On the Cuspless Sub-Riemannian Geodesics in $\mathbb{R}^3 \rtimes S^2$

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Abstract

We study the cuspless curves in three dimensional Euclidean space that minimize the energy functional $\int_0^L \sqrt{\kappa(s)^2 + \beta^2} \, ds$ defined for each spatial arc-length parameterized curve with curvature $\kappa$ and length (free) $L \geq 0$. To compute the stationary curves, we took the Lagrangian approach used by Bryant and Griffiths \cite{5} for similar minimization problems. We have also shown that two other approaches, a direct variational method in $\mathbb{R}^3$ and the Hamiltonian approach using the Pontryagin’s Maximum Principle (PMP) give the same ODE’s for the stationary curves (that are locally minimizing due to PMP). We show that using the spatial arc-length parameterizations is advantageous in our case over the sub-Riemannian arc-length parameterization. The Bryant and Griffiths approach leads us to explicit formulas for the stationary curves. The formulas allow us to extract geometrical properties of the cuspless sub-Riemannian geodesics, such as planarity conditions and explicit bounds on the torsion. Moreover, they allow a study of the symmetries of the associated exponential map, and they allow to numerically solve both the initial and the boundary value problem, and to numerically compute the range of the exponential map. As a result, we characterize necessary conditions on the boundary conditions for which our optimization is well-posed.

1 Introduction

The aim of this paper is to derive expressions for the geodesics in the five dimensional space of positions and directions. This problem is formulated as follows.

On the Sobolev space $W^{2,1}$ of curves in $\mathbb{R}^3$, we define an energy functional $\mathcal{E} : W^{2,1}([0, L], \mathbb{R}^3) \to \mathbb{R}^+$, with $L \in \mathbb{R}^+$ being the length (free) of the curves, by

$$\mathcal{E}(\mathbf{x}) := \int_0^L \sqrt{\kappa(s)^2 + \beta^2} \, ds. \quad (1.1)$$

Here, $s$ denotes the arc-length of curve $\mathbf{x}$ and $\kappa : [0, L] \to \mathbb{R}^+ \cup \{\infty\}$ denotes the absolute curvature of the curve $\mathbf{x}$ at each point, and $\beta$ is a constant. The two dimensional analog of this variational problem was studied as a possible model of the mechanism used by the visual cortex V1 of the human brain to reconstruct curves which are partially hidden or corrupted. The two dimensional model was initially due to Petitot \cite{22, 23} and references therein. Subsequently, Citti and Sarti \cite{7, 27} were the first to introduce the sub-Riemannian structure into the problem \cite{25}. The stationary curves of the problem were derived and studied by Duits \cite{14}, Boscain, Charlot and Rossi in \cite{3}, Sachkov in \cite{18}, and their global optimality is shown by Boscain, Duits, Rossi and Sachkov in \cite{19, 20, 21}. The two dimensional problem relates to a mechanical problem completely solved by Sachkov \cite{19, 25, 26}. It was also studied by Hladky and Pauls in \cite{16} and by Ben-Yosef and Ben-Shahar in \cite{2}. However, many imaging applications such as DW-MRI require an extension to three dimensions \cite{6, 11, 15}, which motivates us to study the three dimensional curve optimization given by Equation (1.1).

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Let \( x_0, x_1 \in \mathbb{R}^3 \) and \( n_0, n_1 \in S^2 = \{ v \in \mathbb{R}^3 \| v \| = 1 \} \). Our goal is to find an arc-length parameterized curve \( s \mapsto x(s) \) such that

\[
x = \arg \inf_{y \in W^{2,1}([0, L], \mathbb{R}^3)} E(y),
\]

where

\[
y(0) = x_0, \quad y(0) = n_0,
\]

\[
y(L) = x_1, \quad y(L) = n_1.
\]

We assume that the boundary conditions \((x_0, n_0)\) and \((x_1, n_1)\) are chosen so that a minimizer exists. Due to rotation and translational invariance of the problem, it is equivalent to the problem with the same functional and boundary conditions \((0, e_z)\) and \((I(x_1 - x_0), R n_1)\), where \( e_z \) denotes the unit vector in the \( z \)-axis in the right handed \( \{x, y, z\} \) coordinate system and \( R \in SO(3) \) such that \( R n_0 = e_z \). Therefore, without loss of generality, we set (unless explicitly stated otherwise) \( x_0 = 0 \) and \( n_0 = e_z \) for the remainder of the article. Hence the problem now is to find a sufficiently smooth arc-length parameterized curve \( s \mapsto x(s) \) such that

\[
x = \arg \inf_{y \in W^{2,1}([0, L], \mathbb{R}^3)} E(y),
\]

where

\[
y(0) = 0, \quad y(0) = e_z,
\]

\[
y(L) = x_1, \quad y(L) = n_1.
\]

We refer to the above problem as \( \mathbf{P}_{\text{curve}} \).

Now we relate the problem \( \mathbf{P}_{\text{curve}} \) to a sub-Riemannian problem \( \mathbf{P}_{\text{mec}} \) on the Lie group quotient

\[
\mathbb{R}^3 \times S^2 := SE(3)/\{ 0 \times SO(2) \}
\]

in the Lie group \( SE(3) \), as was also done for the \( \mathbf{P}_{\text{curve}} \) on \( \mathbb{R}^2 \), cf. \[4, 9\]. We define this sub-Riemannian problem by means of the left-invariant frame (see Figure 1) and its co-frame, which are more formally discussed in Section 3.1. The left-invariant frame consists of the following left invariant vector fields over \( SE(3) \):

\[
A_1 = \cos \tilde{\alpha} \cos \tilde{\beta} \partial_x + (\sin \tilde{\alpha} \cos \gamma + \cos \tilde{\alpha} \sin \tilde{\beta} \sin \gamma) \partial_y + (\sin \tilde{\alpha} \sin \gamma + \cos \tilde{\alpha} \sin \tilde{\beta} \cos \gamma) \partial_z,
\]

\[
A_2 = -\sin \tilde{\alpha} \cos \tilde{\beta} \partial_x + (\cos \tilde{\alpha} \cos \gamma - \sin \tilde{\alpha} \sin \tilde{\beta} \sin \gamma) \partial_y + (\cos \tilde{\alpha} \sin \gamma + \sin \tilde{\alpha} \sin \tilde{\beta} \cos \gamma) \partial_z,
\]

\[
A_3 = \sin \tilde{\beta} \partial_x - \cos \tilde{\beta} \sin \gamma \partial_y + \cos \tilde{\beta} \cos \gamma \partial_z,
\]

\[
A_4 = -\cos \tilde{\alpha} \tan \tilde{\beta} \partial_y + \sin \tilde{\alpha} \partial_3 + \cos \tilde{\alpha} \sec \tilde{\beta} \partial_3,
\]

\[
A_5 = \sin \tilde{\alpha} \tan \tilde{\beta} \partial_x + \cos \tilde{\alpha} \partial_3 - \sin \tilde{\alpha} \sec \tilde{\beta} \partial_3,
\]

\[
A_6 = \partial_3,
\]

where we parametrized \( \mathbb{R}^3 \) by \( \{ x, y, z \} \) and \( SO(3) \) by Euler angles \( \{ \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \} \) with \( \tilde{\alpha} \in [0, 2\pi) \), \( \tilde{\beta} \in [-\pi, \pi) \) and \( \tilde{\gamma} \in [-\pi/2, \pi/2] \) such that

\[
SO(3) \ni R = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \tilde{\gamma} & -\sin \tilde{\gamma} \\
0 & \sin \tilde{\gamma} & \cos \tilde{\gamma}
\end{pmatrix}
\begin{pmatrix}
\cos \tilde{\beta} & 0 & \sin \tilde{\beta} \\
0 & 1 & 0 \\
-\sin \tilde{\beta} & 0 & \cos \tilde{\beta}
\end{pmatrix}
\begin{pmatrix}
\cos \tilde{\alpha} & -\sin \tilde{\alpha} & 0 \\
\sin \tilde{\alpha} & \cos \tilde{\alpha} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The corresponding co-frame is given by the co-vectors \( \{ \omega^1, \ldots, \omega^6 \} \) satisfying

\[
\langle \omega^i, A_j \rangle = \delta^i_j \text{ for } i, j \in \{ 1, \ldots, 6 \},
\]

with \( \delta^i_j \) as the Kronecker delta function.

