abstract. We present the explicit metric forms for higher dimensional vanishing scalar invariant (VSI) Lorentzian spacetimes. We note that all of the VSI spacetimes belong to the higher dimensional Kundt class. We determine all of the VSI spacetimes which admit a covariantly constant null vector, and we note that in general in higher dimensions these spacetimes are of Ricci type III and Weyl type III. The Ricci type N subclass is related to the chiral null models and includes the relativistic gyratons and the higher dimensional pp-wave spacetimes. The spacetimes under investigation are of particular interest since they are solutions of supergravity or superstring theory.

[pacs: 04.20.jb, 04.65+l]

1. Introduction

Recently [1] it was proven that all curvature invariants of all orders vanish in an N-dimensional Lorentzian (VSI) spacetime if and only if there exists an aligned shear-free, non-expanding, non-twisting, geodesic null direction $\ell^a$ along which the Riemann tensor has negative boost order. An analytical form of these conditions are as follows:

(1) $R_{abcd} = 8A_i \ell_{[a}m_{b}m_{c}m_{d]} + 8B_{ijk}m_{[a}m_{b}m_{c}m_{d]} + 8C_{ij}m_{[a}m_{b}m_{c}m_{d]}$

where $i = 1...(N-2)$ (i.e., the Riemann tensor is of algebraic type III N or O [2]), and

(2) $\ell_{a;b} = L_{11}^{a} \ell_{b} + L_{11}^{a} \ell_{b} m_{a}^{i} + L_{i}^{a} m_{a}^{i} \ell_{b}$;

that is, the expansion matrix and the twist matrix are zero (as well as $L_{i0} = 0 = L_{10}$, corresponding to an affinely parametrized geodesic congruence $\ell_{a}$).

This result generalizes a previous theorem in four dimensions (4D) [3]. Indeed, in 4D VSI spacetimes can be classified according to their Petrov type, Segre type and the vanishing or non-vanishing of the spin coefficient $\tau$; all of the corresponding metrics are displayed in [3].

A number of higher dimensional VSI spacetimes are explicitly known [4, 5]. However, we wish to complete this investigation and write down the metric for higher dimensional VSI spacetimes in a canonical form (similar to what is done in 4D). We note that all of the VSI spacetimes have a shear-free, non-expanding, non-twisting geodesic null congruence $\ell = \partial_{v}$, and hence belong to the higher
dimensional Kundt class \([4, 6]\). In this paper, we shall therefore present the explicit metric forms for higher dimensional \(VSI\) spacetimes [6].

1.1. Preliminaries. We shall consider a null frame \(\ell = m_0, n = m_1, m_2, \ldots m_{N-1}\) (\(\ell, n\) null with \(\ell^a \ell_a = n^a n_a = 0, \ell^a n_a = 1, m_i\) real and spacelike \(m_i m_j = \delta_{ij}\), \(i = 2, \ldots, N - 1,\) all other products vanish) in an \(N\)-dimensional Lorentz-signature spacetime, so that

\[
g_{ab} = 2\ell(a n_b) + \delta_{jk} m^j a m^k b. \tag{3}\]

A null rotation about \(n\) is a Lorentz transformation of the form

\[
\hat{n} = n, \quad \hat{m}_i = m_i + z_i n, \quad \hat{\ell} = \ell - z_i m^i - \frac{1}{2} \delta^{ij} z_i z_j n. \tag{4}\]

A null rotation about \(\ell\) has an analogous form. A boost is a transformation of the form

\[
\hat{n} = \lambda^{-1} n, \quad \hat{m}_i = m_i, \quad \hat{\ell} = \lambda \ell, \quad \lambda \neq 0. \tag{5}\]

Using the notation

\[
w_{[a x_b y_c z_d]} \equiv \frac{1}{2} \{w_{[a x_b y_c z_d]} + w_{[c x_d y[a z_b]}\} \equiv \frac{1}{8}\{w_{[a x_d y[a z_b]}\}, \tag{6}\]

we can decompose the Weyl tensor \(C_{abcd}\) and sort its components by boost weight (see Table 1 in [7]). The Weyl scalars also satisfy a number of additional relations, which follow from curvature tensor symmetries and from the trace-free condition.

The boost order of a tensor is a function of the null direction \(k\). We denote boost order by \(\mathcal{B}(k)\). We will call the integer \(1 - \mathcal{B}(k) \in \{0, 1, 2, 3\}\) the order of alignment. The principal type of a Lorentzian manifold is \(I, II, III, N\) according to whether there exists an aligned \(k\) of alignment order 0, 1, 2, 3, respectively (i.e., \(\mathcal{B}(k) = 1, 0, -1, -2,\) respectively) [7, 2]. If no aligned \(k\) exists we will say that the manifold is of type \(G\). If the Weyl tensor vanishes, we will say that the manifold is of type \(O\). It follows that there exists a frame in which the components of the Weyl tensor satisfies:

\[
\begin{align*}
\text{Type I : } & \quad C_{00ij} = 0, \\
\text{Type II : } & \quad C_{00ij} = C_{0ijk} = 0, \\
\text{Type III : } & \quad C_{00ij} = C_{0ijk} = C_{ijkl} = C_{01ij} = 0, \\
\text{Type N : } & \quad C_{00ij} = C_{0ijk} = C_{ijkl} = C_{01ij} = C_{1ijk} = 0.
\end{align*} \tag{7}\]

The general types have various algebraically special subtypes [2]; for example, Type III(a) for \(C_{0111} = 0\).

