

# Wireless multihop networks with stealing : large buffer asymptotics via the ray method

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## WIRELESS MULTIHOP NETWORKS WITH STEALING: LARGE BUFFER ASYMPTOTICS VIA THE RAY METHOD\*

FABRICE GUILLEMIN<sup>†</sup>, CHARLES KNESSL<sup>‡</sup>, AND JOHAN S. H. VAN LEEUWAARDEN<sup>§</sup>

**Abstract.** Wireless networks equipped with CSMA are scheduled in a fully distributed manner. A disadvantage of such distributed control in multihop networks is the hidden node problem, which causes the effect of *stealing*, in which a downstream node steals the channel from an upstream node with probability  $p$ . Aziz, Starobinski, and Thiran [*IEEE/ACM Trans. Networking*, to appear] have recently shown that the  $N$ -hop model with stealing is stable only in the case  $N = 3$  and  $p \in (0, 1]$ . This 3-hop model can be represented as a random walk in the quarter plane. We derive various asymptotic expressions for the stationary large buffer probabilities of the 3-hop model that capture the effect of  $p$ . We mainly rely on the ray method, and we show how this method is related to the compensation approach and singularity analysis.

**Key words.** wireless networks, carrier-sense multiple-access, stealing, random walks in the quarter plane, queueing theory, rare events, asymptotic analysis, ray method

**AMS subject classifications.** 60K25, 68M10, 41A60

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**1. Introduction.** Efficient usage of wireless networks requires decentralized protocols for transmitting data. In wireless mesh networks, most medium access control (MAC) protocols use carrier-sense multiple-access (CSMA), and indeed CSMA schedules the capacity of the network in a distributed manner. The majority of protocols, however, have been developed for single-hop communication, while multihop scenarios, in which multiple links are used to transmit a packet from source to destination, are becoming ever more prominent.

CSMA builds upon the ALOHA protocol, in which nodes wait for some random backoff period before starting a transmission. The length of this backoff period is independent between nodes, ruling out a collision caused by multiple transmissions starting simultaneously. Nevertheless, a collision may occur when a node activates during another node's transmission. The CSMA protocol prescribes that nodes should sense their surroundings for other transmitting nodes. If a node detects a nearby transmitter (within its sensing range), the backoff timer is frozen, deferring the countdown until the channel is sensed clear. Using this mechanism, the number of collisions is greatly reduced. However, CSMA introduces the additional problem of hidden nodes (see [11]). Hidden nodes are not detected by the carrier-sensing mechanism but may still cause collisions. The hidden nodes give rise to the phenomenon of *stealing*, which can be best explained in the context of a multihop network with  $N + 1$  nodes. Node 0 (the source) always has traffic to send. This traffic is sent to node  $N$  (the destination) and will be relayed through nodes  $1, \dots, N - 1$ . Both the transmission range and

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the interference range are limited to nearest neighbors. We assume that the network operates under the CSMA protocol and that nodes have large buffers (which is usually the case). In fact, we assume infinite buffers, so that the network is lossless. Nonzero propagation delays and the hidden node problem will now give rise to the stealing effect.

The stealing effect occurs when a downstream node  $i + 2$  captures the channel from an upstream node  $i$ , even though it accesses the medium later. Denote the probability of the occurrence of stealing by  $p$ . Figure 1 gives an example of the hidden node problem and stealing for  $N = 3$ . Node 0 is transmitting to node 1, and hence node 1 is silenced. However, node 2 is not silenced, and when it decides to start transmitting to the destination node 3, this transmission will cause the ongoing transmission from node 0 to node 1 to fail.

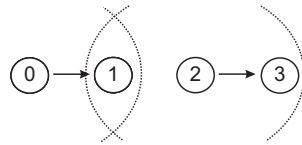


FIG. 1. *The hidden node problem.*

The assumption of infinite buffers (and no losses) gives rise to a potential stability problem, because the buffer contents may become infinite. A striking feature of stealing is that it can have a positive effect on stability. Indeed, in [4] it is shown that the 3-hop model without stealing ( $p = 0$ ) is unstable, while the 3-hop model is stable if the stealing phenomenon occurs (when  $p \in (0, 1]$ ). Hence, stealing has a positive influence in that it leads to a stable system. However, in [4] it is also shown that 4-hop networks are always unstable, even with stealing, and numerical evidence is provided that instability extends to larger networks. This makes the 3-hop network with stealing special, and highly relevant, as the only stable multihop network, particularly when one realizes that more complex topologies can be constructed out of such a linear 3-hop segments. We shall restrict our analysis to the 3-hop network and focus on the stationary distribution. In particular, we shall investigate the impact of the stealing parameter  $p$  on the large buffer asymptotics. Knowledge of the occurrence of such rare events provides structural insights into the behavior of the network and the impact of stealing, and are therefore of interest for dimensioning purposes.

Stealing refers to the event that node  $i + 2$  captures the channel despite having a larger backoff value than node  $i$ ; see [4]. Stealing will be absent ( $p = 0$ ) only in the unrealistic scenario where control messages are instantaneous, while the stealing will be perfect ( $p = 1$ ) if a low transmission rate is used and the control messages are disabled. We shall consider both extreme scenarios. The true value for  $p$  depends on the protocol and messaging and lies in the interval  $(0, 1)$ ; see [4, 10].

We proceed as follows. In section 2 we model the 3-hop network as a two-dimensional Markov chain and show that it falls into the class of *random walks in the quarter plane*. We also provide simulation results that demonstrate the crucial impact that the stealing parameter  $p$  has on the evolution of the buffer contents. We then present our main findings in section 3, where we start in section 3.1 with *perfect stealing* ( $p = 1$ ). In that case, the two-dimensional Markov chain has a surprisingly simple product form solution. In section 3.2 we consider the situation with a *small stealing effect* ( $p \rightarrow 0$ ). For this, we identify a scaling limit of the Markov chain. This scaling limit turns out to be a continuous process in the positive quadrant that

admits a simple product form solution for its stationary density. In section 3.3 we present our key results that give the joint large buffer asymptotics for all  $p \in (0, 1)$ . In section 4 we present some conclusions.

All proofs are relegated to later sections. In proving the joint asymptotics in section 7, we shall use the ray method of geometrical optics. While this will not yield the complete asymptotics, we will show that much of the basic structure can be inferred solely from the balance equations. The singular perturbation approach does make certain assumptions about the forms of the asymptotic expansions and the asymptotic matching between different scales. We will also relate our approach to the compensation approach [1, 3] and to singularity analysis [7, 8, 9].

**2. The 3-hop network.** In this section we shall describe how the 3-hop network can be modeled as a two-dimensional Markov chain, following [4]. Consider a linear topology of four nodes. All packets are generated by node 0 and forwarded to node 3 by successive transmissions via the intermediate nodes 1 and 2 (see Figure 1). We assume that node 0 is saturated, so that there are always packets to be transmitted. The interference range of each node is taken to be nearest neighbors, which for this 3-hop network implies that only one of the first three nodes can be active in a given time slot. Each node is assumed to have an infinite buffer for storage of data packets.

In the case of idealized CSMA without stealing (see [4]), all nonempty nodes have exactly the same probability of being scheduled. When all three nodes have nonempty buffers, each node has a success probability of  $1/3$ . When only nodes 0 and 1 (or nodes 0 and 2) are nonempty, their success probabilities are  $1/2$ , and when nodes 1 and 2 are both empty, a packet will be transmitted from node 0 to node 1 with probability 1. The stealing effect describes a bias towards the downstream node 2. When only node 0 and 1 compete for the channel, the success probability of each node remains  $1/2$ . However, when node 0 and 2 compete together, there is a probability  $p$  that node 2 steals the channel.

Denoting by  $N_k(\tau)$  the number of packets at node  $k$  at the beginning of time slot  $\tau$ , the two-dimensional process  $\{(N_1(\tau), N_2(\tau)), \tau \in \mathbb{N}\}$  is a two-dimensional Markov chain, and in fact a random walk in the quarter plane, as depicted in Figure 2.

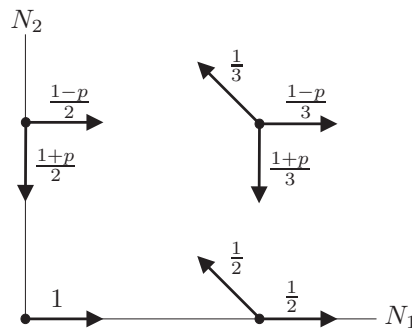


FIG. 2. *The random walk in the quarter plane.*

The random walk takes the following steps:

- In the interior of the state space, the walk steps  $(1, 0)$  w.p.  $\frac{1-p}{3}$ ,  $(0, -1)$  w.p.  $\frac{1+p}{3}$ , and  $(-1, 1)$  w.p.  $\frac{1}{3}$ .
- On the horizontal axis, the walk steps  $(1, 0)$  w.p.  $\frac{1}{2}$  and  $(-1, 1)$  w.p.  $\frac{1}{2}$ .
- On the vertical axis, the walk steps  $(1, 0)$  w.p.  $\frac{1-p}{2}$  and  $(0, -1)$  w.p.  $\frac{1+p}{2}$ .

