real symmetric. The others are real and skew-symmetric and do not give any real constant of the motion, because $u^{\dagger}Mu$ will be purely imaginary if $M = -M^{\dagger}$ and zero if we confine our attention (as we will mostly do) to real quantities.
We therefore obtain two conservation laws generated by M_1 and M_2, and in the case $\omega_1 = \omega_2 = \omega$ two special ones generated by M_5 and M_7. These are, if all quantities are assumed to be real:

$$\left.\begin{aligned} &\frac{d}{dt}(u_1^2 + u_3^2) = 0 \\ &\frac{d}{dt}(u_2^2 + u_4^2) = 0 \\ &\frac{d}{dt}(u_1u_2 + u_3u_4) = 0 \\ &\frac{d}{dt}(u_1u_4 - u_2u_3) = 0 \end{aligned}\right\} \quad \text{if } \omega_1 = \omega_2$$

or

$$\left.\begin{aligned} &\frac{d}{dt}(x^2 + 1/\omega_1^2\dot{x}^2) = 0 \qquad (1) \\ &\frac{d}{dt}(y^2 + 1/\omega_2^2\dot{y}^2) = 0 \qquad (2) \\ &\frac{d}{dt}(xy + 1/\omega^2\dot{x}\dot{y}) = 0 \qquad (3) \\ &\frac{d}{dt}(x\dot{y} - \dot{x}y) = 0 \qquad (4) \end{aligned}\right\} \quad \text{if } \omega = \omega_1 = \omega_2$$

Equation (4) represents the conservation of angular momentum, (1) the conservation of the energy due to the x-component of the motion, (2) is the analogue for the y-component. Equation (3) is the condition, necessary and sufficient to conclude from (1) and (2) that the energy due to the component of the motion in an arbitrary direction is conserved.

Of course the harmonic oscillator, and its conservation laws, can also be treated directly with the second order equations (see for instance Broer-Kobussen [13]). The present treatment is less known and has therefore been included here.

1.5.3 The one-dimensional linearly-damped harmonic oscillator

To find time-dependent real constants of the motion, quadratic in the coordinates and its time derivatives, for the linearly-damped one-dimensional harmonic oscillator described by

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = 0 \tag{1}$$

we transform the equation to

$$\ddot{x} + \omega^2 x = 0;\ \omega^2 = \omega_0^2 - \gamma^2 \tag{2}$$

with the coordinate transformation

$$x = q \exp \gamma t \tag{3}$$

Equation (3) can now be converted into

$$u_t + Au = 0 \tag{4}$$

where

$$\begin{aligned} u &= \text{column } (u_1, u_2) \\ u_1 &= x = q \exp \gamma t \\ u_2 &= 1/\omega\dot{x} = (\omega_0^2 - \gamma^2)^{-\frac{1}{2}}.(\dot{q} + \gamma q)\exp \gamma t \end{aligned} \tag{5}$$

$$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \tag{6}$$

According to equations (1.5.1.16,17) the time-dependent conservation laws are

$$\begin{aligned} &\frac{d}{dt}\,(u^{\dagger}M(t)u) = 0 \\ &M(t) = \exp -tA.M_0.\exp tA \end{aligned} \tag{7}$$

A basis for the set of matrices M(t) is given by the constant matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the time-dependent matrices

$$M_3 = \begin{pmatrix} \sin 2\omega t & \cos 2\omega t \\ \cos 2\omega t & -\sin 2\omega t \end{pmatrix} \qquad M_4 = \begin{pmatrix} -\cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & \cos 2\omega t \end{pmatrix}$$

which are obtained by setting M_0 equal to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ respectively. As M_2 is skew-symmetric, equation (7) does not give a real constant of the motion. The remaining ones are

$$\frac{d}{dt}\,(x^2 + 1/\omega^2\dot{x}^2) \doteq 0 \tag{8}$$

$$\frac{d}{dt}((x^2 - 1/\omega^2\dot{x}^2)\sin 2\omega t + 2/\omega x\dot{x}\cos 2\omega t) \doteq 0 \tag{9}$$

$$\frac{d}{dt}\,((-x^2 + 1/\omega^2\dot{x}^2)\cos 2\omega t + 2/\omega x\dot{x}\sin 2\omega t) \doteq 0 \tag{10}$$

Note that in deriving (8), (9) and (10) we only used equation (2). So these conservation laws are valid also for the ordinary one-dimensional harmonic oscillator.
Using (5) we can explicitly express (8) through (10) in terms of q and $\dot{q}$. So we obtain

$$\frac{d}{dt}\,(1/\omega^2(\omega_0^2 q^2 + 2\gamma q\dot{q} + \dot{q}^2)\exp 2\gamma t) \doteq 0 \tag{11}$$

$$\frac{d}{dt}\,(\{q^2(1-\gamma^2/\omega^2) - 2\gamma/\omega^2 q\dot{q} - 1/\omega^2\dot{q}^2\}\sin 2\omega t.\ \exp 2\gamma t + \{2/\omega q\dot{q} + 4\gamma/\omega q^2\}\cos 2\omega t.\ \exp 2\gamma t) \doteq 0 \tag{12}$$

$$\frac{d}{dt}\,(\{q^2(-1 + \gamma^2/\omega^2) + 2\gamma/\omega^2 q\dot{q} + 1/\omega^2\dot{q}^2\}\cos 2\omega t.\ \exp 2\gamma t + \{2/\omega q\dot{q} + 4\gamma/\omega q^2\}\sin 2\omega t.\ \exp 2\gamma t) \doteq 0 \tag{13}$$

Note that for $\gamma \neq 0$ the equations (11) - (13) take the form (8) - (9).

Thus we have derived three time-dependent quadratic constants of the motion for the linearly-damped oscillator. We will not divert our attention to possible applications, neither give a complete physical interpretation. We only mention that (11) has already been derived by Denman [23] in a different way. He points out that for $\gamma = 0$ the constant of the motion becomes the ordinary energy of the undamped oscillator. We want to emphasize that this does not mean that (11) represents the conservation of the *energy of the damped oscillator*. Such a conclusion would require a more thorough discussion. We have to make then a model in which the friction of the oscillator can be attributed to the interaction of an undamped oscillator with another system (for instance a mechanical system or an electromagnetic field). Then we should prove that Denman's energy is the energy of the undamped oscillator plus an interaction energy of the oscillator and the other system. We will not discuss such a model here.

2. Hamilton's formalism in classical mechanics

2.1 Hamilton's equations

Hamilton's equations of motion can be derived from Euler's equations. The usual derivation starts from a Lagrangian which depends on t, q_i and $\dot{q}_i$ only. If not so this defect can be remedied by the introduction of auxiliary variables. In principle this procedure can be avoided: Thielheim [24] has constructed a formalism, analogous to that of Hamilton, without such a requirement. The result, a rather unusual formalism, in our opinion offers no real advantage for our purposes.

Consider a mechanical system described by the Lagrangian

$$L(q,\dot{q},t) = L(q_1,\ldots,q_n, \dot{q}_1,\ldots,\dot{q}_n t)$$

and Euler's equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{1}$$

We define the *momentum* p_i *canonically conjugated**) to each coordinate q_i, or briefly *momentum*, by

$$p_i = \frac{\partial}{\partial \dot{q}_i} L \tag{2}$$

The momentums p_i defined in this way are functions of the time t, the coordinates q_i and their time-derivatives $\dot{q}_i$. We write

$$p_i = p_i(q_1,\ldots,q_n, \dot{q}_1,\ldots,\dot{q}_n,t) = p_i(q,\dot{q},t) \tag{3}$$

A further requirement is that the transformation of the set of independent variables q_i and $\dot{q}_i$ to the set of variables q_i, p_i can uniquely be inverted. That is, there exists a unique set of inverse-functions $f_i(q,p,t)$ so that

$$\dot{q}_i = f_i(q,p,t) = f_i(q_1,\ldots,q_n, p_1,\ldots,p_n,t). \tag{4}$$

Hamilton's function or the *Hamiltonian* is then defined

$$H(p,q,t) = p_i f_i(q,p,t) - L(q,f(q,p,t),t) \tag{5}$$

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*) *canonical* literally means *according to a certain rule* (canon).

The partial derivatives of H with respect to q and p can now be calculated.

$$\frac{\partial H}{\partial p_k} = f_k(q,p,t) + p_i \frac{\partial f_i}{\partial p_k} - \frac{\partial L}{\partial \dot{q}_i} \frac{\partial f_i}{\partial p_k}$$

$$\frac{\partial H}{\partial q_k} = p_i \frac{\partial f_i}{\partial q_k} - \frac{\partial L}{\partial q_k} - \frac{\partial L}{\partial \dot{q}_i} \frac{\partial f_i}{\partial q_k}$$

Hence by (1), (2) and (4),

$$\frac{\partial H}{\partial p_k} = \dot{q}_k, \tag{6}$$

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k. \tag{7}$$

These equations are *Hamilton's equations*. They are derived here from the definition of the Hamiltonian and Euler's equations. It is readily seen that the proof can be converted. So Euler's equations and Hamilton's equations are equivalent. In general Euler's equations are differential equations of the second order. Hamilton's equations are of the first order. This is the reason why Hamilton's formalism is often easier to handle than Lagrange's formalism.

Although we already mentioned that we do not specify the requirements of decent behaviour, which must be met by the functions and functionals used, we have to make one remark.
It is not always possible to construct, according to the given method, a Hamiltonian when the Lagrangian is given and vice versa. For instance, when the Lagrangian is linear or conjugate-linear in $\dot{q}_i$, the momentum p_i defined by (3) does not depend on $\dot{q}_i$, and there do not exist inverse functions f_i. Starting with the sesquilinear Lagrangian of section (1.3.2), it is not possible to construct a Hamiltonian. Such situations are called *exceptional*. Ryan [25] has shown that, in the exceptional case of a Lagrangian system with the property that the time derivative of a coordinate cannot be expressed in terms of the coordinates and momentums, an alternative procedure may be adopted in setting up Hamilton's formalism. One introduces an equivalent Lagrangian without this defect and uses it instead of the original Lagrangian.

2.2 Hamilton's equations and the associated variational principle

Let the Hamiltonian of a system be given by

$$H(p,q,t) = H(p_1,\ldots,p_n, q_1,\ldots,q_n,t)$$

The equations of motion are:

$$\dot{p}_i = -\frac{\partial}{\partial q_i} H;\ \dot{q}_i = \frac{\partial}{\partial p_i} H \tag{1}$$

We define a function $\hat{L}$ of t, q_i, p_i and $\dot{q}_i$ by

$$\hat{L}(p,q,\dot{q},t) = p_i\dot{q}_i - H(p,q,t) \tag{2}$$

and the functional $\hat{\mathcal{W}}$ by

$$\hat{\mathcal{W}}\{p,q\} = \hat{\mathcal{W}} = \int_{t_1}^{t_2} \hat{L}\, dt \tag{3}$$

Note that $\dot{q}_i$ in (2) is not given by (2.1.4) but is determined by q(t) only. The functional derivatives of $\hat{\mathcal{W}}$ with respect to p_i and q_i are respectively

$$\frac{\delta}{\delta p_i}\hat{\mathcal{W}} = \dot{q}_i - \frac{\partial H}{\partial p_i} \tag{4}$$

$$\frac{\delta}{\delta q_i}\hat{\mathcal{W}} = -\dot{p}_i - \frac{\partial H}{\partial q_i} \tag{5}$$

Hence by (1)

$$\frac{\delta}{\delta p_i}\hat{\mathcal{W}} = \frac{\delta}{\delta q_i}\hat{\mathcal{W}} = 0 \tag{6}$$

Conclusion

The functional $\hat{\mathcal{W}}$ is stationary for infinitesimal variations around solutions of Hamilton's equations, of the argument functions q_i and p_i with fixed values of q_i at the endpoints.

We did not impose boundary conditions for the momentums p_i because $\hat{L}$ does not depend on the derivatives $\dot{p}_i$. In some cases it may be useful to extend the formalism for the same kind of system to $\hat{L}$ depending on $\dot{p}_i$ as well. This can be done by adding to $\hat{L}$ a total derivative of an arbitrary function of p_i, q_i and t (c.f. the gauge transformation of section (1.1)).

In such cases one also imposes the requirement that the endpoints $p_i(t_1)$ and $p_i(t_2)$ remain fixed during the variation.

The treatment here is almost similar to that of the action principle of section (1.1). If distinction is made between both variational principles the variational principle of this section is given the same names as that of section (1.1) with the prefix *canonical*. We want to emphasize here the great difference between the *action principle* and the *canonical action principle*. In case of the canonical action principle p_i and q_i have to be varied independently, in case of the action principle there is only q_i to vary.

As there is no unique Lagrangian for given Euler's equations the Hamiltonian for given Hamilton's equations is not unique either. We will return to this problem in sections (2.3) and (2.4).

2.3 The canonical transformation and its generating function

Let the Hamiltonian of a mechanical system be

$$H(p,q,t) = H(p_1,\ldots p_n,\ q_1,\ldots,q_n,t),$$

and the equations of motion

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}\ ;\ \dot{q}_i = \frac{\partial H}{\partial p_i}\ . \tag{1}$$

Consider the transformation of the coordinates and momentums

$$\begin{aligned}\bar{p}_i &= \bar{p}_i(q,p,t) = \bar{p}_i(p_1,\ldots,p_n,\ q_1,\ldots,q_n,t)\\ \bar{q}_i &= \bar{q}_i(q,p,t) = \bar{q}_i(p_1,\ldots,p_n,\ q_1,\ldots,q_n,t)\end{aligned} \tag{2}$$

Such a transformation is called a *canonical transformation*, or sometimes a *contact transformation*, if a function

$$\bar{H}(\bar{p},\bar{q},t) = \bar{H}(\bar{p}_1,\ldots,\bar{p}_n,\ \bar{q}_1,\ldots,\bar{q}_n,t)$$

exists allowing the equations of motion (1) to assume the form

$$\dot{\bar{p}}_i = -\frac{\partial \bar{H}}{\partial \bar{q}_i}\ ;\ \dot{\bar{q}}_i = \frac{\partial \bar{H}}{\partial \bar{p}_i} \tag{3}$$

In this section we will show that sufficient conditions for the transformati

(2) to be canonical, can be given.

The equations of motion (1) and (3), are equivalent to the variational principles

$$\frac{\delta}{\delta p_i}\hat{\mathcal{W}} = \frac{\delta}{\delta q_i}\hat{\mathcal{W}} = 0 \tag{4}$$

and

$$\frac{\delta}{\delta \bar{p}_i}\hat{\bar{\mathcal{W}}} = \frac{\delta}{\delta \bar{q}_i}\hat{\bar{\mathcal{W}}} = 0 \tag{5}$$

respectively,

with

$$\hat{\mathcal{W}}\{p,q\} = \int_{t_1}^{t_2} (p_i\dot{q}_i - H(p,q,t))dt \tag{6}$$

and

$$\hat{\bar{\mathcal{W}}}\{\bar{p},\bar{q}\} = \int_{t_1}^{t_2} (\bar{p}_i\dot{\bar{q}}_i - \bar{H}(\bar{p},\bar{q},t))dt \tag{7}$$

A sufficient condition for (4) and (5) to be equivalent is that the difference of $\hat{\mathcal{W}}$ and $\hat{\bar{\mathcal{W}}}$ does not depend on p(t) and q(t) for $\bar{p}$ and $\bar{q}$ corresponding to p and q through (2). Therefore, we write

$$\int_{t_1}^{t_2} (p_i\dot{q}_i - H(p,q,t))dt - \int_{t_1}^{t_2} (\bar{p}_i\dot{\bar{q}}_i - \bar{H}(\bar{p},\bar{q},t))dt = \int_{t_1}^{t_2} (\frac{d}{dt}F)dt \tag{8}$$

where F is an arbitrary, in general time-dependent, function of the functions p and q or by means of (2) also a function of $\bar{p}$ and $\bar{q}$, and is called the *generating function of the transformation*. The reason for this name is that once F is given, the transformation equations (2) are completely specified. To carry out the transformation between the two sets of canonical variables p_i, q_i and $\bar{p}_i$, $\bar{q}_i$ it is useful to take F as a function of both the old and the new variables. Besides being a function of the time t, the generating function may be a function of the 4n variables p_i, q_i, $\bar{p}_i$, $\bar{q}_i$. Only 2n of these are independent, because the two sets of variables are connnected by the 2n transformation equations (2). The generating function can therefore be written as a function of independent variables in one of the four given forms

$$F = F_1(\bar{q},q,t);\ G = F_2(\bar{p},q,t);\ F_3(p,\bar{q},t);\ F_4(\bar{p},p,t)$$

The connections between these four forms can be given by a *Legendre transformation*. We will not discuss that here.

The circumstances of the problem will dictate which form has to be chosen. For example, if we are dealing with a *pure coordinate transformation*, or *point transformation* defined by $\bar{q} = \bar{q}(q,t)$ the coordinates q and $\bar{q}$ are not independent variables, and a generating function of the form F_1 must be excluded, whilst any of the others may be used.

If the first form F_1 is a suitable choice, we can put

$$p_i\dot{q}_i - H(p,q,t) - \bar{p}_i\dot{\bar{q}}_i + \bar{H}(\bar{p},\bar{q},t) = \frac{d}{dt}F(\bar{q},q,t)$$

or

$$(-H(p,q,t) + \bar{H}(\bar{p},\bar{q},t) - \frac{\partial}{\partial t}F) + (p_i - \frac{\partial F}{\partial q_i})\dot{q}_i + (-\bar{p}_i - \frac{\partial F}{\partial \bar{q}_i})\dot{\bar{q}}_i = 0 \qquad (9)$$

Since the old and the new coordinates q_i and $\bar{q}_i$ are considered as independent functions of t, the variables q_i, $\bar{q}_i$, $\dot{q}_i$ and $\dot{\bar{q}}_i$ are, for fixed time t, also independent. Therefore (9) has to be satisfied for all values of q_i, $\bar{q}_i$, $\dot{q}_i$ and $\dot{\bar{q}}_i$. If we take for example $\dot{q}_i = \dot{\bar{q}}_i = 0$ we get

$$\bar{H} = H + \frac{\partial}{\partial t}F(\bar{q},q,t) \qquad (10)$$

If we take there upon all variables $\bar{q}_i$ and $\dot{\bar{q}}_i$, except one of these, being zero, we obtain

$$p_i = \frac{\partial}{\partial q_i}F(\bar{q},q,t) \qquad (11)$$

and

$$\bar{p}_i = -\frac{\partial}{\partial \bar{q}_i}F(\bar{q},q,t) \qquad (12)$$

The n equations (11) are n relations involving p_i, q_i, $\bar{q}_i$ and t only. These can be solved, in general, for the n $\bar{q}_i$'s in terms of p_i, q_i and t, thus yielding the first half of the transformation (2). In the same way equation (12) gives the second half of this transformation and (10) provides the connection between the new and the old Hamiltonians $\bar{H}$ and H.

If the independent arguments of F are chosen as q_i and $\bar{p}_i$, the generating function is of the type $F_2 = G(\bar{p},q,t)$. Analogously to equation (9) we obtain

$$(-H + \bar{H} - \frac{\partial}{\partial t} G) + (p_i - \frac{\partial}{\partial q_i} G)\dot{q}_i - \bar{p}_i\dot{\bar{q}}_i - \frac{\partial G}{\partial \bar{p}_i}\dot{\bar{p}}_i = 0$$

or:

$$(-H + \bar{H} - \frac{\partial}{\partial t} G) + (p_i - \frac{\partial}{\partial q_i} G)\dot{q}_i - \frac{d}{dt}(\bar{p}_i\bar{q}_i) + (\bar{q}_i - \frac{\partial}{\partial \bar{p}_i} G)\dot{\bar{p}}_i = 0 \qquad (13)$$

As Hamilton's equations can be derived from the canonical action principle it is possible, without altering the equations of motion, to add to the density $\hat{L}$ of the canonical action $\hat{\mathcal{W}}$ a term $\frac{d}{dt}(\bar{p}_i\bar{q}_i)$, so that we can drop this term in (13).
We want to point out that, after the term $\frac{d}{dt}(\bar{p}_i\ \bar{q}_i)$ has been added, the canonical action principle must be interpreted as one with specified boundary conditions for the momentums (see also section 2.2).

From (13) we obtain now as a sufficient condition for the transformation (2) to be canonical

$$p_i = \frac{\partial}{\partial q_i} G(\bar{p},q,t) \qquad (14)$$

$$\bar{q}_i = \frac{\partial}{\partial \bar{p}_i} G(\bar{p},q,t) \qquad (15)$$

$$\bar{H} = H + \frac{\partial}{\partial t} G(\bar{p},q,t) \qquad (16)$$

Equation (14) can, in general, be solved for $\bar{p}_i$ as a function of q_i, p_i and t, which gives the second half of the transformation (2). The first half of the transformation is then provided by (15).

Starting with generating functions of types F_3 and F_4, we can operate in the same way. We only give the results

$$F_3 = F(p,\bar{q},t) \qquad q_i = -\frac{\partial}{\partial p_i} F(p,\bar{q},t)$$

$$\bar{p}_i = -\frac{\partial}{\partial \bar{q}_i} F(p,\bar{q},t)$$

$$\bar{H} = H + \frac{\partial}{\partial t} F$$

2.3
2.4

$$F_4 = F(p,\bar{p},t) \qquad q_i = -\frac{\partial}{\partial p_i} F(p,\bar{p},t)$$

$$\bar{q}_i = \frac{\partial}{\partial \bar{p}_i} F(p,\bar{p},t)$$

$$\bar{H} = H + \frac{\partial}{\partial t} F$$

In most applications either F_1 or F_2 may be chosen as a generating function of a canonical transformation.

2.4 Examples of canonical transformations

A few simple examples of canonical transformations will be presented here.

The identical transformation

The identical transformation $\bar{p}_i = p_i$, $\bar{q}_i = q_i$, $\bar{H} = H$

is generated by the function

$$G = \bar{p}_i q_i$$

as can be verified by using (2.3.14-16)

Scale transformation of the coordinates

Let $G = \alpha \bar{p}_i q_i$, α a real constant

then $\bar{p}_i = \frac{1}{\alpha} p_i$, $\bar{q}_i = \alpha q_i$, $\bar{H} = H$

Pure coordinate transformation

Let $G = \bar{p}_i f_i(q_1,\ldots,q_n,t)$, then

$$\bar{q}_k = f_k(q_1,\ldots,q_n,t)$$

and

$$\bar{p}_k = p_i \left(\left\| \frac{\partial}{\partial q_i} f_k \right\|^{-1} \right)_{ik}$$

This canonical transformation corresponds to the coordinate transformation of section 1.2 and is often called a *point transformation*. Obviously, the identical transformation and the scale transformation are examples of the point transformation.

The gauge transformation

Let $G = \bar{p}_i q_i + \chi(q_1,\ldots,q_n,t)$

with χ an arbitrary function, then

$$p_i = \bar{p}_i + \frac{\partial}{\partial q_i}\chi$$

$$q_i = \bar{q}_i$$

$$\bar{H} - H = \frac{\partial}{\partial t}\chi$$

This transformation is essentially the same as the gauge transformation discussed in section 1.1., which as we already pointed out, is closely related to the gauge transformation in electrodynamics.

The Hamilton-Jacobi equation

We try to find a canonical transformation, which transforms Hamilton's equations of a given system into

$$\frac{\partial}{\partial \bar{p}_i}\bar{H} = \dot{\bar{q}}_i = 0$$

$$\frac{\partial}{\partial \bar{q}_i}\bar{H} = -\dot{\bar{p}}_i = 0$$

This is the case when the transformed Hamiltonian $\bar{H}$ is a function of t only, for instance zero. Let the canonical transformation be generated by a function $F(q,\bar{q},t)$. Then

$$\bar{H} = H + \frac{\partial}{\partial t}F$$

$$p_i = \frac{\partial}{\partial q_i}F$$

$$\bar{p}_i = -\frac{\partial}{\partial \bar{q}_i}F$$

A sufficient condition for $\bar{H}$ to be zero is

$$H(p,q,t) + \frac{\partial}{\partial t}F(q,\bar{q},t) = 0$$

or using the transformation equations

$$H(\frac{\partial}{\partial q}F(q,\bar{q},t),q,t) + \frac{\partial}{\partial t}F(q,\bar{q},t) = 0$$

This equation is known as the *Hamilton-Jacobi equation*. It does not contain derivatives with respect to $\bar{q}$. The variable $\bar{q}$ appears as an integration constant in the solution of the equation.
A solution of the Hamilton-Jacobi equation is the action functional. This is readily seen by calculating $\frac{d}{dt} F$.

$$\frac{d}{dt} F = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial \bar{q}_i} \dot{\bar{q}}_i$$

Hence by $\dot{\bar{q}}_i = 0$ and $p_i = \frac{\partial}{\partial q_i} F$ and $\bar{H} - H = -H = \frac{\partial F}{\partial t}$

$$\frac{d}{dt} F = - H + p_i \dot{q}_i = L$$

so that $$F = \int L \, dt = \mathcal{W}$$

2.5 The canonical Noether theorem

In order to find an analogy to Noether's theorem for Lagrangian systems, we will investigate in this section infinitesimal canonical transformations for Hamiltonian systems.

As the identical transformation $\bar{p}_i = p_i$; $\bar{q}_i = q_i$; $\bar{H} = H$ is generated by the generating function

$$G = \bar{p}_i q_i ,$$

an infinitesimal canonical transformation is generated by

$$G = \bar{p}_i q_i + \varepsilon \, g(\bar{p},q,t) \tag{1}$$

where ε is a vanishingly small parameter, and where

$$g(\bar{p},q,t) = g(\bar{p}_1,\ldots,\bar{p}_n, q_1,q_2,\ldots,q_n,t)$$

This function g will be called the *generating function of the infinitesimal transformation.*
Using the transformation equations (2.3.14-16) we obtain

$$\bar{p}_i = p_i - \varepsilon \frac{\partial}{\partial q_i} g$$

$$\bar{q}_i = q_i + \varepsilon \frac{\partial}{\partial \bar{p}_i} g$$

$$\bar{H}(\bar{p},\bar{q},t) = H(p,q,t) + \varepsilon \frac{\partial}{\partial t} g \tag{2}$$

As ε is a vanishingly small parameter, terms of second or higher order in ε can be neglected and the bar over $\bar{p}$ in g can be dropped. So we obtain

$$\delta p_i = \bar{p}_i - p_i = -\varepsilon \frac{\partial}{\partial q_i} g(p,q,t) \tag{3}$$

$$\delta q_i = \bar{q}_i - q_i = \varepsilon \frac{\partial}{\partial p_i} g(p,q,t) \tag{4}$$

Analogously to the infinitesimal coordinate transformation, which we called a variation, we call this infinitesimal canonical transformation a *canonical variation*.

Let

$$\delta H = H(\bar{p},\bar{q},t) - H(p,q,t) \tag{5}$$

The first-order term in the Taylor expansion of (5) can be written with (3) and (4) as

$$\delta H = -\varepsilon [g,H]_p \tag{6}$$

where

$$[g,H]_p = \frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i} \tag{7}$$

is the usual *Poisson bracket*.

For p_i's and q_i's which satisfy equations of motion we have

$$\frac{d}{dt} g \triangleq \frac{\partial}{\partial t} g + [g,H]_p \tag{8}$$

Hence by (2), (5), (6) and (8)

$$\bar{H}(\bar{p},\bar{q},t) - H(\bar{p},\bar{q},t) \triangleq \varepsilon \frac{d}{dt} g \tag{9}$$

We call a canonical transformation an *invariant canonical transformation* if $\bar{H}$ is the same function of $\bar{p}$ and $\bar{q}$ as H is of p and q, thus

$$\bar{H}(\bar{p},\bar{q},t) = H(\bar{p},\bar{q},t);\ \bar{H}(p,q,t) = H(p,q,t), \tag{10}$$

The following theorem, which we will call the canonical Noether theorem, is evident now.

Theorem

The generating function of an invariant infinitesimal canonical trans-

formation is a constant of the motion.
Each function g[p,q] that is a constant of the motion is the generating function of an invariant infinitesimal canonical transformation.

The second part of this theorem represents an obvious advantage over Noether's theorem for Lagrangian systems (section 1.4). It means that supplementary investigations like those of Fletcher [16] and Dass [17] are not necessary here.

The relation between the canonical Noether theorem and Noether's theorem will be discussed later (section 5.4) in such a way that the discussion applies to continuous systems as well.

3. Lagrange's formalism in continuum physics

3.1. Euler's equations for continuous systems and the associated variational principle

Let a physical system be fully described by n mutually independent *coordinate functions* or *field variables* q_i, $i = 1,2,\ldots,n$ as a function of the time t and m *independent variables* or *coordinates* x_i, $i = 1,2,\ldots,m$. In most applications the coordinates are spatial variables, but we will also give other examples.

The physical system is called a *Lagrangian system*, if a (in general time-dependent) functional $\mathfrak{L}$, depending on the coordinate functions q_i and their time-derivatives $\dot{q}_i$, $\ddot{q}_i$,... exists so that the equations of motion can be written as

$$\frac{\delta}{\delta q_i}\mathfrak{W}[q] = 0 \tag{1}$$

with

$$\mathfrak{W}[q] = \int_{t_1}^{t_2} \mathfrak{L}\{q,\dot{q},\ddot{q},\ldots,t\}dt \tag{2}$$

Note that for fixed values of time the functions q_i and their time derivatives are independent.

The functional $\mathfrak{L}$ is called the *Lagrangian functional*, the functional $\mathfrak{W}$ the *action* or *action functional*.

In many cases and in all our examples, the functional $\mathfrak{L}$ can be given in the form

$$\mathfrak{L}\{q,\dot{q},\ldots,t\} = \int_G L[q]dx \tag{3}$$

where $\int_G dx$ denotes integration over an arbitrary region in x-space.

Furthermore L[q] may be a *function* of the time t, the coordinates x_i, the coordinate functions q_i and all their derivatives with respect to t and x_i.

If (3) is satisfied equation (1) represents a set of partial differential equations.

The function LLq] is called *the Lagrangian density* or *Lagrange's function*. Sometimes, when this will not cause confusion, we will call both $\mathcal{L}$ and L the *Lagrangian*.

Substitution of (2) into the equation of motion yields

$$\frac{\delta}{\delta q_i}\mathcal{L} - \frac{d}{dt}\frac{\delta}{\delta \dot{q}_i}\mathcal{L} + \frac{d^2}{dt^2}\frac{\delta}{\delta \ddot{q}_i}\mathcal{L} \ldots = 0 \qquad (4)$$

If we express the equation of motion in terms of Lagrange's function by substituting (3) into (4) we obtain

$$\frac{\delta}{\delta q_i}\mathcal{L} = \frac{\partial}{\partial q_i} L - \frac{d}{dx_j}\frac{\partial L}{\partial (q_i)_{x_j}} + \frac{d^2}{dx_j dx_k}\frac{\partial L}{\partial (q_i)_{x_j x_k}} + \ldots\ldots$$

$$\frac{\delta}{\delta \dot{q}_i}\mathcal{L} = \frac{\partial}{\partial \dot{q}_i} L - \frac{d}{dx_j}\frac{\partial L}{\partial (\dot{q}_i)_{x_j}} + \ldots\ldots$$

$$\frac{\delta}{\delta \ddot{q}_i}\mathcal{L} = \frac{\partial}{\partial \ddot{q}_i} L - \frac{d}{dx_j}\frac{\partial L}{\partial (\ddot{q}_i)_{x_j}} + \ldots\ldots$$

where

$$(q_i)_{x_j x_k \ldots} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} \ldots\ q_i$$

If we suppose that L does not depend on derivatives of q_i of second or higher order, the familiar expression for the equations of motion is obtained

$$\frac{\partial}{\partial q_i} L - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dx_k}\frac{\partial L}{\partial (q_i)_{x_k}} = 0 \qquad (5)$$

In view of the analogy with section (1.1) the equations (4) and sometimes (5) are called *Euler's* or *Lagrange's equations*.

It may be noted that the Lagrangian density L can in many cases be chosen as the difference of specific kinetic and potential energy. This fact is not of interest for the formalism. Nevertheless it can be helpful in the search for a Lagrangian density in a specific physical problem. This is illustrated by Broer [26] for the problem of the longitudinal motion of an elastic bar.