2. Higher dimensional \(VSI\) metrics

From [4, 6], it follows that any \(VSI\) metric can be written in the form

\[
ds^2 = 2 du \left[ dv + H(v, u, x^n) du + W_i(v, u, x^n) dx^i \right] + \delta_{ij} dx^i dx^j \tag{8}\]

with \(i, j = 1, \ldots, N - 2\). The metric functions \(H\) and \(W_i\) satisfy the remaining vanishing scalar invariant conditions and the Einstein equations. It is convenient to introduce the null frame

\[
\ell = du, \quad n = dv + H du + W_i m^{i + 1}, \quad m^{i + 1} = dx^i. \tag{9\text{-11}}\]
Hence, the negative boost order conditions of the Riemann tensor yield
\[
\ell^A \ell_{B;A} = \ell^A \ell_{;A} = \ell^{A:B} \ell_{(A;B)} = \ell^{A:B} \ell_{;[A;B]} = 0;
\]
\[i.e., \ell \text{ is geodesic, non-expanding, shear-free and non-twisting (i.e., } L_{ij} \equiv \ell_{ij} = 0 \text{) are higher-dimensional Kundt metrics (or simply Kundt metrics), since they generalise the 4-dimensional Kundt metrics. There is some remaining freedom in the choice of frame; namely null rotations about } \ell, \text{ spins and boosts.}
\]

The generalized Kundt metrics are given by \[\S\], but with a general transverse metric of the form \(g_{ij}(u, x^k)\). From \[\S\], it follows that for any VSI member of the generalized Kundt class, there is a transformation of the form:
\[
(v', u', x^i) = (v, u, f^i(u; x^k)),
\]
which transforms the transverse metric \(g_{ij}(u, x^k)\) to flat space with \(\delta_{ij}\). Hence the VSI spacetimes are Kundt spacetimes of the form \[\S\].

The remaining coordinate freedom preserving the Kundt form are:
\[
\begin{align*}
(1) \quad & (v', u', x^i) = (v, u, f^i(u; x^k)) \text{ and } J'_{ij} \equiv \frac{\partial f^i}{\partial x^j}. \\
& H' = H, \quad W'_i = W_j (J^{-1})_i^j, \quad \delta'_{ij} = \delta_{kl} (J^{-1})_i^k (J^{-1})^l_j. \\
(2) \quad & (v', u', x^i) = (v + h(u, x^k), u, x^i) \\
& H' = H - h, \quad W'_i = W_i - h_i, \quad \delta'_{ij} = \delta_{ij}. \\
(3) \quad & (v', u', x^i) = (v / g_u(u), g(u), x^i) \\
& H' = \frac{1}{g_u} \left( H + v g_u' \right), \quad W'_i = \frac{1}{g_u} W_i, \quad \delta'_{ij} = \delta_{ij}. \\
(4) \quad & (v', u', x^i) = (v, u, f^i(u; x^k)), \text{ where } f^i(u; x^k) \text{ is an isometry of flat space, and } J'_{ij} \equiv \frac{\partial f^i}{\partial x^j}. \\
& H' = H + \delta_{ij} f^i_{,u} f^j_{,u} - W_j (J^{-1})_i^j f^i_{,u}, \quad W'_i = W_j (J^{-1})_i^j - \delta_{ij} f^j_{,u}, \quad \delta'_{ij} = \delta_{kl} (J^{-1})_i^k (J^{-1})^l_j \equiv \delta_{ij}.
\end{align*}
\]

Transformations (1) and (4) are both subsets of the more general transformations \[\S\] (diffeomorphisms and isometries, respectively). The set of transformations (4) therefore consists of translations and rotations of the coordinates \(x^i\).

The linearly independent components of the Riemann tensor with boost weight 1 and 0 are:
\[
\begin{align*}
R_{0101(i+1)} &= -\frac{1}{2} W_{i, vv}, \\
R_{0101} &= -H_{vv} + \frac{1}{4} (W_{i,v}) (W^{i,v}), \\
R_{01(i+1)(j+1)} &= W_i W_{j,v} + W_{[ij], v}, \\
R_{0(i+1)j(j+1)} &= \frac{1}{2} \left[ -W_j W_{i, vv} + W_{i, j,v} - \frac{1}{2} (W_{i,v}) (W_{j,v}) \right].
\end{align*}
\]

Hence, the negative boost order conditions of the Riemann tensor yield
\[
\begin{align*}
W_{i, vv} &= 0, \\
H_{vv} - \frac{1}{4} (W_{i,v}) (W^{i,v}) &= 0.
\end{align*}
\]
from which it follows that

\[(20)\quad W_i(v, u, x^k) = vW_i^{(1)}(u, x^k) + W_i^{(0)}(u, x^k),\]

subject to

\[(22)\quad W_{[i;j],v} = 0,\]

and

\[(23)\quad W_{(i:j),v} - \frac{1}{2}(W_i,v)(W_j,v) = 0.\]

All of these spacetimes are VSI. Further constraints can be made, and the resulting spacetimes invariantly classified.

