Bifurcations of equilibrium sets for a class of mechanical systems with dry friction

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Summary. The presence of dry friction in mechanical systems may induce the existence of an equilibrium set, consisting of infinitely many equilibrium points. The topological structure of the trajectories near an equilibrium set is investigated for systems with one frictional interface. In this case, the equilibrium set will be a compact, connected one-dimensional set in phase space. It is shown in this paper that local bifurcations of equilibrium sets occur near the endpoints of this curve. Based on this result, sufficient conditions for structural stability of equilibrium sets in planar systems are given, and new bifurcations are identified.

Introduction

Many mechanical systems experience sticking behaviour due to dry friction, such that there exist an equilibrium set, that consists of a continuum of equilibrium points, rather than an isolated equilibrium point. This behaviour can accurately be described by a model in terms of a differential inclusion, where the friction force is modelled with a set-valued friction law, that depends solely on the slip velocity, and solutions are considered in the sense of Filippov, see [1].

An equilibrium set of a system with dry friction may be stable or unstable in the sense of Lyapunov. In addition, it may attract all nearby trajectories in finite time, cf. [2]. A natural question is to ask: how do changes in system parameters influence these properties? To answer this question, structural stability and bifurcations of equilibrium sets are studied in this paper. For this purpose, the topological structure of trajectories near an equilibrium set is studied and possible bifurcations are identified.

In this paper, a class of mechanical systems with a single frictional interface is studied. Both the existence of an equilibrium set and the topological structure of nearby trajectories are investigated. Sufficient conditions for structural stability of the equilibrium set are given. At system parameters where the conditions for structural stability are not satisfied, bifurcations are identified that do not occur in smooth systems.

Modelling of a class of mechanical systems with friction

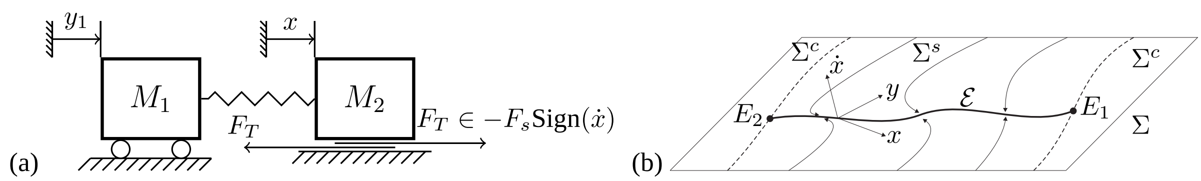

Figure 1: (a) Mechanical system subject to dry friction. (b) Sketch of discontinuity surface of (2) with $n = 3$ and equilibrium set $\mathcal{E}$.

Consider a mechanical system that experiences friction on one interface between two surfaces that move relative to each other in a given direction. Let x denote the displacement in this direction and $\dot{x}$ denote the slip velocity, see Figure 1(a) for an example. For an $n-$dimensional dynamical system this implies that $n-2$ other states y are required besides x and $\dot{x}$. In general, using the states $x, \dot{x}$ and y, the dynamics are described by the following differential inclusion, cf. [1]:

$$\begin{aligned} \ddot{x} - f(x,\dot{x},y) &\in -F_s\text{Sign}(\dot{x}), \\ \dot{y} &= g(x,\dot{x},y), \end{aligned} \quad \text{almost everywhere,} \tag{1}$$

where f and g are sufficiently smooth, $F_s > 0$, and $\text{Sign}(\cdot)$ denotes the set-valued sign function, i.e. $\text{Sign}(p) = p|p|^{-1}$, for $p \neq 0$, and $\text{Sign}(0) = [-1, 1]$. Introducing the state variables $q = (x \quad \dot{x} \quad y^T)^T$, the dynamics of (1) can be reformulated as:

$$\dot{q} \in F(q) = \begin{cases} F_1(q), & h(q) < 0, \\ F_2(q), & h(q) > 0, \\ \text{co}\{F_1(q), F_2(q)\}, & h(q) = 0, \end{cases} \tag{2}$$

where $q \in \mathbb{R}^n$, $\text{co}(a,b)$ denotes the smallest convex hull containing a and b, and F_1, F_2 and h are given by the smooth functions $F_1(q) = \begin{pmatrix} \dot{x} \\ f(x,\dot{x},y) + F_s \\ g(x,\dot{x},y) \end{pmatrix}$, $F_2(q) = \begin{pmatrix} \dot{x} \\ f(x,\dot{x},y) - F_s \\ g(x,\dot{x},y) \end{pmatrix}$, and $h(q) = \dot{x}$. Since $F_s > 0$, the discontinuity boundary $\Sigma = \{q : h(q) = 0\}$ contains a set Σ^c where trajectories cross Σ and a set Σ^s, where trajectories arrive in Σ and subsequently slide along Σ as described by the Filippov solution concept. It is assumed that the functions f and g are such that $\begin{pmatrix} f \\ g \end{pmatrix}$ is proper, $\begin{pmatrix} f(0,0,0) \\ g(0,0,0) \end{pmatrix} = 0$ and $\begin{pmatrix} \frac{\partial f(x,0,y)}{\partial x} & \frac{\partial f(x,0,y)}{\partial y} \\ \frac{\partial g(x,0,y)}{\partial x} & \frac{\partial g(x,0,y)}{\partial y} \end{pmatrix}$ is invertible. Hence, the equilibrium set $\mathcal{E} \subset \Sigma$ of (1) is a one-dimensional curve with endpoints E_1 and E_2, as illustrated in Figure 1(b). In order to study the dynamics in the neighbourhood of the equilibrium set, the endpoints are studied separate from the other points of the equilibrium set.