Next we draw the reader's attention to the fact that our use of the symbol $\frac{d}{dt}$ and $\frac{d}{dx_k}$ does not indicate the ordinary total derivative. The symbol $\frac{d}{dt}$ and $\frac{d}{dx_k}$ denotes differentiation with constant x_i, and constant t and $x_i (i \neq k)$ respectively. When we write $\frac{\partial}{\partial x_k}$ the field variables q_i and the time t are considered to be constant.
Analogously, we write $\frac{\partial}{\partial t}$ when the field variables q_i and the coordinates are considered to be constant:
So*)

$$\frac{\partial}{\partial x_k} = \left(\frac{\partial}{\partial x_k}\right)_{q_i,t,x_i(i\neq k)} \qquad \frac{d}{dx_k} = \left(\frac{\partial}{\partial x_k}\right)_{t,x_i(i\neq k)}$$

$$\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t}\right)_{q_i,x_j} \qquad \frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_{x_j}$$

In general, the equations of motion (1) are invariant with respect to transformations of the form

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{d}{dt}\chi\{q,\dot{q},\ddot{q},\ldots,t\}$$

where χ is an arbitrary functional of q_i, $\dot{q}_i$,... and function of t. In view of the analogy with section (1.1) these transformations are called *gauge transformations*.
Following the rules of functional differentiation we obtain

$$\frac{d}{dt}\chi = \frac{\partial}{\partial t}\chi + \int \frac{\delta\chi}{\delta q_i}\dot{q}_i \, dx + \int \frac{\delta\chi}{\delta\dot{q}_i}\ddot{q}_i \, dx + \ldots\ldots$$

where

$$\frac{\partial}{\partial t}\chi = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\chi\{q,\dot{q},\ldots,t+\varepsilon\} - \chi\{q,\dot{q},\ldots,t\}\right)$$

which relation can be used to obtain the corresponding gauge transformation of the Lagrangian density L

$$L \rightarrow L + \frac{d}{dt}\chi(q,\dot{q},q_x,q_{xt},\ldots,x,t)$$

Here χ is an arbitrary function of the field variables q_i, their derivatives with respect to t and x_i, and x_i and t explicitely. Furthermore

---

*a consequence of this is for instance

$$\frac{\partial}{\partial t}\chi(x,t) = 0$$

3.1
3.2

$$\mathcal{X}\{q,\dot{q},\ldots,t\} = \int \chi(q,\dot{q},q_{x_k},\ldots,x,t)dx$$

3.2. Transformation of the coordinate functions

In section (1.2) we discussed coordinate transformations for mechanical systems. We intend to generalize the concept of coordinate transformation to cases of continuous systems. There are two different ways to proceed. The first is to consider one-to-one mappings of the class of functions $q_i(x,t)$ on a class $\bar{q}_i(x,t)$. The functions $\bar{q}_i$ can then be written as a function of the functions q_i

$$\bar{q}_i(x,t) = f_i(q_1(x,t),\ q_2(x,t),\ldots,x,t)$$

An example of a transformation which cannot be written in this form is the Fourier transformation

$$\bar{q}_i(\bar{x},t) = \int_{-\infty}^{\infty} q_i(x,t)\ \exp\ -ix\bar{x}\ dx$$

The second way illustrated by this example, is to consider a wider class of transformations, in which the new coordinate functions $\bar{q}_i(\bar{x},\bar{t})$ are functionals of $q_i(x,t)$. The new independent variables then appear as parameters in these functionals. In view of further applications in the Hamiltonian formalism we will restrict the transformations of the field variables to transformations in which the time t appears as a parameter. We consider only transformations of the form

$$q_i(x,t) \rightarrow \bar{q}_i(\bar{x},t)$$

which may depend explicitly on time t. We then may write

$$\bar{q}_i(\bar{x},t) = \mathcal{F}_i\{q,\dot{q},\ldots,\bar{x},t\} \tag{1}$$

where $\mathcal{F}_i$ is for fixed values of t and $\bar{x}$ a functional of the field variables q_i and their derivatives with respect to time, and for fixed coordinate functions $q_i(x,t)$ and time a function of $\bar{x}_1,\ldots,\bar{x}_m$.
The transformation (1) is considered to be a one-to-one mapping. Therefore, a unique set of functionals $\tilde{\mathcal{F}}_i$ exists, so that

$$q_i(x,t) = \tilde{\mathcal{F}}_i\{\bar{q},\dot{\bar{q}},\ldots,\bar{x},t\} \tag{2}$$

If there is no risk of confusion we also write instead of (1) and (2)

$$\bar{q}_i = \bar{q}_i\{q\} \tag{3}$$

and

$$q_i = q_i\{\bar{q}\} \tag{4}$$

respectively.

Let now $\mathcal{W}\{q\}$ be the action functional of the problem before transformation, with the equations of motion

$$\frac{\delta}{\delta q_i}\mathcal{W} = 0 \tag{5}$$

We will demonstrate that these equations are equivalent to

$$\frac{\delta}{\delta \bar{q}_i}\bar{\mathcal{W}}\{\bar{q}\} = 0 \tag{6}$$

where apart from a constant term and a constant factor

$$\bar{\mathcal{W}}\{\bar{q}\} = \mathcal{W}\{q\{\bar{q}\}\} \tag{7}$$

Therefore $\mathcal{W}$ and $\bar{\mathcal{W}}$ are numerically equal.
A few preliminary details have to be discussed first.

For functions $f_i(x)$ satisfying the required boundary conditions we have, on the basis of the definition of the functional derivative

$$\bar{\mathcal{W}}\{\bar{q}\{q+\varepsilon f\}\} - \mathcal{W}\{q\}\} = \int \frac{\delta\bar{\mathcal{W}}}{\delta\bar{q}_i}\,(\bar{q}_i\{q+\varepsilon f\} - q_i\{q\})\,d\bar{x} + \mathcal{O}(\varepsilon^2)$$

Without the term of order ε^2, this expression is the *first-order Volterra expansion* (the functional analogue of the Taylor expansion for functions). When we use again the definition of the functional derivative we obtain

$$\bar{\mathcal{W}}\{\bar{q}\{q+\varepsilon f\}\} - \bar{\mathcal{W}}\{\bar{q}\{q\}\} = \varepsilon \int \frac{\delta\bar{\mathcal{W}}}{\delta\bar{q}_i}\,d\bar{x} \int \frac{\delta\bar{q}_i}{\delta q_j}\,f_j\,dx + \mathcal{O}(\varepsilon^2)$$

To avoid confusion we write the dependence of x and $\bar{x}$ explicitly

$$\bar{\mathcal{W}}\{\bar{q}\{q+\varepsilon f\}\} - \bar{\mathcal{W}}\{\bar{q}\{q\}\} = \varepsilon \int f_j(x)dx \int d\bar{x}\,\frac{\delta\bar{\mathcal{W}}}{\delta\bar{q}_i(\bar{x})}\,\frac{\delta\bar{q}_i(\bar{x})}{\delta q_j(x)} + \mathcal{O}(\varepsilon^2)$$

On the other hand, by (7)

$$\bar{\mathcal{W}}\{\bar{q}\{q+\varepsilon f\}\} - \bar{\mathcal{W}}\{\bar{q}\{q\}\} = \mathcal{W}\{q+\varepsilon f\} - \mathcal{W}\{q\} = \varepsilon \int dx\, f_j(x) \frac{\delta \mathcal{W}}{\delta q_j(x)} + \mathcal{O}(\varepsilon^2)$$

So we get

$$\frac{\delta \mathcal{W}}{\delta q_j(x)} = \int d\bar{x}\, \frac{\delta \bar{\mathcal{W}}}{\delta \bar{q}_i(\bar{x})} \frac{\delta \bar{q}_i(\bar{x})}{\delta q_j(x)} \tag{8}$$

We will use the name *chain rule for functional differentiation* for this equation.

From (8) we see that (5) follows from (6) if (1) is satisfied. In an analogous way (6) follows from (3) and (5).
So the equations of motion (5) and (6) are equivalent if the transformation (1) can be uniquely inverted.

When we apply the foregoing to Lagrangian systems, i.e. to those for which a Lagrangian functional exists, a sufficient condition for (7) is

$$\mathcal{L}\{q,\dot{q},\ldots\} - \bar{\mathcal{L}}\{\bar{q},\dot{\bar{q}},\ldots\} = \frac{d}{dt}\mathcal{F}$$

where $\mathcal{F}$ is an arbitrary functional in $q,\dot{q},\ldots$ and $\bar{q},\dot{\bar{q}},\ldots$.
In most cases it will be most convenient to set $\mathcal{F} = 0$.
The additional term $\frac{d}{dt}\mathcal{F}$ corresponds to the gauge transformation already discussed in section (1.1) and (3.1).

3.3. Examples of transformations of coordinate functions

In this section we want to give some examples of the theory developed in the foregoing section

the Fourier transformation

Consider the transformation $q(x,t) \to \bar{q}(\bar{x},t)$ defined by

$$\bar{q} = Fq$$

or

$$\bar{q}(\bar{x},t) = \int_{-\infty}^{\infty} q(x,t) \exp i x\bar{x}\, dx$$

The Lagrangian functional is then assumed to transform in such a way

that

$$\bar{\mathcal{L}}\{\bar{q},\bar{q}_t,\ldots\} = \mathcal{L}\{q,q_t,\ldots\}$$

Let $\mathcal{L}\{q,q_t,\ldots\}$ be given as an integral over a quadratic expression of q and its derivatives:

$$\mathcal{L}\{q,q_t,\ldots\} = \int dx\, L[q]$$

$$L[q] = ..q^*q + ..q^*q_t + \ldots \quad ..q^*q_x + \ldots ,$$

where we denote with $*$ the complex conjugation; then according to *Parseval's theorem*

$$\bar{L}[\bar{q}] = (Fq)^*(\bar{x})(Fq)(\bar{x}) + ..(Fq)^*(\bar{x})(Fq_t)(\bar{x}) + ..(Fq)^*(\bar{x})(Fq_x)(\bar{x}) + \ldots$$

or

$$\bar{\mathcal{L}}\{\bar{q},\bar{q}_t,..\} = \int \bar{L}[\bar{q}]dx$$

The *Laplace*, *Mellin* and *Hankel* transforms can be treated in a similar way.

time-dependent translation of the x-axis

Consider the transformation of the x coordinate

$$x = g(\bar{x},t);\ \bar{x} = \tilde{g}(x,t)$$

The corresponding transformation of the field variable q(x,t) can then be given by

$$\bar{q}(\bar{x},t) = q(g(\bar{x},t),t)$$

$$q(x,t) = \bar{q}(\tilde{g}(x,t),t)$$

It is then readily seen that

$$q_x = \bar{q}_{\bar{x}}\tilde{g}_x$$

and

$$q_t = \bar{q}_t + \bar{q}_{\bar{x}}\,\tilde{g}_t$$

Application of this transformation to the ideal elastic string governed by the equation

$$q_{tt} - c^2 q_{xx} = 0$$

or by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}q_t^2 - \tfrac{1}{2}c^2 q_x^2$$

gives, if we assume $\frac{d\bar{x}}{dx} = \tilde{g}_x = 1$, for the transformed Lagrangian

$$\bar{\mathcal{L}} = \tfrac{1}{2}(\bar{q}_t^2 + \tilde{g}_t \bar{q}_{\bar{x}})^2 - \tfrac{1}{2}c^2 \bar{q}_{\bar{x}}^2$$

or

$$\bar{\mathcal{L}} = \tfrac{1}{2}\,\bar{q}_t^2 + \tilde{g}_t \bar{q}_t \bar{q}_{\bar{x}} - \tfrac{1}{2}(c^2 - \tilde{g}_t^2)\bar{q}_{\bar{x}}^2$$

The transformed equations of motion then are

$$q_{tt} + 2\tilde{g}_t\, \bar{q}_{\bar{x}t} - (c^2 - \tilde{g}_t^2)\bar{q}_{\bar{x}\bar{x}} + \tilde{g}_{tt}\, \bar{q}_{\bar{x}} = 0$$

Another interesting example of a transformation of the field variable is found in the *Liouville transformation* of the one-dimensional wave equation for variable refractive index [27].

3.4 Noether's theorem for continuous systems and local conservation laws

Noether's theorem for mechanical systems was proved in section 1.4. We can formulate an analogous theorem for continuous systems.

Theorem

If there exists an infinitesimal transformation of the functions $q_i(x,t)$ into the functions $\bar{q}_i(x,t)$

$$\bar{q}_i = q_i + \varepsilon\, g_i[q]$$
$$= q_i + \varepsilon\, g_i(q,\dot{q},\ddot{q},\ldots,t),$$

in such a way that it leaves the action functional

$$\mathcal{W} = \int_{t_1}^{t_2} \mathfrak{L}\{q,\dot{q},\ddot{q},\ldots,t\}dt$$

invariant to the first order in ε, not counting a term

$$\int_{t_1}^{t_2} \frac{d}{dt}\,\psi\{q,\dot{q},\ddot{q},\ldots,t\}dt$$

then a relation

$$\frac{d}{dt}(\chi - \psi\{q,\dot{q},\ldots,t\}) = \frac{\delta}{\delta q_i}\mathfrak{W}.g_i[q] \tag{1}$$

exists, and vice versa.

Here χ is a functional depending on q_i, $\dot{q}_i$,..., and g_i, $\dot{g}_i$,... and t.

As was done in section 1.4 we will call such a transformation an *invariant transformation.*

This theorem is in essence the extended version of Noether's theorem [11]. Our decision not to transform the independent variables x_i and t, is not an essential restriction (also see Steudel [19] and section 1.4).

We will not prove the theorem. It can be done in a way entirely analogous to the way used to prove the theorem in section 1.4. The functions L[q] must be replaced by the functionals $\mathfrak{L}\{q,\dot{q},\ldots,t\}$. The remarks made in section 1.4 remain valid here. If, for instance, we restrict ourselves to q_i's which satisfy the equations of motion we obtain

$$\frac{d}{dt}(\chi - \psi) = 0 \tag{2}$$

Relations of this kind are called *global* or *integral conservation laws* (g.l.c), this in contradistinction with the concept of the *local* or *differential conservation law* (l.c.l).
A l.c.l. is a relation of the form

$$\frac{d}{dt} S = \frac{d}{dx_i} F_i \tag{3}$$

where S and the F_i's are functions of the x_i's, t, the q_i's and their derivatives with respect to t as well x_i. S is called the *conserved density*, the quantities F_i constitute the components of the *flux vector.*

In many cases it is possible to deduce a g.c.l. from a l.c.l. by integration over some region G of x-space.
With (3) we obtain in this way

$$\frac{d}{dt}\int_G S\,dx = \int_G \frac{d}{dx_i} F_i\,dx \qquad (4)$$

With Green's first theorem we can transform the right-hand side of (4) into an integral over the boundary of the region. With suitable boundary conditions for the q_i's the right-hand side of (4) will vanish and we obtain a g.c.l.. We will not go into details about the required boundary conditions. We only note that the requirement of suitable boundary conditions in the case of systems of infinite extent corresponds with the decent behaviour required to obtain well-behaving functionals. De Graaf [28], [29] has shown that in suchlike cases any g.c.l also generates a l.c.l. In the case of finite systems it is not always possible to give sufficient physically acceptable boundary conditions which allow a l.c.l. to generate a g.c.l. See for this the following example.

The motion of an *ideal elastic string* with unit phase-velocity can be given by the partial differential equation

$$q_{tt} - q_{xx} = 0 \qquad q = q(x,t)$$

Physically acceptable boundary conditions are for instance

$$q(0,t) = q(1,t) = 0$$

Hence

$$q_t(0,t) = q_t(1,t) = 0$$

The l.c.l.'s for the energy and linear momentum are

$$\frac{d}{dt}(\tfrac{1}{2}q_t^2 + \tfrac{1}{2}q_x^2) - \frac{d}{dx}(q_t q_x) = 0$$

and

$$\frac{d}{dt}(q_t q_x) - \frac{d}{dx}(\tfrac{1}{2}q_t^2 + \tfrac{1}{2}q_x^2) = 0$$

respectively.
It is now readily seen that upon integration from $x = 0$ to $x = 1$ both l.c.l.'s take the form

$$\frac{d}{dt}\int_0^1 (\tfrac{1}{2}q_t^2 + \tfrac{1}{2}q_x^2)dx = 0$$

$$\frac{d}{dt}\int_0^1 (q_x q_t)dx = \tfrac{1}{2}(q_x^2(1,t) - q_x^2(0,t))$$

So we only get a g.c.l. for the linear momentum if we extend the boundary condition, but it is not clear how this can physically be done. Only if the string is infinitely long, we may require $q_x(\pm\infty,t) = 0$ besides $q_t(\pm\infty,0) = 0$ so that there exist a g.c.l. for the energy as well as for the linear momentum.

The example of the string illustrates that l.c.l's are of more interest than g.c.l.'s. A constructive procedure to generally obtain a l.c.l. from a g.c.l. is not known. In view of this difficulty and of those associated with the boundary conditions, it may be profitable to reformulate Noether's process of finding conservation laws, in such a way that the flux-vector components are found automatically. As we will see such a reformulation is possible.

Let the Lagrangian density depend on the derivatives of the coordinate functions $q_i(x,t)$ up to the first order only.
Hence

$$L[q] = L(q_i, \dot{q}_i, (q_i)_{x_k}, x_i, t) \tag{6}$$

The equations of motion are then given by (3.1.5)

$$\Lambda_i L = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dx_k}\frac{\partial L}{\partial (q_i)_{x_k}} = 0 \tag{7}$$

We define the *variation* of the coordinate functions δq_i by the infinitesmally small transformation

$$\bar{q}_i = q_i + \varepsilon f_i[q] = q_i + \delta q_i \tag{8}$$

The *first-order variation* δL of the Lagrangian density L is defined by

$$L[q + \varepsilon f] - L[q] = \delta L + \mathcal{O}(\varepsilon^2) \tag{9}$$

Hence by (6), (8) and (9)

$$\delta L = \delta q_i \, \Lambda_i L + \frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d}{dx_i}\left(\delta q_k \frac{\partial L}{\partial (q_x)_{x_i}}\right) \tag{10}$$

We note that (10) holds for arbitrary coordinate functions q_i and for arbitrary variations δq_i.

Now we suppose that for the special choice $\delta q_i = \varepsilon f_i[q]$ the first order variation δL of L can be written, for arbitrary q_i's, in the form

$$\delta L = \frac{d}{dt}(\phi_0[q]) + \frac{d}{dx_i}(\phi_i[q]) \tag{11}$$

In that case (10) and (11) lead to

$$\frac{d}{dt}(\phi_0[q] - \delta q_i \frac{\partial L}{\partial \dot{q}_i}) + \frac{d}{dx_i}(\phi_i[q] - \delta q_k \frac{\partial L}{\partial (q_k)_{x_i}}) = \delta q_i \Lambda_i L \tag{12}$$

The following theorem is now evident.

Theorem

If there exists an infinitesimal transformation of the functions $q_i(x,t) \to \bar{q}_i(x,t)$

$$\bar{q}_i = q_i + \varepsilon f_i[q]$$

$$f_i[q] = \frac{1}{\varepsilon} \delta q_i = f_i(q_k, \dot{q}_k, (q_k)_{x_i}, x_k, t)$$

which leaves the Lagrangian density

$$L[q] = L(q_i, \dot{q}_i, (q_i)_{x_k}, x_i, t)$$

invariant to first order in ε, not counting a term

$$\frac{d}{dt}(\phi_0[q]) + \frac{d}{dx_i}(\phi_i[q])$$

a relation

$$\frac{d}{dt}(\phi_0[q] - \delta q_i \frac{\partial L}{\partial \dot{q}_i}) + \frac{d}{dx_i}(\phi_i[q] - \delta q_k \frac{\partial L}{\partial (q_k)_{x_i}}) = \delta q_i \Lambda_i L$$

exists.

This theorem is essentially the same as the one mentioned earlier in this section. The first formulation is the best for general use. The formulation in terms of densities is easier to handle in most cases.

From (12) we directly obtain a l.c.l. by restricting the coordinate functions q_i to solutions of the equations of motion (7). To avoid confusion we write

$$\frac{d}{dt}(\phi_0[q] - \delta q_i \frac{\partial L}{\partial \dot{q}_i}) + \frac{d}{dx_i}(\phi_i[q] - \delta q_k \frac{\partial L}{\partial (q_k)_{x_i}}) \doteq 0 \tag{12'}$$

Here the symbol $\doteq$ denotes an equality for solutions of the equations of motion. In our examples we will also write instead of (10)

$$\delta L \triangleq \frac{d}{dt}(\delta q_i \frac{\partial L}{\partial \dot{q}_i}) + \frac{d}{dx_i}(\delta q_k \frac{\partial L}{\partial (q_k)_{x_i}}) \qquad (10')$$

A modification of the derivation of local conservation laws is the following. We consider special transformations of the form (8), which yield

$$\delta L = \frac{d}{dt}(\phi_0[q]) + \frac{d}{dx_i}(\phi_i[q]) + \text{terms that vanish for solutions of}$$

Euler's equations. So

$$\delta L \triangleq \frac{d}{dt}(\phi_0[q]) + \frac{d}{dx_i}(\phi_i[q]) \qquad (11')$$

Equations (10') and (11') together give also a local conservation law of the form (12'). We only have to take care that (10') and (11') are not identical. Otherwise we get a *trivial local conservation law* (see later on in this section).

If we want to derive for special physical systems several local conservation laws, we will for sake of simplicity not refer to the relations given here, but do the same procedure again for the given Lagrangian of the system. Sometimes we will therefore refer to *Noether's method* for constructing local conservation laws instead of *Noether's theorem*.

According to Steudel [19] we will distinguish between four kinds of local conservation laws (l.c.l.).

i) A l.c.l. of *the first kind* is characterized by the identity

$$\frac{d}{dt}\psi_0 + \frac{d}{dx_i}\psi_i = \alpha_k \Lambda_k L, \qquad (13)$$

which holds irrespective of the equations of motion and where α_k is, for solutions of the equations of motion, a non-singular function of the q_k's and its derivatives.

Examples of such l.c.l.'s are found for the ideal elastic string, where $\Lambda L = \Lambda(\frac{1}{2}q_t^2 - \frac{1}{2}q_x^2) = q_{tt} - q_{xx}$ in the conservation of energy and linear momentum:

$$\frac{d}{dt}(\tfrac{1}{2}q_t^2 + \tfrac{1}{2}q_x^2) - \frac{d}{dx}(q_t q_x) = q_t(q_{tt} - q_{xx}) \qquad (14)$$

$$\frac{d}{dt}(q_t q_x) - \frac{d}{dx}(\tfrac{1}{2}q_t^2 + \tfrac{1}{2}q_x^2) = q_x(q_{tt} - q_{xx}) \tag{15}$$

Two other examples for the dynamical system with the equation of motion $u_t + Du_x = 0$, where $u = u(x,t)$ is a real n-dimensional column-vector and D a square real symmetric n×n matrix, are given by

$$\frac{d}{dt}(u^T Mu) + \frac{d}{dx}(u^T MDu) = (u_t + Du_x)^T Mu + u^T M(u_t + Du_x) \qquad \text{with } MD = DM;\ M = M^T \tag{16}$$

and

$$\frac{d}{dt}(u^T DMu_x) - \frac{d}{dx}(u^T DMu_t) = u_t^T M(u_t + Du_x) - (u_t + Du_x)^T Mu_t \qquad \text{with } MD = DM;\ M = -M^T \tag{17}$$

where M is also a square, real n×n matrix, and T denotes transposition. (A systematic derivation of (16) and (17) follows for a more general case in section 3.5.1).

ii) A l.c.l. of *the second kind* is characterized by the equation

$$\frac{d}{dt}\psi_0 + \frac{d}{dx_i}\psi_i \doteq 0 \tag{18}$$

which holds for all solutions $\Lambda_k L = 0$, while it is not possible to express the left-hand side of (18) as a linear combination of the $\Lambda_k L$'s with non-singular coefficients.
For instance, the equation

$$\frac{d}{dt}(u^T Mu_t) + \frac{d}{dx}(u^T DMu_t) \doteq 0 \quad \text{with } MD = DM;\ D = D^T \tag{19}$$

holds for all solutions of $u_t + Du_x = 0$ because

$$\frac{d}{dt}(u^T Mu_t) + \frac{d}{dx}(u^T DMu_t) = (u_t + Du_x)^T Mu_t + u^T M(u_t + Du_x)_t, \tag{20}$$

represents an identity.

iii) A l.c.l. of *the third kind* is characterized by an identity of the form

$$\frac{d}{dt}\psi_0 + \frac{d}{dx_i}\psi_i = 0 \tag{21}$$

which holds for arbitrary coordinate functions.

For instance

$$\frac{d}{dt}(q_x) - \frac{d}{dx}(q_t) = 0 \tag{22}$$

iv) A l.c.l. of *the fourth kind* can be given in the form (18), but the density ψ_0 and the flux vector components ψ_i themselves will vanish for all solutions of the equations of motion.

For instance for the ideal elastic string

$$\frac{d}{dt}(q_t(q_{tt} - q_{xx})) + \frac{d}{dx}(q_x(q_{tt} - q_{xx})) \doteq 0 \tag{23}$$

Local conservation laws of the third and those of the fourth kind are without any physical significance. Hence they will be called *trivial local conservation laws*.

Two l.c.l.'s that differ by a trivial l.c.l. are called *trivially equivalent*.

For instance the l.c.l.'s corresponding to (17) and (20)

$$\frac{d}{dt}(u^{\dagger}DMu_x) - \frac{d}{dx}(u^{\dagger}DMu_t) \doteq 0 \tag{24}$$

and

$$\frac{d}{dt}(u^{\dagger}Mu_t) - \frac{d}{dx}(u^{\dagger}DMu_t) \doteq 0 \tag{25}$$

are trivially equivalent.

With Noether's theorem l.c.l.'s of the *first kind* may be obtained directly. By substituting the equations of motion in the fluxes and/or the densities of l.c.l.'s of the *first kind*, they can be transformed into l.c.l.'s of the *second kind*, which are *trivially equivalent* to the l.c.l. of the first kind.

Fletcher [16] and Dass [17] proved that for any l.c.l. of the second kind there exists a trivially equivalent l.c.l. which can be derived on the basis of Noether's theorem (see also section 1.4).

We will extend the terminology and introduce the concept of *dynamically equivalent* l.c.l.'s:

Two l.c.l.'s are called *dynamically equivalent* if they can be derived from each other through ordinary operations such as differentiation and linear combination.

When for instance the l.c.l. belonging to (16)

$$\frac{d}{dt}(u^T M u) + \frac{d}{dx}(u^T MDu) \doteq 0 \quad MD = DM,\ M = M^T \tag{26}$$

is differentiated with respect to time t and divided by 2, we obtain

$$\frac{d}{dt}(u^T M u_t) + \frac{d}{dx}(u^T MDu_t) \doteq 0 \quad MD = DM;\ \ M = M^T \tag{27}$$

so that (26) and (27) are dynamically equivalent.

We use the term *dynamically equivalent* because, for instance (27) can be derived from (26) without using the dynamics of the system. Two dynamically equivalent l.c.l.'s therefore, contain the same information about the dynamics of the system. This, however, does not imply that a l.c.l., because of its dynamical equivalence to another l.c.l., is meaningless. For instance, the quantum mechanical laws of conservation of linear momentum and energy are dynamically equivalent to the law of conservation of electrical charge for a free Dirac particle (relativistic electron, see section 3.5.1). Despite this fact each of the l.c.l.'s has a distinct *physical meaning*.

We note that the concept of the dynamical equivalence of two l.c.l.'s cannot automatically be extended to global conservation laws, since a g.c.l. corresponding to a given l.c.l. cannot always be indicated.
Consider, for instance the two dynamically equivalent l.c.l.'s for the ideal elastic string

$$\frac{d}{dt}(\tfrac{1}{2}q_t^2 + \tfrac{1}{2}q_x^2) - \frac{d}{dx}(q_t q_x) \doteq 0 \tag{28}$$

and

$$\frac{d}{dt}(q_t q_{xt} + q_x q_{xx}) - \frac{d}{dx}(q_{xt} q_x + q_t q_{xx}) \doteq 0 \tag{29}$$

Let the boundary condition be

$$q(0,t) = q(1,t) = 0$$

Then we get from (28) and (29)

$$\frac{d}{dt}\int_0^1 (\tfrac{1}{2}q_t^2 + \tfrac{1}{2}q_x^2)dx \doteq 0 \tag{30}$$

and

$$\frac{d}{dt}\int_0^1 (q_t q_{xt} + q_x q_{xx})dx \triangleq q_{xt}q_x\Big|_{x=1} - q_{xt}q_x\Big|_{x=0} \tag{31}$$

respectively.

Generally speaking, the right-hand side of (31) will not vanish. Then even although a g.l.c. exists corresponding to the l.c.l. (28), there is no g.l.c. corresponding to (29).

3.5 Applications of Noether's theorem for continuous systems

3.5.1. Sesquilinear Lagrangians - the Dirac equations

Let u and v, column vectors with components u^i and v^i (i = 1,2,...,n), be the coordinate functions of the physical system. The independent variables, the coordinates, are x^i, i = 1,2,...,m.

Analogously to the discussion in section (1.3.2) the sesquilinear Lagrangian density L is defined by*)

$$L = v^\dagger(u_t + A_k \frac{d}{dx^k} u + Bu), \tag{1}$$

where A_k and B are constant n×n matrices satisfying

$$A_k = A_k^\dagger \; ; \; B = -B^\dagger \tag{2}$$

Let an infinitesimal transformation of the coordinate functions be given by

$$\begin{aligned} \bar{u} &= u + \varepsilon f[u] \\ \bar{v} &= v + \varepsilon g[v] \end{aligned} \tag{3}$$

with ε a vanishingly small parameter.

The first order variation δL of L then becomes

$$\frac{1}{\varepsilon}\delta L = g^\dagger[v](u_t + A_k \frac{d}{dx^k} u + Bu) - (v_t + A_k \frac{d}{dx^k} v + Bv)^\dagger f[u]$$
$$+ \frac{d}{dt}(v^\dagger f[u]) + \frac{d}{dx^k}(v^\dagger A_k f[u])$$

---

*The use of superscripts to enumerate the coordinates instead of subscripts as in previous sections is done to conform to customary usage in tensor analysis.

and we obtain

$$\frac{1}{\varepsilon}\,\delta L \doteq \frac{d}{dt}(v^{\dagger}f[u]) + \frac{d}{dx^{k}}\,(v^{\dagger}A_{k}f[u]) \tag{4}$$

and Euler's equations are

$$u_{t} + A_{k}\,\frac{d}{dx_{k}}\,u + Bu = 0 \tag{5}$$

$$v_{t} + A_{k}\,\frac{d}{dx_{k}}\,v + Bv = 0 \tag{6}$$

As in the discussion of section (1.3) we can apply (1) to physical systems described by (5) only, by identifying u and v in the final results, or by substituting for u and v two different simultaneous solutions of (5).
As a starting point in the search for local conservation laws involving a density, quadratic in u, we consider

$$f[u] = Mu \qquad g[v] = N^{\dagger}v \tag{7}$$

where M and N are two constant square matrices, not yet specified. Then

$$\frac{1}{\varepsilon}\,\delta L = v^{\dagger}(N+M)u_{t} + v^{\dagger}(NA_{k} + A_{k}M)\frac{d}{dx^{k}}\,u + v^{\dagger}(NB + BM)u$$

The transformation (7) is an *invariant transformation* if δL vanishes. Therefore we put

$$N+M = 0; \quad NA_{k} + A_{k}M = NB + BM = 0$$

or

$$N = -M; \quad MA_{k} - A_{k}M = MB - BM = 0 \tag{8}$$

which results in $$\delta L = 0 \tag{9}$$

Hence by (4) and (7) - (9)

$$\frac{d}{dt}(v^{\dagger}Mu) + \frac{d}{dx^{k}}\,(v^{\dagger}A_{k}Mu) \doteq 0 \tag{10}$$

Instead of (7) we may choose

$$f[u] = Mu_{t}; \qquad g[v] = N^{\dagger}v_{t}$$

which leads to

$$\frac{1}{\varepsilon}\,\delta L = \frac{d}{dt}(v^{\dagger}Mu_t + v^{\dagger}A_k M\,\frac{d}{dx^k}\,u + v^{\dagger}BMu)$$

if we require

$$N = M;\quad MA_k - A_k M = MB - BM = 0$$

Hence by (4)

$$\frac{d}{dt}(v^{\dagger}A_k M\,\frac{d}{dx^k}\,u + v^{\dagger}BMu) - \frac{d}{dx^k}(v^{\dagger}A_k Mu_t) \doteq 0 \tag{11}$$

$$MA_k - A_k M = MB - BM = 0$$

With $f[u] = M\,\frac{d}{dx^{\ell}}\,u$; $g[v] = N^{\dagger}\frac{d}{dx^{\ell}}\,v$ (ℓ fixed) we find in the same way with N = M

$$\frac{d}{dt}\,(v^{\dagger}M\,\frac{d}{dx^{\ell}}\,u) + \frac{d}{dx^k}\,(v^{\dagger}A_k M\,\frac{d}{dx^{\ell}}\,u - (v^{\dagger}Mu_t + v^{\dagger}A_i M\,\frac{d}{dx^i}\,u + \tag{12}$$

$$+ v^{\dagger}BMu)\delta_k^{\ell}) \doteq 0$$

$$MA_k - A_k M = MB - BM = 0$$

As (10) - (12) are valid for any pair of, generally different, solutions of the equations of motion (5) - (6), they represent *local reciprocity laws*. From these the following local conservation laws can be obtained

$$\frac{d}{dt}(u^{\dagger}Mu) + \frac{d}{dx^k}(u^{\dagger}A_k Mu) \doteq 0 \tag{13}$$

$$\frac{d}{dt}(u^{\dagger}Mu_t) + \frac{d}{dx^k}(u^{\dagger}A_k Mu_t) \doteq 0 \tag{14}$$

$$\frac{d}{dt}(u^{\dagger}M\,\frac{d}{dx^{\ell}}\,u) + \frac{d}{dx^k}(u^{\dagger}A_k M\,\frac{d}{dx^{\ell}}\,u) \doteq 0 \tag{15}$$

which hold if

$$MA_k - A_k M = MB - BM = 0 \tag{16}$$

Here (13) is a l.c.l. of the first kind and is obtained by identifying u and v in (10). Equations (14) and (15) represent l.c.l.'s of the second kind and are obtained by identifying u and v in (11) and (12) and

additionally substituting the equation of motion into density and flux respectively.