From eqn. (22) it follows that \(W_i^{(1)}\) can be written locally as a gradient:

\[(24)\quad W_i^{(1)} = [\phi(u, x^k)]_i.\]

Eq. (23) now simplifies to

\[(25)\quad (e^{-\frac{1}{2}\phi},)_{,ij} = 0,\]

which can be integrated to yield:

\[(26)\quad \phi = -2\ln\left[a_i(u)x^i + C(u)\right],\]

where \(a_i(u)\) and \(C(u)\) are arbitrary functions of \(u\). Now, utilizing the rotations and translations of \(x^i\) we can simplify \(\phi\):

\[(27)\quad \phi = -2\ln[a(u)x^1], \quad \text{or} \quad \phi = -2\ln[C(u)].\]

(Note that rotations in the subspace orthogonal to \(x^1\) are still permitted). Hence, we obtain the two general cases:

(i): \(W_1^{(1)} = \frac{-2}{x^1}; \quad W_i^{(1)} = 0, \quad i \neq 1.\)

(ii): \(W_i^{(1)} = 0.\)

The form of \(H\) is given by eqn. (21), where \(H^{(1)}(u, x^k)\) and \(H^{(0)}(u, x^k)\) are the redefined functions. The spacetimes above are in general of Ricci and Weyl type III.

Further progress can be made by classifying the metric in terms of their Weyl-type (III, N or O) and their Ricci type (N or O), and the form of \(L_{ab}\). In particular, \(H^{(1)}(u, x^k)\) is determined in terms of the functions \(W_i^{(0)}(u, x^k)\) by restricting the metrics to be of Ricci type N. Additional constraints on \(\Psi_{ij}, \Psi_{ijk}, \text{etc.}\) can be obtained by employing the Bianchi and Ricci identities [8]. In addition, the spatial tensors (indices \(i, j\)) can be written in canonical form and remaining coordinate and frame freedom utilized.
2.1. **Kinematics.** The nonzero ‘spin coefficients’ $L_{ij}$ are defined by

$$L_{(a;b)} = L_{a}^{\alpha}L_{b}^{\beta} + L_{i}^{\alpha}m_{b}^{\beta} + L_{b}^{\alpha}m_{a}^{\beta}.$$  

$L = L_{11}$ is the analogue of the spin coefficient $(\gamma + \bar{\gamma})$ in 4-dimensions and the $L_i \equiv L_{1i}$ are the analogues of the spin coefficient $\tau$. From eqn. (28) we have that

$$L = H_{,v},$$

and

$$L_i = \frac{1}{2} W_{i,v} = \frac{1}{2} W_{i}^{(1)}.$$  

Therefore, in the preferred frame (28) $L_1 \neq 0$ and all other $L_i$ are zero (i.e., the rotation used above also simultaneously normalizes the $L_i$). All of the information from the ‘spin coefficients’ can then be summarized by:

$$W_{1}^{(1)} = -2\frac{\epsilon}{x^1}; \quad W_{n}^{(1)} = 0, n = 2, ..., N - 2,$$

where $\epsilon = 0$ corresponds to $W_{i}^{(1)} = 0$ (and to $\tau = 0$ in 4-dimensions) and $\epsilon = 1$ in the case $L_1 \neq 0$ ($\tau \neq 0$). Note that for $\epsilon = 0$ and $H_{,v} = 0$, $\ell_a$ is a null Killing vector.

2.2. **Ricci type.** The Ricci tensor is given by

$$R_{ab} = \Phi_{\ell_a \ell_b} + \Phi_i (\ell_a m_b^{\beta} + \ell_b m_a^{\beta}).$$

The Ricci type is $N$ if $\Phi_i = 0 = R_{i1}$ (otherwise the Ricci type is III; Ricci type O is vacuum). The non-zero components can be simplified and chosen to be constant.

Further progress is most easily made by first performing a type (2) coordinate transformation with $h(u, x^{k})$ satisfying

$$h_{,1} - 2\frac{h_{,\epsilon}}{x^{1}} = W_{1}^{(0)}.$$  

The effect is to transform away $W_{1}^{(0)}$, so that in the new coordinates $W_{1} = -2\epsilon/x^1$ and the remaining metric functions $W_m = W_{m}^{(0)}(u, x^{k})$, $H^{(1)}(u, x^{k})$ and $H^{(0)}(u, x^{k})$ undergo a redefinition. 

That is,

$$W_{1} = -2\frac{\epsilon}{x^1},$$

and

$$W_m = W_{m}^{(0)}(u, x^{k}),$$

where $m$ ranges over $2, \ldots, N - 2$ (i.e., does not include $i = 1$).