The stability condition under which this random walk is ergodic, and thus has a unique stationary distribution, was proved in [4] using Foster’s criterion and a Lyapunov function, as follows.

PROPOSITION 1 (after Aziz, Starobinski, and Thiran [4]). *The random walk  $\{(N_1(\tau), N_2(\tau)), \tau \in \mathbb{N}\}$  is ergodic if and only if  $p \in (0, 1]$ .*

Denote the joint stationary probabilities by  $\pi(n, k) = \mathbb{P}(N_1 = n, N_2 = k) = \lim_{\tau \rightarrow \infty} \mathbb{P}(N_1(\tau) = n, N_2(\tau) = k)$ . The balance equations for the interior of the state space read

$$(2.1) \quad \pi(n, k) = \frac{1}{3}\pi(n + 1, k - 1) + \frac{1 - p}{3}\pi(n - 1, k) + \frac{1 + p}{3}\pi(n, k + 1), \quad n, k \geq 2,$$

$$(2.2) \quad \pi(n, 1) = \frac{1}{2}\pi(n + 1, 0) + \frac{1 - p}{3}\pi(n - 1, 1) + \frac{1 + p}{3}\pi(n, 2), \quad n \geq 2,$$

$$(2.3) \quad \pi(1, k) = \frac{1}{3}\pi(2, k - 1) + \frac{1 - p}{2}\pi(0, k) + \frac{1 + p}{3}\pi(1, k + 1), \quad k \geq 2,$$

while on the boundaries we have

$$(2.4) \quad \pi(n, 0) = \frac{1 + p}{3}\pi(n, 1) + \frac{1}{2}\pi(n - 1, 0), \quad n \geq 2,$$

$$(2.5) \quad \pi(0, k) = \frac{1}{3}\pi(1, k - 1) + \frac{1 + p}{2}\pi(0, k + 1), \quad k \geq 2,$$

and

$$(2.6) \quad \pi(1, 0) = \frac{1 + p}{3}\pi(1, 1) + \pi(0, 0),$$

$$(2.7) \quad \pi(0, 1) = \frac{1}{2}\pi(1, 0) + \frac{1 + p}{2}\pi(0, 2),$$

$$(2.8) \quad \pi(0, 0) = \frac{1 + p}{2}\pi(0, 1).$$

Let  $P(x, y)$  represent the bivariate generating function

$$P(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{P}(N_1 = n, N_2 = k) x^n y^k.$$

From the balance equations it follows that  $P(x, y)$  satisfies the functional equation

$$(2.9) \quad h_1(x, y)P(x, y) = h_2(x, y)P(x, 0) + h_3(x, y)P(0, y) + h_4(x, y)P(0, 0),$$

where

$$(2.10) \quad h_1(x, y) = 6xy - 2(1 - p)x^2y - 2y^2 - 2(1 + p)x,$$

$$(2.11) \quad h_2(x, y) = (1 + 2p)x^2y + y^2 - 2(1 + p)x,$$

$$(2.12) \quad h_3(x, y) = (1 - p)x^2y - 2y^2 + (1 + p)x,$$

$$(2.13) \quad h_4(x, y) = (2 + p)x^2y - y^2 - (1 + p)x.$$

See [6, section 1.3] for a general description of how to derive such functional equations. The functions  $h_j$  are quadratic polynomials in  $x$  or  $y$ . Equation (2.9) cannot be solved directly for  $P(x, y)$ , because it contains other unknown functions  $P(x, 0)$  and  $P(0, y)$ . The approach is then to consider the roots of the kernel  $h_1(x, y)$  w.r.t. one

of the variables  $x, y$ . Substituting such roots into (2.9) yields additional equations between the unknowns  $P(x, 0)$  and  $P(0, y)$  that are free of the term containing the full generating function  $P(x, y)$ . These additional equations in fact give rise to boundary value problems whose solutions lead to a specification of  $P(x, 0)$  and  $P(0, y)$  and hence  $P(x, y)$ . Hence, stationary distributions of two-dimensional one-step random walks in the quarter plane can be obtained by solving functional equations (see [5, 6]). For the present model this will be done in a planned companion paper.

**2.1. The effect of stealing.** Let us now consider the effect of stealing on the evolution of the buffer contents of nodes 1 and 2. Results from our simulation are displayed in Figure 3. Several interesting observations can be made. First, the sizes of both buffers decrease with  $p$ . The case  $p = 0$  is critical, leading to large fluctuations. As the stealing becomes more dominant, the buffer contents decrease. Second, the buffer of node 1 seems to be roughly twice as large as the buffer of node 2. Third, the regeneration periods, defined as the periods between two consecutive times that the buffer empties, tend to become shorter as  $p$  increases.

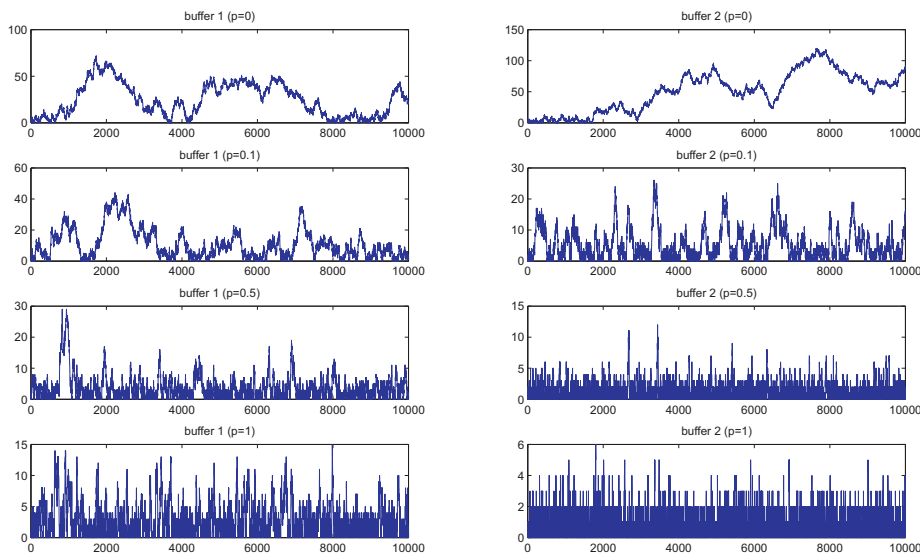


FIG. 3. Buffer evolution for the 3-hop model for 10,000 slots and various stealing probabilities  $p$ .

In this paper we shall focus on the occurrence of large buffer contents, which can be considered as *rare events*. We will derive asymptotic expressions for the probabilities of large buffer contents. It is clear from Figure 3 that large buffer contents become more likely as  $p$  becomes smaller, and that large buffer contents are more likely to occur at node 1 than at node 2. The latter is true because the stealing effect favors the downstream node. To substantiate these numerical observations, we shall derive asymptotic expressions for the probabilities that the buffer contents become large.

We will obtain the asymptotic results directly from the balance equation in (2.1)–(2.8) by using the so-called ray method. In a companion paper we take a different approach, making use of the exact solution to the functional equation (2.9). Solving the functional equation requires sophisticated complex analysis, and the obtained

formal solution is too complicated to invert directly for the stationary distribution. That is why in the companion paper we will employ the functional equation (2.9) to determine the dominant (closest to the origin) singularities of the function  $P(x, y)$ . Subsequently, by investigating  $P(x, y)$  in the neighborhood of these dominant singularities, and by the asymptotic evaluation of complex integrals, we obtain asymptotic expressions for the tail of the probability distribution of  $(N_1, N_2)$ . This approach is sketched in section 7.1.

**3. Main results.** In this section we shall present results for the case of perfect stealing ( $p = 1$ ), the case of a small stealing effect ( $p \approx 0$ ), and then the general case  $p \in (0, 1)$ .

**3.1. Perfect stealing.** For the case of perfect stealing, where  $p = 1$  and hence node 2 always steals the channel, the random walk becomes more tractable. Visual inspection of Figure 2 tells us that the walk can no longer step to the East in the interior of the state space. As it turns out, the random walk allows for a simple product form solution for its stationary distribution.

**THEOREM 1.** *For the case  $p = 1$  the stationary distribution of the random walk has a closed-form solution with  $\pi(0, 0) = \pi(0, 1) = (2 - \sqrt{2})/6$ ,  $\pi(1, 0) = (\sqrt{2} - 1)/3$ , and*

$$(3.1) \quad \pi(n, k) = \left(\frac{1}{\sqrt{2}}\right)^n \left(1 - \frac{1}{\sqrt{2}}\right)^{k+1}, \quad n, k \geq 1,$$

$$(3.2) \quad \pi(n, 0) = \frac{2}{3} \left(1 - \frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}\right)^n, \quad n \geq 1,$$

$$(3.3) \quad \pi(0, k) = \frac{1}{3} \left(1 - \frac{1}{\sqrt{2}}\right)^k, \quad k \geq 1.$$

The marginal distribution follows from Theorem 1, as given next.

