

Finite buffer fluid networks with overflows

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Stijn Fleuren, Erjen Lefeber, Yoni Nazarathy

IWAP 2012, Jerusalem



TU / **e**

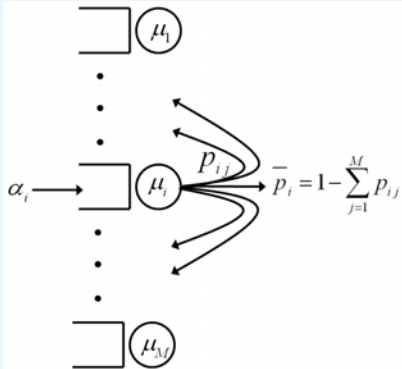
Technische Universiteit
Eindhoven
University of Technology

June 11, 2012

Where innovation starts

Jackson (1957); Goodman, Massey (1984); Chen, Mandelbaum (1991)

Jackson 1957



Problem data

α, μ, P

Traffic equations (stable case)

$$\lambda_i = \alpha_i + \sum_{j=1}^N \lambda_j P_{ji}$$

$$\lambda = \alpha + P' \lambda$$

$$\lambda = (I - P')^{-1} \alpha$$

Theorem (Jackson 1957)

Given an $(M/M/1)^N$ system where every node can be filled and drained, let $\lambda = [\lambda_1, \dots, \lambda_N]'$ be the solution of the throughput equation

$$\lambda = \alpha + P'\lambda$$

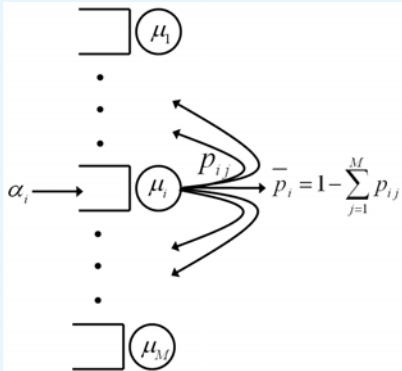
If $\rho_i = \lambda_i/\mu_i$ and $\rho_i < 1$ for all i , then

$$\lim_{t \rightarrow \infty} P(X_1(t) = n_1, \dots, X_N(t) = n_N) = \prod_{i=1}^N (1 - \rho_i) \rho_i^{n_i}$$

for all integers $n_i \geq 0$.

Jackson (1957); Goodman, Massey (1984); Chen, Mandelbaum (1991)

Jackson 1957



Problem data

α, μ, P

Traffic equations (general)

$$\lambda_j = \alpha_j + \sum_{j=1}^N \min(\lambda_j, \mu_j) p_{ji}$$

$$\lambda = \alpha + P' \min(\lambda, \mu)$$

Theorem (Goodman, Massey 1984)

Given an $(M/M/1)^N$ system where every node can be filled and drained, let $\lambda = [\lambda_1, \dots, \lambda_N]'$ be the solution of the throughput equation

$$\lambda = \alpha + P' \min(\lambda, \mu)$$

If $\rho_i = \lambda_i / \mu_i$ and $U = \{i \mid \rho_i < 1\}$, then

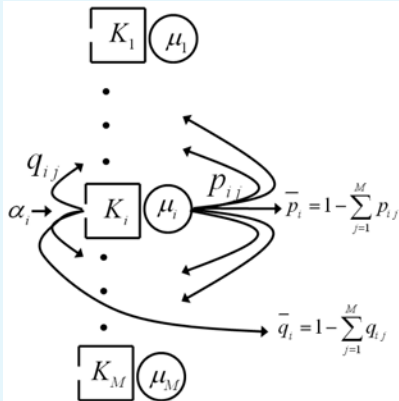
$$\lim_{t \rightarrow \infty} P(X_i(t) = n_i; i \in U) = \prod_{i \in U} (1 - \rho_i) \rho_i^{n_i}$$

for all integers $n_i \geq 0$ with $i \in U$. Moreover, if $j \notin U$ then

$$\lim_{t \rightarrow \infty} P(X_j(t) = n) = 0$$

for all integers $n \geq 0$

Network



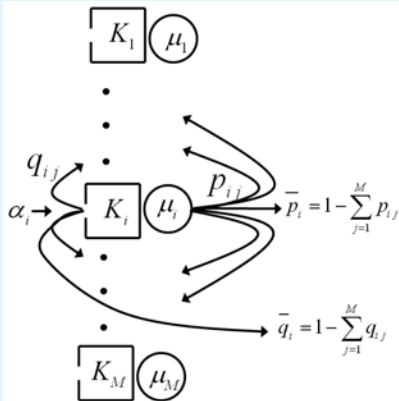
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Our contribution (in progress)