We now consider the sub-Riemannian manifold \((M, H, G_{\beta})\) with \( M = SE(3) \), \( H = \text{span}(\{ A_3, A_4, A_5 \}) \) and \( G_{\beta} = \beta^2 \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3 \). In the geometric control problem \( \mathbf{P}_{\text{mec}} \) on \( SE(3) \), we use the sub-Riemannian arclength parameter \( t \). In \( \mathbf{P}_{\text{mec}} \), we aim for curves
\( \gamma : [0, T] \to SE(3) \), with prescribed boundary conditions \( \gamma(0) = (0, I) \) and \( \gamma(T) = (x_1, R_{n_1}) \), such that
\[
\int_0^T \sqrt{G_{\beta}(\gamma(t), \dot{\gamma}(t))} \, dt = \int_0^T \sqrt{\beta^2(u^3(t))^2 + (u^4(t))^2 + (u^5(t))^2} \, dt \rightarrow \text{minimize (with free } T) \tag{1.6}
\]
with
\[
\dot{\gamma}(t) = \sum_{i=3}^5 u^i(t)A_i|_{\gamma(t)} = \sum_{i=3}^5 \omega^i|_{\gamma(t), \dot{\gamma}(t)}A_i|_{\gamma(t)}
\]
where, \( u^i \in L_1([0, T]) \) for \( i = 3, 4, 5 \) and \( R_{n_1} \in SO(3) \) is any rotation such that \( R_{n_1}e_z = n_1 \). In particular, we only consider the stationary curves for which the absolute curvature is \( L_1 \) rather than \( L_\infty \).

The existence of minimizers for the problem \( P_{\text{mec}} \) is guaranteed by the theorems by Chow-Rashevskii and Filippov on sub-Riemannian structures \( \cite{1} \). We consider those boundary conditions, for which a minimizer of \( P_{\text{mec}} \) does not admit an internal cusp (i.e. an interior point with infinite curvature). Clearly, such minimizers are also geodesics. We have the following important remark about the minimizers.

**Remark 1.0.1.**

- We have that for \( P_{\text{mec}} \), there are no abnormal extremals. It follows from the fact that any sub-Riemannian manifold with a 2-generating distribution does not allow abnormal extremals (see Chapter 20.5.1 in \( \cite{2} \)). This is the case here as we have for \( H := \{A_3, A_4, A_5\} \), \( 6 = \text{dim} H, H = \text{dim}\{A_1, A_2, A_3, A_4, A_5, A_6\} = \text{dim}(SE(3)) \).

- Due to the non-existence of the non abnormal extremals, the minimizers are always analytic \( \cite{2} \).

**Remark 1.0.2.** Note that geodesics of \( P_{\text{mec}} \) may lose local and/or global optimality after the end condition at a conjugate point or global optimality at a Maxwell point. A Maxwell point is a point \( \gamma(t) \) on a sub-Riemannian geodesic \( \gamma \) such that \( \gamma(t) = \tilde{\gamma}(t) \) for another extremal trajectory \( \tilde{\gamma} \) with initial condition satisfying \( \tilde{\gamma}(0) = \gamma(0) \). A conjugate point on the other hand is a critical point of the exponential map defined in Section 3.5 (see Theorem 21.11 in \( \cite{2} \)).

We define a geodesic as follows.

**Definition 1.0.3.** On any manifold with a given metric, a geodesic connecting two points is a locally length minimizing path joining those points with respect to the metric.

For such boundary conditions, a minimizer of \( P_{\text{mec}} \) is also a minimizer of \( P_{\text{curve}} \) and a geodesic as well. Akin to the \( SE(2) \) case \( \cite{3} \), we can use both spatial and sub-Riemannian arc-length parametrizations \( s \) and \( t \), with
\[
\frac{dt}{ds}(s) = \sqrt{\kappa(s)^2 + \beta^2}.
\]

Application of the Pontryagin’s Maximum Principle (PMP) \( \cite{1, 31} \) to \( P_{\text{mec}} \) yields the following ODE for the horizontal part
\[
\dot{\gamma} = \lambda_3A_3|_{\gamma} + \lambda_4A_4|_{\gamma} + \lambda_5A_5|_{\gamma},
\]
and for the vertical part, we obtain the ODE
\[
\frac{d}{dt}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (-\lambda_3\lambda_5, \lambda_3\lambda_4, \lambda_1\lambda_5 - \lambda_2\lambda_4, \lambda_3\lambda_2, -\lambda_3\lambda_4). \tag{1.7}
\]
Here, \( \mathbf{\lambda}(t) = \sum_{i=1}^5 \lambda_i(t)\omega^i|_{\gamma(t)} \) is the momentum expressed in the moving dual frame of reference. Detailed calculations are shown in appendix \( \cite{3} \). We note that the ODE’s for the vertical part

\footnote{By locally length minimizing, we mean that for any point \( X \) on the geodesic, there is a sufficiently small open neighbourhood such that for any point \( Y \) on this geodesic inside this neighbourhood, the geodesic is the shortest path connecting \( X \) and \( Y \).}
take a very simple form when expressed using $s$ parametrization and we subsequently obtain two hyperbolic phase portraits as seen in Section 3. PMP gives us that the stationary curves obtained via these ODE’s are local minimizers at the least, given that the end conditions are close enough. Here it must be noted that an extension of PMP [31] is to be used here which only requires $L_1$ controls (instead of $L_{\infty}$ controls). It then turns out that the resulting geodesics are analytic. Hence we can set smooth PMP controls from the beginning and this results in the same canonical ODE’s.

In this article, we also apply a different method described by Bryant and Griffiths [5] to arrive at the same ODE’s as given by the PMP in a form that simplifies integration to derive the cuspless sub-Riemannian geodesics in $\mathbb{R}^3 \times S^2$.

A necessary condition for the end conditions for minimizers of $P_{\text{mech}}$ is that they are contained within the range of the $\tilde{\text{Exp}}$ map defined later (in Subsection 3.5) for the problem $P_{\text{curve}}$. Roughly speaking, the exponential map $\tilde{\text{Exp}}$ for $P_{\text{curve}}$ maps each pair $(\lambda(0), L)$ of initial momentum and length $L$ to the corresponding end point $(x(L), R(L)) \in SE(3)$. At the end of this article, we include some numerical computations of the range of $\tilde{\text{Exp}}$ of $P_{\text{curve}}$.