When $\epsilon = 0, 1$ the Ricci type N conditions $\Phi_i = 0$ reduce to

$$2H_{(1),1}^{(1)} = \frac{2\epsilon}{x^1} W_{(0)m}^{(0)}, m - W_{(0)m}^{(0)}, m_{1},$$

subject to

$$\Delta W_{n}^{(0)} = W_{n}^{(0)},$$

$$\Delta W_{m}^{(0)} = W_{m}^{(0)}.$$  

It is also possible to choose a gauge with $W_{1}^{(0)}$ non-zero; the corresponding results will be presented elsewhere. 

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\[1\] It is also possible to choose a gauge with $W_{1}^{(0)}$ non-zero; the corresponding results will be presented elsewhere [9].
where $\Delta = \partial_i \partial_i$ is the spatial Laplacian and $m, n \geq 2$. A partial integration of (36)-(39) reduces these constraints to a divergence and a Laplacian that must be satisfied by $W_m^{(0)}$, namely

\begin{equation}
W_m^{(0)} = \epsilon(x^1)^2 \left[ F - \int \frac{4}{(x^1)^3} H^{(1)} dx^1 \right] + (1 - \epsilon) F - 2H^{(1)}
\end{equation}

\begin{equation}
\Delta W_m^{(0)} = \epsilon(x^1)^2 \left[ F_m - \int \frac{4}{(x^1)^3} H^{(1)} dx^1 \right] + (1 - \epsilon) F_m
\end{equation}

where $F = F(u, x^n)$ is an arbitrary function independent of $x^1$. Note when $\epsilon = 0$ (40) defines $H^{(1)}$ with $F$ determined from (41). We note that, in general, $F$ cannot be transformed away.

Therefore, the equations above describe all VSI metrics of Ricci type $N$, with $\epsilon = 0$ or $\epsilon = 1$. In general the Weyl tensor is of type III.

2.3. Weyl-type. The Weyl tensor can be expressed as

\begin{equation}
C_{abcd} = 8\Psi_i \ell_{i(a} m_{b} c m_{d)} + 8\Psi_{ijk} m_{i} c m_{j} k + 8\Psi_{ij} \ell_{i(a} m_{j} c m_{d)}.
\end{equation}

The case $\Psi_{ijk} \neq 0$ is of Weyl type III, while $\Psi_{ijk} = 0$ (and consequently also $\Psi_i = 0$) corresponds to type $N$. Note that $\Psi_{ij}$ is symmetric and traceless. $\Psi_{ijk}$ is antisymmetric in the first two indices with $\Psi_{ij} = 2\Psi_{ijj}$, and in vacuum also satisfies

\begin{equation}
\Psi_{\{ijk\}} = 0.
\end{equation}

2.3.1. Type III. As noted above, in general the Weyl tensor is of type III. Let us treat first the case $\epsilon = 1$:

\begin{equation}
W_1 = -\frac{2}{x^1} v
\end{equation}

\begin{equation}
W_m = W_m^{(0)}(u, x^k)
\end{equation}

\begin{equation}
H = H^{(0)}(u, x^i) + \frac{1}{2} \left( \tilde{F} - W^{(0)m} m \right) v + \frac{v^2}{2(x^1)^2},
\end{equation}

where $\tilde{F} = \tilde{F}(u, x^i)$ is a function satisfying:

\begin{equation}
\tilde{F},_1 = \frac{2}{x^1} W^{(0)m} m, \quad \tilde{F},_m = \Delta W_m^{(0)}
\end{equation}

In addition, we have the Einstein equation

\begin{equation}
x^1 \triangle \left( \frac{H^{(0)}}{x^1} \right) + \left( \frac{W^{(0)m} W_m^{(0)}}{x^1} \right) ,_1 - 2H^{(1)} ,_m W^{(0)m} - H^{(1)} W^{(0)m} ,m
\end{equation}

\begin{equation}
- \frac{1}{4} W_m W_m - W^{(0)m} ,m + \Phi = 0,
\end{equation}

where $W_m W_m = W_m^{(0)} - W_m^{(0)} \Phi$ and $\Phi$ is determined by the matter field (see equation (32)); in the case of vacuum we have $\Phi = 0$.

We consider now the case $\epsilon = 0$:

\begin{equation}
W_1 = 0,
\end{equation}

\begin{equation}
W_m = W_m^{(0)}(u, x^k)
\end{equation}

\begin{equation}
H = H^{(0)}(u, x^i) + \frac{1}{2} \left( F - W^{(0)m} m \right) v,
\end{equation}

\footnotetext{2 Note that, in general, $W_1^{(0)} = 0$ but $W_1 n = -W_{n1} \neq 0$.}
where \( F = F(u, x^n) \) is a function satisfying:

\[
F_{,1} = 0, \quad F_{,m} = \Delta W_n^{(0)}
\]

Finally we have:

\[
\begin{align*}
\triangle H^{(0)} & \quad - \frac{1}{4} W_{mn} W^{mn} - 2 H^{(1),m} W^{(0)m} - H^{(1)} W^{(0)m},m \\
& \quad - W^{(0)m},mn + \Phi = 0,
\end{align*}
\]

\( VSI \) spacetimes with (44)-(46) and (49)-(51) are the higher dimensional analogues of the four-dimensional spacetimes of Petrov (Weyl) type III, PP-type O with \( \tau \neq 0 \) and \( \tau = 0 \), respectively.