**COROLLARY 1.** *For the case  $p = 1$  we have the marginal distributions*

$$(3.4) \quad \mathbb{P}(N_1 = n) = \frac{7\sqrt{2} - 8}{6} \left(\frac{1}{\sqrt{2}}\right)^n, \quad n \geq 1,$$

$$(3.5) \quad \mathbb{P}(N_2 = k) = \left(\frac{1}{3} + \frac{1}{\sqrt{2}}\right) \left(1 - \frac{1}{\sqrt{2}}\right)^k, \quad k \geq 1,$$

$\mathbb{P}(N_1 = 0) = \sqrt{2}/6$ , and  $\mathbb{P}(N_2 = 0) = (2 + \sqrt{2})/6$ .

Note that  $\mathbb{P}(N_2 = k)$  decays faster than  $\mathbb{P}(N_1 = n)$ , which means that large buffer contents are more likely to occur at node 1.

The exact solution in Theorem 1 can alternatively be formulated in terms of the generating functions

$$(3.6) \quad P(x, y) = \frac{2}{3} \frac{2 - \sqrt{2} + (\sqrt{2} - 1)(x + y) - (3 - 2\sqrt{2})xy}{(2 - \sqrt{2}x)(2 - (2 - \sqrt{2})y)}$$

and

$$(3.7) \quad P(x, 0) = \frac{2 - \sqrt{2} + (\sqrt{2} - 1)x}{6 - 3\sqrt{2}x},$$

$$(3.8) \quad P(0, y) = \frac{2 - \sqrt{2} + (\sqrt{2} - 1)y}{6 - 3(2 - \sqrt{2})y},$$

and it is straightforward to check that these functions satisfy the functional equation (2.9). In the companion paper we show how to obtain, in a constructive manner, (3.6) as the solution to (2.9).

**3.2. Small stealing effect.** We now turn to the case  $p \approx 0$ , so that the probability that node 2 steals the channel is small. Using the right space-time scaling, we show that our random walk in the quarter plane converges to a continuous scaling limit whose stationary density obeys a PDE with a simple product form solution. The result is as follows.

**THEOREM 2.** *For the case  $p \rightarrow 0$ , the stationary distribution of the random walk can be approximated as*

$$(3.9) \quad \pi(n, k) \approx p^2 \mathcal{G}(pn, pk), \quad n, k \geq 1,$$

$$(3.10) \quad \pi(n, 0) \approx \frac{2}{3} p^2 \mathcal{G}(pn, 0), \quad n \geq 1,$$

$$(3.11) \quad \pi(0, k) \approx \frac{2}{3} p^2 \mathcal{G}(0, pk), \quad k \geq 1,$$

$$(3.12) \quad \pi(0, 0) \approx \frac{1}{3} p^2 \mathcal{G}(0, 0),$$

where the density  $\mathcal{G}(X, Y)$  satisfies the PDE

$$(3.13) \quad \mathcal{G}_{XX} + \mathcal{G}_{YY} - \mathcal{G}_{XY} + \mathcal{G}_X + \mathcal{G}_Y = 0, \quad X, Y > 0,$$

with the boundary conditions

$$(3.14) \quad \mathcal{G}_X(0, Y) + \mathcal{G}(0, Y) = 0,$$

$$(3.15) \quad \mathcal{G}_X(X, 0) - \mathcal{G}_Y(X, 0) - \mathcal{G}(X, 0) = 0,$$

and with normalized solution

$$(3.16) \quad \mathcal{G}(X, Y) = 2e^{-X}e^{-2Y}.$$

The marginal distributions then readily follow, as given next.

**COROLLARY 2.** *For  $p \rightarrow 0$  we have the approximative marginal distributions*

$$(3.17) \quad \mathbb{P}(N_1 = n) \approx pe^{-pn}, \quad n \geq 1,$$

$$(3.18) \quad \mathbb{P}(N_2 = k) \approx 2pe^{-2pk}, \quad k \geq 1,$$

$\mathbb{P}(N_1 = 0) \approx \frac{2}{3}p$ , and  $\mathbb{P}(N_2 = 0) \approx \frac{4}{3}p$ .

Note that  $\mathbb{P}(N_2 = k)$  decays precisely twice as fast as  $\mathbb{P}(N_1 = n)$ . Theorem 2 and Corollary 2 provide approximations for the stationary distribution of the random walk that are sharp for small values of  $p$ . An example is given in Table 1. The “true” values for  $\mathbb{P}(N_1 = n)$  and  $\mathbb{P}(N_2 = n)$  that are reported in Table 1 (and in all other tables in this paper) are obtained by numerical calculations. Truncating the state space by imposing an upper bound on one of the buffers reduces the random walk in the quarter plane to a random walk on an infinite strip, better known as a quasi-birth-death (QBD) chain. For these Markov chains, fast numerical algorithms are available (see [13]). All numerical results presented were obtained by imposing an upper bound on the second buffer of 500 (which is large enough in order not to see truncation errors in the reported numerical results).



TABLE 1  
Marginal distributions and their approximations in (3.17) and (3.18) for  $p = 0.01$ .

$n$	$\mathbb{P}(N_1 = n)$	$p \exp(-pn)$	$\mathbb{P}(N_2 = k)$	$2p \exp(-2pk)$
5.0000e+00	9.3641e-03	9.5123e-03	1.7892e-02	1.8097e-02
1.0000e+01	8.9136e-03	9.0484e-03	1.6211e-02	1.6375e-02
1.5000e+01	8.4849e-03	8.6071e-03	1.4686e-02	1.4816e-02
2.0000e+01	8.0769e-03	8.1873e-03	1.3302e-02	1.3406e-02
5.0000e+01	6.0099e-03	6.0653e-03	7.3397e-03	7.3576e-03
1.0000e+02	3.6722e-03	3.6788e-03	2.7198e-03	2.7067e-03

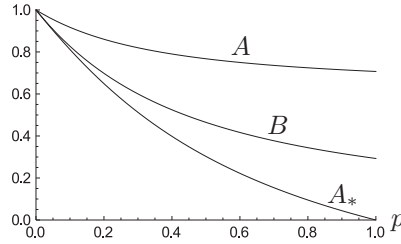


FIG. 4. The decay rates  $A$ ,  $B$ , and  $A_*$  for  $p = (0, 1]$ .

**3.3. Joint asymptotics.** We now present results for the asymptotics of the stationary probabilities  $\pi(n, k)$ , for a fixed  $0 < p < 1$  and  $n$  and/or  $k \rightarrow \infty$ . The results presented will be proved in section 7 using a singular perturbation approach for approximating the difference equations (2.1)–(2.5), and in particular the ray method of geometrical optics.

**THEOREM 3.** For  $n \rightarrow \infty$  and/or  $k \rightarrow \infty$  there holds

$$(3.19) \quad \pi(n, k) \sim C \cdot B^k (A^n + \gamma A_*^n), \quad n, k \geq 1,$$

$$(3.20) \quad \pi(n, 0) \sim C \cdot \frac{2}{3} \cdot A^n,$$

$$(3.21) \quad \pi(0, k) \sim C \cdot \frac{2}{3} \cdot (1 + \gamma) B^k,$$

with  $C$  a constant, and

$$(3.22) \quad A = \frac{1 - p + \sqrt{1 + 2p + 5p^2}}{2(1 + p)},$$

$$(3.23) \quad B = \frac{1 + 3p - \sqrt{1 + 2p + 5p^2}}{2p(1 + p)},$$

$$(3.24) \quad A_* = \frac{(1 - p)(-1 - p + \sqrt{1 + 2p + 5p^2})}{2p^2},$$

$$(3.25) \quad \gamma = \frac{2p^2 - 1 - p + \sqrt{1 + 2p + 5p^2}}{2p(1 + p)}.$$

The constant  $C = C(p)$  satisfies  $C(1) = 1 - 1/\sqrt{2}$  and  $C(p) \sim 2p^2$  as  $p \rightarrow 0$ .

It is readily seen that  $A > B > A_*$  for all  $p \in (0, 1]$ , which again confirms the fact that large buffers are more likely at node 1. From (3.22) and (3.23) we further see that  $A = 1/\sqrt{2}$  and  $B = 1 - 1/\sqrt{2}$  when  $p = 1$ , which agrees with Theorem 1. The functions  $A$ ,  $B$ , and  $A_*$  are plotted in Figure 4.

TABLE 2  
Marginal distribution of  $N_1$  for  $p = 0.3$ .

$n$	$\mathbb{P}(N_1 = n)$	$A^n$	$\mathbb{P}(N_1 = n)A^{-n}$
5.0000e+00	7.0114e-02	3.7055e-01	1.8922e-01
1.0000e+01	2.5870e-02	1.3731e-01	1.8841e-01
1.5000e+01	9.5836e-03	5.0879e-02	1.8836e-01
2.0000e+01	3.5511e-03	1.8853e-02	1.8835e-01
5.0000e+01	9.1928e-06	4.8806e-05	1.8835e-01
1.0000e+02	4.4866e-10	2.3820e-09	1.8835e-01

TABLE 3  
Marginal distribution of  $N_1$  for  $p = 0.9$ .

