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Asymptotic equivalence of the discrete variational functional and a rate-large-deviation-like functional in the Wasserstein gradient flow of the porous medium equation

Manh Hong Duong

Department of Mathematics and Computer Sciences, Technische Universiteit
Eindhoven, The Netherlands.
Email: m.h.duong@tue.nl

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Abstract

In this paper, we study the Wasserstein gradient flow structure of the porous medium equation. We prove that, for the case of q -Gaussians on the real line, the functional derived by the JKO-discretization scheme is asymptotically equivalent to a rate-large-deviation-like functional. The result explains why the Wasserstein metric as well as the combination of it with the Tsallis-entropy play an important role.

Key words. Porous medium equation, Gamma-convergence, Wasserstein gradient flow, variational methods

1 Introduction

1.1 The porous medium equation

In a seminal paper, Otto [Ott01] shows that the porous medium equation

$$\partial_t \rho(t, x) = \Delta \rho^{2-q}(t, x), \quad \text{for } (t, x) \in (0, \infty) \times \mathbf{R}^d \quad \text{and } \rho(0, x) = \rho_0(x), \quad (1)$$

can be interpreted as a gradient flow of the q -Tsallis entropy functional¹,

$$E_q(\rho) = \begin{cases} \frac{1}{1-q} \int_{\mathbf{R}^d} \rho(x) [\rho(x)^{1-q} - 1] dx & \text{if } q \neq 1, \\ \int_{\mathbf{R}^d} \rho(x) \log \rho(x) dx & \text{if } q = 1, \end{cases} \quad (2)$$

with respect to the Wasserstein distance W_2 on the space of probability measures with finite second moment $\mathcal{P}_2(\mathbf{R}^d)$,

$$W_2(\mu, \nu) = \left\{ \inf_{P \in \Gamma(\mu, \nu)} \int_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^2 P(dx dy) \right\}^{\frac{1}{2}}, \quad (3)$$

where $\Gamma(\mu, \nu)$ is the set of all probability measures on $\mathbf{R}^d \times \mathbf{R}^d$ with marginals μ and ν .

This statement can be understood in a variety of different ways. In [Ott01], Otto shows this from a differential geometry point of view by considering $(\mathcal{P}_2(\mathbf{R}^d), W_2)$ as an infinite dimensional Riemannian manifold. However, for the purpose of this paper, the most useful way is that the solution $t \mapsto \rho(t, x)$ can be approximated by the JKO-discretization scheme. Let $T > 0$ be given

¹ E_1 is the Boltzmann-Shannon entropy

and $h > 0$ be a time step. The discrete approximated solution $\rho^{(n)} = \rho(nh); n = 0, \dots, \lfloor \frac{T}{h} \rfloor$ is defined recursively by (see also [JKO98, WW10] and [AGS08] for an exposition of this subject)

$$\begin{aligned} \rho^{(0)} &= \rho_0, \\ \rho^{(n)} &\in \operatorname{argmin}_{\rho \in \mathcal{P}_2(\mathbf{R}^d)} K_h(\rho, \rho^{(n-1)}), \quad K_h(\rho, \rho^{(n-1)}) = \frac{1}{4h} W_2^2(\rho, \rho^{(n-1)}) + \frac{1}{2} (E_q(\rho) - E_q(\rho^{(n-1)})). \end{aligned} \quad (4)$$

The theory of Wasserstein gradient flow has been developed tremendously since [JKO98, Ott01]. At least two different approaches have been investigated. The first one is to explore a larger class of evolution equations that are Wasserstein (generalized/modified) gradient flows. Many equations are now proven to belong to this class [GO01, Gla03, Agu05, GST09, MMS09, Lis09, FG10, Mie11, LMS12]. Recently some attempts have been made to extend the theory to discrete setting [Maa11, Mie12] or to systems that contain also conservative behavior [Hua00, HJ00, DPZ13]. The second approach is to understand why the Wasserstein metric and the combination of it with the entropy functional appear in the setting. This direction recently has been received a lot of attraction. In [ADPZ11, DLZ12, DLR12, PRV11, Léo12], the authors address this question, for the case of the linear diffusion equation, i.e., for $q = 1$, by establishing an intriguing connection between a microscopic many-particle model and the macroscopic gradient flow structure of the diffusion equation. They show that the functional K_h in (4) is asymptotically equivalent as $h \rightarrow 0$ to a discrete rate functional \tilde{J}_h that comes from the large deviation principle of the microscopic model. We now briefly recall the results in these papers in more details. The rate functional $\tilde{J}_h: \mathcal{P}_2(\mathbf{R}^d) \rightarrow [0, +\infty]$ is defined by

$$\tilde{J}_h(\rho|\rho_0) = \inf_{Q \in \Gamma(\rho_0, \rho)} H(Q|\tilde{Q}_{0 \rightarrow h}), \quad (5)$$

where

$$\tilde{Q}_{0 \rightarrow h}(dxdy) = p_h(x, y)\rho_0(dx)dy; \quad p_h(x, y) = \frac{1}{(4\pi h)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4h}},$$

and $H(Q|\tilde{Q}_{0 \rightarrow h})$ is the relative entropy of Q with respect to $\tilde{Q}_{0 \rightarrow h}$

$$H(Q|\tilde{Q}_{0 \rightarrow h}) = \int_{\mathbf{R}^{2d}} \log \left(\frac{dQ}{d\tilde{Q}_{0 \rightarrow h}} \right) dQ, \quad \text{if } \frac{dQ}{d\tilde{Q}_{0 \rightarrow h}} \text{ exists and otherwise being } +\infty.$$

Note that in the case of $d = 1$ and $\rho_0(x) = \mathcal{N}(\mu_0, \sigma_0^2)$, then $\tilde{Q}_{0 \rightarrow h}$ is a bivariate Gaussian $\mathcal{N}(\mu_0, \sigma_0, \mu_0, \sigma_0^2, \theta_h)$, where $\sigma_h^2 = \sigma_0^2 + 2h$, and $\theta_h = \frac{\sigma_0}{\sigma_h}$.

It is proven in [Léo12]² that

$$4h\tilde{J}_h(\cdot, \rho_0) \xrightarrow{\Gamma} W_2^2(\cdot, \rho_0) \quad \text{as } h \rightarrow 0, \quad (6)$$

where Γ is denoted for Gamma-convergence, which is introduced in Section 2. This result is improved to the next order in [ADPZ11, DLZ12, DLR12, PRV11] to give

$$\tilde{J}_h(\cdot|\rho_0) - \frac{1}{4h} W_2^2(\cdot, \rho_0) \xrightarrow{\Gamma} \frac{1}{2} (E_1(\cdot) - E_1(\rho_0)) \quad \text{as } h \rightarrow 0. \quad (7)$$

(6) and (7) indicate the asymptotic Gamma development of the functional $\tilde{J}_h(\cdot|\rho_0)$

$$\tilde{J}_h(\rho|\rho_0) \approx \frac{1}{4h} W_2^2(\rho, \rho_0) + \frac{1}{2} (E_1(\rho) - E_1(\rho_0)) + o(1) \quad \text{as } h \rightarrow 0. \quad (8)$$

It is worth pointing out that the right hand side of (8) is exactly the functional K_h in the variational scheme (4). It is this asymptotic Gamma development that not only provides the

²This result was appeared first in 2007 in his preprint paper [Léo07]

link between the microscopic model and the Wasserstein gradient flow formulation of the linear diffusion equation but also explains why the Wasserstein metric as well as the combination of it with the entropy play a role in the setting.

The aim of this paper is to generalise (6)- (7) to the nonlinear porous medium equation for the class of q -Gaussian measures in 1D. As we see in section 2, this class plays an important role because it is invariant under the semigroup of the porous-medium equation and is isometric to the space of Gaussian measures with respect to the Wasserstein metric. We now describe our result in the next Section.