If the matrix M is anti-Hermitean, $M = -M^{\dagger}$, the density and the flux components in (13) are purely imaginary and do not represent any physical quantity. Consequently, in (13) we must restrict ourselves to matrices M that are Hermitean solutions of (16). Equations (14) and (15) can then be derived from (13) by taking the derivative of (13) with respect to t and x^{ℓ} respectively. So (13), (14) and (13), (15) are *dynamically equivalent.*

When two simultaneous solutions v, w of (5) are substituted for v, u in (10), the following *reciprocity law* is obtained

$$\frac{d}{dt}(v^{\dagger}Mw) + \frac{d}{dx^{k}}(v^{\dagger}A_{k}Mw) = 0 \qquad MA_{k} - A_{k}M = MB - BM = 0 \tag{17}$$

It may readily be seen that the local conservation laws (14) and (15) are obtained from (17) by the substitution $M = I$, $v = u$, $w = u_{t}$ and

$$M = I,\ v = u,\ w = \frac{d}{dx^{\ell}}u \text{ respectively.}$$

We now intend to give an interpretation of (13) - (15) in the case that (5) represents the wave equation for a free relativistic electron - the *Dirac equation.*

In the Dirac equation [30] the field variable or *spinor* is a column vector with dimension 4. The independent variables are the spatial coordinates $x^{1} = x$, $x^{2} = y$, $x^{3} = z$. The matrices $A_{k}, k = 1,2,3$ and B are, according to the *m.k.s. or Giorgi system*, given by

$$A_{k} = c\,\alpha^{k}; \quad B = i\,\frac{mc^{2}}{\hbar}\,\beta \tag{18}$$

where c = the velocity of light in vacuum

$\hbar$ = the Dirac constant = $h/2\pi$

m = the mass of the Dirac particle (electron)

and the *Dirac matrices* α^{k} and β are to satisfy

$$\begin{aligned} &\beta = \beta^{\dagger};\ \alpha^{k} = \alpha^{k\dagger} && \alpha^{k}\beta + \beta\alpha^{k} = 0 \\ &\beta^{2} = I && \alpha^{k}\alpha^{\ell} + \alpha^{k}\alpha^{\ell} = \delta^{k\ell}\,I \end{aligned} \tag{19}$$

where I is the unit matrix of dimension 4. There is no unique solution of equations (19). Nevertheless none of the physical concequences of the Dirac equations depend on the specific form of the matrices α_k, β which satisfy (19). One of the possible ways of choosing the matrices α_k and β is

$$\alpha^k = \begin{pmatrix} 0_2 & \sigma^k \\ \sigma^k & 0_2 \end{pmatrix} \quad k = 1,2,3; \qquad \beta = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}$$

where the matrix elements σ^k are the two-dimensional *Pauli spin matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and moreover

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

It can be shown by tedious, elementary calculations that all matrices commuting with α^k, k = 1,2,3 and β simultaneously can be written as scalar multiples of the identity matrix. Therefore, in the local conservation laws (13) - (15) we have to take M = I. After multiplication with suitable constants we obtain from (13) - (15):

$$\frac{d}{dt}(u^\dagger u) + \frac{d}{dx^k}(u^\dagger c \ \alpha^k u) \doteq 0 \tag{20}$$

which represents the law of conservation of *electrical charge*,

$$\frac{d}{dt}(u^\dagger i\hbar \frac{d}{dt} u) + \frac{d}{dx^k}(u^\dagger i\hbar c \ \alpha^k \frac{d}{dt} u) \doteq 0 \tag{21}$$

which represents the law of *conservation of energy* and

$$\frac{d}{dt}(u^\dagger \frac{\hbar}{i} \frac{d}{dx^\ell} u) + \frac{d}{dx^k}(u^\dagger \frac{\hbar c}{i} \alpha^k \frac{d}{dx^\ell} u) \doteq 0 \qquad \ell = 1,2,3 \tag{22}$$

which represents the law of *conservation of the linear momentum* in the x^ℓ direction.

The covariant (relativistic invariant) formalism can be obtained after the introduction of the γ-matrices

$$\gamma^0 = \beta; \quad \gamma^k = \beta\alpha^k \qquad k = 1,2,3$$

They have the following properties

$$\gamma^0 = \gamma^{0\dagger}; \quad -\gamma^k = \gamma^{k\dagger} \qquad k = 1,2,3$$

$$(\gamma^0)^2 = I; \quad (\gamma^k)^2 = -I \qquad k = 1,2,3$$

or in a more concise notation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

where $\mu,\nu = 0,1,2,3$ and $g^{\mu\nu}$ = diagonal (-1,1,1,1) the *Minkowski metric tensor*.

We now introduce the notation

$$\partial_0 = \frac{1}{c}\frac{d}{dt} \qquad \partial_k = \frac{d}{dk^k} \qquad k = 1,2,3$$

The Lagrangian density L for the Dirac equation can then be written as

$$L = cv^\dagger\gamma^0\gamma^\mu\partial_\mu u + i\,\frac{mc^2}{\hbar}\,v^\dagger\gamma^0 u$$

or equivalently after multiplication with the factor $\frac{i\hbar}{c}$

$$L = i\hbar\, v^\dagger\gamma^0\gamma^\mu\partial_\mu u - mc\, v^\dagger\gamma^0 u \tag{23}$$

The Dirac equation itself reduces to

$$\gamma^\mu\partial_\mu u + i\,\frac{mc}{\hbar}\,u = 0$$

or equivalently

$$i\hbar\,\gamma^\mu\partial_\mu u - mc\,u = 0 \tag{24}$$

Here the summation convention is applied to repeated Greek indices over the range 0,1,2,3. The Minkowski metric tensor can be used to raise or lower indices.

For more details and a justification of the given formalism we refer the reader to [31], [32], [33].

Many authors such as Roman [33], Schweber [34], and Jauch-Rohrlich [35] give expressions for the Lagrangian density of the Dirac equation, equivalent to (23), but where u and v are to represent the same function. The Dirac equation is then derived by independently varying u and $u^\dagger$. This is not correct in our opinion, unless one writes different symbols as we do. It can, nevertheless, be justified if one writes

$$u = a + ib; \quad u^\dagger = a^T - ib^T$$

where a and b are vectors out of a real vector space. The vectors a and b can then be varied independently. The approach based on the sesquilinear Lagrangian appears as less artificial and, in our view, offers the advantage that reciprocity laws can be derived directly.

In the covariant formalism, (17) becomes with M = I

$$\partial_\mu(v^\dagger\gamma^0\gamma^\mu w) \doteq 0 \tag{25}$$

where v and w must satisfy the Dirac equation (24) simultaneously. By direct substitution it can easily be verified that, if v = u satisfies (24), this also goes for

$$w = i\hbar\,\partial_\rho u \qquad (\rho \text{ fixed } 0,1,2,3)$$

We then obtain from (25)

$$i\hbar\,\partial_\mu(u^\dagger\gamma^0\gamma^\mu\partial_\rho u) \doteq 0 \tag{26}$$

This equation is equivalent to (21) and (22) for $\rho = 0$ and $\rho = 1,2,3$ respectively.

The *orbital parity operator* Π_{orb} is defined by

$$\Pi_{orb}u(t,x^1,x^2,x^3) = u(t,-x^1,-x^2,-x^3)$$

It is readily seen that

$$\Pi_{orb}\partial_0 = \partial_0\Pi_{orb}, \quad \Pi_{orb}\partial_k = -\partial_k\Pi_{orb} \qquad k = 1,2,3$$

and that Π_{orb} commutes with the matrices γ.
By substitution in (24) it can be verified that if $v = u$ satisfies (24)

$$w = \gamma^0 \Pi_{orb} u \quad \text{does as well}$$

Substitution in (25) gives

$$\partial_\mu (u^\dagger \gamma^0 \gamma^\mu \gamma^0 \Pi_{orb} u) \doteq 0 \tag{27}$$

which represents the local conservation law belonging to the *conservation of parity*. The operator $\Pi = \gamma^0 \Pi_{orb}$ is usually known as the *parity operator of the Dirac particle*.

In an analogous way we can derive the law of conservation of total angular momentum.
If $v = u$ satisfies (24), this also goes for

$$w = J_{\alpha\beta} u$$

where

$$J_{\alpha\beta} = (x_\alpha \partial_\beta - \partial_\beta x_\alpha - \frac{1}{2} \gamma_\alpha \gamma_\beta)\hbar \tag{28}$$

as is easily verified.
Therefore

$$\partial_\mu (u^\dagger \gamma^0 J_{\alpha\beta} u) \doteq 0 \tag{29}$$

which indeed represents the *conservation of total angular momentum.*

We have so far derived the generally known conservation laws for the Dirac equation. Morgan and Joseph discuss in their paper [36] a rather uncommon formalism with a *tensor Lagrangian*, which describes the Dirac equation, or more accurately the equation

$$\partial_\mu (\gamma^\nu \partial_\nu u + i \frac{mc}{\hbar} u) = 0$$

In that formalism a new conservation law is derived, which can also be found by using our formalism as will be illustrated here. Setting $v = \partial_\sigma u$ and $w = \partial_\mu u$ in the reciprocity law (25) we obtain

$$\partial_\nu (\partial_\sigma u^\dagger \gamma^0 \gamma^\nu \partial_\mu u) \doteq 0 \tag{30}$$

3.5.1
3.5.2

Setting $v = u$ and $w = \partial_\sigma \partial_\mu u$ in (25) we obtain

$$\partial_\nu (u^\dagger \gamma^0 \gamma^\nu \partial_\sigma \partial_\mu) \doteq 0 \tag{31}$$

Hence by (30) and (31)

$$\partial_\nu (\partial_\sigma u^\dagger \gamma^0 \gamma^\nu \partial_\mu u - u^\dagger \gamma^0 \gamma^\nu \partial_\sigma \partial_\mu u) \doteq 0$$

and hence by taking the Hermitean conguate of this equation and adding the two together we obtain

$$\partial_\nu Z^\nu_{\cdot\sigma\mu} \doteq 0$$

where

$$Z^\nu_{\cdot\sigma\mu} = \partial_\sigma u^\dagger \gamma^0 \gamma^\nu \partial_\mu u + \partial_\mu u^\dagger \gamma^0 \gamma^\nu \partial_\sigma u - u^\dagger \gamma^0 \gamma^\nu \partial_\sigma \partial_\mu u + \\ - \partial_\sigma \partial_\mu u^\dagger \gamma^0 \gamma^\nu u$$

Morgan and Joseph call this quantity the *zilch* tensor of the Dirac field, borrowing the word zilch from the theory of the electromagnetic field where it was introduced by Lipkin. This was done because in the Morgan-Joseph theory the conservation of the 'zilch' of the Dirac field and of the electromagnetic field are both due to the invariance of the 'tensor Lagrangian' under space-time translations (see also section 3.5.3). We do not recommend this terminology, since this would suggest that the 'zilch' of the electromagnetic field introduced by Lipkin [37], and the 'zilch' of the Dirac field represent the same physical quantity.
In section (4.3) we will show that this cannot be the case in the sense of transferability. We show that the zilch of an electromagnetic field cannot be transferred additively into the Dirac field.

3.5.2 The non-linear wave equation $q_{tt} - q_{xx} - aq_x q_{xx} = 0$

Noether's theorem is not often applied to other than linear systems. The derived conserved densities and fluxes are mostly quadratic expressions in the coordinates and their derivatives. To illustrate that other uses are also possible we will derive a large number of local conservation laws

for the non-linear partial differential equation

$$q_{tt} - q_{xx} - aq_x q_{xx} = 0 \tag{1}$$

Our results will correspond to those of Kruskal and Zabusky [38], but they are obtained in a different way.

Let the Lagrangian density be:

$$L = \tfrac{1}{2}q_t^2 - \tfrac{1}{2}q_x^2 - \frac{1}{6}aq_x^3 \tag{2}$$

The first order variation δL for the transformation

$$\bar{q} = q + \delta q$$

is then

$$\delta L = (-q_{tt} + q_{xx} + aq_x q_{xx})\delta q + \frac{d}{dt}(q_t \delta q) - \frac{d}{dx}(q_x \delta q + \tfrac{1}{2}aq_x^2 \delta q)$$

Euler's equation is therefore given by (1) and

$$\delta L \doteq \frac{d}{dt}(q_t \delta q) - \frac{d}{dx}(q_x \delta q + \tfrac{1}{2}aq_x^2 \delta q) \tag{3}$$

In the search for local conservation laws, i.e. for invariant transformations, we confine our attention to δq's which are polynomials of q, q_t and q_x. When we order these polynomials according to their degree, the conservation laws appear in a corresponding order.

(i) Suppose $\delta q = \varepsilon$

then $\delta L = 0$

Hence by (3) we get a local conservation law which is the equation of motion itself.

(ii) Let $\delta q = \varepsilon q_t$ then $\frac{1}{\varepsilon}\delta L = \frac{d}{dt}L$

$= \varepsilon q_x$ then $\frac{1}{\varepsilon}\delta L = \frac{d}{dx}L$

$= \varepsilon q$ then $\frac{1}{\varepsilon}\delta L = L$

Consequently there are only two invariant transformations δq of first degree, resulting in the local conservation laws

$$\frac{d}{dt}(\frac{1}{2}\,q_t^2 + \frac{1}{2}\,q_x^2 + \frac{1}{6}\,aq_x^2) - \frac{d}{dx}(q_x q_t + \frac{1}{2}\,aq_x^2 q_t) = 0$$

$$\frac{d}{dt}(q_x q_t) - \frac{d}{dx}(\frac{1}{2}\,q_t^2 + \frac{1}{2}\,q_x^2 + \frac{1}{3}\,aq_x^3) = 0$$

when we take a = 0, the local conservation laws for the energy and linear momentum of the one-dimensional linear wave equation (ideal elastic string) are recognized.

(iii) Let δq depend quadratically on q, q_t and q_x.
It can then easily be seen that there is only one invariant transformation $\delta q = \varepsilon q_t q_x$, which results in the local conservation law

$$\frac{d}{dt}(\frac{1}{6}\,q_x^3 - \frac{1}{2}\,q_x q_t^2 + \frac{1}{12}\,aq_x^4) - \frac{d}{dx}(\frac{1}{6}\,q_t^3 + \frac{1}{2}\,q_t q_x^2 + \frac{1}{3}\,aq_x^3 q_t) \doteq 0$$

In a rather tedious, but straight forward way it can be shown that for

$$\delta q = \varepsilon(q_t^2 + q_x^2 + \frac{1}{3}\,aq_x^3)$$

we also have an invariant transformation with

$$\delta L = \frac{d}{dt}(\frac{2}{3}\,q_t^3) - \frac{d}{dx}(\frac{2}{3}\,q_x^3 + \frac{1}{2}\,aq_x^4 + \frac{1}{10}\,a^2 q_x^5)$$

so that we finally obtain

$$\frac{d}{dt}(\frac{1}{3}\,q_t^3 + q_t q_x^2 + \frac{1}{3}\,aq_t q_x^3) - \frac{d}{dx}(q_t^2 q_x + \frac{1}{3}\,q_x^3 + \frac{1}{3}\,aq_x^4 + \frac{1}{2}\,aq_x^2 q_t^2 + \frac{1}{15}\,a^2 q_x^5) \doteq 0$$

We can continue this process and derive an infinite set of local conservation laws, also found by Kruskal and Zabusky.

Another, more elegant, way to derive these local conservation laws is the following.

Let the equations of motion of a physical system be written

$$\frac{d}{dt} A = \frac{d}{dx} f(B) = f'(B)B_x$$

$$\frac{d}{dt} B = \frac{d}{dx} g(A) = g'(A)A_x$$

Equation (1) is a model of these equations, if we take

$$A = q_t,\ B = q_x;\quad f(B) = B + \tfrac{1}{2}aB^2,\ g(A) = A$$

Local conservation laws now have the form

$$\frac{d}{dt} E(A,B) + \frac{d}{dx} F(A,B) \doteq 0$$

Therefore

$$(E_A f'(B) + F_B)B_x + (E_B g'(A) + F_A)A_x \doteq 0$$

A sufficient condition for this is

$$E_A f'(B) + F_B = E_B g'(A) + F_A = 0$$

and hence

$$E_{AA} f'(B) = E_{BB} g'(A)$$

This equation can be solved by series expansion

$$E = \sum_{ij} e_{ij} A^i B^j;\ f'(B) = \sum_i f_i B^i;\ g'(A) = \sum_i g_i A^i$$

Straightforward substitution then results in

$$\sum_{k=0}^{\ell} (m+2)(m+1) e_{m+2,\ell-k} f_k = \sum_{k=0}^{m} (\ell+2)(\ell+1) e_{m-k,\ell+2} g_k$$

This equation was also found by Kruskal and Zabusky.

3.5.3 The electromagnetic field in vacuum

As we already mentioned, a Lagrangian is not neccessarily defined uniquely. In some cases it may be helpful to use rather unfamiliar forms of the Lagrangian in order to simplify calculations. An interesting example of this is found in the *tensor Lagrangian* for the electromagnetic field in vacuum, introduced by Morgan and Joseph [36]. These authors use the tensor Lagrangian to derive the conservation of the zilch, first found by

Lipkin [37]. As an illustration we will repeat this and extend the application of the theory to the derivation of a new class of conservation laws for the electromagnetic field.

In the Morgan-Joseph theory of the electromagnetic field in vacuum, the components of the skew-symmetric electromagnetic field strength tensor $F_{\alpha\beta}$ are considered to be the coordinate functions. As the components of F happen to be the same as those of the electric and magnetic vectors $\underline{E}$ and $\underline{B}$, Euler's equations have to be equivalent to the whole set of Maxwell's equations in its original form.
In most descriptions of the electromagnetic field in terms of a Lagrangian the components of the four-vector potential $\underline{A}$ are the independent coordinate functions. This means that in such cases one has to worry about the gauge invariance. This problem does not arise here, which is the great advantage of the Morgan-Joseph approach.

In the remaining part of this section we will systematically use the tensor notations and Minkowski metric in the way as introduced for instance by Jaunzemis [31] and Barut [32], with the *Minkowski metric tensor*

$$g^{\mu\nu} = \text{diagonal } (-1,1,1,1)$$

The electromagnetic field is described by the skew-symmetric field strength tensor $F^{\alpha\beta}$ or $F_{\alpha\beta}$ depending on the components of $\underline{E}$ and $\underline{B}$, viz.*)

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & cB_z & -cB_y \\ -E_y & -cB_z & 0 & cB_x \\ -E_z & cB_y & -cB_x & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -cE_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}$$

where c is the velocity of light in vacuum.

The *dual electromagnetic-field strength tensor* $\hat{F}$ is defined by

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*We use the rationalized m.k.s. or Giorgi system.

$$\hat{F}^{\alpha\beta} = \frac{1}{2}\mathcal{E}^{\alpha\beta\mu\nu} F_{\mu\nu} \tag{1}$$

$$F^{\mu\nu} = -\frac{1}{2}\mathcal{E}^{\mu\nu\alpha\beta} \hat{F}_{\alpha\beta} \tag{2}$$

where $\mathcal{E}^{\alpha\beta\mu\nu}$, the *alternator* or *Levi-Civita symbol*, is a tensor skew-symmetric in all indices, with components 0 and ±1, +1 if $\alpha\beta\mu\nu$ is an even permutation of (0,1,2,3).
Written in components we have:

$$\hat{F}^{\mu\nu} = \begin{pmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & E_z & -E_y \\ -cB_y & -E_z & 0 & E_x \\ -cB_z & E_y & -E_x & 0 \end{pmatrix} \qquad \hat{F}_{\mu\nu} = \begin{pmatrix} 0 & -cB_x & -cB_y & -cB_z \\ cB_x & 0 & E_z & -E_y \\ cB_y & -E_z & 0 & E_x \\ cB_z & E_y & -E_x & 0 \end{pmatrix}$$

The two Maxwell's equations div $\underline{E} = 0$ and curl $\underline{B} = \frac{1}{c^2}\underline{E}_t$ can be written

$$\partial_\mu F^{\mu\nu} = 0 \tag{3}$$

or equivalently

$$\partial_\mu \hat{F}_{\nu\gamma} + \partial_\nu \hat{F}_{\gamma\mu} + \partial_\gamma \hat{F}_{\mu\nu} = 0 \tag{4}$$

Maxwell's equations div $\underline{B} = 0$; curl $\underline{E} = -\underline{B}_t$ are written in one of the two following equivalent forms

$$\partial_\mu \hat{F}^{\mu\nu} = 0 \tag{5}$$

$$\partial_\mu F_{\nu\gamma} + \partial_\nu F_{\gamma\mu} + \partial_\gamma F_{\mu\nu} = 0 \tag{6}$$

Now consider Joseph and Morgan's *vector Lagrangian density*

$$L_\mu = L_\mu(F_{\alpha\beta}, \partial_\gamma F_{\alpha\beta}) = \partial_\alpha F^{\alpha\beta}\hat{F}_{\beta\mu} - \partial_\alpha \hat{F}^{\alpha\beta}F_{\beta\mu} \tag{7}$$

which has to be considered as four independent Lagrangian functions L_μ for $\mu = 0,1,2,3$. The field variables are the independent components $F_{\mu\nu}$, $\mu > \nu$ of the field strength tensor.

Although there are twenty-four Euler's equations, only eight are independent. These are just Maxwell's equations. The proof of this (which was not given by Joseph and Morgan) goes as follows.
From (7) we obtain for the first-order variation δL_μ of L_μ:

$$\delta L_\mu = \partial_\alpha(\delta F^{\alpha\beta}\hat{F}_{\beta\mu} - \delta\hat{F}^{\alpha\beta}F_{\beta\mu}) + \delta F_{\gamma\nu}(-\partial^\gamma\hat{F}^\nu{}_\mu - \partial_\alpha\hat{F}^{\alpha\gamma}\delta^\nu{}_\mu) + \delta\hat{F}_{\beta\mu}\partial_\alpha F^{\alpha\beta} + \delta\hat{F}^{\alpha\beta}\partial_\alpha F_{\beta\mu} \tag{8}$$

where $\delta^\nu{}_\mu$ is the Kronecker symbol.
Using the summation rule for the Levi-Civita tensor in four dimensions:

$$\mathscr{E}^{\alpha\beta\gamma\sigma}\mathscr{E}_{\mu\nu\rho\sigma} = \delta^\alpha_\mu(\delta^\beta_\rho\delta^\gamma_\nu - \delta^\beta_\nu\delta^\gamma_\rho) + \delta^\alpha_\nu(\delta^\beta_\mu\delta^\gamma_\rho - \delta^\beta_\rho\delta^\gamma_\mu) + \delta^\alpha_\rho(\delta^\beta_\nu\delta^\gamma_\mu - \delta^\beta_\mu\delta^\gamma_\nu) \tag{9}$$

the last two terms in (8) can be written as

$$\delta\hat{F}_{\beta\mu}\partial_\alpha F^{\alpha\beta} = \delta F_{\gamma\nu}(\partial^\gamma\hat{F}^{\cdot\nu}_\mu + \tfrac{1}{2}\,\partial_\mu\hat{F}^{\nu\gamma})$$

$$\delta\hat{F}^{\alpha\beta}\partial_\alpha F_{\beta\mu} = \delta F_{\gamma\nu}(-\tfrac{1}{2}\,\partial_\mu\hat{F}^{\gamma\nu} - \partial_\alpha\hat{F}^{\alpha\gamma}\delta^\nu_\mu)$$

We finally obtain

$$\delta L_\mu = \partial_\alpha(\delta F^{\alpha\beta}\hat{F}_{\beta\mu} - \delta\hat{F}^{\alpha\beta}F_{\beta\mu}) + \delta F_{\gamma\nu}(2\partial^\gamma\hat{F}^{\cdot\nu}_\mu + \partial_\mu\hat{F}^{\nu\gamma} - 2\partial_\alpha\hat{F}^{\alpha\gamma}\delta^\nu_\mu)$$

Note that not all the $\delta F_{\gamma\nu}$'s are independent, since $F_{\gamma\nu}$ is skew symmetric. Therefore Euler's equations are

$$2\partial^\gamma\hat{F}^{\cdot\nu}_\mu + \partial_\mu\hat{F}^{\nu\gamma} - 2\partial_\alpha\hat{F}^{\alpha\gamma}\delta^\nu_\mu = 2\partial^\nu\hat{F}^{\cdot\gamma}_\mu + \partial_\mu\hat{F}^{\gamma\nu} - 2\partial_\alpha\hat{F}^{\alpha\nu}\delta^\gamma_\mu$$

and these equations are indeed equivalent to Maxwell's equations (4) and (5). Consequently

$$\delta L_\mu \doteq \partial_\alpha(\delta F^{\alpha\beta}\hat{F}_{\beta\mu} - \delta\hat{F}^{\alpha\beta}F_{\beta\mu}) \tag{10}$$

Equations (10) can now be used to derive a number of conservation laws.

$$\text{Let } \delta F^{\alpha\beta} = \varepsilon\,\hat{F}^{\alpha\beta}$$
$$\text{then } \delta\hat{F}^{\alpha\beta} = \varepsilon\,\hat{\hat{F}}^{\alpha\beta} = -\varepsilon\,F^{\alpha\beta} \tag{11}$$

and hence by (7)

$$\delta L_\mu = 0$$

and with (10) we obtain

$$\partial_\alpha S_\mu^{\cdot\alpha} \doteq 0 \tag{12}$$

where

$$S_\mu^{\cdot\alpha} = \hat{F}^{\alpha\beta}\hat{F}_{\beta\mu} + F^{\alpha\beta}F_{\beta\mu} \tag{13}$$

is apart from a constant factor, the *Maxwell energy momentum tensor* as given by Morgan [39].
It is easily seen that $S_\mu^{\cdot\alpha}$ is symmetric in the sense that

$$S_\mu^{\cdot\alpha} = S^\alpha_{\cdot\mu}$$

Let now

$$\delta F^{\alpha\beta} = \varepsilon\, \partial_\nu F^{\alpha\beta} \qquad (\nu \text{ fixed}) \tag{14}$$

then $$\delta \hat{F}^{\alpha\beta} = \varepsilon\, \partial_\nu \hat{F}^{\alpha\beta}$$

and from (7) one obtains

$$\delta L_\mu = \varepsilon\, \partial_\nu L_\mu$$

Hence by (10)

$$\partial_\alpha (Z^\alpha_{\cdot\mu\nu} - \delta^\alpha_\nu (\partial_\gamma F^{\gamma\beta}\hat{F}_{\beta\mu} - \partial_\gamma \hat{F}^{\gamma\beta}F_{\beta\mu})) \doteq 0$$

Moreover, we find the trivially equivalent local conservation law

$$\partial_\alpha Z^\alpha_{\cdot\mu\nu} \doteq 0 \tag{15}$$

where

$$Z^\alpha_{\cdot\mu\nu} = \partial_\nu F^{\alpha\beta}\hat{F}_{\beta\mu} - \partial_\nu \hat{F}^{\alpha\beta}F_{\beta\mu} \tag{16}$$

It can be shown that the tensor Z is symmetric in the first two indices. Consequently, the form of the local conservation law (15) is the same as found by Morgan-Joseph [36] and Kibble [40]. Kibble showed that Z is trace-free in the first two indices, for solutions of Maxwell's equations trace-free in all indices, and has a vanishing divergence in all indices.

He also showed that (16) is essentially the same as the *zilch* introduced by Lipkin [37].

In many books the reader is given the impression that the energy-momentum of a physical system is the conserved quantity belonging to the invariance under space-time translations. The given derivation of the conservation of the energy and momentum illustrates that this impression is generally not correct. This is in fact caused by the special form of the chosen Lagrangian and Euler's equation. In chapter 4 we will discuss an alternative way to identify a distinct conserved quantity with the energy or momentum of the system.

The easiest way to derive the conservation of *angular momentum* is the following

Let $\quad \delta F^{\alpha\beta} = \epsilon\, x^{\gamma} \hat{F}^{\alpha\beta} \qquad (\gamma \text{ fixed})$

then $\quad \delta \hat{F}^{\alpha\beta} = -\epsilon\, x^{\gamma} F^{\alpha\beta}$

Hence by (7) and (13)

$$\delta L_{\mu} = \epsilon\, S_{\mu}^{\cdot\gamma}$$

and hence by (10)

$$\partial_{\alpha}(x^{\gamma} S_{\mu}^{\cdot\alpha}) \doteq S_{\mu}^{\cdot\gamma}$$

Since S is a symmetric tensor we can write

$$\partial_{\alpha}(x^{\gamma} S_{\mu}^{\cdot\alpha} - x_{\mu} S^{\gamma\alpha}) \doteq 0$$

which is the usual form of the local conservation law of the *angular momentum* of the electromagnetic field.

Analogously we can set

$$\delta F^{\alpha\beta} = \epsilon\, x^{\gamma} \partial_{\nu} F^{\alpha\beta} \qquad (\gamma, \nu \text{ fixed})$$

then $\quad \delta \hat{F}^{\alpha\beta} = \epsilon\, x^{\gamma} \partial_{\nu} \hat{F}^{\alpha\beta}$

Hence by (7) and (16)

$$\delta L_{\mu} = \epsilon\, Z^{\gamma}_{\cdot\mu\nu} + \epsilon\, x^{\gamma} \partial_{\nu} L_{\mu}$$

or $$\delta L_\mu \doteq \varepsilon Z^\gamma{}_{\mu\nu}$$

and hence by (10)

$$\partial_\alpha (x^\gamma Z^\alpha{}_{.\mu\nu}) \doteq Z^\gamma{}_{.\mu\nu}$$

Since Z is symmetric in the first two indices we get

$$\partial_\alpha (x^\gamma Z^\alpha{}_{.\mu\nu} - x_\mu Z^{\alpha\gamma}{}_{..\nu}) \doteq 0$$

These set of local conservation laws will be interpreted as the conservation of the *"zilch momentum"*.

4. Additive transferability of constants of the motion

4.1 Introduction

In this chapter we will discuss the transferability of constants of the motion (conserved quantities) from one system to another. It is our purpose to answer such questions as: When can we speak of for instance "energy" of a system A and what is then "energy" for system B, A and B being entirely different systems? We will be interested specifically in the case of an electrically charged particle and an electromagnetic field. As an illustration of our train of thought we will first discuss a simple example in which A and B are identical mechanical systems.

Let A be a three-dimensional electrical dipole fixed in space with the x, y and z components of the dipole strength u^i ($i = 1, 2$ and 3 respectively) which form the vector u = column (u^1,u^2,u^3). The dipole-strength components vary harmonically at frequency ω. The behaviour of the dipole system is then governed by the Lagrangian

$$L_A = \frac{1}{2} u_t^T u_t - \frac{1}{2} \omega^2 u^T u$$

or, since the first-order variation of L_A yields

$$\delta L_A = \frac{d}{dt} (u_t^T \delta u) - (u_{tt} + \omega^2 u)^T \delta u$$

by the equation of motion

$$u_{tt} + \omega^2 u = 0$$

Therefore

$$\delta L_A \doteq \frac{d}{dt} (u_t^T \delta u)$$

When M is some constant square matrix it is easily seen that the Lagrangian is invariant under the transformation

$$\delta u = \varepsilon M u, \quad M = -M^T$$

and invariant up to a gauge transformation under the transformation

$$\delta u = \varepsilon M u_t, \quad M = M^T$$

This leads to the conservation laws

$$\frac{d}{dt}\,(u^T M u_t) \doteq 0, \quad M = -M^T \tag{6}$$

and

$$\frac{d}{dt}\,(\tfrac{1}{2}\,u_t^T M u_t + \tfrac{1}{2}\,\omega^2 u^T M u) \doteq 0, \; M = M^T \tag{7}$$

Since there are six linearly-independent constant symmetric 3×3 matrices and three linearly-independent skew-symmetric 3×3 matrices, equation (6) stands for three and equation (7) for six linearly-independent conservation laws. The physical interpretation of these is similar to the interpretation in the case of the two-dimensional harmonic oscillator (section 1.5.2). Equation (6) yields that the three components of the angular momentum are conserved. The six conservation laws (7) represent the conservation of the energy due to the component of the motion in an arbitrary direction.

When B is an identical system described by

$$L_B = \tfrac{1}{2}\,v_t^T v_t - \tfrac{1}{2}\,\omega^2 v^T v$$

we have the conservation laws

$$\frac{d}{dt}\,(v^T N v_t) \doteq 0; \quad N = -N^T$$

$$\frac{d}{dt}\,(\tfrac{1}{2}\,v_t^T N v_t + \tfrac{1}{2}\,\omega^2 v^T N v) \doteq 0; \quad N = N^T$$

which are caused by the invariant transformations $\delta v = \varepsilon N v$ and $\delta v = \varepsilon N v_t$ respectively.

We will now discuss the constants of the combined motion of A and B. Without coupling between A and B this is trivial. In that case the nine constants of the motion of system A, and the nine of system B remain constants of the combined motion. Since the two three-dimensional oscillating dipoles without any coupling can be considered as one six-dimensional isotropic harmonic oscillator, there must be thirty-six linearly independent quadratic constants of the combined motion. Therefore, eighteen new constants of the motion appear by taking the two systems together. Since these results are rather simple they are not interesting as an illustration of a more general case. In view of this we consider instead the systems with coupling:

$$L_{A+B} = L_A + L_B + L_{AB} \tag{8}$$

Here L_{AB} represents the *interaction Lagrangian* of the systems A and B. We can for instance take

$$L_{AB} = -\frac{1}{4\pi\varepsilon_0|r|^3}\left(u^T v - \frac{3}{|r|^2}(u^T r)(v^T r)\right) \tag{9}$$

where ε_0 stands for the electrical permitivity of vacuum and the column-vector r denotes the position of the dipole B relative to the position of dipole A. This is the Coulomb interaction of two electrical dipoles. (See for instance Panofsky-Phillips [41]).