$n$	$\mathbb{P}(N_1 = n)$	$A^n$	$\mathbb{P}(N_1 = n)A^{-n}$
5.0000e+00	5.7326e-02	1.8695e-01	3.0663e-01
1.0000e+01	1.0717e-02	3.4951e-02	3.0663e-01
1.5000e+01	2.0036e-03	6.5342e-03	3.0663e-01
2.0000e+01	3.7458e-04	1.2216e-03	3.0663e-01
5.0000e+01	1.5993e-08	5.2156e-08	3.0663e-01
1.0000e+02	8.3412e-16	2.7202e-15	3.0663e-01

TABLE 4  
Marginal distribution of  $N_2$  for  $p = 0.3$ .

$k$	$\mathbb{P}(N_2 = k)$	$B^k$	$\mathbb{P}(N_2 = k)B^{-k}$
5.0000e+00	3.8761e-02	7.7941e-02	4.9731e-01
1.0000e+01	3.0864e-03	6.0748e-03	5.0806e-01
1.5000e+01	2.4208e-04	4.7348e-04	5.1128e-01
2.0000e+01	1.8909e-05	3.6904e-05	5.1238e-01
5.0000e+01	4.2444e-12	8.2731e-12	5.1304e-01
1.0000e+02	3.5115e-23	6.8445e-23	5.1304e-01

TABLE 5  
Marginal distribution of  $N_2$  for  $p = 0.9$ .

$k$	$\mathbb{P}(N_2 = k)$	$B^k$	$\mathbb{P}(N_2 = k)B^{-k}$
5.0000e+00	3.1621e-03	3.1806e-03	9.9419e-01
1.0000e+01	1.0057e-05	1.0116e-05	9.9419e-01
1.5000e+01	3.1989e-08	3.2176e-08	9.9419e-01
2.0000e+01	1.0174e-10	1.0234e-10	9.9419e-01
5.0000e+01	1.0533e-25	1.0595e-25	9.9419e-01
1.0000e+02	1.1159e-50	1.1225e-50	9.9419e-01

Some further calculations lead to the following asymptotic result for the marginals.

COROLLARY 3. *For the marginal distributions we find the asymptotic relations*

$$(3.26) \quad \mathbb{P}(N_1 = n) \sim C \left( \frac{2}{3} + \frac{B}{1-B} \right) A^n, \quad n \rightarrow \infty,$$

$$(3.27) \quad \mathbb{P}(N_2 = k) \sim C \left( \frac{2}{3}(1 + \gamma) + \frac{A}{1-A} + \gamma \frac{A_*}{1-A_*} \right) B^k, \quad k \rightarrow \infty.$$

Several results that test the accuracy of (3.26) and (3.27) are provided in Tables 2–5. We see that in all cases the asymptotics kick in fast, which suggests that the asymptotic relations provide sharp approximations to the true values, even for small or moderate values of the buffer content. From the results in Tables 2–5 one can easily infer numerical values of  $C$ . Also, by comparing Tables 2 and 4 and Tables 3 and 5,

one can check that

$$(3.28) \quad \frac{\mathbb{P}(N_1 = n)A^{-n}}{\mathbb{P}(N_2 = n)B^{-n}} \rightarrow \frac{\frac{2}{3} + \frac{B}{1-B}}{\frac{2}{3}(1 + \gamma) + \frac{A}{1-A} + \gamma \frac{A_*}{1-A_*}}, \quad n \rightarrow \infty.$$

We next give some of the higher order terms in the asymptotic expansion of  $\pi(n, k)$ , which show the ultimate deviation from product form behavior.

**THEOREM 4.** *For  $n, k \rightarrow \infty$ , we define the ratio  $R = k/n$ . Then for  $0 < R < \infty$  we have the following:*

(i) *If  $0 < p < (5 - \sqrt{17})/2$ ,  $R > \mathcal{S}_1(p)$ ,*

$$(3.29) \quad \pi(n, k) \sim C \left( A^n B^k + \gamma A_*^n B^k + \frac{1}{\sqrt{n}} f(R) [a(R)]^n [b(R)]^k \right),$$

with

$$(3.30) \quad \mathcal{S}_1(p) = \frac{4p^2 - 11p - 1 + 3\sqrt{1 + 2p + 5p^2}}{2(1 + 2p - 4p^2)},$$

and  $a$  and  $b$  are obtained from solving the system of equations

$$(3.31) \quad 3 - \frac{1-p}{a} = \frac{a}{b} + (1+p)b,$$

$$(3.32) \quad R \left[ \frac{1-p}{a} - \frac{a}{b} \right] = \frac{a}{b} - (1+p)b.$$

(ii) *If  $0 < p < (5 - \sqrt{17})/2$ ,  $0 < R < \mathcal{S}_1(p)$ ,*

$$(3.33) \quad \pi(n, k) \sim C \left( A^n B^k + \frac{1}{\sqrt{n}} f(R) [a(R)]^n [b(R)]^k \right).$$

(iii) *If  $(5 - \sqrt{17})/2 < p < 1$ ,  $R > \mathcal{S}_2(p)$ , with*

$$(3.34) \quad \mathcal{S}_2(p) = \frac{2p^2 - p - 5 + 3\sqrt{1 + 2p + 5p^2}}{2(2 + 4p + p^2)},$$

then there is the asymptotics in (3.29).

(iv) *If  $(5 - \sqrt{17})/2 < p < 1$ ,  $0 < R < \mathcal{S}_2(p)$ ,*

$$(3.35) \quad \pi(n, k) \sim C \left( A^n B^k + \gamma A_*^n B^k + \delta A_*^n B_*^k + \frac{1}{\sqrt{n}} f(R) [a(R)]^n [b(R)]^k \right)$$

with

$$(3.36) \quad B_* = \frac{(1-p)(p-1 + \sqrt{1 + 2p + 5p^2})}{2p(1+p)},$$

$$(3.37) \quad \delta = \frac{1 + 2p^2 + p^3 - 4p^4 + (p^2 + p - 1)\sqrt{1 + 2p + 5p^2}}{2p(p^4 - p^2 - p - 1)}.$$

We ordered the terms on the right-hand sides for  $\pi$  in Theorem 4 so that each term is exponentially smaller than the term that precedes it ( $A_* < A$  and  $B_* < B$ ). We cannot obtain the function  $f(R)$  explicitly without having the full solution for  $\pi(n, k)$ .

Later we shall discuss the behavior of  $f(R)$  for  $R \rightarrow 0$ ,  $R \rightarrow \infty$ , and  $R \rightarrow \mathcal{S}_j(p)$ . We can recast the systems of equations for  $(a, b)$  as

$$(3.38) \quad \begin{aligned} 0 &= (1+p)(R+2)^2 a^3 - 9(R+1)a^2 \\ &\quad + 3(1-p)(2+2R-R^2)a + (1-p)(R-1)(2R+1), \end{aligned}$$

$$(3.39) \quad b = \frac{3(R+1) - (1-p)(2R+1)a^{-1}}{(1+p)(R+2)}.$$

Thus, given  $R = k/n$  and  $p$ , we must solve the above cubic equation (3.38) for  $a$ , and then compute  $b$  from (3.39). When  $p = (5 - \sqrt{17})/2$  we have  $a(0) = A_* = (7 - \sqrt{17})/8$  (and then  $b = B = B_*$ ), when  $R = \mathcal{S}_1(p)$  we have  $a = A_*$  and  $b = B$ , and when  $R = \mathcal{S}_2(p)$  we have  $a = A_*$  and  $b = B_*$ . Also, when  $p = (5 - \sqrt{17})/2$ ,  $\mathcal{S}_1 = \mathcal{S}_2 = 0$ .

The formulas in Theorem 4 must be slightly modified if  $k = O(1)$ ,  $n \rightarrow \infty$ ;  $n = O(1)$ ,  $k \rightarrow \infty$  or  $R \approx \mathcal{S}_j(p)$ ,  $j = 1, 2$ . We thus see that a fair amount of asymptotic information can be obtained without solving fully for  $\pi(n, k)$ , and that the solution is a mixture of three exponentials and the term  $a^n b^k$ , which is geometric for each fixed  $R = k/n$  but not globally geometric. This term represents the second term in the asymptotic expansion of  $\pi$  for case (ii), the third term for the cases (i) and (iii), and only the fourth term for case (iv).

**4. Conclusions.** In this paper we investigated a 3-hop wireless network that is subject to *stealing* (a downstream node captures the channel from an upstream node with probability  $p$ ). This network can be modeled as a random walk in the quarter plane. The latter is a much studied object in probability theory, giving rise to all kinds of interesting mathematics. For the random walk at hand, we derived exact and asymptotic expressions for probabilities that the buffer contents become large. Most of these results were obtained using the ray method, which might prove useful for the asymptotic analysis of other random walks in the quarter plane as well.