1.1 Structure of the article

In Section 2, we apply a classical variational approach on $\mathbb{R}^3$ to solve $P_{\text{curve}}$ and state the equations that are satisfied by the geodesics, in Theorem 2.0.1. Then we move on to the main section of the work, Section 3 where we start with some preliminaries and notations used in the paper in Subsection 3.1. In Subsection 3.2, we describe the problem as a sub-Riemannian problem on $\mathbb{R}^3 \times S^2$ with explanations on the constraints imposed. Then in Subsection 3.3 in Theorem 3.3.2, we derive the expressions for the Lagrangian multipliers in terms of the curvature and torsion of the geodesics. Subsequently in Theorem 3.3.3, we show that along such geodesics, one has covariantly constant momentum w.r.t. Cartan connection. Then we present explicit bounds of the torsion of the cuspless sub-Riemannian geodesics in Corollary 3.3.5. Finally, in Theorem 3.4.3, we derive analytic expressions for the geodesics on the five dimensional coupled space of positions and directions in Section 3.4. Using these explicit expressions, in Theorem 3.4.5, we prove that for co-planar boundary conditions, the unique cuspless sub-Riemannian geodesics of $P_{\text{curve}}$ coincide with the unique cuspless sub-Riemannian geodesics in the two dimensional case as derived in [4, 9]. Having defined the exponential map in Section 3.5, we end the section with a discussion of the symmetries of the exponential map in Section 3.6.

The final section, i.e. Section 4, is dedicated to numerics performed to plot the curves using Mathematica. In Subsection 4.1, we obtain numerical solutions of the initial value problem related to $P_{\text{curve}}$. Here we also discuss the numerical implementation of the range of the exponential map and present some related results. And subsequently in Subsection 4.2, we present some results on the implementation of the shooting algorithm that we used to numerically solve the boundary value problem associated to $P_{\text{curve}}$.

2 The variational approach

Let a stationary curve of $P_{\text{curve}}$ in $\mathbb{R}^3$ be given by $x : [0, L] \to \mathbb{R}^3$ parameterized by arc-length$^4$ denoted by $s$. Let the unit tangent, the unit normal and the unit binormal for this curve be given by $T$, $N$, and $B$ respectively. Assuming the curvature and the torsion of the curve to be given by $\kappa$ and $\tau$, we have the Frenet-Serret equations

$$\frac{d}{ds} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{2.1}$$

$^2$An alpha version of the Mathematica code is available (for academic purposes only). Contact the authors for details.

$^3$Here onwards, $f = \frac{df}{ds}$ for all $f \in C^1([0, L], \mathbb{R})$.
As the energy functional in $P_{\text{curve}}$ is stationary w.r.t. small perturbations along the normal and binormal directions of $x$, we have the following theorem.

**Theorem 2.0.1.** The absolute curvature and the signed torsion of a stationary curve of $P_{\text{curve}}$ satisfy

$$\frac{d^2}{ds^2} \left( \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \right) = \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \left( (\tau(s))^2 + \beta^2 \right)$$

and

$$2\tau(s) \frac{d}{ds} \left( \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \right) + \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \frac{d}{ds} \tau(s) = 0.$$  \hspace{1cm} (2.2)

**Proof.** See Appendix C. \hfill \Box

A direct result of this theorem is the following corollary.

**Corollary 2.0.2.** The torsion for a stationary curve of $P_{\text{curve}}$ (recall Equation (1.3)) in terms of curvature is given by

$$\tau(s) = C \left( 1 + \frac{\beta^2}{\kappa(s)^2} \right)$$

for some constant $C$ and whenever curvature is non zero.

From this result, we get the following corollary.

**Corollary 2.0.3.** Let $[0, L] \ni s \mapsto x(s) \in \mathbb{R}^3$ be a stationary curve of $P_{\text{curve}}$ such that it has finite curvature $\kappa(s)$ for $s \in (0, L)$. If torsion is known to be absent in any interior point of the stationary curve $s \mapsto x(s)$ then the whole curve has no torsion and hence it is a planar curve. And then, Equation (2.2) becomes

$$\frac{d^2}{ds^2} \left( \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \right) = \frac{\beta^2}{\sqrt{(\kappa(s))^2 + \beta^2}} \kappa(s)$$

This confirms the results stated earlier for the two dimension case [12–14].

For general torsion, we can substitute Equation (2.4) in Equation (2.2) and by writing $w(s) := \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}}$, we obtain the following equation:

$$\ddot{w}(s) = Cw(s)^{-3} + \beta^2w(s).$$

We can solve this for $w$ and then solve for $\kappa$ and then apply Equation (2.1) to get the stationary curve. However, these equations are very hard to handle. In the following section, we lift the problem to the five dimensional space $\mathbb{R}^3 \rtimes S^2$ (recall Equality (1.4)) and break the absolute curvature to two signed scalars corresponding to two given directions and obtain a set of easier ODE’s corresponding to Equation (2.2). This will be of help in deriving analytic solutions for the cuspless sub-Riemannian geodesics for $P_{\text{curve}}$ in the subsequent section.

### 3 Lifting the curve optimization to a sub-Riemannian manifold within $SE(3)$

As we can see, the ODE in (2.3) is not easy to solve because of the nonlinear term. In this section, we look for an alternative method to obtain the cuspless sub-Riemannian geodesics explicitly. This method is based on the methods described by Bryant and Griffiths in [5, 6] for solving elastica problems on sub-Riemannian manifolds. Before we start with the details of this method, we present a short review of the background material from differential geometry that we shall employ in this section. Next we lift the problem to a five dimensional Lie group quotient $\mathbb{R}^3 \rtimes S^2$ and derive the Euler Lagrange equations for the cuspless sub-Riemannian geodesics within $\mathbb{R}^3 \rtimes S^2$. 


3.1 Notations and preliminaries

We aim to lift the problem \( \text{P}_{\text{curve}} \) to \( \mathbb{R}^3 \times S^2 \).

As a result of the “Hairy ball theorem” (proved by Brouwer in 1912), \( S^2 \) is not a Lie group but a Lie group quotient \( S^2 = \text{SO}(3)/\text{SO}(2) \). Hence \( \mathbb{R}^3 \times S^2 \) is not a Lie group either. In order to generalize the method in [12–14] for cuspless geodesics on the 2-dimensional case to the 3-dimensional case, we consider the embedding of \( \mathbb{R}^3 \times S^2 \) into the Euclidean motion group \( \text{SE}(3) \).

The group \( \text{SE}(3) \) of three dimensional rigid body motions is given by \( \text{SE}(3) = \mathbb{R}^3 \times \text{SO}(3) \) where \( \text{SO}(3) \) is the group of three dimensional rotation operators. Note that \( \mathbb{R}^3 \times \text{SO}(3) \) is a semidirect product of \( \mathbb{R}^3 \) and \( \text{SO}(3) \) with the group product given by

\[
(x_1, R_1) \cdot (x_2, R_2) = (x_1 + R_1 x_2, R_1 R_2)
\]

for any \( x_1, x_2 \in \mathbb{R}^3 \) and \( R_1, R_2 \in \text{SO}(3) \).