When we consider r constant, its length equal to ℓ and its direction as the z-direction equation (9) can be written as

$$L_{AB} = -u^T D v$$

where

$$D = \alpha \text{ diagonal } (1, 1, -2)$$

$$\alpha = \frac{1}{4\pi\varepsilon_0\ell^3}$$

In this way we have

$$L_{A+B} = \frac{1}{2}u_t^T u_t - \frac{1}{2}\omega^2 u^T u + \frac{1}{2}v_t^T v_t - \frac{1}{2}\omega^2 v^T v - u^T D v$$

and the first-order variation of this expression is

$$\delta L_{A+B} = \frac{d}{dt}(u_t^T \delta u + v_t^T \delta v) - (u_{tt} + \omega^2 u + Dv)^T \delta u - (v_{tt} + \omega^2 v + Du)^T \delta v$$

The equations of motion are therefore

$$u_{tt} + \omega^2 u + Dv = 0 \tag{10}$$

$$v_{tt} + \omega^2 v + Du = 0 \tag{11}$$

and we obtain

$$\delta L_{A+B} \doteq \frac{d}{dt}(u_t^T \delta u + v_t^T \delta v) \tag{12}$$

We will now try to find out what happens, when the transformations $\delta u = \varepsilon M u$, $M = -M^T$ and $\delta v = \varepsilon N v$, $N = -N^T$ are applied to the coupled dipoles. The only real problem then facing up is the question whether this transformation leaves the interaction Lagrangian L_{AB} invariant, c.q. invariant up to a gauge

transformation, or not.
It is readily seen that

$$\delta L_{AB} = 0$$

if and only if the constant skew-symmetric matrices M and N satisfy the relation

$$MD - DN = 0$$

The same condition for the symmetric matrices M and N appears as the condition under which the transformation $\delta u = \varepsilon M u_t$, $\delta v = \varepsilon N v_t$ with $M = M^T$ and $N = N^T$ leaves the interaction Lagrangian invariant up to a gauge transformation

$$\delta L_{AB} = -\varepsilon \frac{d}{dt} (u^T N D v)$$

As conservation laws for the interacting dipoles we thus find

$$\frac{d}{dt} (u^T M u_t + v^T N v_t) \doteq 0 \quad M = -M^T, \; N = -N^T, \; MD = DN \tag{13}$$

and

$$\frac{d}{dt} \left(\frac{1}{2} u_t^T M u_t + \frac{1}{2} \omega^2 u^T M u + \frac{1}{2} v_t^T N v_t + \frac{1}{2} \omega^2 v^T N v + u^T M D v\right) \doteq 0 \tag{14}$$

$$M = M^T, \; N = N^T, \; MD = DN$$

It can easily be seen that solutions for M and N of the matrix equation MD = DN are mutually equal (M = N). So we obtain the commutator problem MD = DM. This problem can be solved in a general way (Gantmacher [22]). A linearly independent basis is then given by the skew-symmetric matrix

$$S = \begin{pmatrix} \cdot & 1 & \cdot \\ -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

and the symmetric matrices

$$G_1 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \quad G_2 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad G_3 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \quad G_4 = \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

We may conclude that there is one conservation law of the form (13) and there are four linearly independent conservation laws of the form (14).

Equation (13) with M = N = S represents the conservation of the total angular momentum in the z-direction of the two dipoles together. Equations (14) with

respectively $M = N = G_i$, $i = 1,2,3,4$ represent the conservation of the total energy of the two dipoles, and the conservation of the energy of the entire system associated with a component of the motion in arbitrary directions in the x-y plane.

A more *formal interpretation* of equation (13) is as follows: while $u^T S u_t$ is a constant of the motion of system A and $v^T S v_t$ of the system B, $u^T S u_t + v^T S v_t$ is a constant of the motion of the coupled system A+B. So the coupling between A and B causes $u^T S u_t$ and $v^T S v_t$ to change in such a way that the sum of both remains constant. The quantities $u^T S u_t$ and $v^T S v_t$ are therefore quantities of the same type and we can give these quantities the same name. If for A the angular momentum in the z-direction is defined as $u^T S u_t$, the quantity $v^T S v_t$ represents the angular momentum in the z-direction of system B. We say: the angular momentum $u^T S u_t$ is *additively transferable* to the conserved quantity $u^T S v_t$ of B (and vice versa). If constants of the motion of different systems can be *transferred additively* to each other they represent the same *physical quantity*, and can be given the same name.

The *formal interpretation* of equation (14) for $M = N = G_I = I$ is more complicated but it illustrates our concept of additively transferable constants of the motion in a more general way. Here we have to deal with three terms instead of two.
Let the total energy of A be defined as $(\frac{1}{2} u_t^T u_t + \frac{1}{2} \omega^2 u^T u) = E_A$, then it follows from (14) with $M = N = G_I = I$ that the energy of A can partly transfer to $E_B = (\frac{1}{2} v_t^T v_t + \frac{1}{2} \omega^2 v^T v)$, constant of the motion of system B, in such a way that $E_{A+B} = E_A + E_B + u^T D v$ remains constant. The quantities E_A, E_B and $E_{AB} = u^T D v$ are therefore of the same type and we may call E_B the *energy of system* B and E_{AB} the *interaction energy* of A and B. We say the energy E_A is *additively transferable* to the system B.

In the special example of this section, where the coupling is caused by a *real physical entity*, the electrical field, we can interpret the interaction energy of A and B as the energy of the electrical field. Nevertheless this interpretation is not fundamental. Later on in this chapter, we will discuss for instance the interaction of an electrically charged particle with an electromagnetic field. Then there is no physical entity to which the interaction energy can be ascribed. Interpretations, like those given in the

foregoing paragraph alsc apply to this situation and are therefore more general.
Similar interpretations of equation (14) as given for $M = N = G_1 = I$ can be given for $M = N = G_i$, $i = 2,3,4$. Therefore we conclude that if there is a Coulomb interaction between the oscillating dipoles A and B, five of the nine constants of the motion of dipole A can be transferred additively into the corresponding constant of the motion of dipole B.

In the search for invariant transformations of the coupled system A+B, we have restricted our attention to those variations of the coordinate functions which are invariant transformations for the uncoupled systems A and B. This procedure necessarily resulted in constants of the motion of the interacting system A+B which can be *decomposed additively* into parts which are constants of the motion of system A and system B respectively, and a part which belongs to the interaction only (and which may be zero sometimes). We do not therefore find any new constants of the motion. We will make the same restriction in the following sections in view of the complexity of the general problem. This is equivalent to looking for additively transferable constants of the motion only.

Here we will demonstrate how a more general approach can be made in the case of the coupled electrical dipoles.

By introducing the six-dimensional vector w

$$w = \text{quasi column } (u,v) \tag{15}$$

the equations of motion (10), (11) can be written

$$w_{tt} + Pw = 0 \tag{16}$$

where the constant symmetric 6×6 matrix P is given by the partitioned matrix

$$P = \begin{pmatrix} \omega^2 I & D \\ D & \omega^2 I \end{pmatrix} \tag{17}$$

where I is the three-dimensional unity matrix.
A Lagrangian function that describes (16) is

$$L = \frac{1}{2} w_t^T w_t - \frac{1}{2} w^T P w \tag{18}$$

and the first-order variation δL yields

$$\delta L \doteq \frac{d}{dt}\,(w_t^T \delta w)$$

Invariant variations now showing up are

$$\delta w = Aw,\ AP = PA,\ A = -A^T$$

and

$$\delta w = Aw_t,\ AP = PA,\ A = A^T$$

This leads to the conservation laws

$$\frac{d}{dt}\,(w^T A w_t) \doteq 0,\ AP = PA,\ A = -A^T \tag{19}$$

and

$$\frac{d}{dt}\,(\tfrac{1}{2}\, w_t^T A w_t + \tfrac{1}{2}\, w^T A P w) \doteq 0,\ AP = PA.\ A = A^T \tag{20}$$

It can easily be verified that all symmetric (skew-symmetric) matrices A satisfying AP = PA can be written

$$A = \begin{pmatrix} M & N \\ N & M \end{pmatrix}$$

where M and N are symmetric (skew-symmetric) 3×3 matrices which commute with the matrix D: MD = DM, ND = DN. A basis for the allowed matrices M and N is therefore S and G_i, i = 1,2,3,4.
If we take N = 0 we find in this way those five linearly independent constants of the motion which have already been found. These can be decomposed into contributions of the sub-systems A and B, and of the interaction.
By putting M = 0, N = S we obtain a conservation law of the form (19). This reads

$$\frac{d}{dt}\,(u^T S v_t - u_t^T S v) \doteq 0 \tag{21}$$

For M = 0, N = G_i, i = 1,2,3,4 four conservation laws of the form (20) are obtained. These can be written

$$\frac{d}{dt}\,(u_t^T G_i v_t + \tfrac{1}{2}\, u^T G_i D u + \omega^2 u^T G_i v + \tfrac{1}{2}\, v^T G_i D v) \doteq 0,\ i = 1,2,3,4 \tag{22}$$

Obviously, the densities of (21) and (22) cannot be decomposed into the sum of constants of the motion of the dipoles A and B and an interaction term. It is nevertheless possible to give an interpretation in terms of the six-dimensional isotropic oscillator, but we will not pursue this matter any further.

4.2. Interaction between an electrically charged particle and an electromagnetic field

4.2.1. Introduction, the electromagnetic field

In this section we will discuss the additive transferability of the energy and linear momentum from an electrically charged particle to an electromagnetic field. The description of the field will be classical. The electrically charged particle will be a relativistic one and will be treated both classically and quantummechanically (Dirac equation).

In section (3.5.3) we introduced a Lagrangian density for the electromagnetic field, but this is not very useful here, since inhomogeneous Maxwell equations have to be considered. Another possibility is the Lagrangian given by Schwinger [42], which is in the notation of section (3.5.3)

$$L = -\frac{\varepsilon_0 c}{2} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4}\varepsilon_0 F^{\mu\nu}F_{\mu\nu} + j^\mu A_\mu \qquad (1)$$

where

c = the vacuum light velocity

ε_0 = the electrical permitivity of vacuum

and j^μ are the non-specified components of the electrical four-current

$j^0 = c\rho$, ρ the electrical charge density

j^i, $i = 1,2,3$ the components of the electrical current density vector.

The components of the skew-symmetric electromagnetic field strength tensor F are considered to be independent of the components of vector A. The coordinate functions are therefore $F^{\mu\nu}$ $(\mu > \nu)$ and A^μ. For independent variations of the coordinate functions, the first-order variation of L is

$$\delta L = -\frac{1}{2}\varepsilon_0 c(\partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{c}F_{\mu\nu})\delta F^{\mu\nu} - \partial_\mu(\varepsilon_0 c F^{\mu\nu}\delta A_\nu) + \varepsilon_0 c(\partial_\mu F^{\mu\nu} + \frac{1}{\varepsilon_0 c} j^\nu)\delta A_\nu \qquad (2)$$

and Euler's equations are therefore

4.2.1
4.2.2

$$\frac{1}{c} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3}$$

$$\partial_\mu F^{\mu\nu} = - \frac{1}{\varepsilon_0 c} j^\nu \tag{4}$$

These are indeed equivalent to the inhomogeneous Maxwell equations and the vector A^μ may be interpreted as the four-vector potential

$$A^0 = \phi$$
$$A^i = (\underline{A})^i \quad i = 1,2,3$$

where ϕ is the scalar potential, and $\underline{A}$ the vector potential of the electromagnetic field defined by

$$\underline{B} = \text{curl } \underline{A}$$

$$\underline{E} = - \text{grad } \phi - \frac{\partial}{\partial t} \underline{A}$$

Finally, we have from (2)

$$\delta L \doteq - \partial_\mu (\varepsilon_0 c \, F^{\mu\nu} \delta A_\nu) \tag{5}$$

It can be shown that the electromagnetic field and its interaction with a current distribution j^ν can also be described by

$$L = - \frac{1}{4} \varepsilon_0 \, F^{\mu\nu} F_{\mu\nu} + j^\nu A_\nu$$

where

$$F_{\mu\nu} = \frac{1}{c} (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

and the field variables are A_μ, $\mu = 0,1,2,3$.
Euler's equations then become

$$\partial_\mu F^{\mu\nu} + \frac{1}{\varepsilon_0 c} j^\nu = 0$$

and the first-order variation δL of L can be written as

$$\delta L \doteq - \partial_\mu (\varepsilon_0 c \, F^{\mu\nu} \delta A_\nu)$$

We shall revert to this equation in section 4.3.

4.2.2. Interaction with a classical relativistic particle

The motion of a classical relativistic particle of rest mass m and charge e in an electromagnetic field described by the four-vector potential A^μ can be

described by the Lagrangian function

$$L = \frac{1}{2} m\dot{z}^{\mu}\dot{z}_{\mu} + e\dot{z}^{\mu}A_{\mu}(z) \tag{6}$$

where the dot means differentiation with respect to the *proper time* τ of the particle. The coordinate functions are therefore $z^{\mu}(\tau)$. The first-order variation of L is

$$\delta L = \frac{d}{d\tau}(m\dot{z}^{\mu}\delta z_{\mu} + eA^{\mu}\delta z_{\mu}) - m\ddot{z}^{\mu}\delta z_{\mu} - e\partial_{\nu}A_{\mu}\dot{z}^{\nu}\delta z^{\mu} + e\partial_{\nu}A_{\mu}\dot{z}^{\mu}\delta z^{\nu} \tag{7}$$

which leads to Euler's equations

$$m\ddot{z}_{\mu} + e(\partial_{\nu}A_{\mu} - \partial_{\mu}A_{\nu})\dot{z}^{\nu} = 0 \tag{8}$$

and leaving

$$\delta L = \frac{d}{d\tau}(m\dot{z}^{\mu}\delta z_{\mu} + eA_{\mu}\delta z^{\mu}) \tag{9}$$

A more extensive discussion of the Lagrangian function for a classical relativistic particle, and the proof that (8) is the equation of motion for such a particle, have been presented in the literature, for instance by Barut [32] and Rohrlich [43].

The dynamics of the particle in interaction with the electromagnetic field is described by the equations (3), (4) and (7) together with the assumption [32]

$$j^{\mu}(x) = e\int_{-\infty}^{\infty}\dot{z}^{\mu}\delta(x-z(\tau))d\tau \tag{10}$$

With this assumption it is rather easy to verify that the equations of motion of the particle (8) and the electromagnetic field (3), (4) can be derived from the action functional

$$\mathcal{W}_{tot} = \mathcal{W}_{em} + \mathcal{W}_{m} + \mathcal{W}_{int} \tag{11}$$

where

$$\mathcal{W}_{em} = \int_{R} dx(-\frac{1}{2}\varepsilon_0 c\, F^{\mu\nu}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) + \frac{1}{4}\varepsilon_0 F_{\mu\nu}F^{\mu\nu}) \tag{12}$$

$$\mathcal{W}_{m} = \int_{-\infty}^{\infty} d\tau(\frac{1}{2} m\dot{z}_{\mu}\dot{z}^{\mu}) \tag{13}$$

and $$\mathfrak{W}_{int} = \int_{-\infty}^{\infty} e\dot{z}^{\mu} A_{\mu}(z) d\tau \tag{14}$$

where R is some region in the four-dimensional x-space, containing the entire path $z(\tau)$ of the particle.
Without using any boundary condition for the variations of $A_{\mu}(x)$, $F^{\mu\nu}(x)$ and $z(\tau)$ it can be seen from (4.2.2)

$$\delta\mathfrak{W}_{em} = \int_R dx(-\frac{1}{2}\varepsilon_0 c(\partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{c}F_{\mu\nu})\delta F^{\mu\nu} - \partial_\mu(\varepsilon_0 c F^{\mu\nu}\delta A_\nu) + \varepsilon_0 c\,\partial_\mu F^{\mu\nu}\delta A_\nu) \tag{15}$$

and from (7)

$$\delta\mathfrak{W}_m = \int_{-\infty}^{\infty} d\tau(\frac{d}{d\tau}(m\dot{z}_\mu \delta z^\mu) - m\ddot{z}^\mu \delta z_\mu) \tag{16}$$

Moreover

$$\delta\mathfrak{W}_{int} = \int_{-\infty}^{\infty} d\tau\, e(\delta\dot{z}^\mu A_\mu(z) + \dot{z}^\mu \partial_\nu A_\mu(z)\delta z^\nu + \dot{z}^\mu \delta A_\mu(z))$$

which can be written as

$$\delta\mathfrak{W}_{int} = \int_{-\infty}^{\infty} d\tau\, e(\frac{d}{d\tau}(\delta z^\mu A_\mu(z) - (\partial_\nu A_\mu(z) - \partial_\mu A_\nu(z))\dot{z}^\nu \delta z^\mu) + \int_R j^\nu(x)\delta A_\nu(x)dx \tag{17}$$

where we have used (10) and have put

$$\delta A_\nu(z) = \int_R \delta A_\nu(x)\delta(x - z(\tau))dx$$

which is true if the whole path $z(\tau)$ is lying in R.
It is now readily seen with (15)-(17) that (11) gives with Hamilton's principle the desired equations of motion. We finally get

$$\delta\mathfrak{W}_{tot} \stackrel{!}{=} -\int_R dx\,\partial_\mu(\varepsilon_0 c F^{\mu\nu}\delta A_\nu) + \int_{-\infty}^{\infty} d\tau\,\frac{d}{d\tau}(m\dot{z}^\mu \delta z_\mu + eA_\mu(z)\delta z^\mu) \tag{18}$$

Now consider the special transformation

$$\delta A_\nu = \varepsilon^\rho(\partial_\rho A_\nu - \partial_\nu A_\rho)$$
$$\delta F^{\mu\nu} = \varepsilon^\rho \partial_\rho F^{\mu\nu} \qquad \delta z^\nu = -\varepsilon^\nu \tag{19}$$

where ε^ρ, $\rho = 0,1,2,3$ are arbitrary infinitesimal constants. Note that this transformation can be regarded as an infinitesimal space-time translation with an additional term in δA_ν, namely $-\varepsilon^\rho \partial_\nu A_\rho$ to ensure the gauge invariance in the final results. It can easily be seen that omission of this term already leads to the non-gauge invariance of the electromagnetic field part alone. It can be verified that

$$\delta\mathfrak{W}_{tot} = \varepsilon^\rho \int_R dx\, \partial_\rho(-\tfrac{1}{2}\varepsilon_0 c F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}\varepsilon_0 F^{\mu\nu}F_{\mu\nu}) - \varepsilon^\rho \int e\,\frac{d}{d\tau}A_\rho(z)d\tau$$

We obtain on the other hand with (18)

$$\delta\mathfrak{W}_{tot} = -\varepsilon^\rho \int_R dx\, \partial_\mu(\varepsilon_0 c F^{\mu\nu}(\partial_\rho A_\nu - \partial_\nu A_\rho)) - \varepsilon^\rho \int_{-\infty}^{\infty} \frac{d}{d\tau}(m\dot{z}_\rho + eA_\rho)d\tau$$

So that

$$\int_R \{\partial_\rho(-\tfrac{1}{2}\varepsilon_0 c F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}\varepsilon_0 F^{\mu\nu}F_{\mu\nu}) + \partial_\mu(\varepsilon_0 c F^{\mu\nu}(\partial_\rho A_\nu - \partial_\nu A_\rho))\}dx + \int_{-\infty}^{\infty} m\ddot{z}_\rho\, d\tau \doteq 0$$

By one of Euler's equations (3) we also may write

$$\int_R dx\, \partial_\mu\, S_\rho{}^\mu + \int_{-\infty}^{\infty} m\ddot{z}_\rho\, d\tau \doteq 0$$

where

$$S_\rho{}^\mu = \varepsilon_0\, F^{\mu\nu}F_{\rho\nu} - \tfrac{1}{4}\varepsilon_0\, F^{\alpha\beta}F_{\alpha\beta}\,\delta^\mu_\rho$$

is apart of constant factor, the energy-momentum tensor of the electromagnetic field as given by Kibble [40] and is because of the identity

$$\tfrac{1}{2}F^{\alpha\beta}F_{\alpha\beta}\delta^\mu_\rho = -F_{\rho\nu}F^{\nu\mu} + \hat{F}^{\mu\nu}\hat{F}_{\nu\rho}$$

equal to

$$S_\rho{}^\mu = \tfrac{1}{2}\varepsilon_0(\hat{F}^{\mu\nu}\hat{F}_{\rho\nu} + F^{\mu\nu}F_{\rho\nu}) \qquad (20)$$

which is equivalent to equation (3.5.3.13)

Equation (20) is in this form also given by Rohrlich [43]. It represents that the total energy and momentum of the electromagnetic field and the particle together at $\dot{x} = \dot{z}(\infty)$ and $\dot{x} = \dot{z}(-\infty)$ are equal. Therefore we can give (20) the

name *"super global conservation law"*. Barut [32] writes that from a variation principle based on an action functional equivalent to (11), the local conservation law of the energy momentum

$$\partial_\mu(S_\rho^{\cdot\mu} + T_\rho^{\cdot\mu}) \doteq 0 \tag{21}$$

can be derived directly.

Here $$T^{\rho\mu} = \int_{-\infty}^{\infty} m\dot{z}^\rho \dot{z}^\mu \delta(x-z(\tau))d\tau \tag{22}$$

is the energy momentum tensor of the particle.
We do not quite see how this can be done. One of the problems that arise is that the domain of integration R is not arbitrary but has to contain the whole path $z(\tau)$. A possible way to derive (21) directly is to start from a Lagrangian density of the form

$$L = (-\frac{1}{2}\varepsilon_0 c F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4}\varepsilon_0 F_{\mu\nu}F^{\mu\nu})\delta(\tau) +$$

$$+ \frac{1}{2} m\dot{z}_\mu \dot{z}^\mu \delta(x-z(\tau)) + e\dot{z}^\mu A_\mu \delta(x-z(\tau))$$

where x^μ, $\mu = 0,1,2,2$ and τ are now the coordinates.
We will not work this out into detail here. The fact that (21) is true can be verified directly, as is done by Barut [32]

A justification of the designation energy momentum tensor for S and T is as follows.
It can be shown that $T_0^{\cdot 0}$ and $T_i^{\cdot 0}$, $i = 1,2,3$ tend to the classical, non-relativistic energy and linear momentum components respectively, at the limit of small velocities.
Therefore $T_0^{\cdot 0}$ is the relativistic generalization of the concept of energy discussed in section (4.1) and analogously $T_i^{\cdot 0}$ with $i = 1,2,3$ are generalizations of the concept of the linear momentum. The relativistic energy and linear momentum are conserved quantities in the absence of the electromagnetic field (section 3.5.3). If coupling with such a field occurs, the quantities are not constant anymore (as can be shown easily). From (21) we see that the mechanical energy $T_0^{\cdot 0}$ and linear momentum $T_i^{\cdot 0}$ can be transferred additively to the electromagnetic energy $S_0^{\cdot 0}$, and linear momentum $S_i^{\cdot 0}$ respectively.
We also see that the coupling between energy and momentum of field and particle does not introduce *"interaction terms"* (on the analogy of the interaction

energy E_{AB} in section 4.1).

4.2.3. Interaction with a Dirac particle

In section 3.5.1. we have presented the Lagrangian density for the Dirac equation. The discussion can be extended to the Dirac equation for a particle in interaction with an electromagnetic field. With the Lagrangian density of section (4.2.1) for the electromagnetic field this leads to the Lagrangian density for the Dirac field and the electromagnetic field together

$$L = L_{em} + L_{Di} + L_{int} \tag{23}$$

with

$$L_{em} = -\frac{\varepsilon_0 c}{2} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{4}\varepsilon_0 F^{\mu\nu}F_{\mu\nu} \tag{24}$$

$$L_{Di} = i\hbar v^\dagger \gamma^0 \gamma^\mu \partial_\mu u - mcv^\dagger \gamma^0 u \tag{25}$$

$$L_{int} = ev^\dagger \gamma^0 \gamma^\mu u A_\mu = j^\mu A_\mu \tag{26}$$

where

$$j^\mu = ev^\dagger \gamma^0 \gamma^\mu u \tag{27}$$

is the electrical current of the Dirac field if u = v and e is the electrical charge of an electron.

It can be easily verified that the first-order variation of L is

$$\delta L = -\partial_\mu(\varepsilon_0 c F^{\mu\nu}\delta A_\nu - i\hbar v^\dagger \gamma^0 \gamma^\mu \delta u) \tag{28}$$

if Euler's equations

$$\frac{1}{c} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{29}$$

$$\partial_\mu F^{\mu\nu} + \frac{1}{\varepsilon_0 c} j^\nu = 0, \tag{30}$$

$$i\hbar\gamma^\mu \partial_\mu u - mcu + e\gamma^\mu u A_\mu = 0, \tag{31}$$

and

$$i\hbar\gamma^\mu \partial_\mu v - mcv + e\gamma^\mu v A_\mu = 0 \tag{32}$$

are satisfied.

Euler's equations (29)-(32) are precisely the equations of motion for the interacting electromagnetic and Dirac fields. Here u and v are two

(generally different) solutions of equation (31).

It is readily verified that the Lagrangian (23) is invariant ($\delta L = 0$) with respect to the inifinitesimal transformation

$$\begin{aligned} \delta u &= i\varepsilon u \\ \delta v &= i\varepsilon v \end{aligned} \tag{33}$$

From (27) and (28) we obtain the local conservation law

$$\partial_\mu (v^\dagger \gamma^0 \gamma^\mu u) = \partial_\mu j^\mu \doteq 0 \tag{34}$$

which represents the conservation of electrical charge if u and v are chosen to be the same function (c.f. equation (3.5.1.25)). So we see that the electrical charge density defined as

$$u^\dagger \gamma^0 \gamma^0 u = u^\dagger u$$

is the density of a local conservation law for both the homogeneous Dirac equation and the Dirac field interacting with an electromagnetic field. It can easily be shown that this is also true if the Dirac field interacts with an arbitrary other system, if the interaction can be written in the form of equation (26).

It is also found that the transformation

$$\begin{aligned} \delta A_\nu &= \varepsilon^\rho (\partial_\rho A_\nu - \partial_\nu A_\rho) \\ \delta F^{\mu\nu} &= \varepsilon^\rho \partial_\rho F^{\mu\nu} \\ \delta u &= \varepsilon^\rho \partial_\rho u \\ \delta v &= \varepsilon^\rho \partial_\rho v \end{aligned} \tag{35}$$

is an invariant transformation for the total Lagrangian L apart from a term of a conservation law:

$$\begin{aligned} \delta L &= \varepsilon^\rho \partial_\rho L - \varepsilon^\rho j^\mu \partial_\mu A_\rho \\ &= \varepsilon^\rho \partial_\mu (L\delta^\mu_\rho - j^\mu A_\rho) + A_\rho \partial_\mu j^\mu \end{aligned}$$

Hence by (34) we obtain

$$\delta L \doteq \varepsilon^{\rho}\, \partial_{\mu}(L\delta^{\mu}_{\rho} - j^{\mu}A_{\rho}) \tag{36}$$

By substitution of (35) into (28) we obtain on the other hand

$$\delta L = \varepsilon^{\rho}\, \partial_{\mu}(-\varepsilon_0 c F^{\mu\nu}(\partial_{\rho}A_{\nu} - \partial_{\nu}A_{\rho}) + i\hbar v^{\dagger}\gamma^0\gamma^{\mu}\,\partial_{\rho}u) \tag{37}$$

Equations (36) and (37) yield a local conservation law of the second kind which using (29) and (31) can be written in the trivially equivalent form

$$\partial_{\mu}\{\varepsilon_0(F^{\mu\nu}F_{\rho\nu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\delta^{\mu}_{\rho}) + v^{\dagger}\gamma^0\gamma^{\mu}(-i\hbar\partial_{\rho} - eA_{\rho})u\} \doteq 0$$

and also in the form

$$\partial_{\mu}(S_{\rho}^{\cdot\mu} + T_{\rho}^{\cdot\mu}) \doteq 0$$

where

$$S_{\rho}^{\cdot\mu} = \varepsilon_0(F^{\mu\nu}F_{\rho\nu} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\delta^{\mu}_{\rho})$$

is the energy-momentum tensor of the electromagnetic field and

$$T_{\rho}^{\cdot\mu} = v^{\dagger}\gamma^0\gamma^{\mu}(-i\hbar\partial_{\rho} - eA_{\rho})u$$

is for $u = v$ the energy-momentum tensor of the Dirac particle in an electromagnetic field (c.f. equation (3.5.1.26)).
In contrast to the case of the interaction of a classical charged particle with an electromagnetic field, here an interaction term $-eu^{\dagger}\gamma^0\gamma^{\mu}uA_{\rho}$ appears in the total energy of the system.

4.3. Lipkin's zilch

In section (3.5.3) we have already derived the conservation of Lipkin's zilch. There the tensor Lagrangian of Morgan and Joseph [36] was used, which is not very suitable to describe the electromagnetic field if this field interacts with a current distribution $j^{\mu}(x,t)$.
The non-homogeneous Maxwell equations can be given in the form

$$\varepsilon_0 c\, \partial_{\mu}F^{\mu\nu} = -j^{\nu} \tag{1}$$

where

$$F_{\mu\nu} = c(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) \tag{2}$$

is the electromagnetic field strength tensor introduced in section (3.5.3). It is readily seen that equation (2) is equivalent to

$$\partial_\mu \hat{F}^{\mu\nu} = 0 \tag{3}$$

and has been pointed out in section (3.5.3) equivalent to

$$\partial_\mu F_{\nu\gamma} + \partial_\nu F_{\gamma\mu} + \partial_\gamma F_{\mu\nu} = 0, \tag{4}$$

where $\hat{F}^{\mu\nu}$ is the dual electromagnetic field strength tensor. In order to obtain a similar expression as (4) for $\hat{F}$ we define

$$\partial_\mu \hat{F}_{\nu\gamma} + \partial_\nu \hat{F}_{\gamma\mu} + \partial_\gamma \hat{F}_{\mu\nu} = j_{\mu\nu\gamma} \tag{5}$$

It is then readily seen that $j_{\mu\nu\gamma}$ is skew-symmetric in all three indices. By multiplication of (5) by the Levi-Civita tensor $\ell^{\mu\nu\gamma\alpha}$, using the relation (3.5.3.2) between F and $\hat{F}$, we obtain

$$\ell^{\mu\nu\gamma\alpha} j_{\mu\nu\gamma} = -6\, \partial_\mu F^{\mu\alpha}$$

Then we see from (1) that

$$j^\alpha = +\frac{1}{6}\, \varepsilon_0 c\, \ell^{\mu\nu\gamma\alpha} j_{\mu\nu\gamma} \tag{6}$$

This relation can be inverted by multiplication of the left and right-hand sides of the equation by $\ell_{\alpha\beta\rho\sigma}$. So we obtain

$$j_{\beta\rho\sigma} = \frac{1}{\varepsilon_0 c}\, j^\alpha\, \ell_{\alpha\beta\rho\sigma} \tag{7}$$

and (1) has been brought into the form (5), where $j_{\mu\nu\rho}$ has to satisfy equations (6) and (7). Maxwell's equations are therefore equivalent to the combination of (1) or (5) and (3) or (4).

Now let the components A^μ be the coordinate functions. Equations (3) and (4) are then identically satisfied and Maxwell's equations are (1) or equivalently (5). A suitable Lagrangian density to describe the field is

$$L = -\frac{1}{4}\, \varepsilon_0 F^{\mu\nu} F_{\mu\nu} + j^\nu A_\nu \tag{8}$$

because the first-order variation δL of L yields

$$\delta L = -\partial_\mu (\varepsilon_0 c\, F^{\mu\nu} \delta A_\nu) + (\varepsilon_0 c\, \partial_\mu F^{\mu\nu} + j^\nu) \delta A_\nu$$

such that Euler's equation is given by (1) and

$$\delta L \doteq -\partial_\mu(\varepsilon_0 c\, F^{\mu\nu}\delta A_\nu) \tag{9}$$

By straight forward calculations it can be shown that for variations

$$\delta A_\nu = \varepsilon^{\alpha\beta}\partial_\alpha \hat{F}_{\nu\beta} \tag{10}$$

with $\varepsilon^{\alpha\beta}$ some infinitesimal arbitrary constant tensor, the first-order variation of L becomes (for solutions of Maxwell's equations)

$$\delta L \doteq \varepsilon^{\alpha\beta}\partial_\mu(\tfrac{1}{2}\varepsilon_0 c\, F^{\rho\nu}\partial_\alpha \hat{F}_{\rho\nu}\delta^\mu_\beta - \varepsilon_0 c\, F_{\beta\nu}\partial_\alpha \hat{F}^{\mu\nu}) + \varepsilon^{\alpha\beta}(\partial_\alpha j^\nu \hat{F}_{\nu\beta} - j^\nu \partial_\alpha \hat{F}_{\nu\beta}) \tag{11}$$

As these calculations are rather tedious, but straightforward, they will be omitted here. From (9) and (10) we obtain

$$\delta L \doteq \varepsilon^{\alpha\beta}\partial_\mu(-\varepsilon_0 c\, F^{\mu\nu}\partial_\alpha \hat{F}_{\nu\beta}) \tag{12}$$

Hence by (11) and (12)

$$\partial_\mu Z^{\cdot\mu}_{\beta.\alpha} \doteq (\partial_\alpha j^\nu)\hat{F}_{\nu\beta} - j^\nu \partial_\alpha \hat{F}_{\nu\beta} \tag{13}$$

where

$$\frac{1}{\varepsilon_0 c} Z^{\cdot\mu}_{\beta.\alpha} = F_{\beta\nu}\partial_\alpha \hat{F}^{\mu\nu} - F^{\mu\nu}\partial_\alpha \hat{F}_{\nu\beta} - \frac{1}{2} F^{\rho\sigma}\partial_\alpha \hat{F}_{\rho\sigma}\delta^\mu_\beta \tag{14}$$

Using techniques involving the Levi-Civita tensor (see also Kibble [40]) it can be verified that

$$F_{\beta\nu}\hat{F}^{\mu\nu} = \frac{1}{4}\hat{F}^{\rho\sigma}F_{\rho\sigma}\delta^\mu_\beta$$

and

$$\hat{F}^{\rho\sigma}\partial_\alpha F_{\rho\sigma} = \partial_\alpha \hat{F}^{\rho\sigma}F_{\rho\sigma}$$

we then also have

$$F_{\beta\nu}\partial_\alpha \hat{F}^{\mu\nu} = \partial_\alpha F_{\beta\nu}\hat{F}^{\nu\mu} + \frac{1}{2} F^{\rho\sigma}\partial_\alpha \hat{F}_{\rho\sigma}\delta^\mu_\beta$$

and substitution of this expression in (14) gives

$$Z^{\cdot\mu}_{\beta.\alpha} = \varepsilon_0 c(\partial_\alpha F_{\beta\nu}\hat{F}^{\nu\mu} - F^{\mu\nu}\partial_\alpha \hat{F}_{\nu\beta})$$

which is the *zilch tensor* defined in equation (3.5.3.16). We may therefore write instead of (13)

$$\partial_\mu Z^\mu_{\cdot\beta\alpha} \doteq (\partial_\alpha j^\nu)\hat{F}_{\nu\beta} - j^\nu \partial_\alpha \hat{F}_{\nu\beta} \tag{15}$$

which gives the conservation of the zilch for a zero current distribution j^μ (homogeneous Maxwell's equations).