For the case of perfect stealing ( $p = 1$ ) we showed that the model possesses a product form solution. For the case  $p \downarrow 0$  we derived a continuous scaling limit whose density obeys a PDE with again a simple product form solution. The exact solutions of both extreme cases can be used to obtain sharp approximation for small and large values of  $p$ .

For the general case  $0 < p < 1$  our asymptotic expressions showed that large buffer contents are more likely to occur at the first relay node. This also explains why the Lyapunov function used in [4] to prove stability for the case  $p \in (0, 1]$  is mainly a function of the buffer contents of the first node. In [4] was also shown that 4-hop networks are always unstable, even with stealing. Nevertheless, it might be possible to obtain insight into how and at what speed the buffer contents grow to infinity.

**5. Proof of Theorem 1.** For the case  $p = 1$  the balance equations for  $n \geq 2$  read

$$(5.1) \quad \pi(n, k) = \frac{1}{3}\pi(n+1, k-1) + \frac{2}{3}\pi(n, k+1), \quad k \geq 2,$$

$$(5.2) \quad \pi(n, 1) = \frac{1}{2}\pi(n+1, 0) + \frac{2}{3}\pi(n, 2),$$

$$(5.3) \quad \pi(n, 0) = \frac{2}{3}\pi(n, 1) + \frac{1}{2}\pi(n-1, 0).$$

We now largely follow the approach in Adan and Weiss [2] for a related model. First we substitute a trial solution  $\pi(n, k) = C \cdot \alpha^n \beta^k$  and  $\pi(n, 0) = C \cdot d \cdot \alpha^n$  into (5.1) and

(5.2), and then divide by  $\alpha^n \beta^{k-1}$  to obtain

$$(5.4) \quad \beta = \frac{1}{3}\alpha + \frac{2}{3}\beta^2,$$

$$(5.5) \quad \beta = \frac{1}{2}d\alpha + \frac{2}{3}\beta^2,$$

and hence  $d = \frac{2}{3}$ . Substituting the same trial solution into (5.3) yields (upon some rewriting)

$$(5.6) \quad \alpha = \alpha\beta + \frac{1}{2}.$$

Note that it follows from (5.6) and  $\alpha < 1$  that  $\beta < \frac{1}{2}$ . Combining (5.4) and (5.6) gives an equation for  $\beta$ ,

$$(5.7) \quad 2\beta^3 - 5\beta^2 + 3\beta - \frac{1}{2} = 0,$$

with solutions  $\frac{1}{2}(2 - \sqrt{2})$ ,  $\frac{1}{2}$ ,  $\frac{1}{2}(2 + \sqrt{2})$ . Therefore, the values of  $\alpha$  and  $\beta$  that lead to a convergent solution of the stationary distribution are given by

$$(5.8) \quad \alpha = \frac{1}{\sqrt{2}}, \quad \beta = 1 - \frac{1}{\sqrt{2}}.$$

We then match this trial solution with the remaining balance equations:

$$(5.9) \quad \pi(0, k) = \frac{1}{3}\pi(1, k - 1) + \pi(0, k + 1), \quad k \geq 2,$$

$$(5.10) \quad \pi(0, 1) = \frac{1}{2}\pi(1, 0) + \pi(0, 2),$$

$$(5.11) \quad \pi(1, 0) = \frac{2}{3}\pi(1, 1) + \pi(0, 0).$$

Substituting the trial solution  $\pi(0, k) = C \cdot \gamma \cdot \beta^k$  into (5.9) yields  $\gamma\beta = \gamma\beta^2 + \frac{1}{3}\alpha$ , and hence (with  $\alpha, \beta$  as in (5.8))

$$\gamma = \frac{\alpha}{3\beta(1 - \beta)} = \frac{2}{6 - 3\sqrt{2}}.$$

Combining (5.10), (5.11), and  $\pi(0, 0) = \pi(0, 1)$  yields  $\pi(0, 0) = \frac{1}{3}C$ . Summing over all probabilities identifies the normalizing constant to be  $C = 1 - \frac{1}{\sqrt{2}}$ , which completes the proof.

**6. Proof of Theorem 2.** We start by scaling the state variables as

$$(6.1) \quad n = \frac{X}{p}, \quad k = \frac{Y}{p}, \quad \pi(n, k; p) = p^2 \mathcal{G}(X, Y; p).$$

The main balance equation (2.1) then becomes

$$(6.2) \quad \begin{aligned} \mathcal{G}(X, Y; p) &= \frac{1}{3}\mathcal{G}(X + p, Y - p; p) + \frac{1-p}{3}\mathcal{G}(X - p, Y; p) \\ &+ \frac{1+p}{3}\mathcal{G}(X, Y + p; p). \end{aligned}$$

Expanding the right-hand side of (6.2) for  $p \rightarrow 0$  leads to

$$(6.3) \quad \mathcal{G} = \mathcal{G} + p \left[ \frac{1}{3}(\partial_X - \partial_Y) - \frac{1}{3}\partial_X + \frac{1}{3}\partial_Y \right] \mathcal{G} \\ + p^2 \left[ \frac{1}{6}(\partial_X^2 - 2\partial_X\partial_Y + \partial_Y^2) + \frac{1}{3}\partial_X + \frac{1}{6}\partial_X^2 + \frac{1}{3}\partial_Y + \frac{1}{6}\partial_Y^2 \right] \mathcal{G} + O(p^3),$$

so that if  $\mathcal{G}(X, Y; p) \sim \mathcal{G}(X, Y)$  as  $p \rightarrow 0$ , the latter function must satisfy the PDE in (3.13) for  $X, Y > 0$ .

We next obtain boundary conditions along  $X = 0$  and  $Y = 0$ . In view of the balance equations (2.2)–(2.5), we must allow for a discontinuity along  $k = 0$  and  $n = 0$ , so that the approximation  $\pi \sim p^2\mathcal{G}$  in (6.1) will apply only for  $n, k \geq 1$ . Along  $k = 0$  and  $n = 0$  let us set

$$(6.4) \quad \pi(n, 0) \sim p^2\mathcal{Q}(x), \quad \pi(0, k) \sim p^2\mathcal{R}(Y).$$

Letting  $p \rightarrow 0$  in (2.4) then yields

$$(6.5) \quad \frac{2}{3}\mathcal{G}(X, 0) = \mathcal{Q}(X),$$

while (2.5) gives

$$(6.6) \quad \frac{2}{3}\mathcal{G}(0, Y) = \mathcal{R}(Y).$$

This relates the boundary values of the diffusion approximation  $\mathcal{G}(X, Y)$  to the actual discrete boundary probabilities  $\pi(n, 0)$  and  $\pi(0, k)$ . By writing (2.2) in terms of  $(X, Y)$  and expanding for small  $p$ , we find to leading order that  $\mathcal{G}(X, 0) = \frac{2}{3}\mathcal{G}(X, 0) + \frac{1}{2}\mathcal{Q}(X)$ , which is consistent with (6.5). At the next order  $O(p)$  we find that  $0 = \frac{1}{2}\mathcal{Q}'(X) - \frac{1}{3}\mathcal{G}_X(X, 0)$ , which is also consistent with (6.5) but provides no new information. In terms of  $(X, Y)$ , (2.4) becomes

$$(6.7) \quad \mathcal{Q}(X) = \frac{1}{2}\mathcal{Q}(X - p) + \frac{1+p}{3}\mathcal{G}(X, p),$$

which leads to (6.5) to leading order in  $p$ , and at the next order we find  $0 = -\frac{1}{2}\mathcal{Q}'(X) + \frac{1}{3}[\mathcal{G}(X, 0) + \mathcal{G}_Y(X, 0)]$ , which leads to the boundary condition  $\mathcal{G}_X(X, 0) = \mathcal{G}_Y(X, 0) + \mathcal{G}(X, 0)$  in Theorem 2.

An alternate way of deriving the boundary condition is to solve (2.4) for  $\pi(n, 0)$ , which gives

$$(6.8) \quad \pi(n, 0) = 2^{1-n}\pi(1, 0) + \frac{1}{3}(1+p) \sum_{l=2}^n 2^{l-n}\pi(l, 1).$$

Then we can use (6.8) to eliminate  $\pi(n+1, 0)$  in (2.2), and to obtain

$$(6.9) \quad \mathcal{G}(X, p) = \frac{1}{3}(1+p)\mathcal{G}(X, 2p) + \frac{1}{3}(1-p)\mathcal{G}(X-p, p) \\ + 2^{-n-1}\pi(1, 0) + \frac{1}{6}(1+p) \sum_{j=0}^{n-1} 2^{-j}\mathcal{G}(X + (1-j)p, p).$$

Expanding (6.9) for  $p \rightarrow 0$  to leading order yields simply  $\mathcal{G}(X, 0) = (\frac{1}{3} + \frac{1}{3} + \frac{1}{3})\mathcal{G}(X, 0)$ , but at the next order we regain the boundary condition along  $Y = 0$ . Note that  $2^{-n}$  is exponentially small for  $X = pn > 0$ .