- ▶ Limiting traffic equations
- ▶ Efficient algorithm for unique solution
- ▶ Limiting deterministic trajectories
- ▶ Limiting sojourn time distribution

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α, μ, P, Q, K

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Scaling yields a fluid system

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A sequence of systems: $N = 1, 2, \dots$

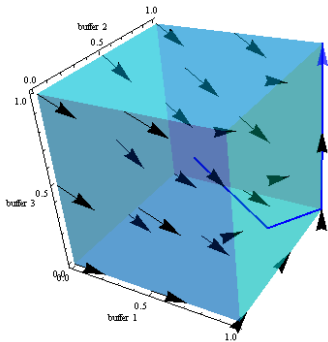
$$\alpha^N = N\alpha$$

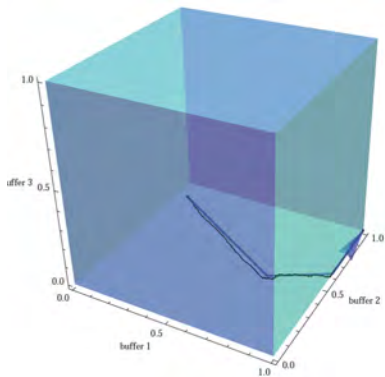
$$\mu^N = N\mu$$

$$K^N = NK$$

Make the jobs **fast** and the buffers **big** by taking $N \rightarrow \infty$.

The proposed limiting model is a deterministic fluid system





$$\lim_{N \rightarrow \infty} \sup_t \left\{ \left| \frac{X^N(t)}{N} - x(t) \right| \right\} = 0$$

outflow rate: $\min(\lambda, \mu)$

overflow rate: $\lambda - \min(\lambda, \mu) = \max(0, \lambda - \mu)$

Traffic equations

$$\lambda_i = \alpha_i + \sum_{j=1}^N \min(\lambda_j, \mu_j) p_{ji} + \sum_{j=1}^N \max(0, \lambda_j - \mu_j) q_{ji}$$

or

$$\lambda = \alpha + P' \min(\lambda, \mu) + Q' \max(0, \lambda - \mu)$$

Question

How to (efficiently) solve traffic equations for given α, μ, P, Q, K ?

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Question

How to (efficiently) solve traffic equations for given α, μ, P, Q, K ?

$$\lambda = \alpha + P' \min(\lambda, \mu) + Q' \max(0, \lambda - \mu)$$

Let $w = \lambda - \min(\lambda, \mu)$ and $z = \mu - \min(\lambda, \mu)$. Then $\lambda = w - z + \mu$ and

$$w \geq 0$$

$$z \geq 0$$

$$w'z = 0$$

Furthermore we obtain for the traffic equation

$$w - z + \mu = \alpha + P'(\mu - z) + Q'w$$

$$(I - Q')w = \alpha - (I - P')\mu + (I - P')z$$

$$w = \underbrace{(I - Q')^{-1}(\alpha - (I - P')\mu)}_q + \underbrace{(I - Q')^{-1}(I - P')}_M z$$

$$\lambda = \alpha + P' \min(\lambda, \mu) + Q' \max(0, \lambda - \mu)$$

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LCP

LCP(q, M): Find z, w such that

$$w - Mz = q \qquad w, z \geq 0 \qquad w'z = 0$$

For our system: $q = (I - Q')^{-1}(\alpha - (I - P')\mu)$, $M = (I - Q')^{-1}(I - P')$

Theorem

LCP(q, M) has **unique solution for all q** iff M is a **P -matrix**, i.e. determinants of all $2^N - 1$ principal submatrices are positive

Observation

No polynomial time algorithm (yet) exists for solving the P -matrix LCP

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Algorithm of Goodman and Massey (1984)

Problem: Solve $\lambda = \alpha + P' \min(\lambda, \mu)$

Observation: If we would know the stable and unstable nodes, we can solve for λ .

Step 1: Assume all queues are unstable, i.e. output rate μ_i , and solve for arrival rate: $\lambda(1)$.

Observation: $\lambda(1)$ is at worst an over-estimate.

Let $I(1) = \{i \mid \lambda_i(1) < \mu_i\}$ denote the set of stable nodes.

Step 2: Assume nodes $i \notin I(1)$ are unstable and solve for the arrival rate: $\lambda(2) \leq \lambda(1)$.

Repeat until: $I(n) = I(n + 1)$.