1.2 Main result

Before introducing the main result of the present paper, we recall corresponding concepts for the q -Gaussian measures.

The q -exponential function and its inverse, the q -logarithmic function, are defined respectively by

$$\exp_q(t) = [1 + (1 - q)t]_+^{\frac{1}{1-q}}, \quad (9)$$

where $[x]_+ = \max\{0, x\}$ and by convention $0^a := \infty$; and

$$\log_q(t) = \frac{t^{1-q} - 1}{1 - q} \quad \text{for } t > 0. \quad (10)$$

Given m , that is specified later on, the m -relative entropy between $Q(dx) = f(x)dx$ and $P(dx) = g(x)dx$

$$H_m(Q||P) = \frac{1}{2 - m} \int [f \log_m f - g \log_m g - (2 - m) \log_m g(f - g)] dx \quad (11)$$

$$= \frac{1}{2 - m} \int [f \log_m f + (1 - m)g \log_m g - (2 - m)f \log_m g] dx. \quad (12)$$

For $v \in \mathbf{R}^d$ and $V \in \text{Sym}^+(d, \mathbf{R})$, which is the set of all symmetric positive definite matrices of size d , the q -Gaussian measure with mean v and covariance matrix V is

$$\mathcal{N}_q(v, V) = C_0(q, d)(\det V)^{\frac{1}{2}} \exp_q \left[-\frac{1}{2} C_1(q, d) \langle x - v, V^{-1}(x - v) \rangle \right] \mathcal{L}^d, \quad (13)$$

where \mathcal{L}^d is the Lebesgue measure on \mathbf{R}^d and $C_0(q, d), C_1(q, d)$ are explicit positive constants depending only on d and q and are given in section 2.

In particular, the q -Gaussian measure in 1D has density

$$\mathcal{N}_q(\mu, \sigma^2) = \frac{C_0(q, 1)}{\sigma} \exp_q \left(-\frac{1}{2} C_1(q, 1) \frac{(x - \mu)^2}{\sigma^2} \right). \quad (14)$$

While in 2D, we call q -bivariate Gaussian $\mathcal{N}_q(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \theta)$ with density

$$\nu(x, y) = \frac{C_0(q, 2)}{\sigma_1 \sigma_2 \sqrt{1 - \theta^2}} \exp_q \left\{ -\frac{1}{2} C_1(q, 2) \frac{1}{1 - \theta^2} \left[\frac{(x - \mu_1)^2}{\sigma_1^2} + \frac{(y - \mu_2)^2}{\sigma_2^2} - \frac{2\theta(x - \mu_1)(y - \mu_2)}{\sigma_1 \sigma_2} \right] \right\} \quad (15)$$

which corresponds to the mean vector $\boldsymbol{\mu}$ and covariance matrix Σ

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \theta \sigma_1 \sigma_2 \\ \theta \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}.$$

It is shown in section 2 that, if the initial data is $\rho_0(x) = \mathcal{N}_q(\mu_0, C\sigma_0^2)$, then the solution to the porous medium equation at time t is $\rho(t, x) = \mathcal{N}_q(\mu_0, C(t + \sigma_0^{3-q})^{\frac{2}{3-q}})$, where C is an explicit constant given in section (27).

We are now in the position to introduce the main result.

Theorem 1.1. Let $d = 1$, $q \in Q_d \equiv (0, 1) \cup \left(1, \frac{d+4}{d+2}\right)$ and $\mathcal{N}_0 = \mathcal{N}_q(\mu_0, C\sigma_0^2)$ be given.

We set $m = 3 - \frac{2}{q}$, $\sigma_h^2 = (h + \sigma_0^{3-q})^{\frac{2}{3-q}}$ and

$$Q_{0 \rightarrow h} = \mathcal{N}_m(\mu_0, C\sigma_0^2, \mu_0, C\sigma_h^2, \frac{\sigma_0}{\sigma_h}), \quad (16)$$

For $\mathcal{N}_q = \mathcal{N}_q(\mu, C\sigma^2)$, we define the functional $J_h(\mathcal{N}_q, \mathcal{N}_0)$ by

$$J_h(\mathcal{N}_q | \mathcal{N}_0) := \inf_{Q \in \mathcal{Q}} H_m(Q || Q_{0 \rightarrow h}), \quad (17)$$

where

$$\mathcal{Q} := \{ \mathcal{N}_m(\mu_0, C\sigma_0^2, \mu, C\sigma^2, \theta) \mid \theta \in \mathbf{R} \}.$$

Then there exist explicit constants $a = a(\sigma_0, q)$ and $b = b(\sigma_0, q)$, which are given respectively in (64) and (65), such that the following statements hold

1. $a(\sigma_h^2 - \sigma_0^2)^{\frac{1}{q}} J_h(\cdot | \mathcal{N}_0) \xrightarrow{\Gamma} W_2^2(\cdot, \mathcal{N}_0)$ as $h \rightarrow 0$.
2. $ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\cdot | \mathcal{N}_0) - \frac{b}{(\sigma_h^2 - \sigma_0^2)} W_2^2(\cdot, \mathcal{N}_0) \xrightarrow{\Gamma} E_q(\cdot) - E_q(\mathcal{N}_0)$ as $h \rightarrow 0$.
3. If $0 < q < 1$, then $ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\cdot | \mathcal{N}_0) - \frac{1}{2h} W_2^2(\cdot, \mathcal{N}_0) \xrightarrow{\Gamma\text{-lim inf}} E_q(\cdot) - E_q(\mathcal{N}_0)$ as $h \rightarrow 0$

When $q \rightarrow 1$, then $a \rightarrow 4$, $b \rightarrow \frac{1}{2}$, $\sigma_h^2 - \sigma_0^2 \rightarrow h$ and we recover (6)-(7) for the diffusion equation.

1.3 Discussion

Given \mathcal{N}_q and \mathcal{N}_0 , both $H_m(\mathcal{N}_q || \mathcal{N}_0)$ and $J_h(\mathcal{N}_q | \mathcal{N}_0)$ are always non-negative and are equal to 0 if and only if $\mathcal{N}_q = \mathcal{N}_0$. By its definition, $J_h(\mathcal{N}_q | \mathcal{N}_0)$ measures the deviation of \mathcal{N}_q from the solution of the porous medium equation at time h given the initial data \mathcal{N}_0 . Hence minimizing J_h means to find the best approximation of the solution at a time h . As a consequence of the Gamma convergence, minimizers of the functionals converge to minimizer of the Gamma-limit functional. Thus the main theorem explains why the Wasserstein metric is involved and why we should minimize the combination of it with the Tsallis entropy.

For the linear diffusion equation, by Sanov's theorem the relative entropy is the *static* rate functional of the empirical process of many i.i.d particles. While the functional \tilde{J}_h in (5) is the rate functional after time h of the empirical process of many-Brownian motions. Hence it has a clear microscopic interpretation. The m -relative entropy H_m and the functional J_h are defined and have similar properties as H and \tilde{J}_h . However, it is unclear whether J_h can be proven to be the rate functional of some microscopic stochastic process. In a recent paper [BLMV13], the authors introduce a stochastic particle system, the Ginzburg-Landau dynamics, and show that the porous medium equation is the hydrodynamic limit of the system. Moreover, the m -relative entropy is the static rate functional of the invariant measures. It would be interesting to study whether the functional \tilde{J}_h is actually (or related to) the rate functional at time h of the Ginzburg-Landau dynamics. Another question for future research is to extend the main theorem to a larger class of measures.

1.4 Organisation of the paper

The rest of the paper is organized as follows. In section 2, we first recall the definition of Gamma convergence and relevant properties of the q -Gaussians. Next we compute the functional J_h in section 3. Finally, the proof of the main theorem is given in section 4.

2 Preliminaries

We first recall the definition of Gamma convergence for the reader's convenience.