The boundary condition along  $X = 0$  can be derived similarly. By expanding (2.3) for  $p \rightarrow 0$ , we regain (6.6) at the leading order, and at  $O(p)$  the equation is automatically satisfied. However, if we expand (2.5), we obtain (6.6) to leading order and then, at  $O(p)$ , the boundary condition  $\mathcal{G}_X(0, Y) + \mathcal{G}(0, Y) = 0$ . Alternately, we can solve (2.5) for  $\pi(0, k)$ , which gives

$$(6.10) \quad \pi(0, k) = \frac{1}{3} \sum_{l=k-1}^n \pi(1, l) \left(\frac{1+p}{2}\right)^{1+l-k}.$$

Using (6.10) to eliminate  $\pi(0, k)$  in (2.3), we obtain

$$(6.11) \quad \begin{aligned} \mathcal{G}(p, Y) &= \frac{1+p}{3}\mathcal{G}(p, Y+p) + \frac{1}{3}\mathcal{G}(2p, Y-p) \\ &+ \frac{1-p}{6} \sum_{j=0}^{\infty} \left(\frac{1+p}{2}\right)^j \mathcal{G}(p, Y+(j-1)p). \end{aligned}$$

To leading order (6.11) is automatically satisfied as  $p \rightarrow \infty$ , and at the next order we regain the boundary condition along  $X = 0$ .

Finally, consider the corner probability  $\pi(0, 0)$ . From (2.8) we obtain  $\pi(0, 0) \sim \frac{1}{2}p^2\mathcal{R}(p) \sim \frac{1}{2}p^2\mathcal{R}(0)$ , so that  $\pi(0, 0) \sim \frac{1}{3}p^2\mathcal{G}(0, 0)$ . This result is consistent with (2.6) and (2.7) if we use  $\pi(1, 1) \sim p^2\mathcal{G}(0, 0)$ ,  $\pi(1, 0) \sim \frac{2}{3}p^2\mathcal{G}(0, 0)$ , and  $\pi(0, 1) \sim \pi(0, 2) \sim p^2\mathcal{R}(0) = \frac{2}{3}p^2\mathcal{G}(0, 0)$ .

**6.1. Approach using the functional equation.** An alternate approach to the heavy traffic limit  $p \rightarrow 0$  is to consider the functional equation in (2.9), with the scaling  $x = 1 - pZ$  and  $y = 1 - pW$ . Then we obtain

$$(6.12) \quad h_1(x, y) = 2p^2[-W^2 - Z^2 + ZW - W - Z] + O(p^3),$$

$$(6.13) \quad h_2(x, y) \sim -3pW,$$

$$(6.14) \quad h_3(x, y) \sim 3p(W - Z).$$

In this limit,  $P(x, y) \sim \bar{P}(Z, W)$ , where

$$(6.15) \quad \bar{P}(Z, W) = \int_0^\infty \int_0^\infty e^{-ZX} e^{-WY} \mathcal{G}(X, Y) dXdY.$$

Thus,  $\bar{P}$  is the double Laplace transform of the limiting density  $\mathcal{G}$ . With this scaling we also obtain the Laplace transforms of the boundary values of  $\mathcal{G}(X, Y)$  as

$$(6.16) \quad Q_*(Z) = p^{-1} \sum_{k=0}^{\infty} (1-pZ)^{X/p} \pi(k, 0) \sim \frac{2}{3} \int_0^\infty e^{-ZX} \mathcal{G}(X, 0) dX,$$

$$(6.17) \quad R_*(Z) \sim \frac{2}{3} \int_0^\infty e^{-WY} \mathcal{G}(0, Y) dY,$$

and (2.9) becomes (after dividing by  $p^2$  and taking  $p \rightarrow 0$ )

$$(6.18) \quad \begin{aligned} 2[-W^2 - Z^2 + ZW - W - Z]\bar{P}(Z, W) \\ = -3WQ_*(Z) + (W - 3Z)R_*(W). \end{aligned}$$

Note that  $h_4(1, 1) = 0$  so that  $h_4(x, y)P(0, 0)$  becomes  $O(p^3)$  and the corner probability does not appear in (6.18). The solution of (6.18) is

$$(6.19) \quad \bar{P}(Z, W) = \frac{2}{(Z+1)(W+2)}$$

with

$$(6.20) \quad Q_*(Z) = \frac{4}{3(Z+1)}, \quad R_*(W) = \frac{4}{3(W+2)},$$

which is equivalent to the product form solution in Theorem 2.

**7. Proofs of Theorems 3 and 4.** In this section we give the proofs of Theorems 3 and 4 using the ray method. While the ray method will not yield the complete asymptotics, we will show that much of the basic structure can be inferred solely from the balance equations. We shall obtain the leading term up to a multiplicative constant, and then discuss some of the higher order terms. We will also relate our approach to the compensation approach and to singularity analysis. The compensation approach can be used to obtain the exact solution if it is a finite or infinite mixture of exponentials, and has been used by Adan, Wessels, and Zijm to analyze the classic shortest queue problem (see [1, 3]).

In the ray method we seek an asymptotic solution of (2.1) in the form

$$(7.1) \quad \pi(n, k) \sim e^{\Phi(n, k)} L(n, k),$$

where  $\Phi$  satisfies the “eiconal” equation

$$(7.2) \quad 3 = e^{\Phi_n - \Phi_k} + (1-p)e^{-\Phi_n} + (1+p)e^{\Phi_k}$$

and  $L$  satisfies the “transport” equation

$$(7.3) \quad 0 = e^{\Phi_n - \Phi_k} \left[ L_n - L_k + \left( \frac{1}{2} \Phi_{nn} - \Phi_{nk} + \frac{1}{2} \Phi_{kk} \right) \right] \\ + (1-p)e^{-\Phi_n} \left[ -L_n + \frac{1}{2} \Phi_{nn} L \right] + (1+p)e^{\Phi_k} \left[ L_k + \frac{1}{2} \Phi_{kk} L \right].$$

We will see that  $\Phi$  has linear growth in  $n$  and  $k$ , and  $L$  has algebraic growth; thus the asymptotic limit in (7.1) fails to satisfy (2.1) exactly only in that (7.3) neglects derivatives of  $L$  of order  $\geq 2$  and derivatives of  $\Phi$  of order  $\geq 3$ .

Equation (7.2) is a nonlinear PDE of the first order, which can be solved by the method of characteristics. The characteristic curves, or “rays,” can be obtained by solving the system

$$(7.4) \quad \dot{n} = (1-p)e^{-\Phi_n} - e^{\Phi_n - \Phi_k}, \quad \dot{k} = e^{\Phi_n - \Phi_k} - (1+p)e^{\Phi_k},$$

$$(7.5) \quad \dot{\Phi}_n = 0, \quad \dot{\Phi}_k = 0,$$

$$(7.6) \quad \dot{\Phi} = \Phi_n \left[ (1-p)e^{-\Phi_n} - e^{\Phi_n - \Phi_k} \right] + \Phi_k \left[ e^{\Phi_n - \Phi_k} - (1+p)e^{\Phi_k} \right].$$

Here, “ $\dot{\cdot}$ ” denotes a derivative along a ray.



We consider three main families of rays: rays that start from the  $k$ -axis ( $n = 0$ ), rays that start from the  $n$ -axis ( $k = 0$ ), and rays that start from the origin  $(n, k) = (0, 0)$  at various slopes. We shall also consider the reflections of the first two ray families. The rays from the axes, as well as from their reflections, will lead to purely exponential solutions to (2.1).

First we consider rays from  $k = 0$ . These must take into account the boundary conditions in (2.2) and (2.4). Along  $k = 0$  we allow for a discontinuity in (7.1), setting

$$(7.7) \quad \pi(n, 0) \sim e^{\Phi(n,0)} \tilde{L}(n).$$

Then, with (7.3) and (7.7), (2.2) leads to

$$(7.8) \quad L(n, 0)e^{\Phi_k(n,0)} = \frac{1}{2}\tilde{L}(n)e^{\Phi_n(n,0)} + \left[ \frac{1-p}{3}e^{\Phi_k(n,0)-\Phi_n(n,0)} + \frac{1+p}{3}e^{2\Phi_k(n,0)} \right] L(n, 0),$$

and (2.4) gives

$$(7.9) \quad \tilde{L}(n) = \frac{1+p}{3}e^{\Phi_k(n,0)}L(n, 0) + \frac{1}{2}e^{-\Phi_n(n,0)}\tilde{L}(n).$$

Using (7.2) with  $k = 0$  and comparing this to (7.8), we conclude that  $\tilde{L}(n) = \frac{2}{3}L(n, 0)$ , and then (7.9) gives

$$(7.10) \quad \frac{2}{3} = \frac{1+p}{3}e^{\Phi_k(n,0)} + \frac{1}{3}e^{-\Phi_n(n,0)},$$

which is the desired boundary condition for  $\Phi$  along  $k = 0$ .