It is readily seen that the right-hand side of (15) can also be written as

$$\partial_\alpha(\hat{F}_{\nu\beta} j^\nu) + \partial^\rho(2A^\sigma \partial_\alpha j_{\beta\rho\sigma}) - 2A^\sigma \partial^\rho \partial_\alpha j_{\beta\rho\sigma} \tag{16}$$

Since in (3) the interaction Lagrangian has the form

$$L_{int} = j^\nu(x)A_\nu(x)$$

the first-order variation of L can, δA_ν being given by (10), be written as

$$\delta L_{int} = \delta j^\nu(x)A_\nu(x) + j^\nu(x)\delta A_\nu(x) \tag{17}$$

where the second term on the right-hand side is equal to (16). When we require δA_ν together with δj^ν to constitute an invariant transformation, δj^ν, must satisfy

$$\delta j^\sigma = 2A^\sigma \partial^\rho \partial_\alpha j_{\beta\rho\sigma} = 2\,\mathcal{E}^{\cdot\cdot\rho\sigma}_{\nu\beta} A_\sigma \partial_\rho \partial_\alpha j^\nu \qquad (\alpha, \beta \text{ fixed}) \tag{18}$$

with the expression (4.2.22) for the current vector in a Dirac field

$$j^\sigma = ev^\dagger \gamma^0 \gamma^\sigma u$$

we have

$$\delta j^\sigma = e\delta v^\dagger \gamma^0 \gamma^\sigma u + ev^\dagger \gamma^0 \gamma^\sigma \delta u \tag{19}$$

So a neccessary condition for the transformation δu, δv, δA^ν to be an invariant one is

$$\delta v^\dagger \gamma^0 \gamma^\sigma u + v^\dagger \gamma^0 \gamma^\sigma \delta u = 2\,\mathcal{E}^{\cdot\cdot\rho\sigma}_{\nu\beta} A_\sigma \partial_\rho \partial_\alpha (v^\dagger \gamma^0 \gamma^\nu u) \tag{20}$$

It can be verified rather easily that no pairs δu, δv can be found which satisfy (20).

Therefore: *The zilch of the electromagnetic field cannot be transferred additively into the Dirac field.*

A similar treatment, leading to the same conclusion, can be given for a zero-spin Boson field.

As the current density vector for a classical particle is more complicated than for a Dirac or Boson field. We cannot hope to give a similar discussion for the transferability of the zilch into a classical particle. But we have succeeded in constructing a non-relativistic mechanical model in which one of the components of the zilch can be transferred additively.

Consider a flat plate at $x = 0$, consisting of a great number of mutually independent mass points harmonically bound to their equilibrium positions. When we assume that the equilibrium positions of the masses are distributed homogeneously all over the plate, the polarization density on the plate can be defined by

$$p = \text{column } (p^2,p^3) = \text{column } (\Delta x,\Delta y)$$

where column $(\Delta x,\Delta y)$ is the deviation of the masses from their equilibrium position, and where we have assumed that we have a unit charge density on the plate. The vector p will be constant as a function of y and z if the initial conditions and all forces are homogeneous in the y and z direction (and this will be assumed here).
Let there be a homogeneous magnetic field in the x direction. The polarization density then satisfies the equation *)

$$p_{tt} + Sp_t + p = 2 \text{ column } (E^2,E^3)\Big|_{x=0} \tag{21}$$

where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$p = \text{column } (p^2,p^3)$

E^2,E^3 is the electrical field emitted by the plate in the y and z direction respectively.

To obtain equation (21) a suitable choice of units and a suitable choice of the magnetic field strength has been made. The factor 2 has been kept to

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*) We use the superscripts 1,2,3 to enumerate the components of vector quantities in the x, y and z direction respectively.

simplify further equations

We can rewrite equation (21) into the form

$$y_t + Ay = \text{column } (0,0,2E^2,2E^3)$$

where

$$A = -A^\dagger = \begin{pmatrix} 0 & -I \\ I & S \end{pmatrix} \qquad \begin{aligned} 0 &= \text{diagonal } (0,0) \\ I &= \text{diagonal } (1,1) \end{aligned}$$

and $$y = \text{column } (p^2,p^3,p_t^2,p_t^3)$$

The electromagnetic field emitted by the plate in the constant magnetic field will consist of a homogeneous plane wave in the +x direction and one in the -x direction. Broer [20] has described such plane waves in a suitable way for our present purpose. This leads to the equation

$$u_t + Du_x + \text{column } (j^2,j^3,j^2,j^3) = 0$$

where

$$D = \text{quasi diagonal } (I, -I)$$

$$u = \text{column } (E^2+B^3,\ E^3-B^2,\ E^2-B^3,\ E^3+B^2)$$

and where j^2,j^3 are the components in the y and z direction of the electrical current density.
E^2, B^2, E^3, B^3 are the electrical and magnetic field components of the emitted field in the y and z direction respectively.

Note that we use, as has been done in [20], the system of electromagnetic units, in which the electrical and magnetic premittivity ε_0, μ_0 of vacuum are equal to 1.
The connection between the equations of motion of the electromagnetic field and those of the plate is given by

$$j = \frac{d}{dt}\, p\, \delta(x)$$

so that the equations of motion of the field and of the plate can be written in the form

$$y_t + Ay - C^\dagger u\Big|_{x=0} = 0 \tag{22}$$

$$u_t + Du_x + Cy\ \delta(x) = 0 \tag{23}$$

where

$$C = \begin{pmatrix} 0 & I \\ 0 & I \end{pmatrix} \tag{24}$$

and again

$$A = -A^\dagger = \begin{pmatrix} 0 & -I \\ I & S \end{pmatrix} \qquad D = D^\dagger = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad S = -S^\dagger = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

With the methods of chapter 3 it is easily verified that the sesquilinear Lagrangian density

$$L = v^\dagger u_t + v^\dagger D u_x + (z^\dagger y_t + z^\dagger A y)\delta(x) + (v^\dagger C y - z^\dagger C^\dagger u)\delta(x) \tag{25}$$

has (22) as Euler's equation for y and z, and (23) as Euler's equation for u and v.

Furthermore (25) yields

$$\delta L \doteq \frac{d}{dt}\,(v^\dagger \delta u + z^\dagger \delta y\ \delta(x)) + \frac{d}{dx}\,(u^\dagger D\ \delta u) \tag{26}$$

Now consider the infinitesimal transformation

$$\delta u = \varepsilon M u;\ \delta v = -\varepsilon N^\dagger v,\ \delta y = \varepsilon N y,\ \delta z = -\varepsilon N^\dagger z \tag{27}$$

It is readily seen from (25) that (27) is an invariant transformation with

$$\delta L = 0 \tag{28}$$

if M and N are constant 4×4 matrices satisfying

$$MD = DM;\ NA = AN;\ CN - MC = NC^\dagger - C^\dagger M = 0 \tag{29}$$

After identifying u, v and y,z we find with (26)-(28) the local conservation law valid for the interacting systems taken together

$$\frac{d}{dt}\,(u^\dagger M u + y^\dagger N y\ \delta(x)) + \frac{d}{dx}\,(u^\dagger D M u) \doteq 0 \tag{30}$$

Restricting M and N to the class of Hermitean matrices, it can be seen that the only solution of (29) for M and N are the same scalar multiples of the unit matrix. We can take

$$M = N = I$$

Equation (30) can then be written as

$$\frac{d}{dt}\,(\underline{E}^2 + \underline{B}^2 + \frac{1}{2}\,(p^\dagger p + p_t{}^\dagger p_t)\delta(x)) + \frac{d}{dx}\,(2(\underline{E}\times\underline{B})^1) \doteq 0 \qquad (31)$$

which clearly represents the conservation of energy of electromagnetic field and the plate together.

Analogous to (27) we can also consider infinitesimal transformations of the form

$$\delta u = \varepsilon M u_t,\ \delta v = \varepsilon M^\dagger v_t,\ \delta y = \varepsilon N y_t,\ \varepsilon z = \varepsilon N^\dagger z_t \qquad (32)$$

This gives

$$\delta L = \varepsilon\,\frac{d}{dt}\,(v^\dagger M u_t + v^\dagger D M u_x + (z^\dagger N y_t + z^\dagger A N y)\delta(x) + (v^\dagger N C y - z^\dagger C^\dagger M u)\delta(x)) \doteq 0 \qquad (33)$$

by introducing the same condition (29).
We obtain with (26), after setting u = v and z = y, the local conservation law

$$\frac{d}{dt}\,(u^\dagger M u_t + y^\dagger N y_t\,\delta(x)) + \frac{d}{dx}\,(u^\dagger D N u_t) \doteq 0 \qquad (34)$$

In (34) M and N may only be anti-Hermitean matrices. Otherwise (34) is the time derivative of (30). With this restriction the only solution of (29) is

$$M = N = \text{quasi diagonal } (S,S)$$

or identical scalar multiples.
The equivalent form of (34) then becomes

$$\frac{d}{dt}\,((\underline{E}\times\underline{E}_t + \underline{B}\times\underline{B}_t)^1 + Y\,\delta(x)) - \frac{d}{dx}\,(\underline{B}.\underline{E}_t - \underline{E}.\underline{B}_t) \doteq 0 \qquad (35)$$

$$Y = p^2 p_t^3 - p^3 p_t^2 + p_t^2 p_{tt}^3 - p_t^3 p_{tt}^2$$

The electromagnetic part of the density of (35) is one of Lipkin's zilches as has been shown by Broer [20]. The flux of (35) is the x-component of the to Lipkin's zilch corresponding flux. This zilch component can therefore be transferred additively into the mechanical quantity Y.

To obtain a possible interpretation of the quantity Y, the mechanical equivalence of the zilch,we consider the equations of motion of the plate, without the emitted electromagnetic field

$$p_{tt}^2 = p_t^3 - p^2;\ p_{tt}^3 = -p_t^2 - p^3$$

These equations will be used to simplify the expression for Y:

$$\frac{1}{2} Y = -\frac{1}{2}(p_t^2 p_t^2 + p_t^3 p_t^3 + p^2 p^2 + p^3 p^3)$$
$$+ p^2(p_t^3 + \frac{1}{2} p^2) - p^3(p_t^2 - \frac{1}{2} p^3)$$

The vector $\underline{A}$ with components $A^1 = 0$, $A^2 = -\frac{1}{4} p^3$, $A^3 = \frac{1}{4} p^2$ represents the vector potential of the constant magnetic field in the x-direction. So, the quantity $\frac{1}{2} Y$ is a linear combination of the energy and the canonical angular momentum of the plate. The mechanical equivalence of the zilch satisfies therefore a conservation law for the mechanical system alone, which is trivially equivalent to a linear combination of the conservation laws of energy and angular momentum.

Conclusion

Lipkin's zilch cannot be transferred additively into a Dirac or zero-spin Boson field. Classical models in which Lipkin's zilch can be transferred additively are possible. It is not obvious therefore what physical significance can be attributed to the zilch.

5. Hamilton's formalism in continuum physics

5.1. Hamilton's equations for continuous systems

It is possible to construct a formalism for continuous systems, analogous to the Hamilton formalism for mechanical systems which was discussed in section 2.1.. Below we will summarize this generalization to continuous systems.

Let the Lagrangian functional of a system depend on the time t, the coordinate functions q_i, and the first-order derivatives with respect to t, only

$$\mathcal{L} = \mathcal{L}\{q,\dot{q},t\}$$

Then the equations of motion are

$$\frac{\delta}{\delta q_i}\mathcal{L} - \frac{d}{dt}\frac{\delta}{\delta \dot{q}_i}\mathcal{L} = 0 \tag{1}$$

If we now define the *momentum function*, more briefly *momentum*, p_i *canonically conjugated* to the coordinate function q_i, by

$$p_i = \frac{\delta}{\delta \dot{q}_i}\mathcal{L}\{q,\dot{q},t\} \tag{2}$$

we obtain momentum functions p_i which depend on the time t, and the (for fixed time t independent) functions q_i and $\dot{q}_i$. We require that the relation (2) can be inverted, so that it defines a unique set of functionals by which $\dot{q}_i$ can be expressed in terms of t, p_i and q_i. We write

$$\dot{q}_i(x,t) = \mathcal{L}_i\{p,q,x,t\} \tag{3}$$

The *Hamilton functional* $\mathcal{H}\{p,q,t\}$ is then defined by

$$\mathcal{H}\{p,q,t\} = -\mathcal{L}\{q,\dot{q},t\} + \int dx\, p_i(x,t)\dot{q}_i(x,t)$$

where for $\dot{q}$ equation (3) has to be substituted, and $\int dx$ denotes $\iint\ldots\int dx_1\ldots dx_m$

The functional derivatives of $\mathcal{H}$ with respect to q_i and p_i can now be calculated

$$\frac{\delta}{\delta p_i(x,t)}\mathcal{H} = -\int \frac{\delta \mathcal{L}}{\delta \dot{q}_j(x',t)} \cdot \frac{\delta \dot{q}_j(x',t)}{\delta p_i(x)}\,dx' + \dot{q}_i(x) + \int p_j(x',t)\,\frac{\delta \dot{q}_j(x',t)}{\delta p_i(x,t)}\,dx'$$

$$\frac{\delta}{\delta q_i(x,t)}\mathcal{H} = -\frac{\delta \mathcal{L}}{\delta q_i(x,t)} - \int\left(\frac{\delta \mathcal{L}}{\delta \dot{q}_j(x',t)} \cdot \frac{\delta \dot{q}_j(x',t)}{\delta q_i(x,t)} + p_j(x',t)\,\frac{\delta \dot{q}_j(x',t)}{\delta q_i(x,t)}\right)dx' \qquad (4)$$

Hence by (1)-(4)

$$\frac{\delta}{\delta p_i}\mathcal{H} = \dot{q}_i; \quad \frac{\delta}{\delta q_i}\mathcal{H} = -\dot{p}_i \qquad (5)$$

These equations are also called *Hamilton's equations*, and are equivalent to Euler's equations.

As in section (2.2), here too a variational principle for the equations (5) exists, which can be formulated as

$$\frac{\delta}{\delta p_i}\hat{\mathcal{W}}\{p,q\} = \frac{\delta}{\delta q_i}\hat{\mathcal{W}}\{p,q\} = 0 \qquad (6)$$

with
$$\hat{\mathcal{W}} = \int_{t_1}^{t_2} \hat{\mathcal{L}}\{p,q,\dot{q},t\}dt \qquad (7)$$

and
$$\hat{\mathcal{L}}\{p,q,\dot{q},t\} = \int p_i(x,t)\dot{q}_i(x,t)dx - \mathcal{H}\{p,q\} \qquad (8)$$

The functionals $\hat{\mathcal{W}}$ and $\hat{\mathcal{L}}$ will be called *canonical action (functional)* and *canonical Lagrangian (functional)* respectively.
The proof and the interpretation of the variational principle goes in entirely the same way as developed in our discussion in sections 2.1 and 2.2.. This analogy is caused by the fact that differentiation and functional differentiatio have most properties in common.

If the Lagrangian functional $\mathcal{L}$ can be written as an integral of a Lagrangian density $L(q,\dot{q},q_x,\ldots,x,t)$, the Hamilton functional $\mathcal{H}$ can also be written as an integral over a *Hamilton density* H

$$\mathcal{H} = \int_{t_1}^{t_2} H(p,q,q_x,\ldots,x,t)dt$$

with $$H = p_i \dot{q}_i(p,q,x,t) - L(q,\dot{q},q_x,\ldots,x,t)$$

In that case Hamilton's equations (5) are written as

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} + \frac{d}{dx_k}\frac{\partial H}{\partial (q_i)_{x_k}} + \ldots.$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} - \frac{d}{dx_k}\frac{\partial H}{\partial (p_i)_{x_k}} + \ldots.$$

Analogous to the Poisson bracket in classical mechanics, a generalized Poisson bracket can be defined in continuum physics.
Let $\mathcal{A}\{p,q\}$ and $\mathcal{B}\{p,q\}$ be functionals depending on the coordinate functions $q_i(x)$ and the momentum functions $p_i(x)$ $(i = 1,2,\ldots,n)$. Then the *generalized Poisson bracket* of $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\{\mathcal{A}\{p,q\},\mathcal{B}\{p,q\}\}_p = \int dx\left(\frac{\delta \mathcal{A}}{\delta q_i(x)}\frac{\delta \mathcal{B}}{\delta p_i(x)} - \frac{\delta \mathcal{A}}{\delta p_i(x)}\cdot\frac{\delta \mathcal{B}}{\delta q_i(x)}\right) \tag{9}$$

As is the case in classical mechanics, the use of the (generalized) Poisson bracket can be of advantage in a group theoretical or algebraic approach of the dynamics and transformations of the variables p and q (see for instance Rosen [5]). We will only use the Poisson bracket for shortening formulas. With the chain rule for differentiation of functionals, it can for instance be seen that if $\mathcal{G}\{p,q,t\}$ is some time-dependent functional

$$\frac{d}{dt}\mathcal{G}\{p,q,t\} = \frac{\partial}{\partial t}\mathcal{G} + \{\mathcal{G},\mathcal{H}\}_p \tag{10}$$

5.2 Canonical transformations and generating functionals

Let the dynamics of a physical continuous system be described by the canonical functional

$$\hat{\mathcal{W}}\{p,q\} = \int_{t_1}^{t_2}\left(\int p_i\dot{q}_i dx - \mathcal{H}\{p,q,t\}\right)dt \tag{1}$$

where

$$p_i = p_i(x_1,x_2,\ldots,x_n,t) = p_i(x,t)$$

and

$$q_i = q_i(x_1,\ldots,x_n,t) = q_i(x,t)$$

The equations of motion are then given by

$$\frac{\delta}{\delta q_i}\hat{\mathfrak{W}} = \frac{\delta}{\delta p_i}\hat{\mathfrak{W}} = 0 \tag{2}$$

Now consider the transformation

$$\bar{q}_i = \bar{q}_i\{q_1,\ldots,q_n,\ p_1,\ldots,p_n,t\}$$

$$\bar{p}_i = \bar{p}_i\{q_1,\ldots,q_n,\ p_1,\ldots,p_n,t\}$$

or briefly

$$\begin{aligned}\bar{q}_i &= \bar{q}_i\{q,p\}\\ \bar{p}_i &= \bar{p}_i\{q,p\}\end{aligned} \tag{3}$$

and the inverse transformation

$$\begin{aligned}q_i &= q_i\{\bar{q},\bar{p}\}\\ p_i &= p_i\{\bar{q},\bar{p}\}\end{aligned} \tag{4}$$

where $\bar{p}_i = \bar{p}_i(\bar{x}_1,\ldots,\bar{x}_n,t) = p_i(\bar{x},t)$

$$\bar{q}_i = \bar{q}_i(\bar{x}_1,\ldots,\bar{x}_n,t) = \bar{q}_i(\bar{x},t)$$

Analogous to the coordinate function transformation discussed in section (3.2) we can state that the equations of motion (2) are equivalent to

$$\frac{\delta}{\delta \bar{q}_i}\hat{\bar{\mathfrak{W}}} = \frac{\delta}{\delta \bar{p}_i}\hat{\bar{\mathfrak{W}}} = 0 \tag{5}$$

with $$\hat{\bar{\mathfrak{W}}}\{\bar{p},\bar{q}\} = \hat{\mathfrak{W}}\{p\{\bar{p},\bar{q}\},\ q\{\bar{p},\bar{q}\}\}$$

The transformation (3), (4) is called a *canonical transformation* if there exists a functional $\bar{\mathfrak{H}}\{\bar{p},\bar{q}\}$ such that

$$\hat{\bar{\mathfrak{W}}} = \int_{t_1}^{t_2}\left(\int \bar{p}_i\bar{q}_i\,d\bar{x} - \bar{\mathfrak{H}}\{\bar{p},\bar{q},t\}\right)dt \tag{6}$$

with (5) yields equations of motion equivalent to the original equations (2).

A sufficient condition for the transformation to be canonical is that the canonical Lagrangians belonging to $\hat{\mathfrak{W}}$ and $\hat{\bar{\mathfrak{W}}}$ are equivalent.

So

$$\int p_i \dot{q}_i dx - \mathcal{H}\{p,q,t\} - \int \bar{p}_i \dot{\bar{q}}_i d\bar{x} - \bar{\mathcal{H}}\{\bar{p},\bar{q},t\} = \frac{d}{dt}\mathcal{F} \tag{7}$$

where $\mathcal{F}$ is an arbitrary, in general time-dependent functional of the functions p and q or by means of (4) also a functional of $\bar{p}$ and $\bar{q}$ and is called the *generating functional*.

As in section (2.3) the argument functions of $\mathcal{F}$ are not independent in view of the transformation (3), (4). In specific cases we may choose

$$\mathcal{F} = \mathcal{F}\{q,\bar{q},t\}$$

Then

$$\frac{d}{dt}\mathcal{F} = \frac{\partial}{\partial t}\mathcal{F} + \int \frac{\delta \mathcal{F}}{\delta q_i} \dot{q}_i dx + \int \frac{\delta \mathcal{F}}{\delta \bar{q}_i} \dot{\bar{q}}_i d\bar{x} \tag{8}$$

where

$$\frac{\partial}{\partial t}\mathcal{F}\{q,\bar{q},t\} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\mathcal{F}\{q,\bar{q},t+\epsilon\} - \mathcal{F}\{q,\bar{q},t\})$$

Substitution of (8) into (7) yields

$$\int dx\, \dot{q}_i (p_i - \frac{\delta \mathcal{F}}{\delta q_i}) + \int d\bar{x}\, \dot{\bar{q}}_i (-\bar{p}_i - \frac{\delta \mathcal{F}}{\delta \bar{q}_i}) - \mathcal{H} + \bar{\mathcal{H}} - \frac{\partial}{\partial t}\mathcal{F} = 0 \tag{9}$$

A sufficient condition for (9) to hold and thus for the transformation to be canonical is:

$$p_i = \frac{\delta}{\delta q_i}\mathcal{F}\{q,\bar{q},t\} \tag{10}$$

$$\bar{p}_i = -\frac{\delta}{\delta \bar{q}_i}\mathcal{F}\{q,\bar{q},t\} \tag{11}$$

$$\bar{\mathcal{H}} - \mathcal{H} = \frac{\partial}{\partial t}\mathcal{F}\{q,\bar{q},t\} \tag{12}$$

Analogously we find

$$p_i = \frac{\delta}{\delta q_i} \mathcal{G}\{\bar{p},q,t\} \tag{13}$$

$$\bar{q}_i = \frac{\delta}{\delta \bar{p}_i} \mathcal{G}\{\bar{p},q,t\} \tag{14}$$

$$\bar{\mathcal{H}} - \mathcal{H} = \frac{\partial}{\partial t} \mathcal{G}\{\bar{p},q,t\} \tag{15}$$

and

$$q_i = - \frac{\delta}{\delta \bar{p}_i} \mathcal{F}\{p,\bar{q},t\} \tag{16}$$

$$\bar{p}_i = - \frac{\delta}{\delta \bar{q}_i} \mathcal{F}\{p,\bar{q},t\} \tag{17}$$

$$\bar{\mathcal{H}} - \mathcal{H} = \frac{\partial}{\partial t} \mathcal{F}\{p,\bar{q},t\} \tag{18}$$

and also

$$q_i = - \frac{\delta}{\delta p_i} \mathcal{F}\{\bar{p},p,t\} \tag{19}$$

$$\bar{q}_i = \frac{\delta}{\delta \bar{p}_i} \mathcal{F}\{\bar{p},p,t\} \tag{20}$$

$$\bar{\mathcal{H}} - \mathcal{H} = \frac{\partial}{\partial t} \mathcal{F}\{\bar{p},p,t\} \tag{21}$$

We observe that in this section the domains of the arguments of the functions p,q and $\bar{p},\bar{q}$ respectively are generally different. For that reason we introduced in a recent paper [27] the concept of a *double functional*, where we had in mind a double integral over x and $\bar{x}$ space e.g.

$$\mathcal{G}\{\bar{p},q\} = \iint dx\, d\bar{x}\; G(\bar{p}(\bar{x}),\bar{p}_{\bar{x}}(\bar{x}),\ldots,q(x),q_x(x),\ldots,x,\bar{x}) \tag{22}$$

and we defined the functional derivatives $\frac{\delta^* \mathcal{G}}{\delta^* \bar{p}}$ and $\frac{\delta^* \mathcal{G}}{\delta^* q}$ implicitly by

$$\lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \mathcal{G}\{\bar{p}+\varepsilon \bar{f},q\} = \iint dx\, d\bar{x}\, \frac{\delta^* \mathcal{G}}{\delta^* \bar{p}}\, \bar{f} \tag{23}$$

$$\lim_{\mu \to 0} \frac{d}{d\mu} \mathcal{G}\{\bar{p},q+\mu f\} = \iint dx\, d\bar{x}\, \frac{\delta^* \mathcal{G}}{\delta^* q}\, f \tag{24}$$

Here the functional derivatives are functions of x as well as of $\bar{x}$, and the relation to the functional derivatives as used in this section is

$$\frac{\delta}{\delta\bar{p}}\mathcal{G} = \int dx\, \frac{\delta^*\mathcal{G}}{\delta^*\bar{p}}; \quad \frac{\delta}{\delta q}\mathcal{G} = \int d\bar{x}\, \frac{\delta^*\mathcal{G}}{\delta^* q} \tag{25}$$

which can, non uniquely, be inverted by

$$\frac{\delta^*}{\delta^*\bar{p}}\mathcal{G} = \frac{\delta}{\delta\bar{p}}\mathcal{G}\cdot\delta(x-\bar{x}); \quad \frac{\delta^*}{\delta^* q}\mathcal{G} = \frac{\delta}{\delta q}\mathcal{G}\,\delta(x-\bar{x}) \tag{26}$$

Using (25) or (26) the transformation equations (10)-(21) become identical with those given in [27]. Therefore, the formulation given in this section and that given in [27] are equivalent. The treatment of this section generally leads to less complicated formulas for the canonical transformations. An additional advantage is that we do not have to formally distinguish between canonical transformations and generalized canonical transformations as we did in [27].

5.3 Examples of canonical transformations

The usefulness of the generating functional formalism developed in section 5.2 will be demonstrated by several examples.

Time-dependent translation of the x-axis

Let $\mathcal{G}\{\bar{p},q,t\} = \iint d\bar{x}dx\ \bar{p}(\bar{x},t)q(x,t)\delta(x-g(\bar{x},t))$

$$= \int d\bar{x}\ \bar{p}(\bar{x},t)q(g(\bar{x},t),t)$$

$$= \int dx\ q(x,t)\bar{p}(\tilde{g}(x,t),t)\frac{d}{dx}\tilde{g}(x,t)$$

where $\tilde{g}(x,t) = \bar{x} \leftrightarrow x = g(\bar{x},t)$

Then equations (5.2.13-15) yield

$$p(x,t) = \frac{\delta}{\delta q}\mathcal{G} = \bar{p}(\tilde{g}(x,t),t)\frac{d}{dx}\tilde{g}(x,t)$$

$$\bar{q}(\bar{x},t) = \frac{\delta}{\delta\bar{p}}\mathcal{G} = q(g(\bar{x},t),t)$$

$$\bar{\mathcal{H}} - \mathcal{H} = \frac{\partial}{\partial t}\mathcal{G} = \int d\bar{x}\ \bar{p}(\bar{x},t)q_x(g(\bar{x},t),t)\frac{d}{dt}g(\bar{x},t)$$

We notice that this transformation is the one discussed in section (3.3). If we take $g(\bar{x},t) = \bar{x}$, the *identical transformation* is obtained.

Specializing the transformation formula for $g(\bar{x},t) = \bar{x}+vt$ (uniform motion of the x axis) we obtain

$$p(x,t) = \bar{p}(x-vt,t)$$

$$\bar{q}(\bar{x},t) = q(\bar{x}+vt,t)$$

$$\bar{\mathcal{H}} - \mathcal{H} = v\int d\bar{x}\ \bar{p}(\bar{x},t)q_x(\bar{x}+vt,t)$$

and their inverse $\quad \bar{p}(\bar{x},t) = p(\bar{x}+vt,t)$

$$q(x,t) = \bar{q}(x-vt,t)$$

Fourier and related transformations

Let $\quad \mathcal{G}(\bar{p},q,t) = \iint d\bar{x}dx\ \bar{p}(\bar{x},t)q(x,t)f(x,\bar{x},t)$

where $f(x,\bar{x},t)$ is some function of x, $\bar{x}$ and t.
From (5.2.13-15) we obtain

$$p(x,t) = \int \bar{p}(\bar{x},t)f(x,\bar{x},t)d\bar{x}$$

$$\bar{q}(\bar{x},t) = \int q(\bar{x},t)f(x,\bar{x},t)dx$$

$$\bar{\mathcal{H}} - \mathcal{H} = \iint d\bar{x}dx\ \bar{p}(\bar{x},t)q(x,t)\frac{d}{dt}f(x,\bar{x},t)$$

Several possible choices for f yield well-known transformations e.g.

$f = \delta(x-\bar{x})$	*identical* transformation
$f = \delta(x-g(\bar{x}))$	*pure coordinate* (x) transformation
$f = \exp i\bar{x}x$	*Fourier* transformation of q
$f = \exp \bar{x}x$	*Laplace* transformation
$f = x^{\bar{x}-1}$	*Mellin* transformation
$f = J_\nu(\bar{x}x)(\bar{x}x)^{\frac{1}{2}}$	*Hankel* transformation

where J_ν is the Besselfunction of order ν

Point transformation

Let $\mathcal{G}\{\bar{p},q,t\} = \int \bar{p}(x,t)f(q(x,t))dx$

where $f(q)$ is some function of q, then

$$p(x,t) = \bar{p}(x,t)f_q(q(x,t)) \qquad \bar{\mathcal{H}} = \mathcal{H}$$
$$\bar{q}(x,t) = f(q(x,t))$$

This transformation may be considered as a generalization of the pure coordinate transformation discussed in section (2.4) and is therefore also called a *point transformation.*

A further generalization can be made by writing

$$\mathcal{G}\{\bar{p},q,t\} = \int \bar{p}(\bar{x},t).\mathcal{F}\{q,\bar{x},t\}d\bar{x}$$

where $\mathcal{F}\{q,\bar{x},t\}$ is a functional of $q(x,t)$ for fixed values of $\bar{x}$, and t, being a function of $\bar{x}$ and t for fixed functions $q(x,t)$.
Taking for example

$$\mathcal{F}\{q,\bar{x},t\} = \int q(x,t)f(x,\bar{x},t)dx$$

we obtain the Fourier and related transformations discussed above.
Another example is

$$\mathcal{F}\{q,\bar{x},t\} = q(-\bar{x},t)$$

which results in

$$\mathcal{G}\{\bar{p},q,t\} = \int \bar{p}(\bar{x},t)q(-\bar{x},t)d\bar{x}$$

and hence by (5.2.13-15)

$$p(x,t) = \bar{p}(-x,t)$$
$$\bar{q}(\bar{x},t) = q(-\bar{x},t) \qquad \bar{\mathcal{H}} = \mathcal{H}$$

This transformation can also be considered to be generated by

$$\mathcal{G} = \iint \bar{p}(\bar{x},t)q(x,t)f(\bar{x},x,t)dxd\bar{x}$$

where $f(\bar{x},x,t) = \delta(\bar{x}+x)$

In chapter 6. we will discuss the case of

$$\mathcal{F}\{q,\bar{x},t\} = \tilde{q}(\bar{x},t)$$

where $\tilde{q}(\bar{x},t)$ is the inverse function of $q(x,t)$, i.e. inverse with respect to the x dependence.