Definition [Bra02] Let X be a metric space. We say that a sequence $f_n: X \rightarrow \overline{\mathbf{R}}$ Γ -converges in X to $f: X \rightarrow \overline{\mathbf{R}}$, denoted by $f_n \xrightarrow{\Gamma} f$, if for all $x \in X$ we have

1. (lower bound part) For every sequence x_n converging to x

$$\liminf_{n \rightarrow \infty} f_n(x_n) \geq f(x), \quad (18)$$

2. (upper bound part) There exists a sequence x_n converging to x such that

$$\lim_{n \rightarrow \infty} f_n(x_n) = f(x). \quad (19)$$

If f_n satisfy the lower (or upper, respectively) bound part then we write $f_n \xrightarrow{\Gamma\text{-lim inf}} f$ (or $f_n \xrightarrow{\Gamma\text{-lim sup}} f$ respectively).

2.1 Properties of q -Gaussian measures

The q -exponential function and q -logarithmic function satisfy the following properties

$$\exp_q(\log_q x) = \log_q(\exp_q x) = x, \quad (20)$$

and

$$\begin{aligned} \log_q(xy) &= \log_q x + \log_q y + (1-q) \log_q(x) \log_q y \\ &= \log_q x + x^{1-q} \log_q y \\ &= \log_q y + y^{1-q} \log_q x. \end{aligned} \quad (21)$$

The constants $C_0(q, d)$ and $C_1(q, d)$ in (13) are given by

$$C_1(q, d) = \frac{2}{2 + (d+2)(1-q)}, \quad (22)$$

and

$$C_0(q, d) = \begin{cases} \frac{\Gamma(\frac{2-q+d}{1-q} + \frac{d}{2})}{\Gamma(\frac{2-q}{1-q})} \left(\frac{(1-q)C_1(q, d)}{2\pi} \right)^{\frac{d}{2}} & \text{if } 0 < q < 1, \\ \frac{\Gamma(\frac{1}{q-1})}{\Gamma(\frac{1}{q-1} - \frac{d}{2})} \left(\frac{(q-1)C_1(q, d)}{2\pi} \right)^{\frac{d}{2}} & \text{if } 1 < q < \frac{d+4}{d+2}. \end{cases} \quad (23)$$

2.2 q -Gaussian measures and solutions of the porous medium equation

The porous medium equation (1) has a self-similar solution, which is called the Barenblatt-Pattle solution, of the form

$$\begin{aligned} \rho_q(x, t) &:= \left[At^{-d\alpha(1-q)} - B|x|^2 t^{-1} \right]_+^{\frac{1}{1-q}} \\ &= \left[A - B|x|^2 t^{-2\alpha} \right]_+^{\frac{1}{1-q}} t^{-d\alpha}, \end{aligned} \quad (24)$$

where

$$\alpha = \alpha(q, d) := \frac{1}{d(1-q) + 2}, B = B(q, d) := \frac{(1-q)\alpha}{2(2-q)}, \quad (25)$$

and A is a normalization constant

$$\int_{\mathbf{R}} \rho_q(x, t) \mathcal{L}^d(dx) = 1.$$

More precisely,

$$A := C_0^{2\alpha(1-q)} \left[\frac{\alpha}{(2-q)C_1} \right]^{d\alpha(1-q)}. \quad (26)$$

It is straightforward to see that

$$\rho_q(x, t) = \mathcal{N}_q(0, Ct^{2\alpha}I_d),$$

where

$$C := \frac{(2-q)C_1}{\alpha} A. \quad (27)$$

It is well known that a solution to the diffusion equation is obtained by a convolution of an initial data with the diffusion kernel. Hence if the initial data is a Gaussian measure $\mathcal{N}(v, V)$ then the solution at time t is again a Gaussian $\mathcal{N}(v, V_t)$, which is given by

$$\mathcal{N}(v, V_t) = \mathcal{N}(v, V) * \mathcal{N}(0, 2tI_d) = \mathcal{N}(v, V + 2tI_d).$$

In [Tak12, OW10], the authors show that a similar statement holds for the porous medium equation on the space of q -Gaussian measures. That is, a solution to the porous medium equation with an initial data being a q -Gaussian measure is again a q -Gaussian for all time. Moreover, to find a solution at time $t > 0$, it reduces to solving an ordinary differential equation for the covariance matrix. Let Θ be a map on $\text{Sym}^+(d, \mathbf{R})$ defined by

$$\Theta(V) := (\det V)^{-\alpha(1-q)} V. \quad (28)$$

We note that $\rho_q(x, t) = \mathcal{N}_q(0, C\Theta(tI_d))$.

Proposition 2.1 ([Tak12]). *For any $q \in Q_d$ and $V \in \text{Sym}^+(d, \mathbf{R})$, we set the time-dependent matrix V_t as*

$$\Theta(V_t) = \Theta(V) + \sigma(t)I_d, \quad \frac{d}{dt}\sigma(t) = 2\alpha(\det \Theta(V_t))^{-\frac{1-q}{2}}. \quad (29)$$

Then $\mathcal{N}_q(v, C\Theta(V_t))$ is a solution to the porous medium equation.

Remark 2.2. *The assertion also holds true for $q = 1$.*

We will work out this theorem in 1D in more detail. We calculate relevant variables.

$$\alpha = \frac{1}{3-q}; \quad \Theta(V) = (\det V)^{-\alpha(1-q)} V = V^{1-\alpha(1-q)} = V^{\frac{2}{3-q}}; \quad \det \Theta(V_t) = V_t. \quad (30)$$

The ODE becomes

$$\Theta(V_t) = \Theta(V) + \sigma(t)I_d, \quad \frac{d}{dt}\sigma(t) = 2\alpha(\Theta(V_t))^{-\frac{1-q}{2}}.$$

Solving this ODE we get

$$\Theta(V_t) = \left[\alpha(3-q)t + \Theta(V)^{\frac{3-q}{2}} \right]^{\frac{2}{3-q}} = (t + V)^{\frac{2}{3-q}}.$$

So if $\rho_0(x) = \mathcal{N}_q(v, CV^{\frac{2}{3-q}})$ then $\rho(t, x) = \mathcal{N}_q(v, C(t + V)^{\frac{2}{3-q}})$ for all $t > 0$. In other words, if $\rho_0(x) = \mathcal{N}_q(\mu_0, C\sigma_0^2)$ then $\rho(t, x) = \mathcal{N}_q(\mu_0, C(t + \sigma_0^{3-q})^{\frac{2}{3-q}})$ for all $t > 0$.

2.3 q -Gaussian and the Wasserstein metric

The q -Gaussian measures have another important property stated in the following Proposition.

Proposition 2.3 ([Tak12]). *For any $q \in Q_d$, the space of q -Gaussian measures is convex and isometric to the space of Gaussian measures with respect to the Wasserstein metric.*

Hence

$$W_2(\mathcal{N}_q(\mu, \Sigma), \mathcal{N}_q(\nu, V))^2 = W_2(\mathcal{N}(\mu, \Sigma), \mathcal{N}(\nu, V))^2 = |\mu - \nu|^2 + \text{tr}\Sigma + \text{tr}V - 2\mathbf{tr}\sqrt{\Sigma^{\frac{1}{2}}V^{\frac{1}{2}}}.$$

In particular

$$W_2(\mathcal{N}_q(\mu_1, \sigma_1^2), \mathcal{N}_q(\mu_2, \sigma_2^2))^2 = (\mu_1 - \mu_2)^2 + (\sigma_1 - \sigma_2)^2. \quad (31)$$

3 Computing the functional J_h

In this section, we compute the functional J_h explicitly. The following Proposition is true for any dimensional d . For clarity, we use a boldface font for vectors.