### 2.3.2. Type III(a)

Further subclasses can be considered [2]. We discuss the subclass III(a) which is defined by vanishing \( C_{011(n+1)} \). In the \( \epsilon = 1 \) case

\[
C_{0112} = H^{(1),1}
\]

(54)

\[
C_{011(m+1)} = H^{(1),m} - \frac{1}{2} \frac{W^{(0)m}}{x^1},
\]

where \( H^{(1)} \) is the coefficient of \( v \) in (46) and \( m = 2, \ldots, N - 2 \). Vanishing of the Weyl components (54) implies \( H^{(1),m1} = 0 \) for consistency; these constraints give already

\[
W_m^{(0)} = \tilde{W}_m(u, x^n) \left( \frac{x^1}{2} \right)^2 + \tilde{\tilde{W}}_m(u, x^n),
\]

where the functions \( \tilde{W}_m, \tilde{\tilde{W}}_m \) do not depend on \( x^1 \). The Weyl components (54) vanish themselves when the functions \( W_m^{(0)} \) satisfy:

\[
W_m^{(0)m},m1 = \frac{2}{x^1} W_m^{(0)m},m
\]

(56)

\[
W_m^{(0)m},mn = \Delta W_m^{(0)} - \frac{W_m^{(0)1}}{x^1}
\]

(57)

Inserting here (55) we get the following conditions for the functions \( \tilde{W}_m(u, x^n), \tilde{\tilde{W}}_m(u, x^n) \):

\[
\begin{align*}
\tilde{W}_m^{mn} &= \Delta \tilde{W}_m \\
\tilde{W}_m^{m} &= \Delta \tilde{\tilde{W}}_m = 0
\end{align*}
\]

(58)

(59)

The class is characterized by

\[
\begin{align*}
W_1 &= - \frac{2}{x^1} v, \\
W_m &= \tilde{W}_m(u, x^n) \left( \frac{x^1}{2} \right)^2 + \tilde{\tilde{W}}_m(u, x^n), \\
H &= H^{(0)}(u, x^i) + \frac{1}{2} f(u, x^n) v + \frac{v^2}{2(\Delta x^1)^2}
\end{align*}
\]

(60)

(61)

(62)

where the function \( f(u, x^n) \) does not depend on \( x^1 \) and it is such that \( f_{,m} = \tilde{W}_m(u, x^n) \). A necessary (but not sufficient) condition for a metric to be in this subclass is that the functions \( \tilde{W}_m, \tilde{\tilde{W}}_m \) satisfy equations (58), (59). Furthermore,
they have to be such that (some of) the boost weight $-1$ $C_{1ijk}$ Weyl components are non-vanishing in order for the considered metric to be truly of type III(a).

An example of a spacetime in this subclass is constructed as follows. For odd $N$, consider the functions

\begin{align}
\tilde{W}_n &= -p_n(u)x^{n+1}Q^{\frac{n}{n+2}}, \\
\tilde{W}_{n+1} &= p_n(u)x^nQ^{\frac{n}{n+2}},
\end{align}

where $n$ only takes on even values of $m$ and $Q = \delta_{mmt}x^mx^{m'}$. The $p_n$ are arbitrary functions of $u$. For even $N$, consider the same functions except for $n = N - 2$:

\begin{equation}
\tilde{W}_{N-2} = 0
\end{equation}

Note that none of these functions depend on $x^1$, as required. The spacetimes are then characterized by equations (60)-(62), with $\tilde{W}_m(u, x^n) = f(u, x^n) = 0$. The choice of functions $W_m$ in $N$ odd (even) dimensions coincides with those of [4] for $N - 1$ even (odd) dimensions. However, these spacetimes are more general than the relativistic gyratons, since the functions $W_1$ and $H$ can depend on $v$; these solutions might be referred to as *Kundt gyratons*, in analogy with the pp-waves (no $v$-dependence) and Kundt waves ($v$-dependence) but which have the same (vanishing) $W_m$ functions.

Another example of a spacetime in this subclass with $\tilde{W}_m = 0$ is given by the six-dimensional metric:

\begin{align}
W_1 &= -\frac{2}{x^1}v, \\
W_2 &= f_2(u)x^3x^4, \\
W_3 &= f_3(u)x^2x^4, \\
W_4 &= f_4(u)x^2x^3, \\
H &= H^{(0)}(u, x^1) + \frac{v^2}{2(x^1)^2},
\end{align}

In the final type III(a) example, $H$ contains a linear dependence on $v$ (the two examples above can be generalized to include a linear dependence on $v$ as well). From equations (60)-(62), we consider the symmetric and antisymmetric parts of $\tilde{W}_{m,n}$. We impose the condition

\begin{equation}
(\tilde{W}_{m,n} - \tilde{W}_{n,m})_\ell = 0,
\end{equation}

along with the constraint (59) and recall that $f, \tilde{W}_m$ and $\tilde{W}_m$ contain no $x^1$ dependence. This results in the vanishing of all components, $C_{011(n+1)}$, of the Weyl tensor, whereas the remaining boost weight -1 components, $C_{1ijk}$, are related to the symmetric part of $\tilde{W}_{m,n}$. We note that, if instead of (71), we required that $\tilde{W}_{(m,n)} = 0$ (and also (59)), the resulting Weyl tensor is of reduced type N.