The Hamilton-Jacobi equation for continuous systems

A paraphrase of the arguments employed in section 2.4 to obtain the *Hamilton-Jacobi* equation for mechanical systems shows that solutions $\mathcal{F}\{q,\bar{q},t\}$ of the functional differential equation

$$\mathcal{H}\{\frac{\delta}{\delta q}\mathcal{F}\{q,\bar{q},t\},q,t\} + \frac{\partial}{\partial t}\mathcal{F}\{q,\bar{q},t\} = 0$$

and the action $\mathcal{W}$ are equal. Also, the functional $\mathcal{F}$ is the functional generating the canonical transformation with

$$\bar{\mathcal{H}} = 0$$

and therefore the transformation equations (5.2.10-11) i.e.

$$p(x) = \frac{\delta}{\delta q(x)}\mathcal{F}\{q,\bar{q},t\}; \quad \bar{p}(x) = -\frac{\delta}{\delta \bar{q}(x)}\mathcal{F}\{q,\bar{q},t\}$$

directly yield solutions of the equations of motion of the system in terms of arbitrary functions $\bar{p}(x)$ and $\bar{q}(x)$ depending on x only.

Recently, this extended Hamilton-Jacobi formalism has also been discussed by Rosen [44]. He illustrates the use of the formalism for systems described by the Hamilton functional

$$\mathcal{H}\{p,q\} = \int dx\ \tfrac{1}{2}(p^2 + (\frac{\partial}{\partial x}\tfrac{1}{i}\,q)(\frac{\partial}{\partial x}\tfrac{1}{i}\,q) + m^2q^2)$$

5.4 The canonical Noether theorem

As pointed out in section (5.3), an identical canonical transformation is generated by the functional

$$\mathcal{G}\{\bar{p},q,t\} = \int \bar{p}(x,t)q(x,t)dx$$

For an arbitrary, but infinitesimally small transformation we may write

$$\mathcal{G}\{\bar{p},q,t\} = \int \bar{p}(x,t)q(x,t)dx + \varepsilon g\{\bar{p},q,t\} \tag{1}$$

$$p_i(x,t) = \bar{p}_i(x,t) + \varepsilon \frac{\delta}{\delta q_i(x)} g\{\bar{p},q,t\}$$

$$\bar{q}_i(\bar{x},t) = q_i(x,t) + \varepsilon \frac{\delta}{\delta \bar{p}_i} g\{\bar{p},q,t\}$$

$$\bar{\mathcal{H}}\{\bar{p},\bar{q},t\} = \mathcal{H}\{p,q,t\} + \varepsilon \frac{\partial}{\partial t} g\{\bar{p},q,t\}$$

Here $g\{\bar{p},q,t\}$ is some functional of $\bar{p}_i$ and q_i. It will be called the *generating functional of the infinitesimal transformation.* The parameter ε is assumed to be vanishingly small, which implies that terms of second and higher order in ε are neglected. For that reason the bar over $\bar{p}$ in g can be dropped and we may write

$$\delta p_i = \bar{p}_i - p_i = -\varepsilon \frac{\delta}{\delta q_i} g \tag{2}$$

$$\delta q_i = \bar{q}_i - q_i = \varepsilon \frac{\delta}{\delta p_i} g \tag{3}$$

$$\bar{\mathcal{H}}\{\bar{p},\bar{q},t\} - \mathcal{H}\{p,q,t\} = \varepsilon \frac{\partial}{\partial t} g \tag{4}$$

We will now calculate

$$\delta \mathcal{H} = \mathcal{H}\{\bar{p},\bar{q},t\} - \mathcal{H}\{p,q,t\} \tag{5}$$

By a first-order Volterra expansion, using also (2) and (3) one obtains

$$\delta \mathcal{H} = \varepsilon \int dx \left(- \frac{\delta \mathcal{H}}{\delta p_i} \frac{\delta g}{\delta q_i} + \frac{\delta \mathcal{H}}{\delta q_i} \frac{\delta g}{\delta p_i}\right)$$

or

$$\delta \mathcal{H} = -\varepsilon \{g,\mathcal{H}\}_p \tag{6}$$

$\{g,\mathcal{H}\}_p$ denoting the generalized Poisson bracket introduced in section (5.1). Hence by (4)-(6) and (5.1.10)

$$\bar{\mathcal{H}}\{\bar{p},\bar{q},t\} - \mathcal{H}\{\bar{p},\bar{q},t\} = \varepsilon \left(\frac{\partial}{\partial t} g + \{g,\mathcal{H}\}_p\right) \triangleq \varepsilon \frac{d}{dt} g \tag{7}$$

Therefore

$$\bar{\mathcal{H}}\{\bar{p},\bar{q},t\} = \mathcal{H}\{\bar{p},\bar{q},t\} \tag{8}$$

if and only if $\frac{d}{dt} g\{p,q,t\} \doteq 0$ (9)

Transformations yielding (8) are called *invariant canonical transformations.* We can therefore state the following theorem, which will be called *the canonical Noether theorem.*

Theorem

The generating functional of an infinitesimal invariant canonical transformation is a constant of the motion.
Each constant of the motion generates an infinitesimal invariant canonical transformation.

In contrast to Noether's theorem the inversion of the canonical Noether theorem is simple. Hence we will investigate in the next section the relation between Noether's theorem and the canonical Noether theorem.

5.5 Noether's theorem and the canonical formalism

In this section we want to discuss the relation between Noether's theorem (section 1.4 c.q. 3.4) and the canonical Noether theorem (section 2.5 c.q. 5.4).

Let the Hamilton functional of a given physical system be given by

$$\mathcal{H} = \mathcal{H}\{p,q,t\}$$

The *canonical Lagrangian functional* $\hat{\mathcal{L}}$ is then given by (5.1.8).

$$\hat{\mathcal{L}}\{p,q,\dot{q},t\} = \int p_i(x,t)\dot{q}_i(x,t)dx \quad -\mathcal{H}\{p,q,t\} \tag{1}$$

If we vary p and q mutually independent we obtain for the first-order variation $\delta\hat{\mathcal{L}}$ of $\hat{\mathcal{L}}$

$$\delta\hat{\mathcal{L}}\{p,q,\dot{q},t\} = \frac{d}{dt}\int p_i(x,t)\delta q_i(x,t)dx +$$

$$+ \int((-\dot{p}_i - \frac{\delta}{\delta q_i}\mathcal{H})\delta q_i + (\dot{q}_i - \frac{\delta}{\delta p_i}\mathcal{H})\delta p_i)dx$$

or using Hamilton's equations of motion

$$\delta\hat{\mathcal{L}}\{p,q,\dot{q},t\} = \frac{d}{dt}\int p_i\delta q_i\, dx \tag{2}$$

Let us now suppose that the variation of p and q is a canonical one, then

$$\delta p_i = -\frac{\delta}{\delta q_i}\mathcal{G} \qquad \delta q_i = \frac{\delta}{\delta p_i}\mathcal{G}$$

where $\mathcal{G} = \mathcal{G}\{p,q,t\}$ is some decent infinitesimal functional in which the parameter ε has been included.

For such a canonical variation we derived in section 5.4

$$\delta\mathcal{H} = -\{g,\mathcal{H}\}_p \tag{4}$$

and

$$\bar{\mathcal{H}}\{\bar{p},\bar{q},t\} - \mathcal{H}\{\bar{p},\bar{q},t\} = \frac{\partial}{\partial t}g + \{g,\mathcal{H}\}_p \tag{5}$$

From (1) we then immediately get

$$\delta\hat{\mathcal{L}} = \frac{d}{dt}\int p_i\delta q_i\,dx - \int(\dot{q}_i\frac{\delta}{\delta q_i}g + \dot{p}_i\frac{\delta}{\delta p_i}g)dx + \{g,\mathcal{H}\}_p \tag{6}$$

$$= \frac{d}{dt}\left(\int p_i\delta q_i\,dx - g\right) + \frac{\partial}{\partial t}g + \{g,\mathcal{H}\}_p$$

and hence by (5)

$$\delta\hat{\mathcal{L}} = \frac{d}{dt}\left(\int p_i\delta q_i\,dx - g\right) + (\bar{\mathcal{H}}\{\bar{p},\bar{q},t\} - \mathcal{H}\{\bar{p},\bar{q},t\}) \tag{7}$$

Conclusion

An invariant canonical variation of the coordinate functions q_i and momentum functions p_i leaves the canonical Lagrangian invariant up to a gauge transformation.

By combining (2) and (7) and using the canoncial Noether theorem we obtain the proof of Noether's inverse theorem in terms of the canonical Lagrangian. We intend to do the same for the normal Lagrangian in the canonical formalism.

The Lagrangian functional $\mathcal{L} = \mathcal{L}\{q,\dot{q},t\}$ is defined by

$$\mathcal{L}\{q,\dot{q},t\} = \hat{\mathcal{L}}\{f\{q,\dot{q},x,t\},\ q,\dot{q},t\} \tag{8}$$

where

$$p_i(x,t) = f_i\{q,\dot{q},x,t\} \tag{9}$$

is the solution of

$$\dot{q}_i = \frac{\delta}{\delta p_i}\mathcal{H} \tag{10}$$

which we will call *the coupling equation*.

From (1) it is easily verified that

$$\frac{\delta}{\delta p_i}\hat{\mathcal{L}} = \dot{q}_i - \frac{\delta}{\delta p_i}\mathcal{H} \tag{11}$$

$$\frac{\delta}{\delta q_i}\hat{\mathcal{L}} = -\frac{\delta}{\delta q_i}\mathcal{H} \tag{12}$$

and

$$\frac{\delta}{\delta \dot{q}_i}\hat{\mathcal{L}} = p_i \tag{13}$$

Here and in the following p_i has to be considered as functional of q and $\dot{q}$ as given by (9). Therefore p has to be varied through q and $\dot{q}$.

$$\delta\mathcal{L} = \int dx\left(\frac{\delta\hat{\mathcal{L}}}{\delta p_i}\,\delta p_i + \frac{\delta\hat{\mathcal{L}}}{\delta q_i}\,\delta q_i + \frac{\delta\hat{\mathcal{L}}}{\delta \dot{q}_i}\,\delta\dot{q}_i\right)$$

where

$$\delta p_i = \int dy\left(\frac{\delta p_i(x)}{\delta q_j(y)}\,\delta q_j(y) + \frac{\delta p_i(x)}{\delta \dot{q}_j(y)}\,\delta\dot{q}_j(y)\right)$$

Hence by (11)-(13)

$$\delta\mathcal{L} = \int dx\left(\dot{q}_i(x) - \frac{\delta\mathcal{H}}{\delta p_i(x)}\right)\delta p_i(x) + \int dx\left(-\frac{\delta\mathcal{H}}{\delta q_i(x)}\,\delta q_i(x) + p_i(x)\delta\dot{q}_i(x)\right)$$

and after some rearrangements

$$\delta\mathcal{L} = \int dx\ \dot{q}_i(x)\delta p_i(x) + \int dx\ p_i(x)\delta\dot{q}_i(x) - \delta\mathcal{H}$$

where

$$\delta\mathcal{H} = \int dx\left(\frac{\delta}{\delta q_i(x)}\,\delta q_i(x) + \frac{\delta}{\delta p_i(x)}\,\delta p_i(x)\right) \tag{14}$$

We also may write

$$\delta\mathcal{L} = -\delta\mathcal{H} + \frac{d}{dt}\int dx\ p_i(x)\delta q_i(x) + \int dx\left(\dot{q}_i(x)\delta p_i(x) - \dot{p}_i(x)\delta q_i(x)\right) \tag{15}$$

Now suppose that δq_i is the coordinate-function part of a canonical variation, i.e. there exists a functional $\mathcal{h}\{p,q,t\}$ such that

$$\delta q_i(x) = \frac{\delta}{\delta p_i(x)}\mathcal{h}\{p,q,t\} \tag{16}$$

We then have

$$\delta\mathcal{L} = -\delta\mathcal{H} + \frac{d}{dt}\Big(\int p_i \delta q_i(x)dx - h\Big) + \frac{\partial}{\partial t}h$$

$$+ \int \dot{q}_i(x)\,\frac{\delta}{\delta q_i(x)}h\,dx + \int \dot{q}_i(x)\delta p_i(x)dx$$

or

$$\delta\mathcal{L} - \frac{d}{dt}\Big(\int p_i \frac{\delta}{\delta p_i}h\,dx - h\Big) = \frac{\partial}{\partial t}h - \delta\mathcal{H} + \int \dot{q}_i\Big(\delta p_i + \frac{\delta}{\delta q_i}h\Big)dx$$

Hence by (16), the coupling equation (10) and the definition of the generalized Poisson bracket, it is easily seen that

$$\delta\mathcal{L} - \frac{d}{dt}\Big(\int p_i \frac{\delta}{\delta p_i}h\,dx - h\Big) = \frac{\partial}{\partial t}h + \{h,\mathcal{H}\}_p$$

Until this point we did not use Hamilton's equation for p_i. We may conlude therefore that $\delta\mathcal{L}$ can be written as

$$\delta\mathcal{L} = \frac{d}{dt}\Big(\int p_i \frac{\delta}{\delta q_i}h\,dx - h\Big)$$

if

$$\frac{\partial}{\partial t}h + \{h,\mathcal{H}\}_p = 0$$

that is if h is a constant of the motion.
We have now proved *Noether's inverse theorem* for conservation laws of the form

$$\frac{d}{dt}h\{\mathcal{L}\{q,\dot{q},x,t\},\ q,t\} \doteq 0$$

in the case of systems for which there exists both a Hamilton and a Lagrangian functional.

Note that in equation (15) we have identified $\frac{d}{dt}\delta q$ and $\delta\dot{q}$. This is allowed because of $\dot{q} = \frac{d}{dt}q$ in $\mathcal{L}$, whereas p has been fixed in terms of q and $\dot{q}$ by the coupling equation (10). In case the coupling equation would have been used to fix $\dot{q}$ in terms of q and p the identification of $\frac{d}{dt}\delta q$ and $\delta\dot{q}$ would in general not be permissible. To illustrate the kind of intricacies into which one may run by uncorrect reasoning we will calculate the difference

between $\delta\dot{q}$ and $\frac{d}{dt}\delta q$ if $\dot{q}$ is defined by the coupling equation (10).

Let

$$\delta q_i = \frac{\delta}{\delta p_i} g\{p,q,t\}$$

$$\delta p_i = -\frac{\delta}{\delta q_i} g\{p,q,t\}$$

and

$$\delta\dot{q}_i(x) = \delta\left(\frac{\delta \mathcal{H}}{\delta p_i(x)}\right)$$

According to the first-order Volterra expansion of the function $\frac{\delta}{\delta p_i(x)}\mathcal{H}$ which is for fixed x a functional of the coordinate and momentum functions, we obtain

$$\begin{aligned}
\delta\dot{q}_i(x) &= \int\left(\frac{\delta^2 \mathcal{H}}{\delta p_j(y)\delta p_i(x)}\,\delta p_j(y) + \frac{\delta^2 \mathcal{H}}{\delta q_j(y)\delta p_i(x)}\,\delta q_j(y)\right)dy \\
&= \int\left(-\frac{\delta^2 \mathcal{H}}{\delta p_j(y)\delta p_i(x)}\,\frac{\delta g}{\delta q_j(y)} + \frac{\delta^2 \mathcal{H}}{\delta q_j(y)\delta p_i(x)}\,\frac{\delta g}{\delta p_j(y)}\right)dy \\
&= \int\left(\frac{\delta}{\delta p_i(x)}\left[\frac{\delta \mathcal{H}}{\delta q_j(y)}\,\frac{\delta g}{\delta p_j(y)} - \frac{\delta \mathcal{H}}{\delta p_j(y)}\,\frac{\delta g}{\delta q_j(y)}\right]\right. \\
&\qquad \left. - \frac{\delta \mathcal{H}}{\delta q_j(y)}\,\frac{\delta^2 g}{\delta p_i(x)\delta p_j(y)} + \frac{\delta \mathcal{H}}{\delta p_j(y)}\,\frac{\delta^2 g}{\delta p_i(x)\delta q_j(y)}\right)dy \\
&= -\frac{\delta}{\delta p_i(x)}\{g,\mathcal{H}\}_p + \left\{\frac{\delta g}{\delta p_i(x)},\mathcal{H}\right\}_p
\end{aligned} \tag{16}$$

We also can set

$$\begin{aligned}
\frac{d}{dt}\delta q_i &= \frac{d}{dt}\left(\frac{\delta g}{\delta p_i(x)}\right) = \frac{\partial}{\partial t}\frac{\delta g}{\delta p_i(x)} + \left\{\frac{\delta g}{\delta p_i(x)},\mathcal{H}\right\}_p \\
&= \frac{\delta}{\delta p_i(x)}\frac{\partial}{\partial t} g + \left\{\frac{\delta g}{\delta p_i(x)},\mathcal{H}\right\}_p
\end{aligned} \tag{17}$$

An alternative way of finding an expression for $\frac{d}{dt}\delta q_i$ is to start with

$$\frac{d}{dt}\delta q_i = \frac{d}{dt}(\bar{q}_i - q_i) = \frac{\delta}{\delta \bar{p}_i}\bar{\mathcal{H}} - \frac{\delta}{\delta p_i}\mathcal{H}$$

We will not work this out because it would be rather laborious and because it would yet give the same result.
Hence by (16) and (17)

$$\delta\dot{q}_i - \frac{d}{dt}\delta q_i = -\frac{\delta}{\delta p_i(x)}\left[\frac{\partial}{\partial t}\mathcal{G} + \{\mathcal{G},\mathcal{H}\}_p\right]$$

we conclude that $\delta\dot{q}_i$ and $\frac{d}{dt}\delta q_i$ in the sense as above are one and the same quantity, if the generating functional of the canonical variation is a constant of the motion.

6. Exchanging the dependent and the independent variables

6.1. Introduction

In many problems in physics it is not obvious what physical quantities must be taken as the independent variables, the coordinates. In describing the dynamics of a fluid one may use *"local" coordinates* (c.q. *Eulerian coordinates*) or *"material" coordinates* (c.q. *Lagrangian* coordinates) to give a correct description of the motion.
If on the one hand local coordinates are used, one would like to know what material element m^α is at a certain position and at a given moment, the unknown field variables or coordinate functions being $m^\alpha(x,t)$.
On the other hand, however, one would look for the position of each material element m^α at each moment, if material coordinates are used. The coordinate functions are $x^i(m,t)$.
The conversion from local to material coordinates or reversely would therefore involve a transposition of dependent and independent variables. We shall describe in this chapter such a transposition as a transformation of the coordinate functions in Lagrange's formalism and as a canonical transformation in Hamilton's formalism. Here we will discuss the problem in its general form for n-dimensions. Recently we studied this problem into a greater detail and in one dimension too [45].

6.2. The inversion of a set of functions

In the following we will need the concept of *the inverse function* or *the set of inverse functions* which can be introduced as follows.

Let $q(m)$ represent a set of n functions q^i of the independent variables $m^1,\ldots,m^n$, $i = 1,\ldots,n$.
We write

$$x^i = q^i(m) = q^i(m^1,\ldots,m^n)$$

or

$$x = q(m)$$

If the set of functions q^i constitutes a one-to-one mapping of m-space onto x-space, the *inverse functions* $\tilde{q}^i$ can be defined as

$$\tilde{q}^i(x^1,\ldots,x^n) = m^i \leftrightarrow x^i = q^i(m^1,\ldots,m^n) \quad i = 1,2,\ldots,n$$

or in an abridged notation

$$\tilde{q}(x) = m \leftrightarrow x = q(m) \tag{1}$$

When the definition is applied twice we obtain

$$\tilde{\tilde{q}}(m) = x \leftrightarrow \tilde{q}(x) = m \leftrightarrow q(m) = x$$

so that

$$\tilde{\tilde{q}}(m) = q(m) \tag{2}$$

From the fact that q as well as $\tilde{q}$ are one-to-one maps, it readily follows

$$x = q(\tilde{q}(x)) \tag{3}$$

and

$$m = \tilde{q}(q(m)) \tag{4}$$

Equation (3) can be used to relate the derivatives of the functions q_i and $\tilde{q}_i$.
Differentiation of (3) with respect to x_k gives, if we use the chain rule for differentiation of functions

$$\delta^i_k = \left(\frac{d}{dm^j} q^i(m)\right)_{m=\tilde{q}(x)} \frac{d}{dx^k} \tilde{q}^j(x)$$

where $\delta^i{}_k = 0$ if $i \neq k$ and $\delta^i{}_k = 1$ if $i = k$ (the *Kronecker symbol*).

The matrix $\left(\frac{d}{dm^j} q^i(m)\right)_{m=\tilde{q}(x)}$ is therefore the inverse of the matrix $\frac{d}{dx^k} \tilde{q}^i(x)$

We write

$$\frac{d}{dx^j} \tilde{q}^i(x) = \left(\left\| \left(\frac{d}{dm^\ell} q^k(m)\right)_{m=\tilde{q}(x)} \right\|^{-1} \right)^i_j \tag{5}$$

With equation (5) it is clear how $\tilde{q}^i(x)$ changes if the arguments x^j change. Analogously we are interested to know how the $\tilde{q}^i(x)$ change if the functions $q^j(m)$ undergo an infinitesimal change.
Let therefore

$$y^i = q^i(m) + \varepsilon f^i(m) \qquad i = 1,2,\ldots,n$$

where ε is a vanishingly small parameter and $f^i(m)$ represents an arbitrary decent function of the arguments $m^1,\ldots,m^n$.
We may simply write

$$y = q(m) + \varepsilon f(m) \tag{6}$$

The inversion of (6) is given by

$$m = \widetilde{q+\varepsilon f}\ (y)$$

and we suppose

$$\widetilde{q+\varepsilon f}\ (y) = \tilde{q}(y) + \varepsilon g(y) \tag{7}$$

Since $\tilde{q}(y)$ is a functional of the functions $q(m)$, which depends on the parameter y, we may write

$$g(y) = \int \frac{\delta \tilde{q}(y)}{\delta q(m)} f(m)dm \tag{8}$$

Substitution of (7) in (6) gives

$$y = q(\tilde{q}(y) + \varepsilon g(y)) + \varepsilon f(\tilde{q}(y) + \varepsilon g(y))$$

By first-order Taylor expansion, using also (3), we obtain

$$\frac{d}{dm^i} q^j(m)\Big|_{m=\tilde{q}(y)} g^i(y) + f^j(\tilde{q}(y)) = 0$$

and therefore

$$g^i(y) = -f^i(\tilde{q}(y)) \left(\left\| \frac{d}{dm^\ell} q^k(m) \right\|_{m=\tilde{q}(y)}^{-1} \right)_i^j \tag{9}$$

so that

$$\frac{\tilde{q}^j(y)}{q^i(m)} = - \left(\left\| \frac{d}{dm^\ell} q^k(m) \right\|^{-1} \right)_i^j \delta(m-\tilde{q}(y)) \tag{10}$$

In the foregoing all the functions q and $\tilde{q}$ depend on the material or spatial variables m^i, x^i respectively with $i = 1,\ldots,n$. In dynamical theory we have to consider functions depending on the time t as well. All functions then become functions of t. The equations we derived will still hold because the time t may be considered as a parameter. In our notation we will therefore suppress the explicit dependence on t unless confusion might arise.

We wish to derive a relation between the time-derivatives of $\tilde{q}(x)$ and $q(m)$. Consequently, we have to write the time dependence explicitly. Instead of (1) we write

$$\tilde{q}(x,t) = m \leftrightarrow x = q(m,t) \tag{11}$$

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6.3

or

$$\tilde{q}^i(x^1,\ldots,x^n,t) = m^i \leftrightarrow x^j = q(m^1,\ldots,m^n,t) \quad i,j = 1,2,\ldots,n$$

The time derivative of the functions $q^i(m,t)$ can be defined by

$$q(m,t+\varepsilon) = q(m,t) + \varepsilon \frac{d}{dt} q(m,t) \tag{12}$$

where ε is a vanishingly small parameter.

Let now

$$z = q(m,t+\varepsilon) \tag{13}$$

then it follows from (11)

$$m = \tilde{q}(z,t+\varepsilon) \tag{14}$$

and hence by (12)-(14)

$$m = \tilde{q}(q(m,t) + \varepsilon \frac{d}{dt} q(m,t),\ t+\varepsilon)$$

By first-order Taylor expansion and by using (4) one obtains

$$\frac{d}{dx^j} \tilde{q}^i(x)\Big|_{x=q(m,t)} \frac{d}{dt} q^j(m) = - \frac{d}{dt} \tilde{q}^i(x)\Big|_{x=q(m,t)} \tag{15}$$

or in reverse

$$\frac{d}{dt} q^j(m) = - \frac{d}{dt} \tilde{q}^i(x)\Big|_{x=\tilde{q}(m,t)} \left(\left\| \frac{d}{dx^\ell} \tilde{q}^k(x) \right\|^{-1}_{x=\tilde{q}(m,t)} \right)^j_i \tag{16}$$

Note that $\frac{d}{dt} q^j$ is calculated with fixed m, whereas $\frac{d}{dt} \tilde{q}^k$ is calculated with fixed x. In fact this is the reason of the minus sign in (15) and (16).

6.3. Exchanging the dependent and independent variables as a canonical transformation in Hamilton's formalism

Let a physical system be characterized by n mutually independent coordinate functions $q^1,\ldots,q^n$ and the canonically conjugated momentum functions $p_1,\ldots,p_n$, all depending on the time t and the mutually independent coordinates $m^1,\ldots,m^n$.

We write $q^i(m^1,\ldots,m^n,t) = q^i(m,t) = q^i(m)$

$$p^i(m^1,\ldots,m^n,t) = p^i(m,t) = p^i(m)$$

we will interchange the roles played by q and m in the formalism. In other

words, we will discuss the transformation

$$\bar{q}(\bar{m}) = \tilde{q}(\bar{m})$$

For further convenience we will write x instead of $\bar{m}$:

$$\bar{q}(x) = \tilde{q}(x) \tag{1}$$

or

$$\bar{q}^i(x^1,\ldots,x^n,t) = \tilde{q}^i(x^1,\ldots,x^n,t),\ i = 1,2,\ldots,n$$

This transformation exists if the functions $q^i(m)$ can be inverted uniquely, or if they form a one-to-one mapping. In that case we have a pure coordinate function transformation which in Hamilton's formalism can be generated by the generating functional

$$G\{\bar{p},q\} = \int \bar{p}_i(x,t)\tilde{q}^i(x)dx \tag{2}$$

because (5.2.14) directly gives

$$\bar{q}^i(x) = \frac{\delta}{\delta \bar{p}_i(x)} G = \tilde{q}^i(x) \tag{3}$$

To complete the canonical transformation we can calculate the corresponding transformation of the momentum functions.
Equation (5.2.13) gives

$$p_i(m) = \frac{\delta}{\delta q^i(m)} G\{\bar{p},q\}$$

From the chain rule for differentiation of functionals one obtains

$$p_i(m) = \int \frac{\delta G}{\delta \tilde{q}^j(x)} \frac{\delta \tilde{q}^j(x)}{\delta q^i(m)} dx$$

From (2) it is seen that $\frac{\delta}{\delta \tilde{q}^j(x)} G = \bar{p}_i(x)$

We already derived (6.2.10) i.e.

$$\frac{\delta \tilde{q}^j(x)}{\delta q^i(m)} = -\left(\left\| \frac{dq^k(m)}{dm^\ell} \right\|^{-1} \right)^j_i \delta(m-\tilde{q}(x))$$

so that

$$p_i(m) = -\int \bar{p}_i(x) \left(\left\| \frac{d}{dm^\ell} q^k(m) \right\|^{-1} \right)^j_i \delta(m-\tilde{q}(x))dx \tag{4}$$

which yields

$$p_i(m) = -\bar{p}_i(q(m)) \left(\left\| \frac{d}{dm^\ell} q^k(m) \right\|^{-1} \right)^j_i \det \left(\frac{d}{dm^\ell} q^k(m) \right) \qquad (5)$$

Since the generating functional $\mathcal{G}$ does not depend explicitly on the time

$$\bar{\mathcal{H}}\{\bar{q},\bar{p}\} = \mathcal{H}\{q,p\} \qquad (6)$$

where $\bar{\mathcal{H}}$ is the Hamilton functional for the transformed canonically conjugated variables $\bar{p},\bar{q}$ and $\mathcal{H}$ analogously for the untransformed system.
Equations (3), (5) and (6) describe the complete canonical transformation, which constitutes a transposition of the coordinates and coordinate functions.

In most practical cases we will operate with Hamiltonian densities $H[p,q,t]$ and $\bar{H}[\bar{p},\bar{q},t]$, such that

$$\bar{\mathcal{H}} = \int \bar{H}\, dx \quad \text{and} \quad \mathcal{H} = \int H\, dm$$

In those cases a sufficient condition for (6) to be valid is

$$\bar{H} = H \det\left(\frac{d}{dx^j} \tilde{q}^i(x)\right) \qquad (7)$$

Let $H[p,q,t]$ be known and depend on p_i, q^i, $\frac{d}{dm^k} q^i$, m^i and t only, with $i,k = 1,2,\ldots,n$.
A suitable Hamiltonian density $\bar{H}[\bar{p},\bar{q},t]$ can then be found by substitution of the following equations into $H[p,q,t]$ of (7)

$$p_i(m) = -\bar{p}_j(x) \frac{d}{dx^i} \tilde{q}^j(x) \left(\det\left(\frac{d}{dx^j} \tilde{q}^i(x)\right)\right)^{-1}$$

$$q^i(m) = x^i$$

$$\frac{d}{dm^k} q^i(m) = \left(\left\| \frac{d}{dx^\ell} \bar{q}^j(x) \right\|^{-1} \right)^i_k$$

and

$$m^i = \bar{q}^i(x)$$

as can be seen from equations (5) and (6.2.5) and the assumptions that $q(m) = x$; $m = \tilde{q}(x) = \bar{q}(x)$.

6.4. Exchanging the dependent and independent variables as a transformation of the coordinate functions in Lagrange's formalism

As we have seen, the transposition of dependent and independent variables is a pure coordinate function transformation. This means that the transformed coordinate functions $\bar{q}^i(x)$ do not depend on the untransformed momentums $p_i(m)$. So the transposition can also be formulated in terms of Lagrange's formalism.

Let the original system by described by a Lagrangian density $L[q,t]$ which depends on $q^i(m)$, $\frac{d}{dt} q^i(m)$, $\frac{d}{dm^k} q^i(m)$, m^i and t only $(i,k = 1,2,\ldots,n)$.

As is readily seen a suitable Lagrangian density $\bar{L}[\bar{q},t]$ is then given by

$$\bar{L}[\bar{q},t] = L[q,t] \det\left(\frac{d}{dx^j} \bar{q}^i(x)\right)$$

and the substitution

$$q^i(m) = x^i$$

$$\frac{d}{dm^k} q^i(m) = \left(\left\| \frac{d}{dx^\ell} \bar{q}^j(x) \right\|^{-1} \right)^i_k$$

$$\frac{d}{dt} q^i(m) = -\left(\left\| \frac{d}{dx^\ell} \bar{q}^k(x) \right\|^{-1} \right)^i_j \frac{d}{dt} \bar{q}^j(x)$$

and $$m^i = \bar{q}^i(x)$$

The transposition of dependent and independent variables can be used to transform the equations of motion of a physical system into equations which are simpler to handle.

Barbashov & Chernikov [46] use an equation for a model of a Born-Infeld type field that is invariant for a transposition of the dependent and independent variables.

The equation is

$$(1-\phi_t^2)\phi_{xx} + 2\phi_x\phi_t\phi_{xt} - (1+\phi_x^2)\phi_{tt} = 0$$

and can be derived from the Lagrangian density

$$L[\phi] = (1+\phi_x^2 - \phi_t^2)^{\frac{1}{2}}$$

Suppose that the functions $\phi(x,t)$ can be inverted.
Application of the theory of this section then gives for the Lagrangian density $\bar{L}[x]$ for the coordinate function $x(\phi,t)$

$$\bar{L}[x] = x_\phi\left(1 + \left(\frac{1}{x_\phi}\right)^2 - \left(\frac{x_t}{x_\phi}\right)^2\right)^{\frac{1}{2}}$$

$$= (1 + x_\phi^2 - x_t^2)^{\frac{1}{2}}$$

So that the equations of motion may also be written

$$(1-x_t^2)x_{\phi\phi} + 2x_\phi x_t x_{\phi t} - (1+x_\phi^2)x_{tt} = 0$$

An other simple example, the longitudinal motion of an elastic bar, has recently been studied by us [45]. In the next (final) chapter we will discuss in an analogous way the motion of a three-dimensional inviscid compressible fluid.

7. The motion of an inviscid compressible fluid

7.1. Introduction

To describe the three dimensional motion of an inviscid compressible fluid (liquid or gas) we introduce a fixed rectangular coordinate system (x^1,x^2,x^3). We will refer to the coordinate triple (x^1,x^2,x^3) as *position* and denote it by x. Consider now a typical point (or particle) moving with the fluid. Let this point be characterized by the position $x_0 = (x_0^1,x_0^2,x_0^3)$ which it occupies at time $t = 0$. Suppose the point has moved to position x at time t. Then x is determined as a function of x_0 and t. Consequently the flow may be represented by the function

$$x = X(x_0,t)$$

or

$$x^i = X^i(x_0^1,x_0^2,x_0^3,t) \qquad i = 1,2,3$$

With x_0 fixed and varying t this is the equation for the particle trajectory. For fixed time t it determines a mapping of the region initially occupied by the fluid into the region occupied at time t. We will assume that initially distinct points remain distinct throughout the entire motion. Therefore the transformation $x = X(x_0,t)$ possesses an inverse

$$x_0 = \tilde{X}(x,t)$$

$$x_0 = \tilde{X}^\alpha(x^2,x^2,x^3,t) \qquad \alpha = 1,2,3$$

and the Jacobian of the transformation $x = X(x_0,t)$

$$J = \frac{\partial(x^1,x^2,x^3)}{\partial(x_0^1,x_0^2,x_0^3)} = \det\left(\frac{d}{dx_0^\alpha}x^i\right) = \left(\det\frac{d}{dx^i}x_0^\alpha\right)^{-1}$$

is defined. The Jacobian represents the dilatation of an infinitesimal volume as it follows the flow. This means that it is proportional to the specific volume $v = \rho^{-1}$ when ρ is the mass density. This leads to the introduction of a new set of variables

$$dm^\alpha = \rho_0^{\frac{1}{3}}\,dx_0^\alpha = v_0^{-\frac{1}{3}}\,dx_0^\alpha \tag{1}$$

where $\rho_0 = v_0^{-1}$ represents the mass density in the fluid at time $t = 0$ at the position x_0.