Proposition 3.1. *Let $P(d\mathbf{x}) = g(\mathbf{x})d\mathbf{x}$ be absolutely continuous with respect to the Lebesgue measure in \mathbf{R}^d . Let \mathcal{Q} be the set of all Borel measures $Q = f(\mathbf{x})d\mathbf{x}$ in \mathbf{R}^d satisfying*

$$\int r_i(\mathbf{x}) \cdot f(\mathbf{x})d\mathbf{x} = a_i, \quad i \in \{1, 2, \dots, N\}, \quad (32)$$

where $r_i(\mathbf{x}): \mathbf{R}^d \rightarrow \mathbf{R}$ are given functions and $a_i \in \mathbf{R}$.

Assume that there is a measure $Q^* \in \mathcal{Q}$ that has a density satisfying the equation

$$\log_m f^*(\mathbf{x}) = \log_m g(\mathbf{x}) + \sum_{i=1}^N \lambda_i r_i(\mathbf{x}), \quad (33)$$

for some $\lambda_i \in \mathbf{R}$. Then there holds

$$H_m(Q\|P) = H_m(Q^*\|P) + H_m(Q\|Q^*) \quad \text{for all } Q \in \mathcal{Q}. \quad (34)$$

and as a consequence, Q^* is the unique minimiser of $H_m(Q\|P)$ over all $Q \in \mathcal{Q}$.

Proof. We have

$$\begin{aligned} (2-m)H_m(Q\|P) &\stackrel{(11)}{=} \int f \log_m f + (1-m)g \log_m g - (2-m)f \log_m g \\ &= \int f \log_m f + (1-m)f^* \log_m f^* - (2-m)f \log_m f^* \\ &\quad + \int (2-m)f(\log_m f^* - \log_m g) - (1-m)(f^* \log_m f^* - g \log_m g) \\ &\stackrel{(11)}{=} (2-m)H_m(Q^*\|Q) + \int (2-m)f(\log_m f^* - \log_m g) - (1-m)(f^* \log_m f^* - g \log_m g) \end{aligned} \quad (35)$$

We rewrite the second line in (35) using (32) and (33)

$$\begin{aligned}
& \int (2-m)f(\log_m f^* - \log_m g) - (1-m)(f^* \log_m f^* - g \log_m g) \\
& \stackrel{(32)}{=} (2-m) \int f \sum_{i=1}^N \lambda_i r_i - (1-m) \int (f^* \log_m f^* - g \log_m g) \\
& = (2-m) \sum_{i=1}^N \lambda_i \int f r_i - (1-m) \int (f^* \log_m f^* - g \log_m g) \\
& \stackrel{(33)}{=} (2-m) \sum_{i=1}^N \lambda_i \int f^* r_i - (1-m) \int (f^* \log_m f^* - g \log_m g) \\
& = (2-m) \int f^* \sum_{i=1}^N \lambda_i r_i - (1-m) \int (f^* \log_m f^* - g \log_m g) \\
& \stackrel{(33)}{=} (2-m) \int f^*(\log_m f^* - \log_m g) - (1-m) \int (f^* \log_m f^* - g \log_m g) \\
& = \int f^* \log_m f^* - (2-m)f^* \log_m g + (1-m)g \log_m g \\
& \stackrel{(11)}{=} (2-m)H_m(Q^* \| P) \tag{36}
\end{aligned}$$

From (35) and (36) we obtain (34). \square

Remark 3.2. *The property that the relative entropy and the m -relative entropy satisfy a generalized Pythagorean relation is well-known in the literature, see for instance [Csi75] and [OW10] for similar relationship.*

We now apply this Proposition to the q -bivariate measures.

Proposition 3.3. *Let $P = \mathcal{N}_m(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \theta)$ be given. We define*

$$\mathcal{Q} := \left\{ \mathcal{N}_m(\nu_1, \xi_1^2, \nu_2, \xi_2^2, \tilde{\theta}) \mid \tilde{\theta} \in \mathbf{R} \right\} \tag{37}$$

The minimizer of the minimization problem

$$\min_{Q \in \mathcal{Q}} H_m(Q \| P), \tag{38}$$

is given by

$$Q^* = \mathcal{N}_m(\nu_1, \xi_1^2, \nu_2, \xi_2^2, \eta), \tag{39}$$

where

$$\frac{\eta}{(1-\eta^2)^{\frac{3-m}{2}}} = \frac{\theta}{(1-\theta^2)^{\frac{3-m}{2}}} \left(\frac{\xi_1 \xi_2}{\sigma_1 \sigma_2} \right)^{2-m}. \tag{40}$$

Proof. We actually prove a stronger statement, namely that Q^* is the minimiser of $H(\cdot \| P)$ over $\tilde{\mathcal{Q}}$, which is defined as the set of all $Q(d\mathbf{x}) = f(\mathbf{x})d\mathbf{x}$ satisfying

$$\begin{aligned}
\int f(x, y) dx dy &= 1, \\
\int x f(x, y) dx dy &= \nu_1, & \int y f(x, y) dx dy &= \nu_2, \\
\int x^2 f(x, y) dx dy &= \xi_1^2 + \nu_1^2, & \int y^2 f(x, y) dx dy &= \xi_2^2 + \nu_2^2.
\end{aligned}$$

Since $Q^* \in \mathcal{Q} \subset \widetilde{\mathcal{Q}}$, Q^* is also the unique minimizer on \mathcal{Q} . Let $g(\mathbf{x})$ and $f^*(\mathbf{x})$ respectively denote the densities of P and Q^* . By Proposition 3.1, f^* satisfies the following equation

$$\log_m f^*(\mathbf{x}) = \log_m g(\mathbf{x}) + \sum_{i=1}^5 \lambda_i r_i(\mathbf{x}), \quad (41)$$

where $r_1 = 1, r_2 = x, r_3 = y, r_4 = x^2, r_5 = y^2$. To get the equation (40), we simply equalise the coefficients of xy in two sides of (41) and obtain

$$\frac{\theta}{\sigma_1 \sigma_2 (1 - \theta^2)} \left[\frac{C_0(m, d)}{\sigma_1 \sigma_2 \sqrt{1 - \theta^2}} \right]^{1-m} = \frac{\eta}{\xi_1 \xi_2 (1 - \eta^2)} \left[\frac{C_0(m, d)}{\sigma_1 \sigma_2 \sqrt{1 - \eta^2}} \right]^{1-m},$$

which is equivalent to

$$\frac{\eta}{(1 - \eta^2)^{\frac{3-m}{2}}} = \frac{\theta}{(1 - \theta^2)^{\frac{3-m}{2}}} \left(\frac{\xi_1 \xi_2}{\sigma_1 \sigma_2} \right)^{2-m}.$$

□

We now compute the m -entropy of a m -Gaussian and the m -relative entropy between two m -Gaussians.