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Algorithm

Problem: Solve $\lambda = \alpha + P' \min(\lambda, \mu) + Q' \max(0, \lambda - \mu)$

Step 1: Assume all nodes have infinite capacity, i.e. $K_i = \infty$ and no overflow, and solve for arrival rate: $\lambda(1)$.

Observation: $\lambda(1)$ is at worst an under-estimate.

Let $J(1) = \{j \mid \lambda_j(1) > \mu_j\}$ denote the overflowing nodes.

Step 2: Assume nodes $j \notin J(1)$ have infinite capacity and solve for the arrival rate: $\lambda(2) \geq \lambda(1)$.

Repeat until: $J(n) = J(n + 1)$.

Number of iterations

Worst case: $O(N^2)$.

Practice (max 800 nodes): $O(\log(N))$

Algorithm

Worst case: $O(N^5)$.

Practice: $O(N^3 \log(N))$

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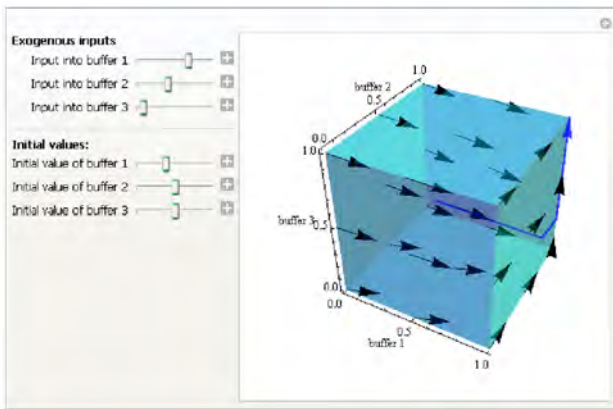
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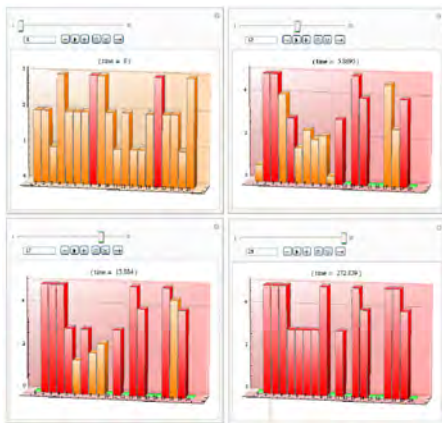
Algorithm can also be used for determining transient behavior

See also <http://demonstrations.wolfram.com/DynamicsOfADeterministicOverflowFluidNetwork/>



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Definition

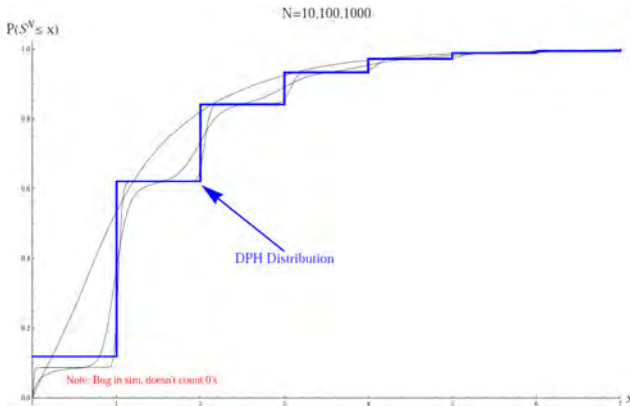
Sojourn time: time in system of customer arriving to steady state FCFS system

Definition

S^N : sojourn time of customer in N th scaled system

Problem

We want to find the limiting distribution of S^N , i.e. $P(S^N \leq x)$, for $N \rightarrow \infty$.



Observation

Sojourn times scale to a discrete distribution

$$F = \{1, \dots, s\}$$

$$\lambda_i > \mu_i \text{ for } i \in F$$

$$\bar{F} = \{s + 1, \dots, N\}$$

$$\lambda_i < \mu_i \text{ for } i \in \bar{F}$$

Observation

Time through $i \in F \approx NK_i/(N\mu_i) = K_i/\mu_i$, time through $i \notin F \approx 1/(N\mu_i) \approx 0$.