We denote our domain of interest by \( \mathbb{R}^3 \times S^2 \) instead of \( \mathbb{R}^3 \times S^2 \) to emphasize the indirect product structure induced by the embedding into \( \mathbb{R}^3 \times \text{SO}(3) \) (recall Equation (1.4)). The embedding of \( \mathbb{R}^3 \times S^2 \) into \( \mathbb{R}^3 \times \text{SO}(3) \) is given by

\[
(x, n) \leftrightarrow (x, R_n)
\]

with \( (x, n) \in \mathbb{R}^3 \times S^2 \) and \( (x, R_n) \in \mathbb{R}^3 \times \text{SO}(3) \) such that \( R_n e_z = n \) with \( e_z \) being the unit vector in \( \mathbb{R}^3 \) along the z-axis. Note that if \( R_n e_z = n \) then for any \( Q \in \{ Q_0 \in \text{SO}(3) | Q_0 e_z = e_z \} \equiv \text{SO}(2) \), \( R_n Q e_z = n \). Hence we consider

\[
\mathbb{R}^3 \times S^2 = (\mathbb{R}^3 \times \text{SO}(3))/\{0\} \times \text{SO}(2),
\]

where we consider \( \{0\} \times \text{SO}(2) \) as a subgroup of \( \mathbb{R}^3 \times \text{SO}(3) \). So each element \( (x, n) \in \mathbb{R}^3 \times S^2 \) represents a left coset \( \{ (x, R_n) | R_n \in \text{SO}(3) \} \) in the quotient given above.

We shall denote by \( e_x, e_y, \) and \( e_z \), the unit vectors in \( \mathbb{R}^3 \) parallel to respectively the x, y and z axes. And by \( R_{\theta}^i \), we shall denote an anticlockwise rotation by an angle \( \theta \) about the \( i \) axis for \( i \in \{ x, y, z \} \). Then for any rotation \( R_n \in \text{SO}(3) \), we have \( R_n = R_x^\alpha R_y^\beta R_z^\gamma \) with \( \alpha, \beta, \gamma \) denoting the corresponding Euler angles for \( R_n \), where \( \alpha \in [0, 2\pi) \) can be freely chosen.

Considering \( \text{SE}(3) \) as a Lie group, we denote the tangent space at the unity element \( e := (0, 1) \) by \( T_e(\text{SE}(3)) \). We parameterize the elements \( g \in \text{SE}(3) \) by \( g = ((x, y, z)^T, R_x^\alpha R_y^\beta R_z^\gamma) \). Using the differential operator definition of tangent vectors, we choose the following basis vectors for the vector space \( T_e(\text{SE}(3)) \):

\[
A_1 = \partial_{\alpha}\big|_e, A_2 = \partial_{\beta}\big|_e, A_3 = \partial_{\gamma}\big|_e, A_4 = \partial_{\alpha}\big|_e, A_5 = \partial_{\beta}\big|_e, A_6 = \partial_{\gamma}\big|_e.
\]

The Lie bracket in \( T_e(\text{SE}(3)) \) is given by

\[
[A_i, A_j] = \sum_{k=1}^{6} c_{ij}^k A_k.
\]

The \( c_{ij}^k \) for \( i, j, k \in \{1, \ldots, 6\} \) are called the structure constants of the Lie algebra \( T_e(\text{SE}(3)) \) of \( \text{SE}(3) \), and they are antisymmetric in the lower indices. For explicit computation of the structure constants, see [10,12,14].

Next, we introduce the left invariant vector fields.

**Definition 3.1.1.** For any Lie group \( G \), let \( (L_g)_* \) denote the push-forward of the left multiplication by \( g \in G \). For any vector field \( V \) over \( G \), such that the tangent vector corresponding to \( V \) at any \( h \in G \) given by \( V_h \), it is called left invariant if \( (L_g)_*(V_h) = V_{gh} \forall g, h \in G \).

Let the vector space of all sections of the tangent bundle \( T(\text{SE}(3)) \) over \( \text{SE}(3) \) be denoted by \( \Gamma(\text{SE}(3)) \). Let \( L(\text{SE}(3)) := \{ V \in \Gamma(\text{SE}(3)) | (L_g)_*(V_h) = V_{gh} \forall g, h \in G \} \).

**Remark 3.1.2.** We have, as a well known result, \( L(\text{SE}(3)) \cong T_e(\text{SE}(3)) \).
Figure 1: Illustrations of the left invariant frame representing a moving frame of reference along a curve on $\mathbb{R}^3 \rtimes S^2$. The spatial velocity and the angular velocity are depicted in the frame to highlight the constraints between the spatial and angular frame.

\textbf{Corollary 3.1.3.} Let us define $A_i \in \mathcal{L}(\text{SE}(3))$ for $i = 1, \ldots, 6$ by $T_g(\text{SE}(3)) \ni A_i|_g := L^*_g(A_e)$ for all $g \in G$. Then $\{A_i\}_{i=1}^6$, explicitly given by the formulas (1.3), defines a basis of $\mathcal{L}(\text{SE}(3))$ corresponding to the basis of $T_e(\text{SE}(3))$ given by (3.1).

Figure 1 illustrates the reference frame in $\text{SE}(3)$ provided by the left invariant vector fields. It also shows a relation of the frame with the spatial and angular velocities of the curve $s \mapsto x(s)$.

We denote the co-tangent space at any $g \in \text{SE}(3)$ by $T^*_g(\text{SE}(3))$ and the co-tangent bundle over $\text{SE}(3)$ by $T^*(\text{SE}(3))$. Let $\mathcal{L}^*(\text{SE}(3))$ denote the dual vector space to $\mathcal{L}(\text{SE}(3))$. We define a basis of $\mathcal{L}^*(\text{SE}(3))$ denoted by $\{\omega^i\}_{i=1}^6$ as

$$\langle \omega^i, A_j \rangle = \delta^i_j$$

for $i, j \in \{1, \ldots, 6\}$,

where $\delta^i_j$ is the Kronecker delta function. Next we present the notion of horizontal curves.

\textbf{Definition 3.1.4.} Let $\text{span}\{A_i|_g\}_{i=3,4,5}$ be the horizontal part of the tangent space $T_g(\text{SE}(3))$ at any $g \in \text{SE}(3)$. A horizontal curve then is defined as a function $\gamma : \mathbb{R} \to \text{SE}(3)$ such that the vector field tangent to it $\dot{\gamma} \in \text{span}\{A_i|_\gamma\}_{i=3,4,5}$. Or equivalently, $\gamma$ is horizontal iff $\langle \omega^i|_\gamma, \dot{\gamma} \rangle = 0$ for $i \in \{1, 2, 6\}$. Note that the derivative of $\gamma$ is taken w.r.t. spatial arc-length $s$ parametrization, a convention that we will follow in the remainder of this paper\footnote{Such a convention can only be applied to cuspless geodesics, i.e. we consider only those stationary solutions of $P_{\text{mec}}$ (1.6) which satisfy $u^3(t) \neq 0$ for all $t \in (0, T)$.}

The significance of horizontality in our context is that horizontal curves $\gamma = (x, R)$ lie in the kernel of the map $f : C^\infty(\mathbb{R}, \text{SE}(3)) \to C^\infty(\mathbb{R}, \mathbb{R}^3)$ given by $(x, R) \mapsto \dot{x} - Re_z$ and have the property that the $\mathbb{R}^3$ part of the curves do not twist about their own tangent direction. Thus they uniquely represent smooth curves on $\mathbb{R}^3$.

For sake of notational convenience, we use the same notations for denoting a point on a manifold and a function from $\mathbb{R}$ to the manifold whenever the meaning is clear from the context. For example $(x, R) \in \text{SE}(3)$, or $(x, R) : \mathbb{R} \to \text{SE}(3)$, given by $s \mapsto (x(s), R(s))$, depending on the context.