The flow then may be represented either by the functions

$$x = q(m,t)$$
$$\text{or} \qquad x^i = q^i(m^1,m^2,m^3,t) \quad i = 1,2,3 \tag{2}$$

or by the inverse functions

$$m = \tilde{q}(x,t)$$
$$m^\alpha = \tilde{q}^\alpha(x^1,x^2,x^3,t) \quad \alpha = 1,2,3 \tag{3}$$

The independent variables x_0 or m in the original description (2) of the fluid flow are called *material* or *Lagrangian coordinates*. If (3) is used as a description of the flow the names *local* or *Eulerian coordinates* are used.

In view of our earlier remark on the transformation from the variables x_0 into x, the Jacobian of the transformation $x = q(m,t)$ is now equal to the specific volume v of the fluid

$$\frac{\partial(x^1,x^2,x^3)}{\partial(m^1,m^2,m^3)} = \det\left(\frac{d}{dm^\alpha} q^i\right) = \left(\det \frac{d}{dx^i} \tilde{q}^\alpha\right)^{-1} = v = \rho^{-1}$$

To avoid confusion in notation we will always use v in the Lagrangian description and ρ in the Eulerian description of the fluid motion

$$v = \det \frac{d}{dm^\alpha} q^i(m,t) = \det\left(q^i{}_{,\alpha}\right) \tag{4}$$

$$\rho = \det \frac{d}{dx^i} \tilde{q}^\alpha(x,t) = \det\left(\tilde{q}^\alpha{}_{,i}\right) \tag{5}$$

We will moreover use Latin characters for indices corresponding to the local coordinates x and coordinate functions q, and Greek letters for the indices corresponding to the material coordinates m and coordinate functions $\tilde{q}$.

7.2. Remarks on notation

In the foregoing section we already introduced the notation

$$q^i{}_{,\alpha} = \frac{d}{dm^\alpha} q^i(m,t)$$

and

$$\tilde{q}^\alpha{}_{,i} = \frac{d}{dx^i} \tilde{q}^\alpha(x,t)$$

Latin indices are used for local variables, Greek indices for material variables.

The inverse matrices to $q^i_{,\alpha}$ and $\tilde{q}^\alpha_{,i}$ are defined by

$$q^i_{,\alpha}\, M^\alpha_j = \delta^i_j \qquad q^i_{,\alpha} M^\beta_i = \delta^\beta_\alpha \tag{1}$$

and

$$\tilde{q}^\alpha_{,i} X^j_\alpha = \delta^j_i \qquad \tilde{q}^\alpha_{,i} X^i_\beta = \delta^\alpha_\beta \tag{2}$$

We observe that M^α_i is numerically equal to $\tilde{q}^\alpha_{,i}$ and X^i_α to $q^i_{,\alpha}$. The matrix element M^α_i is a function of the original matrix elements $q^i_{,\alpha}$, so that because of (1)

$$\frac{\partial}{\partial q^j_{,\beta}} (q^i_{,\alpha} M^\alpha_k) = 0$$

and therefore

$$\frac{\partial M^\alpha_k}{\partial q^j_{,\beta}} = -M^\beta_k\, M^\alpha_j \tag{3}$$

In the same way the matrix elements X^k_γ satisfy

$$\frac{\partial X^k_\gamma}{\partial \tilde{q}^\beta_j} = -X^k_\beta\, X^j_\gamma \tag{4}$$

The functions $q^i(m,t)$ and $\tilde{q}^\alpha(x,t)$ completely specify the flow of the fluid. It may nevertheless be useful to consider quantities such as the velocity which is defined in material coordinates (local coordinate functions) as

$$\underline{u}(m,t) = u^i \underline{e}_i$$

with

$$u^i(m,t) = \dot{q}^i(m,t) \tag{5}$$

The vectors $\underline{e}_i$ are the constant rectangular base unit-vectors in the rectangular x-space. The components u^i can also be formulated in terms of the material coordinate functions by means of equation (6.2.16) which reads

$$\dot{q}^i = -\dot{\tilde{q}}^\alpha X^i_\alpha$$

so that

$$\dot{q}^i = u^i = u^i(\tilde{q}(x,t),t) = -\dot{\tilde{q}}^\alpha X^i_\alpha \tag{6}$$

To simplify notation it is useful to introduce components u^α of the velocity with respect to base vectors $\underline{e}_\alpha$ of m-space. We define

$$u^\alpha(x,t) = -\dot{\tilde{q}}^\alpha(x,t) \tag{7}$$

$$\underline{e}_\alpha = \chi^i_\alpha \underline{e}_i \tag{8}$$

It is then clear that

$$\underline{u} = u^i \underline{e}_i = u^\alpha \underline{e}_\alpha$$

and that

$$(\underline{e}_\alpha . \underline{e}_\beta) = \chi^i_\alpha \chi^j_\beta \delta_{ij} = \mu_{\alpha\beta} = \mu_{\beta\alpha} \tag{9}$$

may be interpreted as the metric tensor of the curvi-linear m-space.

In the rectangular x-space the metric tensor is the Kronecker symbol

$$(\underline{e}_i . \underline{e}_j) = \delta_{ij}$$

and nothing is to be gained by the introduction of a reciprocal basis in x-space, but to do this in m-space may be of advantage.
We define the reciprocal base vectors $\underline{e}^\alpha$

$$\underline{e}^\alpha = \tilde{q}^\beta_{,i} \delta^{ij} \underline{e}_j \tag{10}$$

The following properties are easily verified

$$\begin{aligned} (\underline{e}^\alpha . \underline{e}_\beta) &= \delta^\alpha_\beta \\ (\underline{e}^\alpha . \underline{e}^\beta) &= \tilde{q}^\alpha_{,i} \tilde{q}^\beta_{,j} \delta^{ij} = \mu^{\alpha\beta} = \mu^{\beta\alpha} \\ \mu^{\alpha\beta} \mu_{\alpha\gamma} &= \delta^\beta_\gamma \end{aligned} \tag{11}$$

Here $\mu^{\alpha\beta}$ represents the reciprocal metric tensor of the curvi-linear m-space. The base vectors $\underline{e}_\alpha$ and the reciprocal base vectors $\underline{e}^\alpha$ are mutually related through

$$\underline{e}^\alpha = \mu^{\alpha\beta} \underline{e}_\beta ; \qquad \underline{e}_\alpha = \mu_{\alpha\beta} \underline{e}^\beta \tag{12}$$

We also may define the components u_α of the velocity $\underline{u}$ with respect to the basis $\underline{e}^\alpha$

$$u_\alpha = \mu_{\alpha\beta} u^\beta$$

From this it is readily seen

$$\underline{u} = u^{\alpha}\underline{e}_{\alpha} = u_{\beta}\underline{e}^{\beta} = u^{i}\underline{e}_{i} \tag{13}$$

If there is some vector field $\underline{w} = w^{i}\underline{e}_{i}$ we can analogously define the components w^{α} and w_{α}. If we do so, is is readily seen

$$u^{\alpha}w_{\alpha} = u_{\alpha}w^{\alpha} = u^{i}w^{j}\delta_{ij} \tag{14}$$

So, the scalar product of two vectors is invariant for the choice of basis. For a local theory, as the one we are dealing with, this may be useful.

As we did with the velocity, all quantities describing the fluid can be formulated as functions of the local coordinates x and also of the material coordinates m. For instance

$$f = f(x,t) = f(q(m,t),t)$$

The chain-rule for differentiation of functions then yields

$$\frac{d}{dm^{\alpha}} f(q(m,t),t) = \left(\frac{d}{dx^{i}}f(x,t)\right)_{x=q(m,t)} \cdot \frac{d}{dm^{\alpha}} q^{i}(m,t)$$

or after writing the right-hand side as a function of the local coordinate x

$$f_{,\alpha} = f_{,i}\chi^{i}_{\alpha} \tag{15}$$

and the inverse transformation is

$$f_{,i} = f_{,\alpha}M^{\alpha}_{i} \tag{16}$$

Of special interest for the dynamical theory of the fluid flow are the specific volume v, defined by (7.1.4) and the mass density ρ defined by (7.1.5). We will therefore derive several useful equations involving ρ and v before discussing some kinematic relations and the dynamical theory of the flow.

From the definition (7.1.4) we obtain

$$v = \det(q^{i}_{,\alpha}) = \mathcal{E}^{\alpha\beta\gamma}q^{1}_{,\alpha}q^{2}_{,\beta}q^{3}_{,\gamma} = \frac{1}{6}\mathcal{E}^{\alpha\beta\gamma}\mathcal{E}_{ijk}q^{i}_{,\alpha}q^{j}_{,\beta}q^{k}_{,\gamma}$$

where $\mathcal{E}^{\alpha\beta\gamma}$ and $\mathcal{E}_{ijk}$ are 3-dimensional *Levi-Civita tensors*.

From this we see

$$\frac{\partial v}{\partial q^i_{,\alpha}} q^i_{,\beta} = \delta^\alpha_\beta v$$

and therefore

$$\frac{\partial v}{\partial q^i_{,\alpha}} = v M^\alpha_i \tag{17}$$

Differentiating (17), using also (3), we obtain

$$\frac{\partial^2 v}{\partial q^i_{,\alpha} \partial q^j_{,\beta}} = v(M^\alpha_i M^\beta_j - M^\alpha_j M^\beta_i) \tag{18}$$

We see that the right-hand side of (18) is skew-symmetric in the indices α,β as well as in the indices i,j. An important consequence of this is for instance

$$(vM^\alpha_i)_{,\alpha} = \frac{d}{dm^\alpha} \frac{\partial v}{\partial q^i_{,\alpha}} = v(M^\alpha_i M^\beta_j - M^\alpha_j M^\beta_i) q^j_{,\alpha\beta} = 0 \tag{19}$$

Analogously the mass density yields

$$\rho = \det(\tilde{q}^\alpha_{,i}) = \mathcal{E}^{ijk} \tilde{q}^1_{,i} \tilde{q}^2_{,j} \tilde{q}^3_{,k} = \frac{1}{6} \mathcal{E}^{ijk} \mathcal{E}_{\alpha\beta\gamma} \tilde{q}^\alpha_{,i} \tilde{q}^\beta_{,j} \tilde{q}^\gamma_{,k}$$

$$\frac{\partial \rho}{\partial \tilde{q}^\alpha_{,i}} = \rho X^i_\alpha \tag{20}$$

and

$$\frac{\partial^2 \rho}{\partial \tilde{q}^\alpha_{,i} \partial \tilde{q}^\beta_{,j}} = \rho(X^i_\alpha X^j_\beta - X^j_\alpha X^i_\beta) \tag{21}$$

and consequently

$$\frac{d}{dx^i} \frac{\partial \rho}{\partial \tilde{q}^\alpha_{,i}} = \rho(X^i_\alpha X^j_\beta - X^j_\alpha X^i_\beta) \tilde{q}^\beta_{,ij} = 0 \tag{22}$$

7.3. The continuity equation and the circulation theorem

7.3.1. The continuity equation

After the preparatory discussion in the foregoing section we can derive an important *kinematic relation*, namely the continuity equation for the compressible fluid. First we will do this in terms of material coordinates. We calculate

$$\left(\frac{d}{dt}\, v\right)_m = \frac{\partial v}{\partial q^i_{,\alpha}}\, \dot{q}^i_{,\alpha}$$

Hence by (7.2.19)

$$\left(\frac{d}{dt}\, v\right)_m = \left(\frac{\partial v}{\partial q^i_{,\alpha}}\, \dot{q}^i\right)_{,\alpha} \tag{1}$$

This relation represents the continuity equation written in the form of a local*) conservation law. We obtain a more usual form of it with (7.2.17), (7.2.5) and (7.2.16).

$$\left(\frac{d}{dt}\, v\right)_m = v M^\alpha_i u^i_{,\alpha} = v u^i_{,i} = v \operatorname{div} \underline{u} \tag{2}$$

In this form the continuity equation can be found in many text books [47], [48].

In local coordinates we have

$$\left(\frac{d}{dt}\, \rho\right)_x = \left(\frac{\partial \rho}{\partial \tilde{q}^\alpha_{,i}}\, \dot{\tilde{q}}^\alpha_{,i}\right) = \left(\frac{\partial \rho}{\partial \tilde{q}^\alpha_{,i}}\, \dot{\tilde{q}}^\alpha\right)_{,i}$$

$$= \left(\rho X^i_\alpha \dot{\tilde{q}}^\alpha\right)_{,i} = -\left(\rho u^i\right)_{,i} = -\operatorname{div}(\rho \underline{u}) \tag{3}$$

which is equivalent to (2) as is shown in literature, [47], [48].

We note that (1) as well as (3) assume the form of a local conservation law. Because (1) and (3) are satisfied irrespective of the equations of motion, (1) and (3) are *kinematic relations* or *trivial conservation laws*. Following Steudel's classification (section 3.4) they are local

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*) The adjective *local* is used here to indicate the contrast with *global* in *global* conservation law (section 3.4).

conservation laws of the *third kind*.
The continuity equation here is not the result of the equations of motion, but of the fact that the position q^i of an individual particle is a well defined concept, which implies conservation of mass and thus continuity.

7.3.2. The circulation theorem

An important theorem in fluid dynamics is the *circulation theorem* or *Kelvin's theorem* which states

$$(\frac{d}{dt}\,\Gamma)_m \triangleq 0$$

where the *velocity-circulation* Γ can be defined as

$$\Gamma = \iint_S \text{curl } \underline{u}.d\underline{\sigma} \qquad \underline{u} = \dot{q}^i \underline{e}_i$$

and where the integration is performed over some unclosed surface S in the fluid (See for instance Landau-Lifshitz [47]). In the index notation curl $\underline{u}.d\underline{\sigma}$ can be written as

$$\begin{aligned}\text{curl } \underline{u}.d\underline{\sigma} &= (\text{curl } \underline{u})^i (d\underline{\sigma})^j \delta_{ij} \\ &= \mathscr{E}^{ik\ell}\, u^m_{,k} \delta_{m\ell} \mathscr{E}^{jnp}\, dx^q dx^r \delta_{nq} \delta_{pr} \delta_{ij}\end{aligned}$$

Using the formalism of section 7.1 and 7.2 it may be seen

$$\Gamma = \iint C_{\alpha\beta}\, dm^\alpha dm^\beta$$

where

$$C_{\alpha\beta} = -C_{\beta\alpha} = (\dot{q}^i_{,\beta} q^j_{,\alpha} - \dot{q}^i_{,\alpha} q^j_{,\beta})\delta_{ij} \tag{4}$$

By taking the time derivative of (4) we obtain

$$\begin{aligned}\frac{d}{dt} C_{\alpha\beta} &= (\ddot{q}^i_{,\beta} q^j_{,\alpha} - \ddot{q}^i_{,\alpha} q^j_{,\beta})\delta_{ij} \\ &= ((\ddot{q}^i q^j_{,\alpha})_{,\beta} - (\ddot{q}^i q^j_{,\beta})_{,\alpha})\delta_{ij} \qquad (5) \\ &= (\ddot{q}^i q^j_{,\alpha} \delta^\mu_\beta - \ddot{q}^i q^j_{,\beta} \delta^\mu_\alpha)_{,\mu} \delta_{ij}\end{aligned}$$

and we have a trivial local conservation law from which the density is the *circulation tensor* $C_{\alpha\beta}$.
We note that the right-hand side of equation (5) may be interpreted as the *acceleration-circulation* and therefore this equation states that the (material) time-derivative of the velocity-circulation is equal to the acceleration-circulation.

We emphasize that these results too, have been obtained by using the kinematics of the fluid only. The use of the dynamics of the fluid now allows equation (5) to be simplified.

Let there be an acceleration potential h such that the dynamics of the fluid flow yields

$$\ddot{q}^i \doteq -h_{,j}\delta^{ij} \tag{6}$$

then equation (5) reads

$$\frac{d}{dt} C_{\alpha\beta} \doteq -h_{,\alpha\beta} + h_{,\beta\alpha} = 0 \tag{7}$$

and we obtain the *circulation theorem* or *Kelvin's theorem* in differential form.
In the next section we will show that indeed there exists an acceleration potential and that it is equal to the enthalpy of the fluid.

Summarizing the results we may say that the local conservation law (5) is a trivial one. The flux term of (5) vanishes identically if an acceleration potential exists. Equation (7) is trivially equivalent to the trivial local conservation law (5).

7.4. The dynamics of the compressible fluid in material coordinates

We will describe here the dynamics of a three-dimensional compressible fluid by means of a Lagrangian functional , c.q. a Lagrangian density L. This can, as mentioned already, be done in terms of either material or local coordinates. It will become clear that it is more convenient to use material coordinates and we will therefore do this first. The use of local coordinates will be discussed in section 7.7..

For conservative mechanical systems $\mathfrak{L}$ may be chosen as the difference of the kinetic energy and the potential energy of the system. In continuum mechanics the potential energy is the energy of deformation of the medium. In a fluid this quantity depends, per definition, only on the specific volume, not on other characteristics of the strain (see also Eckhart [50]).
Therefore we set

$$\mathfrak{L} = \int (\tfrac{1}{2}\dot{q}^i\dot{q}^j\delta_{ij} - E(v))dm \tag{1}$$

or

$$L = \tfrac{1}{2}\dot{q}^i\dot{q}^j\delta_{ij} - E(v) \tag{2}$$

Here the field variables $q^i(m,t)$ denote the position of the material element m of the fluid at time t, δ_{ij} being the *Kronecker* symbol.

The first order variation of L is given by *)

$$L = \frac{d}{dt}(\dot{q}^i\delta q^j\delta_{ij}) - (\ddot{q}^i\delta_{ij} - \frac{d}{dm^\alpha}(\frac{dE}{dv}\cdot\frac{\partial v}{\partial q^i_{,\alpha}}))\delta q^i + \\ - \frac{d}{dm^\alpha}(\frac{dE}{dv}\cdot\frac{\partial v}{\partial q^i_{,\alpha}}\delta q^i)$$

and Euler's equations are therefore

$$\ddot{q}^i - \frac{d}{dm^\alpha}(\frac{dE}{dv}\frac{\partial v}{\partial q^j_{,\alpha}}\delta^{ij}) = \ddot{q}^i - (v\frac{dE}{dv}M^\alpha_j\delta^{ij})_{,\alpha} = 0 \tag{3}$$

so that

$$\delta L \doteq \frac{d}{dt}(\dot{q}^i\delta q^j\delta_{ij}) - \frac{d}{dm^\alpha}(\frac{dE}{dv}\frac{\partial v}{\partial q^i_{,\alpha}}\delta q^i) \\ \doteq \frac{d}{dt}(\dot{q}^i\delta q^j\delta_{ij}) - (v\frac{dE}{dv}M^\alpha_i\delta q^i)_{,\alpha} \tag{4}$$

The equations of motion (3) can, using (7.2.19) be written

---

*) Using material coordinates $\frac{d}{dt}$ means differentiation with respect to t at constant m.

$$\frac{d}{dt}(\dot{q}^i) = (\frac{dE}{dv})_{,\alpha} \frac{\partial v}{\partial q^j_{,\alpha}} \delta^{ij}$$

or again with (7.2.17) and (7.2.16) as

$$\frac{d}{dt}(\dot{q}^i) = v(\frac{dE}{dv})_{,j} \delta^{ij} = -(E - v \frac{dE}{dv})_{,j} \delta^{ij} \tag{5}$$

so that

$$\frac{d}{dt} \underline{u} = v \text{ grad } \frac{dE}{dv} = - \text{ grad } (E - v \frac{dE}{dv}) \tag{6}$$

This expression does indeed represent the equations of motion of a compressible fluid if a pressure P is defined as

$$P = - \frac{dE}{dv} \tag{7}$$

and this is the case when E is the *internal energy* for an *isentropic* flow. The quantity

$$h = E - v \frac{dE}{dv} \tag{8}$$

is the *enthalpy* or *heat content* *). See for this Landau-Lifshitz [47], Serrin [48] or Morgenau-Murphy [49] and note that

$$\frac{dh}{dv} = -v \frac{d^2E}{dv^2}$$

A Hamilton functional describing the dynamics of the isentropic flow can now be obtained rather easily from the definition of the momentum functions p_i

$$p_i = \frac{\delta}{\delta \dot{q}_i} \mathcal{L} = \dot{q}^i$$

and of the Hamilton functional

$$\mathcal{H}\{p,q\} = \int p_i \dot{q}^i dm - \mathcal{L} = \int (\frac{1}{2} p_i p_j \delta^{ij} + E(v)) dm \tag{9}$$

---

*) From a pure theoretical point of view we can also consider an isothermal flow. In that case E should be the *Helmholtz free energy* or *work content* and h the *Gibbs thermodynamic potential*. Inviscid isothermal flows are not physically realistic however.

Obviously $\mathcal{H}$ represents the total energy of the fluid.
Hamilton's equations are

$$\frac{d}{dt} q^i = \frac{\delta}{\delta p_i} \mathcal{H} = p_j \delta^{ij} = p^i \tag{10}$$

$$\frac{d}{dt} p_i = - \frac{\delta}{\delta q^i} \mathcal{H} = \frac{d}{dm^\alpha} \left(\frac{dE}{dv} \frac{\partial v}{\partial q^i_{,\alpha}} \right) = \left(v \frac{dE}{dv} M^\alpha_i \right)_{,\alpha} = -h_{,i} \tag{11}$$

It is readily seen that equations (10) and (11) together are equivalent to Euler's equations (3) and that equation (11) represents the conservation of *linear momentum* of the fluid. Furthermore (11) shows that the enthalpy h plays the role of an *acceleration potential*.

7.5. Local conservation laws of the compressible fluid

In section 7.2 we have already derived a few local conservation laws of the compressible fluid i.c. the continuity equation and the circulation theorem. The continuity equation is a trivial local conservation law and the circulation theorem is trivially equivalent to a trivial local conservation law. It can be shown that the invariant transformations generated by these conservation laws (section 5.4) are given by

$$\delta q^i = 0, \quad \delta p^i = 0$$

In this section we will derive in a systematic way a number of linear independent local conservation laws of the inviscid compressible fluid. We do this in terms of Lagrange's formalism. Therefore we return to equations (7.4.2) and (7.4.4)

$$L = \frac{1}{2} \dot{q}^i \dot{q}^j \delta_{ij} - E(v) \tag{1}$$

$$\delta L = \frac{d}{dt} (q^i \delta q^j \delta_{ij}) - \left(v \frac{dE}{dv} M^\alpha_i \delta q^i \right)_{,\alpha} \tag{2}$$

and will discuss successively some elementary transformations of the coordinate functions q^i.

Translation in t-space

It is easily seen that L is invariant, up to a gauge transformation, for infinitesimal translations in t-space. This leads to the *conservation* of

total energy.

Let

$$\delta q^i = \varepsilon \dot{q}^i \tag{3}$$

then

$$\delta L = \varepsilon \frac{d}{dt}\left(\frac{1}{2}\dot{q}^i\dot{q}^j\delta_{ij} - E(v)\right)$$

Hence by (2) and (3)

$$\delta L \doteq \varepsilon \frac{d}{dt}(\dot{q}^i\dot{q}^j\delta_{ij}) - \left(v \frac{dE}{dv} M_i^\alpha \dot{q}^i\right)_{,\alpha}$$

In this way we obtain the local conservation law

$$\frac{d}{dt}\left(\frac{1}{2}\dot{q}^i\dot{q}^j\delta_{ij} + E(v)\right) - \left(v \frac{dE}{dv} M_i^\alpha \dot{q}^i\right)_{,\alpha} \doteq 0 \tag{4}$$

which because of (7.2.19) can be written

$$\frac{d}{dt}\left(\frac{1}{2}\underline{u}^2 + E(v)\right) - v \operatorname{div}\left(\frac{dE}{dv}\underline{u}\right) \doteq 0 \tag{5}$$

Conservation of total energy can be found in many text books in this form [45].

Translation in q-space

In section 7.4 we already noticed one of Hamilton's equations

$$\frac{d}{dt} p_i = \left(v \frac{dE}{dv} M_i^\alpha\right)_{,\alpha}$$

assumes the form of a local conservation law. It is readily seen that this local conservation law is a consequence of the invariance of the Lagrangian ($\delta L = 0$) for infinitesimal translations in q-space

$$\delta q^i = \varepsilon a^i$$

where a is a constant vector. This conservation law describes the conservation of the *linear momentum* of the fluid.

We can generalize the treatment when we take

$$\delta q^i = \varepsilon a^i(m) \tag{6}$$

where a^i depends explicitly on m only.

From (1) and (6) follows the first-order variation δL

$$\frac{1}{\varepsilon}\,\delta L = -\frac{dE}{dv}\frac{\partial v}{\partial q^i_{,\alpha}}\,a^i_{,\alpha} = -v\,\frac{dE}{dv}\,M^\alpha_i\,a^i_{,\alpha} \tag{7}$$

Therefore (6) is an invariant transformation if $a^i(m)$ satisfies

$$M^\alpha_i\,a^i_{,\alpha} = a^i_{,i} = 0$$

and this condition is satisfied if a^i can be written as

$$a^i = \ell^{ijk}b_{k,j} = \ell^{ijk}M^\beta_j b_{k,\beta} \tag{8}$$

b_k being some arbitrary function of the coordinates m. We obtain in this way

$$\delta L = 0$$

But, on the other hand from (2) it follows that

$$\frac{1}{\varepsilon}\,\delta L \doteq \frac{d}{dt}(\dot{q}^i\ell^{j\ell k}b_{k,\beta}M^\beta_\ell\delta_{ij}) - (v\,\frac{\partial E}{\partial v}\,M^\alpha_i\ell^{ijk}M^\beta_j b_{k,\beta})_{,\alpha}$$

and we obtain the local conservation law of the first kind

$$\frac{d}{dt}(\underline{q}_t\ \mathrm{curl}\ \underline{b}) - (v\,\frac{dE}{dv}\,(\mathrm{curl}\ \underline{b})^i M^\alpha_i)_{,\alpha} \doteq 0 \tag{9}$$

Obviously this is a more *general* form of the conservation law for the *linear momentum*.

Rotations in q-space

For many physical systems the conservation of *angular momentum* is a consequence of the invariance of the system for infinitesimal rotations. Therefore we put here

$$\delta\underline{q} = \varepsilon\,\underline{\lambda}\times\underline{q} \quad \text{or} \quad \delta q^i = \ell^{ijk}\lambda_j q^\ell\delta_{k\ell} \tag{10}$$

where $\underline{\lambda}$ is some in general m-dependent vector with components $\lambda_j = \lambda_j(m)$. It can be easily verified that

$$\delta L = 0$$

if

$$\lambda_{j,i} = \lambda_{i,j}$$

i.e. $$\lambda_{j,\alpha} M^{\alpha}_{i} = \lambda_{i,\alpha} M^{\alpha}_{j} \tag{11}$$

and we obtain the local conservation law

$$\frac{d}{dt}(\dot{q}^{i} \ell^{jmk} \lambda_{m} q^{\ell} \delta_{k\ell} \delta_{ij}) - (v \frac{dE}{dv} M^{\alpha}_{i} \ell^{ijk} \lambda_{j} q^{\ell} \delta_{k\ell})_{,\alpha} \doteq 0 \tag{12}$$

This equation can also be written

$$\frac{d}{dt}(\underline{\lambda} \cdot (\underline{q} \times \underline{u})) - v \underline{\lambda} \cdot (\underline{q} \times \text{grad} \frac{dE}{dv}) \doteq 0 \tag{13}$$

which is indeed the usual form of the conservation law of the *angular momentum*.

Translations and rotations in m-space

In the formalism we are dealing with transformations of the coordinates must be translated into transformations of the coordinate functions. Infinitesimal translations and rotations in m-space are in this way described by

$$\delta q^{i} = \varepsilon \lambda^{\beta} q^{i}_{,\beta}$$

and

$$\delta q^{i} = \varepsilon \ell^{\beta\mu\nu} a_{\mu} m^{\alpha} \delta_{\alpha\nu} q^{i}_{,\beta}$$

where λ^{β} and a_{μ} are constants.

A more general approach is given by the transformation

$$\delta q^{i} = \varepsilon \lambda^{\beta} q^{i}_{,\beta} \tag{14}$$

where $\lambda^{\beta} = \lambda^{\beta}(m)$ is a function of m.

Using (7.2.1) and (7.2.17) it is readily seen that (14) leaves the Lagrangian L invariant up to a gauge transformation if we require

$$\lambda^{\beta}_{,\beta} = 0 \tag{15}$$

Then

$$\frac{1}{\varepsilon} \delta L = (\frac{1}{2} \lambda^{\beta} \dot{q}^{i} \dot{q}^{j} \delta_{ij} - \lambda^{\beta} E(v))_{,\beta}$$

From (2) we see

$$\frac{1}{\varepsilon} \delta L \doteq \frac{d}{dt}(\lambda^{\beta} \dot{q}^{i} q^{j}_{,\beta} \delta_{ij}) - (v \frac{dE}{dv} \lambda^{\beta})_{,\beta}$$

and end up with a local conservation law of the first kind

$$\frac{d}{dt}(\lambda^\beta \dot{q}^i q^j_{,\beta}\delta_{ij}) + (\lambda^\beta(E(v) - v\frac{d}{dv}E(v) - \frac{1}{2}\dot{q}^i\dot{q}^j\delta_{ij}))_{,\beta} \doteq 0$$

or using 7.4.8

$$\frac{d}{dt}(\lambda^\beta \dot{q}^i q^j_{,\beta}\delta_{ij}) + (\lambda^\beta(h - \frac{1}{2}\dot{q}^i\dot{q}^j\delta_{ij}))_{,\beta} \doteq 0 \qquad (16)$$

Expression (16) is as we will show *trivially equivalent* to a local conservation law which is closely related to the circulation theorem (section 7.3.2).

Let $\lambda^\beta = 2A^{\alpha\beta}{}_{,\alpha}$; $A^{\alpha\beta} = -A^{\beta\alpha}$ (17)

Then $\lambda^\beta{}_{,\beta} = 0$ (c.f. equation 15)

If we take $A^{\alpha\beta} = \frac{1}{4}(a^\beta m^\alpha - a^\alpha m^\beta)$ (18)

or

$$A^{\alpha\beta} = \frac{1}{6}a_\mu m^\rho \delta_{\rho\nu}(\varepsilon^{\beta\mu\nu}m^\alpha - \varepsilon^{\alpha\mu\nu}m^\beta) \qquad (19)$$

where a^α and a_α, $\alpha = 1,2,3$ are arbitrary constants, we have

$$\lambda^\beta = a^\beta \qquad (20)$$

and

$$\lambda^\beta = \varepsilon^{\beta\mu\nu}a_\mu m^\alpha \delta_{\alpha\nu} \qquad (21)$$

respectively.
So, the infinitesimal rotations and translations can be formulated in terms of both $A^{\alpha\beta}$ and λ^β.

Substitution of (17) into (16) yields

$$\frac{d}{dt}(A^{\alpha\beta}_{,\alpha}\dot{q}^i q^j_{,\beta}\delta_{ij}) + (A^{\alpha\beta}_{,\alpha}(h - \dot{q}^i q^j\delta_{ij})_{,\beta} \doteq 0 \qquad (22)$$

Because A is skew-symmetric

$$\frac{d}{dt}(A^{\alpha\beta}_{,\alpha}\dot{q}^i q^j_{,\beta}\delta_{ij}) = \frac{d}{dt}(A^{\alpha\beta}\dot{q}^i q^j_{,\beta}\delta_{ij})_{,\alpha} - \frac{d}{dt}(A^{\alpha\beta}\dot{q}^i_{,\alpha}q^j_{,\beta}\delta_{ij}) \qquad (23)$$

Furthermore

$$\frac{d}{dt}(A^{\alpha\beta}\dot{q}^i q^j_{,\beta}\delta_{ij})_{,\alpha} = (A^{\alpha\beta}\dot{q}^i\dot{q}^j_{,\beta}\delta_{ij})_{,\alpha} = (A^{\alpha\beta}\ddot{q}^i q^j_{,\beta}\delta_{ij})_{,\alpha}$$

$$= A^{\alpha\beta}_{,\alpha}\dot{q}^i\dot{q}^j_{,\beta}\delta_{ij} + (A^{\alpha\beta}\ddot{q}^i q^j_{,\beta}\delta_{ij})_{,\alpha} \qquad (24)$$

$$= (A^{\alpha\beta}_{,\alpha}\frac{1}{2}\dot{q}^i\dot{q}^j\delta_{ij})_{,\beta} + (A^{\alpha\beta}\ddot{q}^i q^j_{,\beta}\delta_{ij})_{,\alpha}$$

Equations (23) and (24) are mathematical identities and therefore conservation laws of the third kind (section 3.4). Hence by (22)-(24) we obtain the trivially equivalent local conservation law

$$\frac{d}{dt}(A^{\alpha\beta}\dot{q}^i_{,\alpha}q^j_{,\beta}\delta_{ij}) \doteq (A^{\alpha\beta}\ddot{q}^i q^j_{,\beta}\delta_{ij})_{,\alpha} + (A^{\alpha\beta}_{,\alpha}h)_{,\beta} \qquad (25)$$

Because of the equations of motion (7.4.10-11) the right-hand side of (25) reads

$$-(A^{\alpha\beta}h_{,i}q^i_{,\beta})_{,\alpha} + (A^{\alpha\beta}_{,\alpha}h)_{,\beta} = -(A^{\alpha\beta}h_{,\beta})_{,\alpha} + (A^{\alpha\beta}_{,\alpha}h)_{,\beta}$$

$$= -A^{\alpha\beta}h_{,\alpha\beta} + A^{\alpha\beta}_{,\alpha\beta}h = 0$$

and we obtain from (25)

$$\frac{d}{dt}(A^{\alpha\beta}\dot{q}^i_{,\alpha}q^j_{,\beta}\delta_{ij}) \doteq 0$$

Hence by 7.3.4 and the fact that $A^{\alpha\beta}$ is skew-symmetric

$$\frac{d}{dt}(A^{\alpha\beta}C_{\alpha\beta}) \doteq 0$$

Thus we find a generalization of the circulation theorem.