For job at entrance of buffer $i \in F$:

- ▶ enters buffer i w.p. $\approx \mu_i/\lambda_i$
- ▶ routed to entrance of buffer j w.p. $\approx (1 - \mu_i/\lambda_i)q_{ij}$
- ▶ leaves system w.p. $\approx (1 - \mu_i/\lambda_i)\bar{q}_i$

Job at entrance of buffer $i \in \bar{F}$:

- ▶ routed according to P almost immediately

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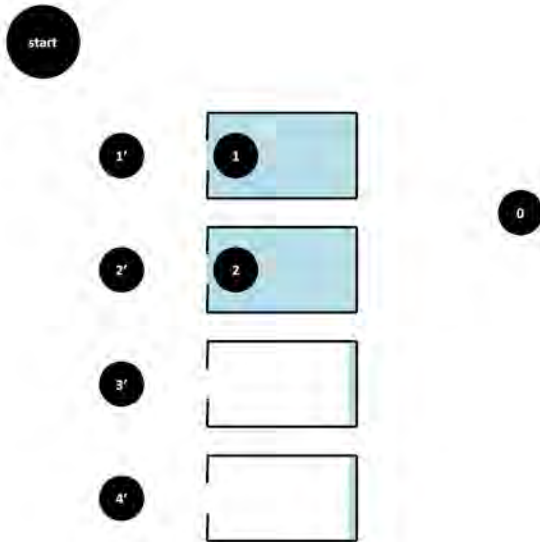
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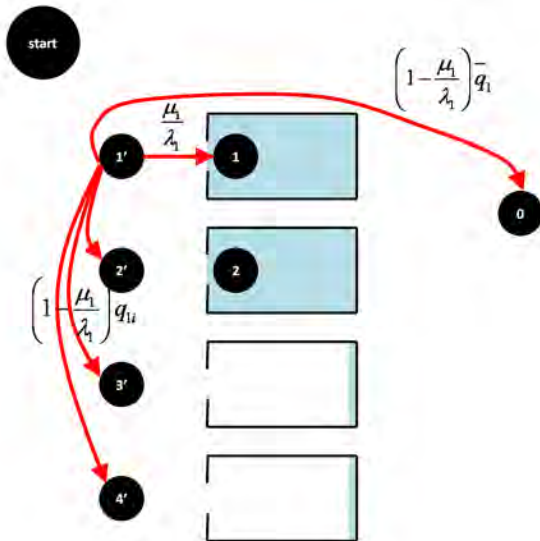
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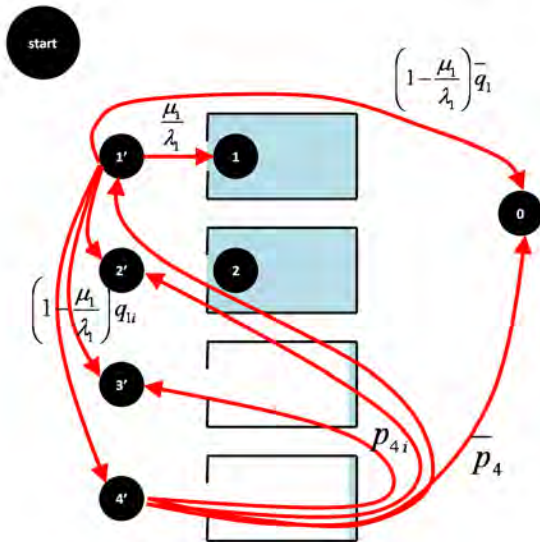
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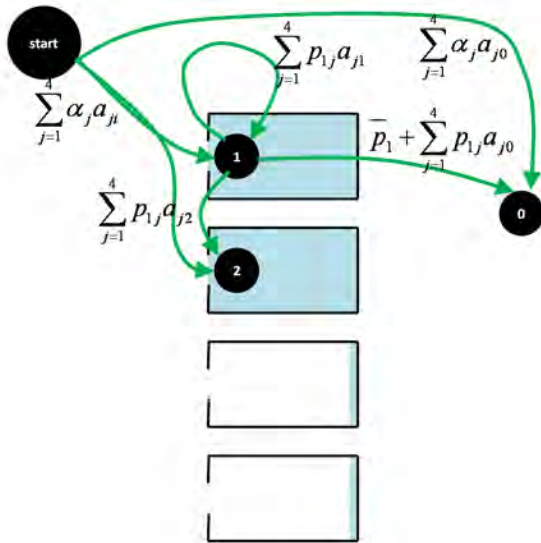
Fast chain on
 $\{0, 1, 2, 1', 2', 3', 4'\}$



Fast chain on
 $\{0, 1, 2, 1', 2', 3', 4'\}$



a_{ij} absorption probability in $j \in \{0, 1, 2\}$ starting in i'



Slow chain on
 $\{0, 1, 2\}$, transitions
based on fast chain.

Finite buffer networks with overflows.

Contributions

- ▶ Limiting traffic equations
- ▶ Efficient algorithm for unique solution
- ▶ Limiting deterministic trajectories
- ▶ Limiting sojourn time distribution

Future work

Work on limits (Chen and Mandelbaum, 1991)