\textbf{Definition 3.1.5.} Given a manifold $M$, a sub-Riemannian geometry on $M$ is made of a horizontal sub-bundle $H$ of the tangent bundle $TM$ over $M$ and an inner product $G$, over $H$. We denote this sub-Riemannian manifold by $(M, H, G)$, cf. [27].
Let $\beta > 0$ be the constant in (1.1) that balances the relative cost for bending and stretching of the curve $x$ in $P_{\text{curve}}$. In order to lift $P_{\text{curve}}$ to $\mathbb{R}^3 \times S^2 \hookrightarrow SE(3)$, we define the metric tensor $G_\beta$, the manifold $M$ and the horizontal space $H$ as

$$\begin{align*}
G_\beta &:= \beta^2 \omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4 + \omega^5 \otimes \omega^5, \\
M &= SE(3), \\
H &= \ker(\omega^1) \cap \ker(\omega^2) \cap \ker(\omega^6),
\end{align*}$$

(3.3)

we have the canonical metric $d$ on the sub-Riemannian manifold $(M, H, G_\beta)$ as

$$d(g, h) := \inf_{\gamma \in \tilde{C}[0, L], \, SE(3), \, \dot{\gamma} \in H, \gamma(0) = g, \, \gamma(L) = h, \, L \geq 0} \left( \int_0^L \sqrt{G_\beta(\dot{\gamma}(s), \dot{\gamma}(s)) \, ds} \right)$$

(3.4)

for any given pair $g, h \in M$, where $\tilde{C}[0, L], \, SE(3)$ denotes the set of absolutely continuous curves.

From Equations (1.3) and (3.4), we deduce that lifting the problem $P_{\text{curve}}$ boils down to finding the cuspless sub-Riemannian geodesics connecting two given points $g$ and $h$ on the sub-Riemannian manifold $(M, H, G_\beta)$, where (akin to the two dimensional case [4, 9]) we assume that $g$ and $h$ are chosen such that they can be connected by a cuspless geodesic.

The following theorem is useful to relate the horizontal curves with the curvature and torsion of the spatial part of this curve.

**Theorem 3.1.6.** Let $\gamma : I \subseteq \mathbb{R} \to SE(3)$ be a horizontal curve with unit speed in the spatial part i.e. $\dot{\gamma}(s) = A_3|_{\gamma(s)} + a(s)A_4|_{\gamma(s)} + b(s)A_5|_{\gamma(s)}$ for all $s \in I$. Then the curvature magnitude $\kappa$ and the torsion $\tau$ for the spatial part of the curve are given by

$$\begin{align*}
\kappa(s) &= \sqrt{(a(s))^2 + (b(s))^2} \quad \text{and} \\
\tau(s) &= \frac{a(s)b(s) - b(s)a(s)}{(a(s))^2 + (b(s))^2}.
\end{align*}$$

(3.5) (3.6)

**Proof.** First of all we note that the curve $\gamma$ when projected on $\mathbb{R}^3$ has the unit tangent vector at any $s \in I$ as $T(s) := A_3|_{\gamma(s)}$. Then we use the covariant derivative of the left invariant vector fields in the following way

$$\frac{d}{ds} A_i|_{\gamma(s)} = \nabla \gamma A_i|_{\gamma(s)} = \sum_{j,k \in \{1, \ldots, 6\}} c_{jk}^i(\omega^j|_{\gamma(s)}, \dot{\gamma}(s))A_k|_{\gamma(s)}.$$  

(3.7)

Note that the covariant derivatives of the left invariant vector fields corresponding to the $\mathbb{R}^3$ part of $SE(3)$ do not involve the other left invariant vector fields, and hence the normal and the binormal in $\mathbb{R}^3$ are obtained by using (3.7) and the Serret-Frenet equations. Curvature and torsion follow from the same equations. \hfill \square

### 3.2 The extended manifold

Let us return to the minimization problem $P_{\text{curve}}$ (1.3) and its extension $P_{\text{mec}}$ (1.6), where we assume that curves are parameterizable by spatial arc-length. We take the Lagrangian approach by Bryant and Griffiths [3] to obtain the cuspless sub-Riemannian geodesics. To that end, we consider an extension of the manifold $SE(3)$ to the 15 dimensional manifold

$$Z := SE(3) \times (\mathbb{R}^+ \cup \{\infty\})^2 \times \mathbb{R}^+ \times T^*(SE(3))$$

...
Lemma 3.3.1. The following holds along cuspless sub-Riemannian geodesics in \((M, H, \mathcal{G}_3)\) (i.e. lifts of the stationary curves of \(P_{\text{curve}}\)):

\[ w_1(s)\dot{w}_2(s) - w_2(s)\dot{w}_1(s) = w_1(0)\dot{w}_2(0) - w_2(0)\dot{w}_1(0) =: W \text{ for all } s \in [0, L].\]
Proof. The torsion of any stationary curve must satisfy both \([2.4]\) and \([3.6]\). From \([2.4]\), we have for \(s \in [0, L]\) and some constant \(C\), that

\[
\tau(s) = C \left(1 + \frac{\beta^2}{\kappa(s)^2}\right) = \frac{C}{\|w(s)\|^2}.
\]

On the other hand, we have from \([3.6]\),

\[
\tau(s) = \frac{\kappa_1(s)\kappa_2(s) - \kappa_2(s)\kappa_1(s)}{(\kappa_1(s))^2 + (\kappa_2(s))^2} = \frac{w_1(s)\dot{w}_2(s) - w_2(s)\dot{w}_1(s)}{\|w(s)\|^2}.
\]

So we conclude that the constant \(C = w_1(s)\dot{w}_2(s) - w_2(s)\dot{w}_1(s) = W\) for all \(s \in [0, L]\). \(\Box\)

**Theorem 3.3.2.** Curves \(s \mapsto \mathbf{r}(s) \in \mathbb{R}^3\) with unit speed parameter \(s\) corresponding to the stationary points of the energy functional \(E\) (we call them stationary curves of \(P_{\text{curve}}\)) having corresponding arclength \(L\) (free) and curvature \(\kappa\) subject to boundary conditions \(\mathbf{r}(0) = \mathbf{0} \in \mathbb{R}^3\), \(\dot{\mathbf{r}}(0) = \mathbf{e}_s \in S^2\), \(\mathbf{r}(L) = \mathbf{x}_1 \in \mathbb{R}^3\) and \(\dot{\mathbf{r}}(L) = \mathbf{n}_1 \in S^2\), satisfy the following condition for \(\psi\) in the extended space:

\[
\frac{d}{d\tau} \left( \int_{N_r} \psi \right) \bigg|_{\tau = 0} = 0
\]

for a one parameter family \(\tau \mapsto N_r\) of all perturbations of the curve in the extended manifold such that \(N_0 = \{\Gamma(s) | 0 \leq s \leq \tau_{\text{max}}\}\) corresponds to the stationary curve in \(\mathbb{R}^3\).