7.6. Remarks on the linearized theory of compressible fluid dynamics

To set up linear equations of motion we assume the compressible fluid to be an *ideal elastic medium* i.e. a medium with constant *Young's modulus*

$$Y_0 = \frac{d^2}{dv^2}E(v)$$

In section (7.1) we have taken as the reference situation for the mass distribution the situation at t = 0. We further assume here that this mass distribution is an equilibrium distribution i.e. the specific volume v_0 is a constant. For the internal energy we may then write

$$E(v) = E(v_0) - (v-v_0)P_0 + \frac{1}{2}(v-v_0)^2\gamma_0 \tag{1}$$

where

$$P_0 = -\left(\frac{d}{dv}E(v)\right)_{v=v_0} \tag{2}$$

is the equilibrium pressure.

We furthermore assume that we may write

$$q^i(m,t) = q^i(m,0) + \varepsilon\, \xi^i(m,t) \tag{3}$$

where ε is a small parameter.
From (7.1.1) we then see

$$q^i_{,\alpha}(m,0) = v_0^{\frac{1}{3}}\,\delta^i_\alpha \tag{4}$$

We introduce a new independent variable according to

$$x^\alpha = v_0^{\frac{1}{3}}\, m^\alpha \tag{5}$$

(Note that the physical dimension of x^α is a length so that x^α denotes the position of the mass element m at t = 0) and

$$w^i(x,t) = \xi^i(v_0^{-\frac{1}{3}}x,t)$$

Then

$$\dot{q}^i = \varepsilon\, \dot{\xi}^i(m,t) = \varepsilon\, \dot{w}^i(x,t) \tag{6}$$

and

$$\begin{aligned} v = \det q^i_{,\alpha} &= \det(q^i_{,\alpha}(0) + \varepsilon\, \xi^i_{,\alpha}(m,t)) \\ &= \det v_0^{\frac{1}{3}}\left(\delta^i_\alpha + \varepsilon\, \frac{d}{dx^\alpha} w^i(x,t)\right) \\ &= v_0(1+\varepsilon\ \mathrm{div}\ \underline{w}) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{7}$$

Therefore

$$\frac{v-v_0}{v_0} = \varepsilon\ \mathrm{div}\ \underline{w} + \mathcal{O}(\varepsilon^2) \tag{8}$$

It is not difficult to see that, when a suitable term of order ε^2 is added to the right-hand side of (3)

$$q^i(m,t) = q^i(m,0) + \varepsilon\, \xi^i(m,t) + \mathcal{O}(\varepsilon^2) \tag{3'}$$

it can be arranged that instead of (8) we obtain

$$\frac{v-v_0}{v_0} = \varepsilon\, \operatorname{div} \underline{w} + \mathcal{O}(\varepsilon^2) \tag{8'}$$

Analagously we get

$$\dot{q}^i = \varepsilon\, \dot{w}^i(x,t) + \mathcal{O}(\varepsilon^2) \tag{6'}$$

From (1) and (8') we then obtain

$$E(v) = E(v_0) - \varepsilon\, p_0 \operatorname{div} \underline{w} + \tfrac{1}{2}\varepsilon^2 c^2 (\operatorname{div} \underline{w})^2 + \mathcal{O}(\varepsilon^3) \tag{9}$$

where

$$c = v_0 \gamma_0^{\frac{1}{2}} \tag{10}$$

will be identified later as the *small-amplitude speed of acoustical waves.*

Substitution of (5), (6') and (9) into the Lagrangian functional (7.3.1) gives

$$\mathfrak{L} = \int L\, dx$$

$$L = \varepsilon^2 v_0^{-1} (\tfrac{1}{2} \underline{w}_t)^2 - \tfrac{1}{2} c^2 (\operatorname{div} \underline{w})^2) - E(v_0) + \varepsilon\, p_0 \operatorname{div} \underline{w} + \mathcal{O}(\varepsilon^3)$$

As constant factors, constant terms and terms which can be written as a divergence, do not contribute to the equation of motion we may also write when terms containing third or higher powers of ε are neglected

$$L = \tfrac{1}{2}(\underline{w}_t)^2 - \tfrac{1}{2} c^2 (\operatorname{div} \underline{w})^2 \tag{11}$$

From this we derive Euler's equation which is the *acoustical wave equation*

$$\underline{w}_{tt} - c^2 \operatorname{grad} \operatorname{div} \underline{w} = 0 \tag{12}$$

We note that if we should have required, as Broer [26] did for the one-dimensional case, the equilibrium pressure p_0 being zero, we can obtain equations (11) and (12) from (6) and (8) instead of (6') and (8'). Because

of our method of derivation the results have a larger field of physical application.

The Hamilton functional corresponding to (11) is

$$\mathcal{H} = \int H \, dx \ , \quad H = \frac{1}{2} \underline{p}^2 + \frac{1}{2} c^2 (\operatorname{div} \underline{w})^2 \tag{13}$$

and Hamilton's equations are

$$\underline{p}_t = - \frac{\delta}{\delta \underline{w}} \mathcal{H} = c^2 \operatorname{grad} \operatorname{div} \underline{w} \tag{14}$$

$$\underline{w}_t = \frac{\delta}{\delta \underline{p}} \mathcal{H} = \underline{p} \tag{15}$$

Eliminating $\underline{p}$ from (14) and (15) we obtain, as should be expected, Euler's equation (12).
If the same linearization procedure is applied to the continuity equation (7.2.20), it takes the form

$$\frac{d}{dt} \operatorname{div} \underline{w} = \operatorname{div} \underline{w}_t$$

which represents a mathematical identity (conservation law of the third kind).

Often in practical situations one is not so much interested in positions of the individual fluid particles as in the "velocity" $p = \underline{w}_t$ and the "specific volume" $\phi = \operatorname{div} \underline{w}$ (we use apostrophes because the *true* velocity of the fluid is given by $\underline{q}_t = \varepsilon \, \underline{w}_t$ and the *true* specific volume by $v = v_0(1+\varepsilon \operatorname{div} \underline{w})$. Under such circumstances it can be useful to introduce a coordinate-function transformation of the form

$$\begin{aligned} \phi(x) &= \operatorname{div} \underline{w}(x) \\ \underline{A}(x) &= \operatorname{curl} \underline{w}(x) \end{aligned} \tag{16}$$

where $\underline{A}$ is some solenoidal ($\operatorname{div} \underline{A} = 0$) vector field and therefore counting only for two independent coordinate functions.
Let the momentum functions, canonically conjugated to ϕ and $\underline{A}$ be ψ and $\underline{B}$, where $\underline{B}$ is a solenoidal vector field too. The transformation (16) is then one half of the canonical transformation generated by

$$\mathcal{G} = \int dx (\underline{B} \cdot \operatorname{curl} \underline{w} + \psi \operatorname{div} \underline{w})$$

because the transformation formulas (5.2.13) - (5.2.15) yield

$$\underline{p} = \frac{\delta}{\delta \underline{w}} \mathcal{G} = - \text{ grad } \psi + \text{curl } \underline{B} \tag{17}$$

$$\phi = \frac{\delta}{\delta \psi} \mathcal{G} = \text{div } \underline{w} \qquad \underline{A} = \frac{\delta}{\delta \underline{B}} \mathcal{G} = \text{curl } \underline{w} \tag{18}$$

$$\bar{\mathcal{H}} = \mathcal{H} \tag{19}$$

For the transformed Hamilton functional we get

$$\bar{\mathcal{H}}\{\underline{B},\psi,\underline{A},\phi\} = \int \bar{H} \, dx \tag{20}$$

$$\bar{H} = \frac{1}{2} (\text{curl } \underline{B})^2 + \frac{1}{2} (\text{grad } \psi)^2 + \frac{1}{2} c^2 \phi^2 - \text{grad } \psi . \text{ curl } \underline{B}$$

As also

$$\text{grad } \psi . \text{curl } \underline{B} = \text{div}(\underline{B} \times \text{grad } \psi) + \underline{B} . \text{curl grad } \psi = \text{div}(\underline{B} \times \text{grad } \psi)$$

and a divergence term does not contribute to the equations of motion, we may also write

$$\bar{H} = \frac{1}{2}(\text{curl } \underline{B})^2 + \frac{1}{2}(\text{grad } \psi)^2 + \frac{1}{2} c^2 \phi^2 \tag{21}$$

Hamilton's equations then take the form

$$\phi_t = \frac{\delta}{\delta \psi} \bar{\mathcal{H}} = - \text{ div grad } \psi \tag{22}$$

$$\underline{A}_t = \frac{\delta}{\delta \underline{B}} \bar{\mathcal{H}} = -(\text{div grad})\underline{B} \tag{23}$$

$$\psi_t = - \frac{\delta}{\delta \phi} \bar{\mathcal{H}} = - c^2 \phi \tag{24}$$

$$\underline{B}_t = - \frac{\delta}{\delta \underline{A}} \bar{\mathcal{H}} = 0 \tag{25}$$

From (22) and (24) we see that ψ has to satisfy the scalar wave equation

$$\psi_{tt} = c^2 \text{ div grad } \psi \tag{26}$$

From (25) it follows that $\underline{B}$ is constant as function of time

$$B(x,t) = \underline{B}(x,0)$$

so that

$$\underline{p} = \underline{w}_t = - \text{ grad } \psi + \text{curl } \underline{B} = - \text{ grad } \psi + \text{curl } \underline{B}(x,0)$$

and

$$\phi = \text{div } \underline{w} = - \frac{1}{c^2} \psi_t$$

where ψ has to satisfy (26).
Therefore ψ can be interpreted as the *velocity potential*. If the initial conditions are such that curl $\underline{w}_t(x,0) = 0$ it is easily seen that curl curl $B_0(x) = 0$, and therefore at any time curl $w_t = 0$. The flow is then called *irrotational* and we have a *potential flow* as we did not take viscosity into account.

We finally note that we did not find a canonical transformation which simplifies the equations of motion of the non-linear theory as equations (22)-(25) do for the linearized theory.

7.7. The dynamics of the compressible fluid in local coordinates

In order to obtain a Hamilton functional or a Lagrangian functional, which describes the dynamics of the inviscid compressible fluid in local coordinates, we apply the transformation of the coordinate functions

$$\bar{q}(x,t) = \tilde{q}(x,t) \tag{1}$$

(we use the notation $\bar{m} = x$ here)
to the Lagrangian (7.4.1) or Hamiltonian (7.4.9) in the way discussed in section (6.3) and (6.4), with the following results *)

$$\bar{\mathcal{H}} = \int dx \left(\frac{1}{2\rho} \bar{p}_\alpha \bar{p}_\beta \bar{q}^\alpha_{,i} \bar{q}^\beta_{,j} \delta^{ij} + \rho\, E\left(\frac{1}{\rho}\right)\right) \tag{2}$$

and

$$\bar{\mathcal{L}} = \int dx \int \left(\frac{1}{2} \rho\, \dot{\bar{q}}^\alpha \dot{\bar{q}}^\beta \chi^i_\alpha \chi^j_\beta \delta_{ij} - \rho\, E\left(\frac{1}{\rho}\right)\right) \tag{3}$$

The coordinate functions are $\bar{q}^\alpha(x,t) = \tilde{q}^\alpha(x,t)$ and the canonically conjugated momentum functions $\bar{p}^\alpha(x,t)$. The mass density ρ is given by

$$\rho = \det(\bar{q}^\alpha_{,i})$$

---

*) Using local coordinates $\frac{d}{dt} q = \dot{q}$ means differentiation with respect to t at constant x.

To simplify our notation we will in the following omit the bar on q and p. This will not, with the index notation lead to confusion because Latin superscripts are used for q(m,t) and Greek ones for $\bar{q}(x,t)$.

Hamilton's equations of motion then are

$$\dot{q}^\alpha = \frac{\delta}{\delta \rho_\alpha} \bar{\mathcal{H}} = \frac{1}{\rho} p_\beta q^\alpha_{,i} q^\beta_{,j} \delta^{ij} \tag{4}$$

or with (7.2.2)

$$p_\beta = \rho \dot{q}^\alpha \chi^i_\alpha \chi^i_\beta \delta_{ij} \tag{5}$$

and

$$\dot{p}_\alpha = -\frac{\delta}{\delta q^\alpha} \bar{\mathcal{H}} = \frac{d}{dx^j} \left(\frac{1}{\rho} p_\alpha p_\beta q^\beta_{,i} \delta^{ij} + \frac{\partial \rho}{\partial q^\alpha_{,j}} \left(-\frac{1}{2\rho^2} p_\gamma p_\beta q^\gamma_{,i} q^\beta_{,k} \delta^{ik} + h\right)\right. \tag{6}$$

Here

$$h = \frac{d}{d\rho} (\rho E(\frac{1}{\rho})) \tag{7}$$

is the *enthalpy* or *heat-content* of the fluid (see also equation (7.4.8)).

It is easily seen that equations (4) and (6) taken together are equivalent to Euler's equation belonging to (3). The first Hamilton equation (4) can easily be interpreted. Multiplication of (4) with $\rho \chi^k_\alpha \underline{e}_k$ yields with (7.2.2), (7.2.6) and (7.2.7)

$$-\rho u^k \underline{e}_k = p_\beta q^\beta_{,j} \underline{e}_k \delta^{kj}$$

If we introduce the (as in 7.2.10) *reciprocal base vector* of m-space by

$$\underline{e}^\beta = q^\beta_{,i} \underline{e}_j \delta^{ij}$$

we obtain as a result

$$\underline{p} = p_\beta \underline{e}^\beta = -\rho \underline{u} \tag{8}$$

In the appendix to this chapter we will show that the second of Hamilton's equations (6) can be written in the form

$$\frac{d}{dt} \underline{u} + (\underline{u}.\text{grad})\underline{u} + \text{grad } h = 0$$

which is equivalent to (7.4.6) because of the identity

$$(\frac{d}{dt}\,\underline{u})_m = (\frac{d}{dt}\,\underline{u})_x + (\underline{u}.\mathrm{grad})\underline{u}$$

So (2) and (3) give the same equations of motion as (7.4.1) and (7.4.9).

Broer [26] showed that if the linearization of the equations of motion of the one-dimensional ideal-elastic bar is carried out both in material and local coordinates the same equations are obtained.
This is related to the fact that the linear approximation of the coordinate functions in terms of local and material coordinates differ by a constant factor only. We will show that the same applies to the equations of the three-dimensional fluid motion.

When we assume that the fluid is an ideal-elastic medium we can write instead of (7.6.1)

$$E(\frac{1}{\rho}) = E(v_0) - p_0 v_0(\frac{\rho_0}{\rho} - 1) + \frac{1}{2} v_0^2 \gamma_0 (\frac{\rho_0}{\rho} - 1)^2$$

where $\rho = v^{-1}$ and $\rho_0 = v_0^{-1}$.
Linear equations are obtained by putting

$$q^\alpha(x,t) = q^\alpha(x,0) + \varepsilon\, \rho_0^{\frac{1}{3}}\, w^\alpha(x,t) + \mathcal{O}(\varepsilon^2)$$

and therefore we have

$$q^\alpha_{,i}(x,0) = \rho_0^{\frac{1}{3}}\, \delta^\alpha_i + \mathcal{O}(\varepsilon^2)$$

and

$$q^\alpha_{,i}(x,t) = \rho_0^{\frac{1}{3}}\, (\delta^\alpha_i + \varepsilon\, w^\alpha_{,i} + \mathcal{O}(\varepsilon^2))$$

$$\dot{q}^\alpha(x,t) = \varepsilon\, \rho_0^{\frac{1}{3}}\, \dot{w}^\alpha + \mathcal{O}(\varepsilon^2)$$

This leads to

$$\rho = \det(q^\alpha_{,i}) = \rho_0\, \det(\delta^\alpha_i + \varepsilon\, w^\alpha_{,i}) = \rho_0(1+\varepsilon\ \mathrm{div}\ \underline{w}) + \mathcal{O}(\varepsilon^2)$$

or when the term of order ε^2 in $q\ (x,t)$ is suitably chosen to

$$\rho = \rho_0(1+\varepsilon\ \mathrm{div}\ \underline{w}) + \mathcal{O}(\varepsilon^3)$$

Also we have

$$X^i_\alpha = \left(\left\| q^\alpha_{,i} \right\|^{-1} \right)^i_\alpha = \rho_0^{-\frac{1}{3}} \delta^i_\alpha + \mathcal{O}(\epsilon)$$

After substituting these relations in the expression (3) for $\bar{\mathcal{L}}$, and after leaving out terms of order ϵ^3 and higher, a factor $\epsilon^2 \rho_0$ and terms in the density which can be written as a divergence, we obtain

$$\bar{\mathcal{L}} = \int dx \left(\frac{1}{2} \underline{w}_t^2 - \frac{1}{2} c^2 (\text{div } \underline{w})^2 \right)$$

where as in (7.6.10) $c = v_0 \gamma_0^{\frac{1}{2}} = \gamma_0^{\frac{1}{2}} \rho_0^{-1}$ is the *small amplitude speed of acoustical waves*.
We obtain in this way the same Lagrangian (7.6.11) which leads to the same acoustical wave equation (7.6.12).

Treatments as presented in section (7.5) can also be given when we start from the Lagrangian (3) or Hamiltonian (2) in terms of the local coordinates x. This is a relatively awkward procedure because of the complexity of the Lagrangian and Hamiltonian.

Both in material coordinates and in local coordinates the Hamilton functional represents the total energy of the fluid. In material coordinates E(v) can be chosen a quadratic function. In that case Hamilton's equations are linear, and the quantization rules of for instance Rosen [5] can be applied. Doing so the canonical transformation of material coordinates into local coordinates can be used to translate the results of the quantization into the results for local coordinates.

Appendix to Chapter 7 (section 7.7)

In this appendix we show that the second Hamilton equation (7.7.6) is equivalent to the usual equation of motion of the inviscid compressible fluid. Equation (7.7.6) reads

$$\dot{p}_\alpha = -\frac{\delta}{\delta q^\alpha}\bar{\mathcal{H}} = \frac{d}{dx^i}\left(\frac{1}{\rho}\, p_\alpha p_\beta q^\beta_{,j}\delta^{ij} + \frac{\partial \rho}{\partial q^\alpha_{,i}}\left(-\frac{1}{2\rho^2}\, p_\gamma p_\beta q^\gamma_{,j} q^\beta_{,k}\delta^{jk} + h\right)\right) \tag{1}$$

with (7.2.20)

$$\frac{\partial \rho}{\partial q^\alpha_{,i}} = \rho\, \chi^i_\alpha \tag{2}$$

this becomes

$$\dot{p}_\alpha = \frac{d}{dx^i}\left(\frac{1}{\rho}\, p_\alpha p_\beta q^\beta_{,j}\delta^{ij} - \frac{1}{2\rho}\, p_\gamma p_\beta q^\gamma_{,j} q^\beta_{,k}\, \chi^i_\alpha \delta^{jk} + \rho\, h\, \chi^i_\alpha\right) \tag{3}$$

Now we multiply this equation with

$$\frac{1}{\rho}\, q^\alpha_{,\ell}\underline{e}^\ell = \frac{1}{\rho}\, q^\alpha_{,\ell}\underline{e}_m\delta^{m\ell} = \frac{1}{\rho}\,\underline{e}^\alpha \tag{4}$$

(c.f. equation (7.2.10))
and will investigate each term separately. The left-hand side becomes

$$\frac{1}{\rho}\,\dot{p}_\alpha\underline{e}^\alpha = \frac{d}{dt}\left(\frac{1}{\rho}\, p_\alpha\underline{e}^\alpha\right) - p_\alpha\,\frac{d}{dt}\left(\frac{1}{\rho}\, q^\alpha_{,\ell}\right)\underline{e}^\ell \tag{5}$$

With (7.7.8)

$$\underline{p} = p_\alpha\underline{e}^\alpha \tag{6}$$

and (2), (5) becomes

$$\frac{1}{\rho}\,\dot{p}_\alpha\underline{e}^\alpha = \frac{d}{dt}\left(\frac{1}{\rho}\,\underline{p}\right) + \frac{1}{\rho}\, p_\alpha\chi^i_\beta\,\dot{q}^\beta_{,i}q^\alpha_{,\ell}\underline{e}^\ell - \frac{1}{\rho}\, p_\alpha\dot{q}^\alpha_{,\ell}\underline{e}^\ell$$

Hence by (4) and (6)

$$\frac{1}{\rho}\,\dot{p}_\alpha\underline{e}^\alpha = \frac{d}{dt}\left(\frac{1}{\rho}\,\underline{p}\right) + \frac{1}{\rho}\,\chi^i_\beta\,\dot{q}^\beta_{,i}\underline{p} - \frac{1}{\rho}\, p_\alpha\dot{q}^\alpha_{,\ell}\underline{e}^\ell \tag{7}$$

For the second term on the right-hand side of (7) we can write

$$\frac{1}{\rho}\,\chi^i_\beta\,\dot{q}^\beta_{,i}\underline{p} = \frac{1}{\rho^2}\,\underline{p}\left(\frac{d}{dx^i}\left(\rho\,\chi^i_\beta\dot{q}^\beta\right) - \frac{d}{dx^i}\left(\rho\,\chi^i_\alpha\right)\dot{q}^\alpha\right) \tag{8}$$

The second term on the right-hand side of (8) vanishes because of (2) and (7.2.22):

$$\frac{1}{\rho} X^i_\beta \dot{q}^\beta_{,i} \underline{p} = \frac{1}{\rho^2} \underline{p} \frac{d}{dx^i} (\rho X^i_\beta \dot{q}^\beta) \tag{9}$$

From the first Hamilton equation (7.7.4) follows

$$\rho X^i_\beta \dot{q}^\beta = p_\alpha q^\alpha_{,j} \delta^{ij}$$

such that

$$\frac{d}{dx^i}(\rho X^i_\beta \dot{q}^\beta) = \frac{d}{dx^i} (p_\alpha q^\alpha_{,j} \delta^{ij})$$

Hence by (4) and

$$(\underline{e}^i . \underline{e}^j) = \delta^{ij} \tag{10}$$

$$\frac{d}{dx^i} (\rho X^i_\beta \dot{q}^\beta) = \frac{d}{dx^i} (p_\alpha q^\alpha_{,j} \delta^{ij}) = \text{div } \underline{p} \tag{11}$$

Hence by (7), (9) and (11) we obtain

$$\frac{1}{\rho} \dot{p}_\alpha \underline{e}^\alpha = \frac{d}{dt}(\frac{1}{\rho} \underline{p}) + \frac{1}{\rho^2} \underline{p}.\text{div } \underline{p} - \frac{1}{\rho} \dot{p}_\alpha q^\alpha_{,\ell} \underline{e}^\ell \tag{12}$$

The first term on the right-hand side of (3) becomes

$$\frac{1}{\rho} \frac{d}{dx^i} (\frac{1}{\rho} p_\alpha p_\beta q^\beta_{,j} \delta^{ij}) \underline{e}^\alpha = \frac{1}{\rho} p_\beta q^\beta_{,j} \frac{d}{dx^i} (\frac{1}{\rho} p_\alpha) \underline{e}^\alpha \delta^{ij} + \frac{1}{\rho^2} p_\alpha \frac{d}{dx^i} (p_\beta q^\beta_{,j} \delta^{ij}) \underline{e}^\alpha$$

Hence by (6) and (11)

$$\frac{1}{\rho} \frac{d}{dx^i} (\frac{1}{\rho} p_\alpha p_\beta q^\beta_{,j} \delta^{ij}) \underline{e}^\alpha = \frac{1}{\rho} p_\beta q^\beta_{,j} \frac{d}{dx^i} (\frac{1}{\rho} p_\alpha) \underline{e}^\alpha \delta^{ij} + \frac{1}{\rho^2} \underline{p}.\text{div } \underline{p}$$

$$= \frac{1}{\rho} p_\beta q^\beta_{,j} \delta^{ij} \frac{d}{dx^i} (\frac{1}{\rho} p_\alpha \underline{e}^\alpha) - \frac{1}{\rho^2} p_\beta q^\beta_{,j} p_\alpha \frac{d}{dx^i} (q^\alpha_{,k}) \underline{e}^k \delta^{ij} + \frac{1}{\rho^2} \underline{p}.\text{div } \underline{p}$$

Hence by (4) and (6)

$$\frac{1}{\rho}\frac{d}{dx^i}\left(\frac{1}{\rho}\,p_\alpha p_\beta q^\beta_{,j}\delta^{ij}\right)\underline{e}^\alpha = \left(\frac{1}{\rho}\,\mathrm{p.grad}\right)\frac{1}{\rho}\,\underline{p} + \frac{1}{\rho^2}\,\underline{p}.\mathrm{div}\,\underline{p} - \frac{1}{\rho^2}\,p_\beta q^\beta_{,j} p_\alpha q^\alpha_{,ki}\underline{e}^k\delta^{ij} \tag{13}$$

The second term on the right-hand side of (3) becomes

$$-\frac{1}{\rho}\frac{d}{dx^i}\left(\frac{1}{2\rho}\,p_\gamma p_\beta q^\gamma_{,j} q^\beta_{,k}\delta^{jk}\,\chi^i_\alpha\right)q^\alpha_{,\ell}\underline{e}^\ell = \frac{d}{dx^i}\left(-\frac{1}{2\rho^2}\,p_\gamma p_\beta q^\gamma_{,j} q^\beta_{,k}\delta^{jk}\right)\underline{e}^i \tag{14}$$

which is easily verified when (2) and (7.2.18) are used.
With the chain rule for differentiations of functions we also may write for the right hand side of (14)

$$-\frac{1}{\rho}\,p_\gamma q^\gamma_{,j}\frac{d}{dx^i}\left(\frac{1}{\rho}\,q^\beta_{,k}p_\beta\right)\underline{e}^i\delta^{jk}$$

or whith (7.7.5) and (7.2.2)

$$-\frac{1}{\rho}\,p_\gamma q^\gamma_{,j}\frac{d}{dx^i}\left(\dot{q}^\beta\,\chi^k_\beta\right)\underline{e}^i\,\delta^j_k$$

$$= -\frac{1}{\rho}\,p_\gamma q^\gamma_{,j}\dot{q}^\beta_{,i}\,\chi^k_\beta\,\delta^j_k\,\underline{e}^i - \frac{1}{\rho}\,p_\gamma q^\gamma_{,j}\dot{q}^\beta\left(\frac{\partial}{\partial q^\alpha_{,\ell}}\,\chi^k_\beta\right)q^\alpha_{,\ell i}\,\delta^j_k\,\underline{e}^i$$

With (7.2.2) and (7.2.4) this becomes

$$= -\frac{1}{\rho}\,p_\gamma\dot{q}^\gamma_{,i}\,\underline{e}^i + \frac{1}{\rho}\,p_\alpha\dot{q}^\beta\chi^\ell_\beta\,q^\alpha_{,\ell i}\,\underline{e}^i$$

or with (7.7.4) and (7.2.2)

$$= -\frac{1}{\rho}\,p_\gamma\dot{q}^\gamma_{,i}\,\underline{e}^i + \frac{1}{\rho^2}\,p_\alpha p_\gamma q^\gamma_{,k} q^\alpha_{,ij}\delta^{jk}\,\underline{e}^i \tag{15}$$

The third term on the right-hand side of equation (3) becomes

$$\frac{1}{\rho}\frac{d}{dx^i}\left(\rho\,h\,\chi^i_\alpha\right)q^\alpha_{,k}\,\underline{e}^k$$

and can also be written as

$$\chi^i_\alpha\frac{d}{dx^i}(h)q^\alpha_{,k}\,\underline{e}^k + \frac{1}{\rho}\,q^\alpha_{,k}\frac{d}{dx^i}\left(\rho\,\chi^i_\alpha\right)h\,\underline{e}^k$$

or with (7.2.2), (2) and (7.2.18)

$$= \frac{d}{dx^i}(h)\underline{e}^i = \text{grad } h \qquad (16)$$

Substitution of (12), (13), (15) and (16) in the equation obtained by combining (3) and (4) brings the second Hamilton equation of the compressible fluid into the form

$$\frac{d}{dt}\left(\frac{1}{\rho}\underline{p}\right) = \left(\frac{1}{\rho}\underline{p}.\text{grad}\right)\frac{1}{\rho}\underline{p} + \text{grad } h$$

or with (7.7.8) $\underline{p} = -\rho\underline{u}$ we obtain Euler's equation of motion of the inviscid compressible fluid

$$-\frac{d}{dt}\underline{u} = (\underline{u}.\text{grad})\underline{u} + \text{grad } h$$

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Samenvatting

In dit proefschrift worden een aantal onderwerpen uit de klassieke mechanika en de generalisatie daarvan voor continue fysische systemen besproken. De volgende onderwerpen krijgen speciale aandacht:

(i) Het Noether theorema als methode ter bepaling van behoudswetten.
(ii) Sesquilineaire Lagrange funkties en funktionalen die zeer geschikt zijn om eerste-orde gewone en partiële differentiaalvergelijkingen te beschrijven.
(iii) De additieve overdraagbaarheid van bewegingsconstanten.
(iv) Kanonieke transformaties voor continue fysische systemen met bijbehorende genererende funktionalen.

Als toepassing van de theorie worden, gebruik makende van materiële (Lagrange) coördinaten, de bewegingsvergelijkingen van een niet visceus samendrukbaar gas besproken zowel in de Lagrange als in de Hamiltonse vorm.
Met behulp van het Noether theorema worden een aantal behoudswetten bepaald. De overgang van materiële naar locale (Eulerse) coördinaten wordt beschreven als een voorbeeld van een kanonieke transformatie, i.c. een punttransformatie.

Kurzfassung

Diese Arbeit betrifft mehrere Themen der klassischen Mechanik und deren Veralgemeinerung für die Dynamik der kontinuierlichen Systeme.
Den folgenden Themen wird besondere Aufmerksamkeit gewidmet:

(i) Der Noetherische Satz zur Bestimmung von Erhaltungssätzen.
(ii) Sesquilineare Lagrange Funktionen und Funktionalen die zur Beschreibung gewöhnlicher und partieller Differentialgleichungen des ersten Grades sehr geeignet sind.
(iii) Die additive Uebertragbarkeit von Bewegungskonstanten.
(iv) Kanonische Transformationen für kontinuierliche Systeme und ihre erzeugende Funktionalen.

Die Theorie wird in der Strömungslehre für die Bewegung eines zusammendrückbaren Mediums ohne innerliche Reibung angewendet. Die Bewegungsgleichungen werden mit der Benutzung materieller (Lagrangischer) Koordinaten in der Lagrangischen und auch in der Hamiltonischen Form erötert. Mit Hilfe des Noetherischen Satzes werden mehrere Erhaltungssätze bestimmt. Der Uebergang von materiellen Koordinaten zu lokalen (Eulerischen) Koordinaten wird als ein Beispiel der kanonischen Transformationen betrachtet und zwar als eine Punkttransformation.

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Personally I accept the full responsibility for all the errors left over.

Biographical note

The author of this thesis was born on 19440830 in Eindhoven. He obtained his professional degree in physical engineering at the Eindhoven University of Technology on 19681129. He then joined the scientific staff of this University to work in the theoretical physics section.

Stellingen

1. Als bij een proefschrift meer dan de zes wettelijk voorgeschreven stellingen worden opgenomen, terwijl een van deze stellingen betrekking heeft op de nutteloosheid van stellingen, moet er getwijfeld worden aan de oprechtheid van de promovendus.

2. Bij groefmaten voor O-ringen dient tevens de hardheid van de te gebruiken ring te worden vermeld.

B.W.L.M.Sessink, N.F.Verster, Design of elastomer O-ring vacuum seals, to be published in Vacuum.

3. Het is wenselijk dat de technische hogeschool eenvoudig drukwerk ten behoeve van proefschriften van haar medewerkers tegen kostprijs verzorgt.

4. Het gebruik van academische titels binnen de academie is overbodig, buiten de academie niet ter zake en vaak misleidend.

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6. Stelling 2 behorende bij het proefschrift van J.de Graaf is fout.

J.de Graaf, Linear dynamical systems in Hilbert Space, proefschrift, Technische Hogeschool Eindhoven, Eindhoven, 1970.

W.J.M.van Breugel, Behoudswetten van eerste orde dynamische systemen onder extra voorwaarden, afstudeerverslag, Technische Hogeschool Eindhoven, 1972.

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