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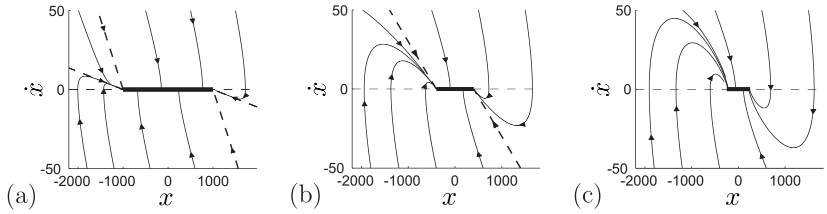

Figure 2: Exemplary system showing a focus-node bifurcation. The equilibrium set $\mathcal{E}$ is given by a bold line, the eigenvectors of stable or unstable eigenvalues of A are represented with dashed lines. (a) $a_{21} = -0.001$, (b) $a_{21} = -0.0025$, (c) $a_{21} = -0.004$.

Bifurcations

A change of the topological structure of the phase portrait of a system A under parameter variation is called a bifurcation, where the topological structure is defined as follows.

Definition 1 ([1]) *We say that the trajectories of two systems in open domains G_1 and G_2, respectively, have the same* topological structure *if there exist a topological map between G_1 and G_2.*

A *topological map* between domains G_1 and G_2 is a homeomorphism from G_1 to G_2 which carries, as does its inverse, trajectories into trajectories and preserves the direction of time.

In most existing bifurcation results for differential inclusions, see e.g. [1, 3–5], parameter changes are considered that induce perturbations of the function F in (2). Hence, in these studies the first component of F is perturbed, which implies that the case where the discontinuity surface coincides with the set where the first element of F is zero is considered non-generic by these authors. This implies that the existence of an equilibrium set in (2) is non-generic. However, parameter changes for the specific system (1) will only yield perturbations of f and g in (2). Hence, for the class of systems under study, i.e. mechanical systems with set-valued friction, equilibrium sets will occur, generically. In this work, structural stability and bifurcations are studied for these systems, where f and g change due to parameter variations.

The functions f and g are considered to depend smoothly on system parameters. If the topological structure of the phase portrait of (1) does not change under small perturbations of f and g, then (1) is called structurally stable. This property excludes the possibility of bifurcations.

The following result gives conditions under which bifurcations can only occur near the endpoints of the equilibrium set $\mathcal{E}$, as proven in [6].

Theorem 1 *If $\frac{\partial g}{\partial y}\big|_p$ has no eigenvalue λ with real$(\lambda) = 0$ for any $p \in \mathcal{E}$, then local bifurcations of the equilibrium set of* (1) *only occur near the endpoints E_1 or E_2.*

This result significantly simplifies the further study of structural stability and bifurcations of equilibrium sets for this class of mechanical systems with friction.

For planar systems, the dynamics near the endpoints E_1 and E_2 are studied. In one of the smooth domains, the vector field has an equilibrium point at $E_k,\ k = 1, 2$, and its local dynamics is studied using $A_k := \frac{\partial F_k}{\partial q}\big|_{q=E_k},\ k = 1, 2$. In the opposite domain, the vector field is pointing towards Σ. Using these properties, the following result is proven in [6].

Theorem 2 *Consider a system given by* (1) *with $n = 2$. If $A_k, k = 1, 2$, have distinct, nonzero eigenvalues, then the equilibrium set of the system is structurally stable for perturbations in f.*

This result excludes the occurrence of bifurcations for most system parameters. If system parameters are such that the condition on A_k is violated, then bifurcations may occur. For example, consider the system $\ddot{x} \in a_{21}x + a_{22}\dot{x} - F_s\text{Sign}(\dot{x})$, where $F_s = 1$ and $a_{22} = -0.1$. For this system, the eigenvalues of $A_k,\ k = 1, 2$, become identical when $a_{21} = -0.0025$. At this point, a bifurcation occurs as shown in Figure 2. We will refer to this bifurcation as a focus-node bifurcation. At the bifurcation point, the stable manifolds of trajectories converging to $E_k,\ k = 1, 2$, collide, and subsequently disappear. We note that all trajectories converge to the equilibrium set in finite time for parameters a_{21} below the bifurcation value.

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