\footnote{There is no compact way to write down the $C_{1ijk}$ in a similar fashion to that in the III(a) $\epsilon = 0$ case (equation (70)), so we shall not explicitly display them here.}
In the $\epsilon = 0$ case

\begin{equation}
C_{011(n+1)} = H^{(1)}_{,n}
\end{equation}

where $H^{(1)}$ is the coefficient of $v$ in (51) and $n = 1, \ldots, N - 2$. The Weyl components vanish when the functions $W_{m}^{(0)}$ satisfy:

\begin{align}
W_{m}^{(0)}_{,m1} &= 0 \\
W_{m}^{(0)}_{,mn} &= \Delta W_{n}^{(0)}
\end{align}

These conditions can be written more compactly:

\begin{equation}
\partial_{m} W_{mn} = 0
\end{equation}

(and are consequently expressed in a form similar to the Maxwell equations in Euclidean space). The following class is obtained:

\begin{align}
W_{1} &= 0, \\
W_{m} &= W_{m}^{(0)}(u, x^{k}) \\
H &= H^{(0)}(u, x^{i})
\end{align}

A necessary (but not sufficient) condition for a metric to be in this subclass is that the functions $W_{m}$ satisfy equations (73), (74). Furthermore, the functions $W_{m}$ have to be such that (some of the) boost weight $-1$ Weyl components are non-vanishing:

\begin{equation}
C_{1(l+1)(n+1)(m+1)} = \frac{1}{2} (W_{m,n} - W_{n,m})_{,d} \neq 0
\end{equation}

This is equivalent to:

\begin{equation}
\partial_{d} W^{mn} \neq 0.
\end{equation}

In general, there is no Lorentz transformation such that these boost weight $-1$ components can be set to zero; hence (76)-(78) is in general of Weyl type III (and not of type N; i.e., in this subclass $\Psi_{d} = 0$ does not imply $\Psi_{ijk} = 0$). The higher dimensional relativistic gyraton spacetime in [5] is an example of a spacetime in this subclass. A five-dimensional example of another spacetime in this subclass is given by:

\begin{align}
W_{1} &= 0, \\
W_{2} &= f_{2}(u)x^{1}x^{3}, \\
W_{3} &= f_{3}(u)x^{1}x^{2}, \\
H &= H^{(0)}(u, x^{i})
\end{align}

Here, $f_{2}$ and $f_{3}$ are arbitrary functions of $u$. We note that metrics in this subclass have a covariantly constant null vector (see later).

2.3.3. Type N. The spacetime is of Weyl type N if:

\begin{equation}
\Psi_{ijk} = 0 = C_{1kij}, \Psi_{i} = 0 = C_{101i}.
\end{equation}

Further constraints on $\Psi_{ij}$, $\Psi_{ijk}$, etc. can be obtained by employing the Bianchi and Ricci identities.

Further progress can then be made by requiring the vanishing of boost weight -1 components of the Weyl tensor; using the above results and equations (40) and
A. COLEY, A. FUSTER, S. HERVIK AND N. PELAVAS

For $\epsilon = 1$ we have:

\begin{align}
W_1 &= -2\frac{v}{x^1}, \\
W_m &= x^n B_{nm}(u) + C_m(u), \\
H &= \frac{v^2}{2(x^1)^2} + H^{(0)}(u, x^i).
\end{align}

And (48) simplifies to

\begin{align}
\triangle (\frac{H^{(0)}}{x^1}) - \frac{1}{(x^1)^2} \sum W_m^2 - 2 \sum_{m<n} B_{mn}^2 + \Phi = 0,
\end{align}

where $W_m$ is given by (87). In the case $B_{nm} = C_m \equiv 0$ one obtains the higher dimensional Kundt waves in [4]. There are further subclasses in which further simplification occurs (in which certain components can be set to zero, and others can be set constant).

**Generalized pp-waves.** In the case $\epsilon = 0$ we have

\begin{align}
W_1 &= 0, \\
W_m &= x^1 C_m(u) + x^n B_{nm}(u), \\
H &= H^{(0)}(u, x^i),
\end{align}

and (53) reduces to:

\begin{align}
\triangle H^{(0)} - \frac{1}{2} \sum C_m^2 - 2 \sum_{m<n} B_{mn}^2 + \Phi = 0,
\end{align}

$B_{nm} = B_{[nm]}$ in (87) and (91). In (91) a type (2) coordinate transformation has been used to remove a gradient term and in (92) a type (3) transformation was used to eliminate the linear $v$ dependence. As in the four-dimensional case the term $x^1 C_m(u)$ can be transformed away (at the expense of introducing a non-vanishing $W_1$). However, unlike the four-dimensional case, terms linear in $x^n$ in $W_m$ cannot be transformed away (even if a non-zero $W_1$ is allowed).