Along the stationary curves \(\Gamma = (g, \kappa_1, \kappa_2, \sigma, \lambda_1, \ldots, \lambda_6)\) in \(Z\), the Lagrange-multipliers are given by

\[
\begin{align*}
\lambda_1(s) &= -\dot{w}_1(s), \\
\lambda_2(s) &= -\dot{w}_2(s), \\
\lambda_3(s) &= \beta \sqrt{1 - \|\mathbf{w}(s)\|^2}, \\
\lambda_4(s) &= -w_2(s), \\
\lambda_5(s) &= +w_1(s), \\
\lambda_6 &= 0,
\end{align*}
\]

expressed in the “normalized” curvature

\[
\mathbf{w}(s) = (w_1(s), w_2(s)) := \frac{1}{\sqrt{(\kappa_1(s))^2 + \beta^2}} (\kappa_1(s), \kappa_2(s)) = \cosh(\beta s, \mathbf{w}(0)) + \frac{\sinh(\beta s)}{\beta} \mathbf{w}(0),
\]

with \(\kappa = \sqrt{(\kappa_1)^2 + (\kappa_2)^2}\). They satisfy the following preservation-laws

\[(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2 = \epsilon^2 \beta^2 \quad \text{(co-adjoint orbits)},
\]

\[
\beta^{-2}(\lambda_3)^2 + (\lambda_4)^2 + (\lambda_5)^2 = 1,
\]

with

\[
\epsilon = \sqrt{1 - \|\mathbf{w}(0)\|^2 + \beta^{-2}\|\mathbf{w}(0)\|^2} > 0
\]

a constant along the curve.

Proof. For minimization of the energy functional given by Equation \([1.1]\), the corresponding functional to be minimized for curves in \(Z\) is \(\int \psi\). Following the theory on optimizing Lagrangians set out in \([3.0]\), we consider a one parameter family of perturbations of the stationary curve in \(Z\) and consider the variation of the functional. For a stationary curve \(\Gamma\) in \(Z\),

\[
\frac{d}{d\tau} \left( \int_{N_r} \psi \right) \bigg|_{\tau = 0} = 0
\]

Hence

\[
\frac{d}{d\tau} \left( \int_{N_r} \psi \right) \bigg|_{\tau = 0} = \int_{N_0} \mathcal{L}_{\dot{\mathbf{r}}} \psi = \int_{N_0} \left( \int_{\partial N_0} \frac{\partial}{\partial \tau} \right) d\psi + \int_{N_0} \frac{\partial}{\partial \tau} \psi = \int_{\partial N_r} \frac{\partial}{\partial \tau} d\psi = 0
\]

where \(\mathcal{L}\) is the Lie derivative along the vector \(\frac{\partial}{\partial \tau}\) and \(j\) is the contraction operator. Here we used \(\mathcal{L}_{\frac{\partial}{\partial \tau}} = X_{\parallel} d\omega + d(X_{\parallel} \omega)\) for vectors \(X\) and tensors \(\omega\), the Stokes’ theorem \(\int_{\partial N_r} d\omega = \int_{\partial N_r} \omega\), and the fact that we have fixed boundary conditions implying \(\int_{\partial N} \frac{\partial}{\partial \tau} \psi = 0\).
This tells us that for the stationary curves in $Z$, as $\frac{\partial}{\partial r}$ is an arbitrary vector field for perturbation directions in $Z$, 
\[ X|d\psi = 0 \text{ for all } X \in T(Z). \]
Calculating for the 15 basis vectors \( \{\partial_{\lambda_1}, \ldots, \partial_{\lambda_6}, \partial_{\kappa_1}, \partial_{\kappa_2}, A_1, \ldots, A_6\} \) of \( T(Z) \), using Cartan's structural equations,
\[ d(\omega^k) = -\sum_{i,j} c^k_{ij} \omega^i \otimes \omega^j \]
with structure constants \( \{c^k_{ij}\}_{i,j,k \in \{1, \ldots, 6\}} \), we obtain the following Pfaffian system of equations:
\[
\begin{align*}
\partial_{\lambda_1} d\psi &= \theta^i = 0 & i \in \{1, \ldots, 6\}, \\
\partial_{\lambda_2} d\psi &= (\sqrt{\kappa^2 + \beta^2} - \lambda_3 + \lambda_4 \kappa_2 - \lambda_5 \kappa_1) dr = 0, \\
\partial_{\lambda_3} d\psi &= (\kappa_1/\sqrt{\kappa^2 + \beta^2} - \lambda_3) \sigma dr = 0, \\
\partial_{\lambda_4} d\psi &= (\kappa_2/\sqrt{\kappa^2 + \beta^2} + \lambda_4) \sigma dr = 0, \\
\partial_{\lambda_5} d\psi &= \partial_{\lambda_1} - \lambda_2 \omega^6 + \lambda_3 \omega^5 = 0, \\
\partial_{\lambda_6} d\psi &= \partial_{\lambda_2} - \lambda_3 \omega^4 + \lambda_1 \omega^6 = 0, \\
\partial_{\kappa_1} d\psi &= \partial_{\lambda_1} - \lambda_3 \omega^5 + \lambda_2 \omega^4 = 0, \\
\partial_{\kappa_2} d\psi &= \partial_{\lambda_4} - \lambda_5 \omega^6 + \lambda_6 \omega^5 - \lambda_2 \omega^3 + \lambda_3 \omega^2 = 0, \\
\partial_{A_1} d\psi &= \partial_{A_2} - \lambda_5 \omega^4 + \lambda_4 \omega^6 - \lambda_3 \omega^1 + \lambda_1 \omega^3 = 0, \\
\partial_{A_6} d\psi &= \partial_{A_3} - \lambda_4 \omega^5 + \lambda_5 \omega^4 - \lambda_1 \omega^2 + \lambda_2 \omega^1 = 0, \\
\end{align*}
\]
with $ds = \sigma dr$ and $\sigma = \|\partial_r x(r)\|$. The first six equations give back the horizontality constraints. The next three equations give us
\[
\begin{align*}
\lambda_3 &= \frac{\beta^2}{\sqrt{\kappa^2 + \beta^2}} = \beta \sqrt{1 - \|w(s)\|^2}, \\
\lambda_5 &= \frac{\kappa_1}{\sqrt{\kappa^2 + \beta^2}} = w_1, \\
\lambda_4 &= -\frac{\kappa_2}{\sqrt{\kappa^2 + \beta^2}} = -w_2.
\end{align*}
\]
From the rest, we get the following coupled ODE’s.
\[
\begin{align*}
\dot{\lambda}_1 &= -\kappa_1 \lambda_3 = -\beta^2 \lambda_5, & \dot{\lambda}_4 &= -\kappa_1 \lambda_6 + \lambda_2, \\
\dot{\lambda}_2 &= -\kappa_2 \lambda_3 = \beta^2 \lambda_4, & \dot{\lambda}_5 &= -\kappa_2 \lambda_6 - \lambda_1, \\
\dot{\lambda}_3 &= \kappa_1 \lambda_1 + \kappa_2 \lambda_2, & \dot{\lambda}_6 &= 0.
\end{align*}
\]
Hence we have
\[
\dot{\lambda}_4 = -\kappa_1 \lambda_6 - \kappa_2 \lambda_3 \text{ and } \dot{\lambda}_5 = -\kappa_2 \lambda_6 + \kappa_1 \lambda_3.
\]
Eliminating $\lambda_3$ from these, and using the relation $w_1 \dot{k}_1 + w_2 \dot{k}_2 = (1 - \|w\|^2)^{-\frac{3}{2}} \beta w \cdot \mathbf{\dot{w}}$ we get
\[
\lambda_6 = -\frac{\kappa_1 \dot{\lambda}_4 + \kappa_2 \dot{\lambda}_5}{\kappa_1 \dot{k}_1 + \kappa_2 \dot{k}_2} = \frac{w_1 \dot{w}_2 + w_2 \dot{w}_1}{w_1 \dot{k}_1 + w_2 \dot{k}_2} = \frac{(1 - \|w\|^2)^{\frac{3}{2}}}{\beta w \cdot \mathbf{\dot{w}}} \frac{d}{ds} W = 0 \text{ using Lemma 3.3.1.}
\]
Thus we get the ODE’s
\[
\dot{\lambda}_4 = \lambda_2 \text{ and } \dot{\lambda}_5 = -\lambda_1.
\]
After derivation and substituting with $\mathbf{w}$, we get the following system
\[
\mathbf{\dot{w}}(s) = \beta^2 \mathbf{w}(s) \text{ for all } s \in [0, L].
\]
Figure 2: Phase portraits corresponding to the components of \( w \) satisfying the second order differential equation given by (3.18) along the optimal curves. Several orbits are shown with arrows. The various colours (i.e. cases for \( c \)) signify different shapes of the resulting optimal curves.