Proposition 3.4. *We have*

$$E_m(\mathcal{N}_m(\boldsymbol{\mu}, \Sigma)) - E_m(\mathcal{N}_m(\boldsymbol{\mu}, V)) = (2 - m)C_1(m, d) \left(\frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}} \right)^{1-m} \log_m \frac{(\det V)^{\frac{1}{2}}}{(\det \Sigma)^{\frac{1}{2}}}, \quad (42)$$

and

$$H_m(\mathcal{N}_m(\boldsymbol{\mu}, \Sigma), \mathcal{N}_m(\boldsymbol{\nu}, V)) = \frac{1}{2}C_1(m, d) \left(\frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}} \right)^{1-m} \left[\text{tr}(V^{-1}\Sigma) + \langle \boldsymbol{\mu} - \boldsymbol{\nu}, V^{-1}(\boldsymbol{\mu} - \boldsymbol{\nu}) \rangle + 2 \log_m \frac{(\det \Sigma)^{\frac{1}{2}}}{(\det V)^{\frac{1}{2}}} - d \right] \quad (43)$$

Remark 3.5. *When $m = 1$, we recover the corresponding formula for the Gaussian measures.*

Proof. We denote by $f(\mathbf{x})$ and $g(\mathbf{x})$ respectively the densities of $\mathcal{N}_m(\boldsymbol{\mu}, \Sigma)$ and $\mathcal{N}_m(\boldsymbol{\nu}, V)$. Using the explicit formula of f and g as in (13) and by a straightforward calculation we get

$$\begin{aligned} \int f \log_m f d\mathbf{x} &= \log_m \left[\frac{C_0(m, d)}{(\det \Sigma)^{\frac{1}{2}}} \exp_m \left(-\frac{d}{2}C_1(m, d) \right) \right] \\ &\stackrel{(21)}{=} \left(-\frac{d}{2}C_1(m, d) \right) + \log_m \frac{C_0(m, d)}{(\det \Sigma)^{\frac{1}{2}}} + (1 - m) \left(-\frac{d}{2}C_1(m, d) \right) \log_m \frac{C_0(m, d)}{(\det \Sigma)^{\frac{1}{2}}} \end{aligned} \quad (44)$$

$$\begin{aligned} \int g \log_m g d\mathbf{x} &= \log_m \left[\frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}} \exp_m \left(-\frac{d}{2}C_1(m, d) \right) \right] \\ &\stackrel{(21)}{=} \left(-\frac{d}{2}C_1(m, d) \right) + \log_m \frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}} + (1 - m) \left(-\frac{d}{2}C_1(m, d) \right) \log_m \frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}} \end{aligned} \quad (45)$$

$$\int f \log_m g d\mathbf{x} = \log_m \left[\frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}} \exp_m \left(-\frac{1}{2}C_1(m, d)(\text{tr}(V^{-1}\Sigma) + \langle \boldsymbol{\mu} - \boldsymbol{\nu}, V^{-1}(\boldsymbol{\mu} - \boldsymbol{\nu}) \rangle) \right) \right]. \quad (46)$$

$$\stackrel{(21)}{=} \left(-\frac{1}{2}C_1(m, d)(\text{tr}(V^{-1}\Sigma) + \langle \boldsymbol{\mu} - \boldsymbol{\nu}, V^{-1}(\boldsymbol{\mu} - \boldsymbol{\nu}) \rangle) \right) + \log_m \frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}} \quad (47)$$

$$+ (1 - m) \left(-\frac{1}{2}C_1(m, d)(\text{tr}(V^{-1}\Sigma) + \langle \boldsymbol{\mu} - \boldsymbol{\nu}, V^{-1}(\boldsymbol{\mu} - \boldsymbol{\nu}) \rangle) \right) \log_m \frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}} \quad (48)$$

Hence

$$\begin{aligned} \int (f \log_m f - g \log_m g) d\mathbf{x} &= \left(1 - (1-m)\frac{d}{2}C_1(m, d)\right) \left[\log_m \frac{C_0(m, d)}{(\det \Sigma)^{\frac{1}{2}}} - \log_m \frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}}\right] \\ &= \left(1 - (1-m)\frac{d}{2}C_1(m, d)\right) \left(\frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}}\right)^{1-m} \log_m \frac{(\det V)^{\frac{1}{2}}}{(\det \Sigma)^{\frac{1}{2}}}. \end{aligned} \quad (49)$$

and

$$\int (g - f) \log_m g d\mathbf{x} = \frac{1}{2}C_1(m, d) (\text{tr}(V^{-1}\Sigma) + \langle \mu - \nu, V^{-1}(\mu - \nu) \rangle - d) \left[1 + (1-m) \log_m \frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}}\right] \quad (50)$$

$$= \frac{1}{2}C_1(m, d) (\text{tr}(V^{-1}\Sigma) + \langle \mu - \nu, V^{-1}(\mu - \nu) \rangle - d) \left[\frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}}\right]^{1-m} \quad (51)$$

Since

$$1 - (1-m)\frac{d}{2}C_1(m, d) = (2-m)C_1(m, d),$$

We get

$$\begin{aligned} E_m(\mathcal{N}_m(\boldsymbol{\mu}, \Sigma)) - E_m(\mathcal{N}_m(\boldsymbol{\mu}, V)) &= \int (f \log_m f - g \log_m g) d\mathbf{x} \\ &= (2-m) \left(\frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}}\right)^{1-m} \log_m \frac{(\det V)^{\frac{1}{2}}}{(\det \Sigma)^{\frac{1}{2}}}, \end{aligned} \quad (52)$$

and

$$\begin{aligned} H_m(\mathcal{N}_m(\boldsymbol{\mu}, \Sigma), \mathcal{N}_m(\boldsymbol{\nu}, V)) &\stackrel{(21)}{=} \frac{1}{2-m} \int [f \log_m f - g \log_m g - (2-m) \log_m g(f-g)] d\mathbf{x} \\ &= \frac{1}{2}C_1(m, d) \left[\frac{C_0(m, d)}{(\det V)^{\frac{1}{2}}}\right]^{1-m} \left[\text{tr}(V^{-1}\Sigma) + \langle \mu - \nu, V^{-1}(\mu - \nu) \rangle + 2 \log_m \frac{(\det V)^{\frac{1}{2}}}{(\det \Sigma)^{\frac{1}{2}}} - d\right]. \end{aligned} \quad (53)$$

□

Proposition 3.6. Let $\mathcal{N}_q(\mu_0, C\sigma_0^2), \mathcal{N}_q(\mu, C\sigma^2)$ be given. Set $\sigma_h^2 = (h + \sigma_0^{3-q})^{\frac{2}{3-q}}$ and let $\mathcal{N}_q(\mu_0, C\sigma_h^2)$ be the solution at time h . Let

$$Q_{0 \rightarrow h} = \mathcal{N}_m(\mu_0, C\sigma_0^2, \mu_0, C\sigma_h^2, \theta_h), \quad \text{where } \theta_h = \frac{\sigma_0}{\sigma_h}, \quad (54)$$

be the m -bivariate with mean vector

$$\boldsymbol{\mu}_{0 \rightarrow h} = \begin{pmatrix} \mu_0 \\ \mu_0 \end{pmatrix}, \quad \Sigma_{0 \rightarrow h} = \begin{pmatrix} C\sigma_0^2 & C\theta_h\sigma_0\sigma_h \\ C\theta_h\sigma_0\sigma_h & C\sigma_h^2 \end{pmatrix} = \begin{pmatrix} C\sigma_0^2 & C\sigma_0^2 \\ C\sigma_0^2 & C\sigma_h^2 \end{pmatrix}. \quad (55)$$

Define

$$\mathcal{Q} = \{\mathcal{N}_m(\mu_0, C\sigma_0^2, \mu, C\sigma^2, \theta) | \theta \in \mathbf{R}\} \quad (56)$$

and

$$Q^* = \operatorname{argmin}_{Q \in \mathcal{Q}} H_m(Q || Q_{0 \rightarrow h}) \quad (57)$$

Then

$$Q^* = \mathcal{N}_m(\mu_0, C\sigma_0^2, \mu, C\sigma^2, \eta_h), \quad (58)$$

which is a m -bivariate with the mean vector and covariance matrix

$$\boldsymbol{\mu}^* = \begin{pmatrix} \mu_0 \\ \mu \end{pmatrix}, \quad \Sigma^* = \begin{pmatrix} C\sigma_0^2 & C\eta_h\sigma_0\sigma \\ C\eta_h\sigma_0\sigma & C\sigma^2 \end{pmatrix}, \quad (59)$$

where η_h satisfies the equation

$$\frac{\eta_h}{(1 - \eta_h^2)^{\frac{3-m}{2}}} = \frac{\theta_h}{(1 - \theta_h^2)^{\frac{3-m}{2}}} \left(\frac{\sigma}{\sigma_h} \right)^{2-m}. \quad (60)$$