Since both Weyl and Ricci tensors have only boost weight -2 components (i.e. both are of type N), VS1 spacetimes with (86)-(88) and (90)-(92) are the higher dimensional generalizations of Kundt and pp-waves (i.e., are the higher dimensional analogues of the four-dimensional spacetimes of Petrov (Weyl) type N, PP-type O with $\tau \neq 0$ and $\tau = 0$, respectively). Higher dimensional pp-wave spacetimes have been studied extensively [10, 11, 12, 13, 14].

2.3.4. **Type O.** The spacetime is of type O if the Weyl tensor vanishes. We consider the case where $\Phi = \Phi(u)$ in equation (52).

For $\epsilon = 1$, the function $H^{(0)}$ must satisfy:

\begin{align}
x^1 \left( \frac{H^{(0)}}{x^1} \right)_{,11} = \frac{1}{(x^1)^2} \sum W_m^2 - \frac{1}{8} \Phi, \quad H^{(0)}_{,mm} = \sum_{m<n} B_{mn}^2 - \frac{1}{8} \Phi
\end{align}

It can be seen that this cannot be accomplished in the case $\Phi = \Phi(u)$. However, the Weyl tensor does vanish for $\Phi = \Phi_0(u)x^1$ and

\begin{align}
H^{(0)} = \frac{1}{2} \sum W_m^2 - \frac{1}{16} \Phi_0(u)x^1 \left[ (x^1)^2 + (x^m)^2 \right] + x^1 F_0(u) + x^1 x^i F_i(u).
\end{align}
Here \( F_0(u), F_i(u) \) are arbitrary functions of \( u \). The last two terms in \( H^{(0)} \) correspond to the Weyl type O reduction of the higher dimensional Kundt waves. This particular form of \( \Phi \) is likely to be the most general one compatible with a vanishing Weyl tensor. It remains to clarify the associated type of matter field for this conformally flat spacetime and the given Ricci tensor. Null Maxwell fields and massless scalar fields can already be excluded, as is the case in four dimensions [15].

The case \( \epsilon = 0 \):

\[
H^{(0)} = \frac{1}{8} \left( \sum C_m^2 (x^1)^2 + \sum C_m C_n x^m x^n \right) + \frac{1}{2} x^1 x^m (C'_m + B_{mn} C_n) \\
+ \frac{1}{2} B_{ml} B_{nl} x^m x^n + x^1 F_i(u) - \frac{1}{16} \Phi \left[(x^1)^2 + (x^m)^2 \right].
\]

Here ‘ means derivative with respect to \( u \) and \( l \neq m, n \); \( F_i(u) \) are arbitrary functions of \( u \). The term \( x^1 F_i(u) \) corresponds to the Weyl type O reduction of generalized pp-waves with \( W_m = 0 \). In the case of matter depending on the spatial coordinates as well, \( \Phi(u, x^i) \), the last term on the right-hand side of equation (96) has to be replaced by some \( H^0_0 \Phi \) such that:

\[
H^0_{\Phi,ii} = -\frac{1}{8} \Phi, \quad H^0_{\Phi,ij} = 0, \quad i, j = 1, \ldots, 8.
\]

2.4. Ricci type O - vacuum. In the vacuum case and Weyl type III or N, equations are the same as for Ricci type N except \( \Phi \) is now zero. Finally, in the case of Weyl type O, we simply have N-dimensional Minkowski space.

3. Discussion

In Table 1 we collect together the results of our analysis, listing the metric functions (and any remaining constraints for the higher dimensional VSI spacetimes) according to the algebraic classification of the Weyl tensor.

In higher dimensions the spacetimes with vanishing zeroth order curvature invariants, called VSI0, have Ricci and Weyl type III, N or O [1]. Spacetimes with vanishing zeroth and first order invariants, called VSI1 spacetimes, were discussed in the case of four dimensions in [16]. It is plausible that in higher dimensions the proper VSI1 spacetimes (i.e., not VSI), have Weyl type N, Ricci type N or O, and admit an aligned geodesic null congruence. More specifically, consider the null frame \( e_a = \{ \ell, n, m_i \} \) where \( i = 2, \ldots, N-2 \). Then assuming the aligned geodesic null congruence has been affinely parametrized, we can write its covariant derivative \( L^i_{\alpha} \) as \( L^i_{\alpha,\beta} = L_{11}^i \ell^\alpha \ell^\beta + L_{14}^i \ell^\alpha m^\beta + L_{13}^i m^\alpha \ell^\beta + L_{43}^i m^\alpha m^\beta \). A decomposition of \( L_{ij} \) into symmetric and antisymmetric parts gives the expansion, shear, and twist of \( \ell \). The VSI1 subclasses will arise from the vanishing of certain combinations of these optical scalars. However, unlike the four dimensional result [16], VSI1 spacetimes do not exist in the higher dimensional Robinson-Trautman class [8, 17]. We expect the corresponding metrics to be contained in the higher dimensional analogue of the Plebański class along with a class containing the higher dimensional Hauser solution [18].