Hence we find the solution:

\[
\mathbf{w}(s) = \cosh(\beta s)\mathbf{w}(0) + \sinh(\beta s)\dot{\mathbf{w}}(0) / \beta.
\]

Deriving w.r.t. \( s \) we get

\[
\dot{\mathbf{w}}(s) = \beta \sinh(\beta s)\mathbf{w}(0) + \cosh(\beta s)\dot{\mathbf{w}}(0).
\]

It readily follows that

\[
\|\mathbf{w}(s)\|^2 - \beta^{-2}\|\dot{\mathbf{w}}(s)\|^2 = 1 - c^2 \text{ for all } s \in [0, L]
\]

for \( c \) as defined earlier. The preservation laws follow.

Figure 2 shows the phase portraits for the orbits of \( w_1 \) and \( w_2 \) w.r.t. \( \dot{w}_1 \) and \( \dot{w}_2 \) describing the evolution of these quantities along the stationary curves obtained.

The system (3.13) can also be written in a compact form using the Cartan connection defined in Appendix A. We have the following remarkable result.

**Theorem 3.3.3.**

- Exponential curves \(^5\) are the autoparallel curves, which are the solutions of \( \nabla_{\dot{\gamma}} \dot{\gamma} = 0 \), w.r.t. Cartan connection \( \nabla \) on the tangent bundle \( T(SE(3)) \) (see Appendix A, Equation (A.2)).

- Horizontal exponential curves are the autoparallel curves w.r.t. connection \( \bar{\nabla} \) (see appendices).

- Along an exponential curve, the tangent vectors are covariantly constant, whereas, along a stationary curve, one has covariantly constant momentum, i.e.

\[
\bar{\nabla}_\gamma \lambda = 0.
\]

See Appendix[A.1] for a proof. We have the following results for the range of the curvature and the torsion of the optimal curve.

---

\(^5\)The exponential curves are given by \( \gamma(s) = \gamma(0) \cdot \text{Exp}(s \sum c_i A_i) \) with \( c_1^2 + c_2^2 + c_3^2 = 1 \) to ensure \( s \) to be the spatial arc-length parameter.
Corollary 3.3.4. The absolute curvature $\kappa$ of the optimal curve increases/decreases as $\|w\|$ increases/decreases. Both $\kappa$ and $\|\dot{w}\|$ along with $\|\ddot{w}\|$ attain the minimum at the point

$$s_{\text{min}} = \frac{1}{2\beta} \ln \frac{\|\beta w(0) - \dot{w}(0)\|}{\|\beta w(0) + \dot{w}(0)\|}$$

if $w(0) \cdot \dot{w}(0) \leq 0$, else $s_{\text{min}} = 0$. If $w(0) \cdot \dot{w}(0) \leq 0$, then the minimum values of $\|w\|$ and $\|\dot{w}\|$ are

$$\sqrt{\frac{1}{2} \left( \sqrt{(1 - c^2)^2 + 4\beta^2 W^2 + 1 - c^2} \right)}$$

and

$$\sqrt{\frac{2}{\beta} \left( \sqrt{(1 - c^2)^2 + 4\beta^{-2} W^2 - 1 + c^2} \right)}$$

respectively, with $W$ and $\epsilon$ given by Lemma 3.3.1 and Equation 3.12 respectively.

Proof. From Theorem 3.3.2, we have

$$\|w\| = \frac{\kappa}{\sqrt{\kappa^2 + \beta^2}} \Rightarrow \frac{d}{ds}\|w\| = \frac{\beta^2 \kappa}{(\kappa^2 + \beta^2)^{3/2}}$$

Therefore, $\dot{\kappa}$ has the same sign as $\frac{d}{ds}\|w\|$ all along the curve, and attains 0 at the same $s \geq 0$.

Now,

$$\frac{d}{ds}\|w\| = \frac{w \cdot \dot{w}}{\|w\|}$$

$$\Rightarrow \frac{d^2}{ds^2}\|w\| = \left( \frac{\|w\|^2 + w \cdot \dot{w}}{\|w\|^3} \right) = \left( \frac{\|w\|^2\|w\|^2 - (w \cdot \dot{w})^2}{\|w\|^4} \right) + \beta^2\|w\|^4 \geq 0.$$

Here we have used $\ddot{w}(s) = \beta^2 w(s)$ (Equation 3.18).

Hence, we conclude that a critical point of $\|w\|$ in $[0, L]$ is a minimum. To obtain $s_{\text{min}}$, we set the derivative to zero and obtain,

$$\frac{d}{ds}\|w(s)\| \big|_{s=s_{\text{min}}} = 0 \Rightarrow w(s_{\text{min}}) \cdot \dot{w}(s_{\text{min}}) = 0$$

Solving this for $s_{\text{min}}$ using Theorem 3.3.2 we get

$$s_{\text{min}} = \frac{1}{2\beta} \ln \frac{\|\beta w(0) - \dot{w}(0)\|}{\|\beta w(0) + \dot{w}(0)\|}$$

Evaluating $\|w\|$ at this point, we get the minimum value. By similar calculations, results for the minimum of $\|\dot{w}\|$ follow.

Note that $s_{\text{min}}$ is a positive real number only if $w(0) \cdot \dot{w}(0) \leq 0$. This is expected as $\|w\|$ is clearly convex, and if its derivative is always positive then the minimum is attained at $s = 0$. □

Corollary 3.3.5. For a cuspless sub-Riemannian geodesic of problem $P_{\text{curve}}$ satisfying $W \neq 0$, the torsion is given by

$$\tau(s) = W \left\| \frac{w}{\|w(s)\|^2} \right\|^2 \text{ for all } s \in [0, L]$$

(3.19)

and satisfies the following bounds

$$\left| W \right| \leq \left\| \tau(s) \right\| \leq \frac{2\left| W \right|}{\sqrt{1 + c^4 - 2c^2 + 4\beta^{-2} W^2 + 1 - c^2}} \text{ for all } s \in [0, L]$$

(3.20)

with $W$ and $\epsilon$ given by Lemma 3.3.1 and Equation 3.12 respectively.

Proof. The expression (3.19) for torsion is readily derived from Lemma 3.3.1. Inequality (3.20) follows by taking the upper and lower bounds on $\|w(s)\|$ from Corollary 3.3.4. □
The following corollary tells us that the variational method also gives us the same equations for the cuspless sub-Riemannian geodesics of problem $P_{\text{curve}}$.

**Corollary 3.3.6.** With the curvature $\kappa$ and torsion $\tau$ given by Theorem 3.3.2 Lemma 3.3.1 and Corollary 3.3.5, we have that $\kappa$ and $\tau$ satisfy Equations (2.2) and (2.3).