Moreover, the m -relative entropy $H_m(Q^* \| Q_{0 \rightarrow h})$ is

$$\begin{aligned} & H_m(Q^* \| Q_{0 \rightarrow h}) \\ &= \frac{1}{2} C_1(m, 2) \left(\frac{C_0(m, 2)}{C \sigma_0 (\sigma_h^2 - \sigma_0^2)^{\frac{1}{2}}} \right)^{1-m} \left[\frac{(\sigma - \sigma_0)^2}{\sigma_h^2 - \sigma_0^2} + \frac{2\sigma_0\sigma(1 - \eta_h)}{\sigma_h^2 - \sigma_0^2} + \frac{(\mu - \mu_0)^2}{C(\sigma_h^2 - \sigma_0^2)} + 2 \log_m \left(\frac{\sigma_0}{\sigma \eta_h} \right)^{\frac{1}{3-m}} - 1 \right] \end{aligned} \quad (61)$$

The difference $E_q(\mathcal{N}_q(\mu, C\sigma^2)) - E_q(\mathcal{N}_q(\mu_0, C\sigma_0^2))$ is

$$E_q(\mathcal{N}_q(\mu, C\sigma^2)) - E_q(\mathcal{N}_q(\mu_0, C\sigma_0^2)) = (2 - q) C_1(q, 1) \left(\frac{C_0(q, 1)}{\sigma_0 \sqrt{C}} \right)^{1-q} \log_q \left(\frac{\sigma_0}{\sigma} \right), \quad (62)$$

and the Wasserstein distance $W_2^2(\mathcal{N}_q(\mu, C\sigma^2), \mathcal{N}_q(\mu_0, C\sigma_0^2))$ is

$$W_2(\mathcal{N}_q(\mu, C\sigma^2), \mathcal{N}_q(\mu_0, C\sigma_0^2))^2 = C(\sigma - \sigma_0)^2 + (\mu - \mu_0)^2. \quad (63)$$

Proof. By Proposition 3.4,

$$\begin{aligned} H_m(Q^* \| Q_{0 \rightarrow h}) &= \frac{1}{2} C_1(m, 2) \left[\frac{C_0(m, 2)}{(\det \Sigma_{0 \rightarrow h})^{\frac{1}{2}}} \right]^{1-m} \\ &\quad \left[\text{tr}(\Sigma_{0 \rightarrow h}^{-1} \Sigma^*) + \langle \mu^* - \mu_{0 \rightarrow h}, \Sigma_{0 \rightarrow h}^{-1} (\mu^* - \mu_{0 \rightarrow h}) \rangle + 2 \log_m \frac{(\det \Sigma_{0 \rightarrow h})^{\frac{1}{2}}}{(\det \Sigma^*)^{\frac{1}{2}}} - 2 \right]. \end{aligned}$$

We now calculate each term in the above formula explicitly.

$$\det \Sigma_{0 \rightarrow h} = C^2 \sigma_0^2 (\sigma_h^2 - \sigma_0^2) = C^2 \sigma_0^2 \sigma_h^2 (1 - \theta_h^2), \quad \Sigma_{0 \rightarrow h}^{-1} = \frac{1}{C(\sigma_h^2 - \sigma_0^2)} \begin{pmatrix} \frac{1}{\theta_h^2} & -1 \\ -1 & 1 \end{pmatrix}.$$

$$\det \Sigma^* = C^2 \sigma_0^2 \sigma^2 (1 - \eta^2).$$

$$\text{tr}(\Sigma_{0 \rightarrow h}^{-1} \Sigma^*) = \frac{\sigma_h^2 + \sigma_0^2 - 2\eta\sigma_0\sigma}{\sigma_h^2 - \sigma_0^2} = \frac{(\sigma - \sigma_0)^2}{\sigma_h^2 - \sigma_0^2} + \frac{2\sigma_0\sigma(1 - \eta)}{\sigma_h^2 - \sigma_0^2} + 1.$$

$$(\mu^* - \mu_{0 \rightarrow h})^T \Sigma_{0 \rightarrow h}^{-1} (\mu^* - \mu_{0 \rightarrow h}) = \frac{(\mu - \mu_0)^2}{C(\sigma_h^2 - \sigma_0^2)}.$$

$$\frac{\det \Sigma_{0 \rightarrow h}}{\det \Sigma^*} = \frac{\sigma_h^2}{\sigma^2} \cdot \frac{1 - \theta_h^2}{1 - \eta^2}.$$

By (60)

$$\frac{\eta_h}{(1 - \eta_h^2)^{\frac{3-m}{2}}} = \frac{\theta_h}{(1 - \theta_h^2)^{\frac{3-m}{2}}} \left(\frac{\sigma}{\sigma_h} \right)^{2-m}.$$

Hence

$$\frac{1 - \theta_h^2}{1 - \eta_h^2} = \left(\frac{\theta_h}{\eta_h} \right)^{\frac{2}{3-m}} \left(\frac{\sigma}{\sigma_h} \right)^{\frac{2(2-m)}{3-m}},$$

and

$$\frac{\sigma_h^2}{\sigma^2} \cdot \frac{1 - \theta_h^2}{1 - \eta_h^2} = \left(\frac{\sigma_h \theta_h}{\sigma \eta_h} \right)^{\frac{2}{3-m}} = \left(\frac{\sigma_0}{\sigma \eta_h} \right)^{\frac{2}{3-m}}.$$

Therefore

$$\frac{(\det \Sigma_{0 \rightarrow h})^{\frac{1}{2}}}{(\det \Sigma^*)^{\frac{1}{2}}} = \left(\frac{\sigma_0}{\sigma \eta_h} \right)^{\frac{1}{3-m}},$$

and (61) follows.

(62) is a direct consequence of the first equality in Proposition 3.4 and (63) has been shown in (31). \square

4 Proof of the main theorem

In this section, we bring all ingredients together to prove the main theorem. Suppose that the assumption of the main theorem is true. We set

$$a(q, \sigma_0) = \frac{2C}{C_1(m, 2)} \left(\frac{C_0(m, 2)}{C\sigma_0} \right)^{m-1} = \frac{2C^{2-m}}{C_1(m, 2)} \left(\frac{C_0(m, 2)}{\sigma_0} \right)^{m-1}. \quad (64)$$

$$b(\sigma_0, q) = \frac{(2-q)C_1(q, 1)}{C} \left(\frac{C_0(q, 1)}{\sigma_0 \sqrt{C}} \right)^{1-q} = \frac{(2-q)C_1(q, 1)}{C^{\frac{3-q}{2}}} \left(\frac{C_0(q, 1)}{\sigma_0} \right)^{1-q}. \quad (65)$$

Let Q^* be the minimizer in (17). By Proposition 3.6, we have

$$Q^* = \mathcal{N}_m(\mu_0, \sigma_0^2, \mu, \sigma^2, \eta_h),$$

where $\eta_h = \eta(h, \sigma)$ satisfies the following equation

$$\frac{\eta_h}{(1 - \eta_h^2)^{\frac{3-m}{2}}} = \frac{\theta_h}{(1 - \theta_h^2)^{\frac{3-m}{2}}} \left(\frac{\sigma}{\sigma_h} \right)^{2-m} = \frac{\frac{\sigma_0}{\sigma_h}}{\left(1 - \frac{\sigma_0^2}{\sigma_h^2}\right)^{\frac{3-m}{2}}} \left(\frac{\sigma}{\sigma_h} \right)^{2-m} = \frac{\sigma_0}{(\sigma_h^2 - \sigma_0^2)^{\frac{3-m}{2}}} \sigma^{2-m}. \quad (66)$$

Since $|\eta_h| \leq 1$ and the right hand side of (66) is positive, it holds that $0 < \eta_h < 1$. Using the relationship $m = 3 - \frac{2}{q}$, we can rewrite the above equation as follows

$$\frac{\eta_h^q}{1 - \eta_h^2} = \frac{\sigma_0^q \sigma^{(2-q)}}{\sigma_h^2 - \sigma_0^2}. \quad (67)$$

We now use the following statement whose proof is straightforward.