As noted earlier, the aligned, repeated, null vector \( \ell \) of [8] is a null Killing vector (KV) if and only if \( H_v = 0 \) and \( W_{i,v} = 0 \) (\( \epsilon = 0 \)) (whence the metric no longer has any \( v \) dependence). Furthermore, since \( L_{AB} := L_{A;B} = L_{(A;B)} \) it follows that
in this case if $\ell$ is a null KV then it is also covariantly constant. Without any further restrictions, the higher dimensional $VSI$ metrics admitting a null KV have Ricci and Weyl type III. The $VSI$ spacetimes are recurrent if $\epsilon = 0$ in $\ell$.

Therefore, in higher dimensions a $VSI$ spacetime which admits a covariantly constant null vector (CCNV) has a metric of the form (76)-(78) and, in general, is of Ricci type III and Weyl type III. These $VSI$ spacetimes are summarized in Table 2 (however, these spacetimes are not the most general spacetimes with a CCNV; see [19]). The subclass of Ricci type N CCNV spacetimes are related to the $(F = 1)$ chiral null models (satisfying eqn. (2.14) subject to eqns. (2.15)-(2.16)) of [11]. The subclass of Ricci type N and Weyl type III(a) spacetimes in which the functions $W_m$ in (77) satisfy the conditions (73), (74), (79) includes the relativistic gyratons [5], as was noted above. The subclass of Ricci type N and Weyl type N spacetimes are the generalized pp-wave spacetimes. These are the analogues of the four-dimensional Brinkmann pp-waves [20] (which are always of PP-type O (Ricci

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>Weyl type</th>
<th>Metric functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>III</td>
<td>$W_1 = 0$</td>
<td>$W_m = W_{m(0)}(u, x^k)$</td>
</tr>
<tr>
<td></td>
<td>$H = H^{(0)}(u, x^i) + \frac{1}{2} \left( F - W^{(0)m}_{,m} \right) v^2; , F(u, x^i)$ defined by (59)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Eq. (53)</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>N</td>
<td>$W_1 = 0$</td>
</tr>
<tr>
<td></td>
<td>$W_m = x^1 C_m(u) + x^n B_{nm}(u)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$H = H^{(0)}(u, x^i)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Eq. (63)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>N</td>
<td>$W_1 = -2x^1 v$</td>
</tr>
<tr>
<td></td>
<td>$W_m = W_{m(0)}(u, x^k)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$H = H^{(0)}(u, x^i) + \frac{1}{2} \left( F - W^{(0)m}_{,m} \right) v^2; , F(u, x^i)$ defined by (47)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Eq. (45)</td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>N</td>
<td>$W_1, W_m$ as in type N; $H^{(0)}$ given by (96), (97)</td>
</tr>
<tr>
<td></td>
<td>Eq. (48)</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: All higher dimensional $VSI$ spacetimes of Ricci type N. In the above, $m = 2, \ldots, N - 2$. Ricci type O (vacuum) spacetimes occur for $\Phi = 0$ in (48), (53), (89), (93).
Table 2: All higher-dimensional VSI spacetimes with a covariantly constant null vector are presented. The $\Phi_i$, where $i = 2, \ldots, N-1$, are given in equation (32).

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>Ricci type</th>
<th>Weyl type</th>
<th>Metric functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>III</td>
<td>III</td>
<td>$W_1 = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$W_m = W^{(0)}(u, x^k)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$H = H^{(0)}(u, x')$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$-\Delta H^{(0)} + \frac{1}{4} W_{mn} W^{mn} + W^{(0)mn} \delta_{mn} = \Phi + \delta_{mn} W_{0m}^0 \Phi^{(n+1)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$W^{(0)0} = 2\Phi_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$-\Delta W^{(0)} + W^{(0)mn} = 2\Phi_{n+1}$</td>
</tr>
<tr>
<td>N</td>
<td>III(a)</td>
<td></td>
<td>$W_1 = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$W_m = W^{(0)}(u, x^k)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$H = H^{(0)}(u, x')$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Eq. (53), $\left( H^{(1)} = 0 \right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$W_m$ satisfy (73), (74), (79)</td>
</tr>
<tr>
<td>N</td>
<td>Generalized pp waves; Eqns. (90) - (93)</td>
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</table>

It is the higher dimensional (generalized) pp-wave spacetimes that have been most studied in the literature. It is known that such spacetimes are exact solutions in string theory [21]. Recently, type-IIB superstrings in pp-wave backgrounds with an R-R five-form field were also shown to be exactly solvable [10]. In the context of string theory, higher dimensional generalizations of pp-wave backgrounds have been considered [11, 12], including string models corresponding not only to the NS-NS but also to certain R-R backgrounds [13, 14], and pp-waves in eleven- and ten-dimensional supergravity theory [22]. In four-dimensions, a wide range of VSI spacetimes have been shown to be exact solutions in string theory [23]. The same is expected in higher dimensions. Indeed, VSI supergravity solutions can be constructed [11, 24] (although attention has been mainly focussed on the plane wave versions of generalized pp-waves), and it is likely that all VSI spacetimes are solutions of superstring theory when supported by appropriate bosonic fields. In addition, the VSI CCNV metrics preserve supersymmetry when embedded in $N = 1$, $D = 10$ supergravity [25]. Whether this is also the case for other supergravity theories and/or more general VSI spacetimes remains to be studied. We shall return to this in future work.

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References