We return to (7.4)–(7.6). From (7.5) we conclude that  $\Phi_n$  and  $\Phi_k$  are both constant along a ray, and we write  $\Phi_n = \log A$  and  $\Phi_k = \log B$ . But the boundary condition in (7.10) implies that  $\Phi_n$  and  $\Phi_k$  are constant globally. Then we can solve for  $(A, B)$  by using (7.2) (i.e.,  $3 = A/B + (1 - p)/A + (1 + p)B$ ) and (7.10) (i.e.,  $2 = (1 + p)B + 1/A$ ), which leads to (3.22) and (3.23).

We next determine the domain that is filled by the rays from the  $n$ -axis. Solving (7.4) leads to

$$(7.11) \quad n = n_0 + \left( \frac{1-p}{A} - \frac{A}{B} \right) t, \quad k = \left( \frac{A}{B} - (1+p)B \right) t,$$

which gives the rays in parametric form. Here,  $n_0$  is where the ray begins on the  $n$ -axis. The rays must enter the state space  $(n, k \geq 0)$  for  $t > 0$ , and it is easy to verify that, for all  $p \in (0, 1]$ ,

$$(7.12) \quad \dot{k} = -1 - \frac{1}{2p} + \frac{p}{2} + \left( \frac{1}{2} + \frac{1}{2p} \right) \sqrt{1 + 2p + 5p^2} > 0.$$

Furthermore, for all  $p \in (0, 1]$ ,

$$(7.13) \quad \dot{n} = \frac{1}{2} - p - \frac{1}{2p} + \left( \frac{1}{2p} - 1 \right) \sqrt{1 + 2p + 5p^2} < 0,$$

so that the rays fill the entire quadrant. By considering in more detail the boundary condition for  $L(n, k)$  along  $k = 0$ , we can show that  $L$  is in fact a constant. (It follows

immediately from (7.3), since  $\Phi_{nn} = \Phi_{kk} = \Phi_{nk} = 0$ , that  $L$  is a constant along a ray.) Also, (7.6) can be written as  $\dot{\Phi} = (\log A)\dot{n} + (\log B)\dot{k}$ , so that  $\Phi = n \log A + k \log B$  and  $e^\Phi = A^n B^k$ . We have thus shown that an asymptotic solution of (2.1), (2.2), and (2.4) is

$$(7.14) \quad \pi(n, 0) \sim \frac{2}{3}CA^n, \quad \pi(n, k) \sim CA^n B^k, \quad k \geq 1.$$

This fails to satisfy the boundary condition along  $n = 0$ ; cf. (2.3) and (2.5).

It remains to consider rays from the  $k$ -axis ( $n = 0$ ) and rays from the origin. But we can show that rays from the  $k$ -axis would correspond to a product form solution with  $\dot{n} < 0$ , so the corresponding ray family would not enter the domain. We shall discuss rays from the origin shortly, but these will lead to a solution exponentially smaller than (7.14).

To satisfy the boundary condition along  $n = 0$  and  $n = 1$  in (2.5) and (2.3) we must consider the scale  $n = O(1)$ ,  $k \rightarrow \infty$ , and construct a ‘‘boundary layer’’ correction to (7.14). Since (7.14) is a product form this essentially corresponds to the second step of the compensation methods of Adan [1]. Setting  $\pi(n, k) \sim B^k g(n)$  for  $n \geq 1$  and  $\pi(0, k) \sim B^k g_0$  and analyzing (2.1), (2.3), and (2.5), we find that  $g(n)$  satisfies the simple difference equation

$$(7.15) \quad 3 = B^{-1}g(n + 1) + (1 - p)g(n - 1) + (1 + p)Bg(n),$$

so that  $g(n)$  is a linear combination of  $A^n$  and  $A_*^n$  (defined in (3.24)). Then (2.3) and (2.5) show that  $g_0 = \frac{2}{3}g(0)$  and that  $g(n)$  must be proportional to  $A^n + \gamma A_*^n$  with  $\gamma$  in (3.25). Thus we have introduced the second exponential  $B^k A_*^n$  into the solution. Also, we have  $A > A_*$  for all  $p$ .

This second exponential can be interpreted in terms of the ray method, as the reflection (in the  $k$ -axis) of the rays that started in the  $n$ -axis. The reflected rays have

$$(7.16) \quad \dot{k} = \frac{A_*}{B} - (1 + p)B = -\frac{1}{p} - \frac{1}{2} - \frac{p}{2} + \left(\frac{1}{p} - \frac{1}{2}\right) \sqrt{1 + 2p + 5p^2}$$

and

$$(7.17) \quad \dot{n} = \frac{1 - p}{A_*} - \frac{A_*}{B} = p + \frac{1}{2p} - \frac{1}{2} + \left(1 - \frac{1}{2p}\right) \sqrt{1 + 2p + 5p^2}.$$

We have  $\dot{n} > 0$  for all  $p \in (0, 1]$  so that the reflected rays enter the positive quadrant. However, we can have  $\dot{k} > 0$  or  $\dot{k} < 0$ . From (7.16) we see that  $\dot{k} = 0$  if  $p = \frac{1}{2}(5 - \sqrt{17}) \equiv p_*$ . Then we obtain  $\dot{k} < 0$  for  $p_* < p < 1$  and  $\dot{k} > 0$  for  $0 < p < p_*$ . The reflected rays all have the same slope, that is, the ratio of the right-hand side of (7.16) to that of (7.17), and this is the same as  $\mathcal{S}_1(p)$  in (3.30). For  $0 < p < p_*$  the reflected rays fill only the sector  $k/n > \mathcal{S}_1(p)$ , while if  $p_* < p < 1$ , they fill the entire positive quadrant. In the latter case the reflected rays are themselves reflected in the  $n$ -axis ( $k = 0$ ). This corresponds to a secondary reflection of the rays that started in the  $n$ -axis, and this will lead to a third exponential appearing in the solution as  $p$  increases past  $p_*$ . For  $p < p_*$  only the two exponentials  $A^n B^k$  and  $A_*^n B^k$  appear. But then there is the transition in the asymptotics of  $\pi(n, k) \sim CA^n B^k$  along the ray  $k/n = \mathcal{S}_1(p)$ . This difference will be asymptotic to the second exponential if  $k/n > \mathcal{S}_1(p)$ , and to the solution that corresponds to rays from the origin if  $k/n < \mathcal{S}_1(p)$ .

Consider first  $p_* < p < 1$ , where the secondary reflection occurs. While  $A^n B^k$  satisfies the boundary condition along the  $n$ -axis, the second exponential does not.

However, it can be compensated for by a third exponential of the form  $A_*^n B_*^k$ . This boundary layer construction leads to the term proportional to  $\delta$  in (3.37). The secondarily reflected rays have

$$(7.18) \quad \dot{n} = \frac{1-p}{A_*} - \frac{A_*}{B} = -1 + \frac{p}{2} - \frac{1}{2p} + \left(\frac{1}{2} + \frac{1}{2p}\right) \sqrt{1+2p+5p^2},$$

and  $\dot{k} = A_*/B_* - (1+p)B_*$ . We can easily show that  $\dot{k} > 0$  for  $p > p_*$  and that  $\dot{n} > 0$  always. It follows that the secondary reflections fill only the sector  $k/n < \mathcal{S}_2(p)$ , and hence no further reflections are possible. This suggests also that the compensation method cannot be carried out further. Indeed, even if we completely ignore the ray geometry, we can try to correct for the third exponential  $A_*^n B_*^k$  by a fourth, of the form  $\tilde{A}^n \tilde{B}^k$ , near the  $k$ -axis. But then we would find that

$$(7.19) \quad \tilde{A} = \frac{1-p}{2(1+p)} \left[ 1 + 3p + \sqrt{1+2p+5p^2} \right]$$

and so  $\tilde{A} > A$ . Hence, the fourth exponential cannot serve as a boundary layer correction to the third.

Finally we discuss rays from the origin. Since by (7.5) these have  $\Phi_n$  and  $\Phi_k$  constant along a ray, we set  $\Phi_n = \log(a)$  and  $\Phi_k = \log(b)$ . In view of the PDE (7.2) we have

$$(7.20) \quad 3 = \frac{a}{b} + (1-p)\frac{1}{a} + (1+p)b.$$

Integrating (7.4) then leads to  $n = t[(1-p)/a - a/b]$  and  $k = t[a/b - (1+p)b]$ , so that a particular ray has the slope

$$(7.21) \quad \frac{k}{n} = \frac{a/b - (1+p)b}{(1-p)/a - a/b}.$$

We can view (7.20) and (7.21) as two equations for the two unknowns  $a$  and  $b$ , which are both functions of  $k/n \equiv R$ . Then (7.6) integrates to  $\Phi = n \log(a) + k \log(b)$  and hence  $e^\Phi = a^n b^k = [a(k/n)]^n [b(k/n)]^k$ . We can eliminate  $b$  to obtain the cubic equation (3.38) for  $a$ . This equation has three real branches, which we plot in Figure 5 for  $p = 0.5, 0.8$ . We see that one branch starts at  $a(0) > 1$ , and the branch with the smallest value of  $a(0)$  becomes negative as  $k/n$  increases past 1. Thus we choose the middle branch, which has  $a'(R) > 0$  and corresponds to  $k/n > 0$  in (7.21).