Assume that x_0 is given. For all $\epsilon > 0$, and for all $h > 0$ sufficiently small, there exists a constant $C = C(\epsilon, x_0)$ such that

$$(h + x_0)^\epsilon - x_0^\epsilon \leq Ch. \quad (68)$$

In particular, if $\epsilon < 1$ then $C = \epsilon x_0^{\epsilon-1}$.

Using (68) for $\epsilon = \frac{2}{3-q}$, $x_0 = \sigma^{3-q}$ and from (67), we get

$$1 - \eta_h = \frac{\eta_h^q}{1 + \eta_h} \frac{\sigma_h^2 - \sigma_0^2}{\sigma_0^q \sigma^{(2-q)}} = \frac{\eta_h^q}{1 + \eta_h} \frac{(h + \sigma_0^{3-q})^{\frac{2}{3-q}} - \sigma_0^2}{\sigma_0^q \sigma^{(2-q)}} \leq \frac{Ch}{\sigma^{(2-q)}}, \quad (69)$$

where $C > 0$ is a constant depending only on σ_0 and q . This implies that for fixed σ , $\lim_{h \rightarrow 0} \eta_h = \lim_{h \rightarrow 0} \eta(h, \sigma) = 1$ and as a sequence of functions $\eta(h, \cdot) \rightarrow 1$ locally uniform.

1. We now prove the first statement of the main theorem. We need to prove

$$a(\sigma_h^2 - \sigma_0^2)^{\frac{1}{q}} J_h(\cdot | \mathcal{N}_0) \xrightarrow{\Gamma} W_2^2(\cdot, \mathcal{N}_0) \quad \text{as } h \rightarrow 0. \quad (70)$$

Let $\mathcal{N}_q(\mu, C\sigma^2)$ be given and we denote it by \mathcal{N}_q for short. By Proposition 3.6, we have

$$a(\sigma_h^2 - \sigma_0^2)^{\frac{1}{q}} J_h(\mathcal{N}_q | \mathcal{N}_0) = C(\sigma - \sigma_0)^2 + (\mu - \mu_0)^2 + 2C\sigma_0\sigma(1 - \eta_h) + C(\sigma_h^2 - \sigma_0^2) \left[2 \log_m \left(\frac{\sigma_0}{\sigma \eta_h} \right)^{\frac{1}{3-m}} - 1 \right], \quad (71)$$

and

$$W_2^2(\mathcal{N}_q, \mathcal{N}_0) = C(\sigma - \sigma_0)^2 + (\mu - \mu_0)^2. \quad (72)$$

For the lower bound part: Assume that $\mathcal{N}_q^h = \mathcal{N}_q(\nu_h, C\xi_h^2) \rightarrow \mathcal{N}_q$. This means that $(\nu_h - \mu)^2 + (\xi_h - \sigma)^2 \rightarrow 0$. Hence we can assume that $0 < \frac{\sigma}{2} \leq \sup_h \xi_h \leq \frac{3}{2}\sigma$.

Let $\eta_h = \eta(h, \xi_h)$ be the solution of (66) where σ is replaced by ξ_h . By (69) we have $\lim_{h \rightarrow 0} \eta_h = 1$, so we can assume that $\xi_h \eta_h$ is uniformly bounded above and away from 0. We now have

$$\begin{aligned} & a(\sigma_h^2 - \sigma_0^2)^{\frac{1}{q}} J_h(\mathcal{N}_q^h | \mathcal{N}_0) \\ &= C(\xi_h - \sigma_0)^2 + (\nu_h - \mu_0)^2 + 2C\sigma_0\xi_h(1 - \eta_h) + C(\sigma_h^2 - \sigma_0^2) \left[2 \log_m \left(\frac{\sigma_0}{\xi_h \eta_h} \right)^{\frac{1}{3-m}} - 1 \right] \\ &\geq C(\xi_h - \sigma_0)^2 + (\nu_h - \mu_0)^2 + C(\sigma_h^2 - \sigma_0^2) \left[2 \log_m \left(\frac{\sigma_0}{\xi_h \eta_h} \right)^{\frac{1}{3-m}} - 1 \right]. \end{aligned}$$

Hence

$$\begin{aligned} & \liminf_{h \rightarrow 0} a(\sigma_h^2 - \sigma_0^2)^{\frac{1}{q}} J_h(\mathcal{N}_q^h | \mathcal{N}_0) \\ &\geq \liminf_{h \rightarrow 0} \left\{ C(\xi_h - \sigma_0)^2 + (\nu_h - \mu_0)^2 + C(\sigma_h^2 - \sigma_0^2) \left[2 \log_m \left(\frac{\sigma_0}{\xi_h \eta_h} \right)^{\frac{1}{3-m}} - 1 \right] \right\} \\ &= C(\sigma - \sigma_0)^2 + (\mu - \mu_0)^2 = W_2^2(\mathcal{N}_q, \mathcal{N}_0). \end{aligned}$$

For the upper bound part: as a recovery sequence, we just simply take the fixed sequence $\mathcal{N}_q^h = \mathcal{N}_q$.

2. We now prove the second statement of the main theorem. We need to prove:

$$ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\cdot | \mathcal{N}_0) - \frac{b}{\sigma_h^2 - \sigma_0^2} W_2^2(\cdot, \mathcal{N}_0) \xrightarrow{\Gamma} E_q(\cdot) - E_q(\mathcal{N}_0) \quad \text{as } h \rightarrow 0. \quad (73)$$

Let $\mathcal{N}_q = \mathcal{N}_q(\mu, \sigma^2)$ be given. Let η_h be the solution of (66). We have

$$ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\mathcal{N}_q | \mathcal{N}_0) = bC \left\{ \frac{(\sigma - \sigma_0)^2}{\sigma_h^2 - \sigma_0^2} + \frac{(\mu - \mu_0)^2}{C(\sigma_h^2 - \sigma_0^2)} + 2 \frac{\sigma_0 \sigma (1 - \eta_h)}{\sigma_h^2 - \sigma_0^2} + \left[2 \log_m \left(\frac{\sigma_0}{\sigma \eta_h} \right)^{\frac{1}{3-m}} - 1 \right] \right\}, \quad (74)$$

$$\frac{b}{\sigma_h^2 - \sigma_0^2} W_2^2(\mathcal{N}_q, \mathcal{N}_0) = bC \left[\frac{(\sigma - \sigma_0)^2}{\sigma_h^2 - \sigma_0^2} + \frac{(\mu - \mu_0)^2}{C(\sigma_h^2 - \sigma_0^2)} \right], \quad (75)$$

$$E_q(\mathcal{N}_q) - E_q(\mathcal{N}_0) = bC \log_q \left(\frac{\sigma_0}{\sigma} \right). \quad (76)$$

Define

$$\begin{aligned} \mathcal{F}_h(\mathcal{N}_q; \mathcal{N}_0) &:= \frac{2\sigma_0 \sigma (1 - \eta_h)}{\sigma_h^2 - \sigma_0^2} + 2 \log_m \left(\frac{\sigma_0}{\sigma \eta_h} \right)^{\frac{1}{3-m}} - 1, \\ \mathcal{F}(\mathcal{N}_q; \mathcal{N}_0) &:= \log_q \left(\frac{\sigma_0}{\sigma} \right). \end{aligned}$$

We need to prove

$$\mathcal{F}_h(\cdot; \mathcal{N}_0) \xrightarrow{\Gamma} \mathcal{F}(\cdot; \mathcal{N}_0). \quad (77)$$

We first prove that $\mathcal{F}_h(\cdot, \mathcal{N}_0) \rightarrow \mathcal{F}(\cdot, \mathcal{N}_0)$ locally uniform. We now rewrite the RHS of \mathcal{F}_h using the relationship between m and q .

Since for any $t > 0$

$$\log_m t^{\frac{1}{3-m}} = \frac{t^{\frac{1-m}{3-m}} - 1}{1-m} = \frac{t^{1-q} - 1}{\frac{2(1-q)}{q}} = \frac{q}{2} \log_q t;$$

Hence

$$2 \log_m \left(\frac{\sigma_0}{\sigma \eta_h} \right)^{\frac{1}{3-m}} = q \log_q \left(\frac{\sigma_0}{\sigma \eta_h} \right). \quad (78)$$

From (66), we get

$$\frac{\eta_h}{(1+\eta_h)^{\frac{3-m}{2}}} = \left(\frac{\sigma_0 \sigma (1-\eta_h)}{\sigma_h^2 - \sigma_0^2} \right)^{\frac{3-m}{2}} \left(\frac{\sigma}{\sigma_0} \right)^{\frac{1-m}{2}}. \quad (79)$$

This implies that

$$\frac{\sigma_0 \sigma (1-\eta_h)}{\sigma_h^2 - \sigma_0^2} = \frac{\eta_h^q}{1+\eta_h} \left(\frac{\sigma_0}{\sigma} \right)^{1-q} = \frac{\eta_h^q}{1+\eta_h} \left((1-q) \log_q \left(\frac{\sigma_0}{\sigma} \right) + 1 \right). \quad (80)$$

From (78) and (80) we obtain

$$\mathcal{F}_h(\mathcal{N}_q; \mathcal{N}_0) = 2 \frac{\eta_h^q}{1+\eta_h} \left(\frac{\sigma_0}{\sigma} \right)^{1-q} + q \log_q \left(\frac{\sigma_0}{\sigma \eta_h} \right) - 1.$$

Now we have the following estimate

$$\begin{aligned} |\mathcal{F}_h - \mathcal{F}| &= \left| 2 \frac{\eta_h^q}{1+\eta_h} \left(\frac{\sigma_0}{\sigma} \right)^{1-q} + q \log_q \left(\frac{\sigma_0}{\sigma \eta_h} \right) - \log_q \left(\frac{\sigma_0}{\sigma} \right) - 1 \right| \\ &= \left| 2 \frac{\eta_h^q}{1+\eta_h} \left(\frac{\sigma_0}{\sigma} \right)^{1-q} - 1 - (1-q) \log_q \left(\frac{\sigma_0}{\sigma} \right) + q \log_q \left(\frac{\sigma_0}{\sigma \eta_h} \right) - q \log_q \left(\frac{\sigma_0}{\sigma} \right) \right| \\ &= \left(\frac{\sigma_0}{\sigma} \right)^{1-q} \left| 2 \frac{\eta_h^q}{1+\eta_h} - 1 + \frac{q}{1-q} \left(\eta_h^{(q-1)} - 1 \right) \right| \\ &\leq \left(\frac{\sigma_0}{\sigma} \right)^{1-q} \left[\left| 2 \frac{\eta_h^q}{1+\eta_h} - 1 \right| + \left| \frac{q}{1-q} \left(\eta_h^{(q-1)} - 1 \right) \right| \right] \\ &\leq \frac{1}{\sigma^{1-q}} C (1-\eta_h) \\ &\stackrel{(69)}{\leq} \frac{Ch}{\sigma^{3-2q}}, \end{aligned} \quad (81)$$

where C is a constant depending only on σ_0 and q .

The locally uniform convergence of \mathcal{F}_h to \mathcal{F} thus follows from the estimate (81).

For the Γ -convergence, we get the lower bound part by the local uniform convergence and the continuity of the entropy. Indeed, let $\mathcal{N}_q := \mathcal{N}_q(\mu, C\sigma^2)$ be given and assume that $\mathcal{N}_q^h := \mathcal{N}_q(\mu_h, C\xi_h^2) \rightarrow \mathcal{N}_q$. Then $\mu_h \rightarrow \mu$ and $\xi_h \rightarrow \sigma$. Hence we can assume without loss of generality that $\frac{\sigma}{2} \leq \sup_h \xi_h \leq \frac{3\sigma}{2}$. We have the following estimate

$$\begin{aligned} \left| \mathcal{F}_h(\mathcal{N}_q^h; \mathcal{N}_q^0) - \mathcal{F}(\mathcal{N}_q; \mathcal{N}_q^0) \right| &\leq \left| \mathcal{F}_h(\mathcal{N}_q^h; \mathcal{N}_0) - \mathcal{F}(\mathcal{N}_q^h; \mathcal{N}_q^0) \right| + \left| \mathcal{F}(\mathcal{N}_q^h; \mathcal{N}_0) - \mathcal{F}(\mathcal{N}_q; \mathcal{N}_q^0) \right| \\ &\stackrel{(81)}{\leq} \frac{Ch}{\xi_h^{3-2q}} + \left| \log_q \left(\frac{\sigma_0}{\xi_h} \right) - \log_q \left(\frac{\sigma_0}{\sigma} \right) \right| \\ &\leq \frac{Ch}{\sigma^{3-2q}} + \left| \log_q \left(\frac{\sigma_0}{\xi_h} \right) - \log_q \left(\frac{\sigma_0}{\sigma} \right) \right| \rightarrow 0. \end{aligned}$$

Therefore

$$\lim_{h \rightarrow 0} \mathcal{F}_h(\mathcal{N}_q^h; \mathcal{N}_q^0) = \mathcal{F}(\mathcal{N}_q; \mathcal{N}_q^0).$$

For the upper part, as a recovery sequence, we can choose the fixed sequence $\mathcal{N}_q^h = \mathcal{N}_q$.

3. We now prove the third statement of the main theorem. Assume that $0 < q < 1$. We need to prove

$$ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\cdot | \mathcal{N}_0) - \frac{1}{2h} W_2^2(\cdot, \mathcal{N}_0) \xrightarrow{\Gamma\text{-lim inf}} E_q(\cdot) - E_q(\mathcal{N}_0) \quad \text{as } h \rightarrow 0.$$

We have

$$\begin{aligned} & ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\cdot | \mathcal{N}_0) - \frac{1}{2h} W_2^2(\cdot, \mathcal{N}_0) \\ &= ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\cdot | \mathcal{N}_0) - \frac{b}{\sigma_h^2 - \sigma_0^2} W_2^2(\cdot, \mathcal{N}_0) + \left(\frac{b}{\sigma_h^2 - \sigma_0^2} - \frac{1}{2h} \right) W_2^2(\cdot, \mathcal{N}_0). \end{aligned}$$

Since $0 < q < 1$, $0 < \frac{2}{3-q} < 1$, using (81) for $\epsilon = \frac{2}{3-q}$, $x_0 = \sigma_0^{3-q}$, we have

$$\sigma_h^2 - \sigma_0^2 = (h + \sigma_0^{3-q})^{\frac{2}{3-q}} - \sigma_0^2 \leq \frac{2}{(3-q)\sigma_0^{1-q}} h.$$

Therefore

$$\frac{b}{\sigma_h^2 - \sigma_0^2} \geq \frac{(3-q)b\sigma_0^{1-q}}{2h} = \frac{1}{2h}.$$

It implies that

$$ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\cdot | \mathcal{N}_0) - \frac{1}{2h} W_2^2(\cdot, \mathcal{N}_0) \geq ab(\sigma_h^2 - \sigma_0^2)^{\frac{1-q}{q}} J_h(\cdot | \mathcal{N}_0) - \frac{b}{\sigma_h^2 - \sigma_0^2} W_2^2(\cdot, \mathcal{N}_0),$$

and the third statement thus follows from the second one.

This completes the proof of the main theorem.

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