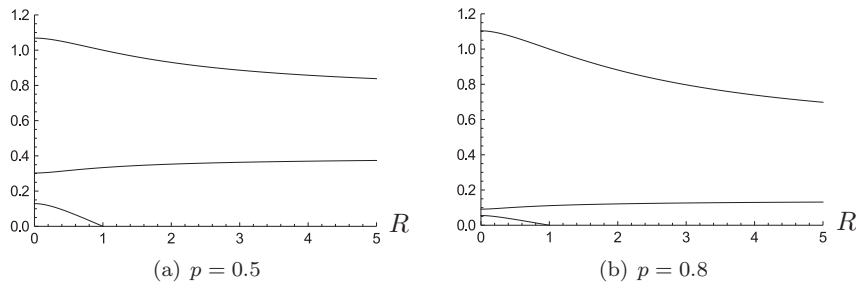


FIG. 5. The three branches of (3.38) for  $R \in [0, 5]$ .

The rays from  $(0, 0)$  correspond to a non-product form solution in (7.1). For this solution  $\Phi$ , we can rewrite, after some calculation, the transport equation (7.3) as

$\dot{L}/L = -1/(2t)$ , which gives the derivative of  $L$  along a ray. Hence we can write the general solution of  $L$  as  $L = f(R)/\sqrt{n}$ . It does not seem possible to determine the function  $f(R)$  by ray-type analysis alone. Its determination would require a careful analysis of the “corner scale,”  $n, k = O(1)$ , but this is equivalent to an exact solution to the full problem.

Rays from the origin thus lead to an asymptotic solution that is smaller than any of the three exponentials but which fills the entire first quadrant for all values of  $p$ . This solution does not take into account the boundary conditions (2.2)–(2.5) along the axes. It is possible to construct boundary layer corrections to the ray solution, which will apply for  $n \rightarrow \infty, k = O(1)$  and  $k \rightarrow \infty, n = O(1)$ . We omit that analysis but comment that it would yield additional information about the function  $f(R)$ , namely that  $f(R) \sim (\text{const.})R$  as  $R \rightarrow 0$  and that  $f(R) \sim (\text{const.}')R^{-3/2}$  as  $R \rightarrow \infty$ . The expansion that applies for  $n \rightarrow \infty$  and  $k = O(1)$  would have the form

$$(7.22) \quad n^{-3/2}(v_1k + v_0)[a(0)]^n[b(0)]^k, \quad k \geq 1,$$

with a suitable modification along  $k = 0$ . From (3.38) we also find that  $a'(0) = 0$ . For  $k \rightarrow \infty$  and  $n = O(1)$  the boundary layer correction to the ray expansion would take the form

$$(7.23) \quad k^{-3/2}(v'_1n + v'_0)[b(\infty)]^k[a(\infty)]^n, \quad n \geq 1,$$

and we have  $b'(\infty) = 0$ .

The asymptotic solution  $n^{-1/2}f(R)[a(R)]^n[b(R)]^k$  will also develop problems near  $R = k/n = \mathcal{S}_1(p)$  if  $p < p_*$ , and near  $R = \mathcal{S}_2(p)$  if  $p > p_*$ , which corresponds to where an extra exponential begins to appear in the asymptotics. We omit the details of the analysis, but mention only that on the scale  $k - n\mathcal{S}_1(p) = O(\sqrt{n})$  we can ultimately approximate the difference equation (2.1) by a parabolic PDE, from whose solution we can infer that  $f(R)$  has a singularity of the form  $f(R) \sim v_*[R - \mathcal{S}_1(p)]^{-1}$  as  $R \rightarrow \mathcal{S}_1(p)$ . We also note that  $a(\mathcal{S}_1(p)) = A_*$  and  $b(\mathcal{S}_1(p)) = B$ , so that the rays from the origin and the exponential solution  $A_*^n B^k$  agree along the critical line  $R = \mathcal{S}_1(p)$ , at least in their exponential orders of magnitude.

**7.1. Asymptotics via an exact solution.** We briefly discuss how such asymptotics would follow from the exact solution if it were available. This approach is taken in our companion paper. Inverting the generating function  $P(x, y, k)$  would yield  $\pi(n, k)$  as the double Cauchy integral,

$$(7.24) \quad \pi(n, k) = \left(\frac{1}{2\pi i}\right)^2 \oint \oint \frac{P(x, y)}{x^{n+1}y^{k+1}} dx dy.$$

By studying the singularities of  $P$ , where  $h_1(x, y) = 0$ , we could evaluate one of the integrals in (7.24) by the residue theorem and then locate the saddle points in the remaining integral. For  $n, k$  simultaneously large the saddle-point equation would have

$$(7.25) \quad -\frac{n}{x} dx - \frac{k}{y} dy = 0,$$

and  $x$  and  $y$  would also be related by  $h_1(x, y) = 0$ ; hence  $3 = (1-p)x + y/x + (1+p)/y$ . By implicitly differentiating  $h_1(x, y) = 0$  w.r.t.  $x$  and then using (7.25) to solve for  $dy/dx$  in terms of  $n/k$ , we obtain

$$(7.26) \quad \frac{y}{x} - (1-p)x = -\frac{y}{x} \frac{n}{k} + \frac{1+p}{y} \frac{n}{k}.$$

But (7.26) and  $h_1 = 0$  correspond precisely to (7.20) and (7.21) if we identify a solution  $(\tilde{x}, \tilde{y})$  of the former system with  $(1/a, 1/b)$ . Thus, the rays from the origin correspond to a saddle-point approximation to (7.24). The exponential solutions arising from rays from the coordinate axes (and their reflections) then correspond to poles in the integral representation. Of course given the explicit form in (7.24), we could also compute the constant  $C$  and the function  $f(R)$  explicitly. Nevertheless, the ray approach gives a simple argument for directly deriving the basic asymptotic structure of  $\pi(n, k)$ .

**7.2. Marginal probabilities.** From Theorem 4 we can also get a more detailed description of the asymptotics of the marginal probabilities. For any  $p$  we obtain

$$(7.27) \quad \mathbb{P}(N_2 = k) \sim C[\alpha_0 B^k + \alpha_1 k^{-3/2}(b(\infty))^k], \quad k \rightarrow \infty,$$

where  $\alpha_0$  can be obtained from Corollary 3 and  $\alpha_1$  can be obtained by summing (7.23) for  $n \geq 1$  and adding the approximation for  $n = 0$  (which we did not give). Note that if we had an integral representation for  $P(1, y)$ , the term involving the  $B^k$  in (7.27) would correspond to a pole at  $y = 1/B$ , while the second term in (7.27) would come from a branch point at  $y = 1/b(\infty)$ .

For the marginal of  $N_1$  we obtain

$$(7.28) \quad \mathbb{P}(N_1 = n) = \begin{cases} C[\beta_0 A^n + \beta_2 n^{-3/2}(a(0))^n], & p < p_*, \\ C[\beta_0 A^n + \beta_1 A_*^n + \beta_2 n^{-3/2}(a(0))^n], & p > p_*. \end{cases}$$

These terms follow from the ray method and would correspond to the poles in  $P(x, 1)$  at  $x = 1/A$  and  $x = 1/A_*$ , and to a branch point at  $x = 1/a(0)$ . The constant  $\beta_0$  is given in Corollary 3,

$$(7.29) \quad \beta_1 = \frac{2}{3}(\gamma + \delta) + \gamma \frac{B}{1 - B} + \delta \frac{B_*}{1 - B_*},$$

and  $\beta_2$  can be expressed in terms of  $v_0$  and  $v_1$  in (7.22). We also note that the perturbation method can be used to infer the ratio  $v_1/v_0$  explicitly in terms of  $p$ .

We conclude with some observations about the constants  $a(0)$  and  $b(\infty)$ . If we solve  $h_1(x, y) = 0$  for  $y$  as a function of  $x$ , we are led to

$$(7.30) \quad y = \frac{1}{2} \left[ 3x - (1 - p)x^2 \pm \sqrt{(3x - (1 - p)x^2)^2 - 4(1 + p)x} \right].$$

The above discriminant has three positive roots  $x_j$  that satisfy  $0 < x_1 < 1 < x_2 < x_3$ , and the middle of these corresponds precisely to  $x_2 = 1/a(0)$ . Similarly, if we solve  $h_1(x, y) = 0$  for  $x$  in terms of  $y$ , that discriminant has three real roots that can be ordered as  $0 < y_1 < y_2 < 1 < y_3$ , and we have  $b(\infty) = 1/y_3$ . If we set  $a = a(0)$  in (7.20) and solve for  $b$ , then  $b = b(0)$  is a double root. Similarly, if we replace  $b$  by  $b(\infty)$ , then  $a = a(\infty)$  is a double root.

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