*Proof.* First we derive Equation (2.2):

$$
\frac{d^2}{ds^2} \left( \frac{\kappa(s)}{\sqrt{\kappa(s)^2 + \beta^2}} \right) - \frac{\kappa(s)}{\sqrt{\kappa(s)^2 + \beta^2}} \left( (\tau(s))^2 + \beta^2 \right)
= \frac{d^2}{ds^2} (\|w(s)\|) - \|w(s)\| \left( \left( \frac{W}{\|w(s)\|} \right)^2 + \beta^2 \right)
= \left( \|\dot{w}(s)\|^2 - \|w(s)\|^2 - (w(s) \cdot \dot{w}(s))^2 \right) + \beta^2 \|w(s)\|^4
- \frac{W^2 + \beta^2 \|w(s)\|^4}{\|w(s)\|^2}
= 0 \text{ (Using } W^2 + (w(s) \cdot \dot{w}(s))^2 = \|\dot{w}(s)\|^2 \|w(s)\|^2). \)

Next we derive Equation (2.3):

$$
2\tau(s) \frac{d}{ds} \left( \frac{\kappa(s)}{\sqrt{\kappa(s)^2 + \beta^2}} \right) + \frac{\kappa(s)}{\sqrt{\kappa(s)^2 + \beta^2}} \frac{d}{ds} \tau(s)
= \frac{2W}{\|w(s)\|^2} \frac{d}{ds} (\|w(s)\|) + \|w(s)\| \frac{d}{ds} \left( \frac{W}{\|w(s)\|} \right)
= 0.
$$

Thus we conclude that the equations given by the two methods (in respectively Section 2 and Section 3) give the same stationary curves.

The cuspless sub-Riemannian geodesic of problem $P_{\text{curve}}$ given in Section 2 can be obtained uniquely for given initial conditions $w(0), \dot{w}(0), C$ and the angle $\theta$ denoting the angle made by the initial normal vector $N(0)$ with the positive $x$-axis. The following corollary gives the correspondence of the initial conditions of the two methods to produce the same cuspless sub-Riemannian geodesics.

**Corollary 3.3.7.** Consider the cuspless sub-Riemannian geodesics given by the Theorem 3.3.2 with initial conditions $w(0)$ and $\dot{w}(0)$. We get the same curves by the method pursued in Section 2 by taking $w(0) = \|w(0)\|, \dot{w}(0) = \frac{w(0)}{\|w(0)\|} (0 \text{ if } \|\dot{w}(0)\| = 0), C = W$ and $\{\cos \theta, \sin \theta\}^T = \frac{\dot{w}(0)}{\|\dot{w}(0)\|}$ (if $\|\dot{w}(0)\| = 0$). In the special case $\|w(0)\| = 0 = \|\dot{w}(0)\|$, the geodesics are straight lines.

### 3.4 The stationary curves

From the formulas, it is clear that a stationary curve cannot extend indefinitely for all initial conditions on $w$ and $\dot{w}$. In fact, for any given initial values of $w$ and $\dot{w}$, the maximum length $s_{\text{max}}$ of a stationary curve, where we have $\kappa(s) \to \infty$ as $s \uparrow s_{\text{max}}$, is given by the following theorem.

**Theorem 3.4.1.** Consider a stationary curve of $P_{\text{curve}}$ with initial values $w(0)$ and $\dot{w}(0)$ such that $\|w(0)\| \leq 1$, then its maximum length is given by

$$
s_{\text{max}} = \frac{1}{2\beta} \ln \frac{1 + c^2 + 2\sqrt{c^2 - \beta^{-2}W^2}}{\|w(0)\| + \beta^{-1} \|\dot{w}(0)\|} \quad (3.21)
$$

with $W$ and $c$ given by Lemma 3.3.1 and Equation (3.12) respectively. For given $w(0)$ and $\dot{w}(0)$, the stationary curve can extend indefinitely iff $\beta w(0) + \dot{w}(0) = 0$. 

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Proof. It is clear from the expression of the curvature in terms of $w$ that we can obtain a real or infinite curvature as long as $\|w\| \leq 1$. But we have seen that $\|w\|$ is convex. So, $s_{\text{max}}$ is the length of the curve at which $\|w\| = 1$ and the curve cannot be extended after this as $\frac{d}{ds}\|w\| > 0$ and the curvature becomes infinity already. Solving the following equation

$$\|w(s)\|^2 = \|\cosh(\beta s)w(0) + \frac{\sinh(\beta s)}{\beta} \dot{w}(0)\|^2 = 1$$

for $s$, we get the required expression.  

A consequence of this theorem is that among the sub-Riemannian geodesics in $(M, H, G_\beta)$ having same values for $\|w(0)\|$ and $\|\dot{w}(0)\|$, the ones having $W = 0$, i.e, the planar sub-Riemannian geodesics have the highest $s_{\text{max}}$.

**Corollary 3.4.2.** For fixed $\|w(0)\|$ and $\|\dot{w}(0)\|$, $s_{\text{max}}$ is maximum at such $w(0)$ and $\dot{w}(0)$ which satisfy $W = 0$ and $w(0) \cdot \dot{w}(0) \leq 0$.

**Proof.** Let $-\pi \leq \theta \leq \pi$ be such that $w(0) \cdot \dot{w}(0) = \|w(0)\|\|\dot{w}(0)\| \cos \theta$ and $W = \|w(0)\|\|\dot{w}(0)\| \sin \theta$. Then along with the condition that $s_{\text{max}} > 0$, we obtain the following implication by deriving Equation (3.21) w.r.t $\theta$

$$\frac{ds_{\text{max}}}{d\theta} = 0 \Rightarrow \sin \theta = 0,$$

which gives us three critical points, $\pm \pi$ and $0$. Comparing the $s_{\text{max}}$ at these critical points, we get $\pm \pi$ as the candidates for $s_{\text{max}}$ to be maximum. Checking the second derivative at $\pm \pi$ we obtain $s_{\text{max}}$ to be maximum at $\theta = \pm \pi$. This implies $W = 0$, and $w(0) \cdot \dot{w}(0) = -\|w(0)\|\|\dot{w}(0)\| \leq 0$.  

The next theorem gives the description of the cuspless sub-Riemannian geodesics in $(SE(3), H, G_\beta)$ having length less than or equal to $s_{\text{max}}$ corresponding to the initial conditions.

**Theorem 3.4.3.** The spatial part of the cuspless sub-Riemannian geodesic in $(SE(3), H, G_\beta)$ curve is given by

$$\mathbf{x}(s) = \tilde{R}(0)^T (\mathbf{z}(s) - \tilde{z}(0))$$

where, $\tilde{R}(0)$ and $\tilde{z}(s) := (\tilde{x}(s), \tilde{y}(s), \tilde{z}(s))$ are given in terms of $w$ and $\dot{w}$ depending on several cases. For all cases, we have

$$\tilde{x}(s) = \frac{1}{\xi} \int_0^s \sqrt{1 - \|w(\tau)\|^2} \, d\tau.$$  

(3.23)

For the case $\dot{w}(0) = 0$, we have

$$\tilde{R}(0) = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right)$$

(3.24)

For the case $\dot{w}(0) \neq 0$, we have

$$\tilde{R}(0) = \frac{1}{\xi \beta} \left( \begin{array}{ccc} -\dot{w}_1(0) & -\dot{w}_2(0) & \sqrt{\xi^2 \beta^2 - \|w(0)\|^2} \\ \frac{\xi \beta}{\|w(0)\|^2} w_1(0) \sqrt{\xi^2 \beta^2 - \|w(0)\|^2} & \frac{\xi \beta}{\|w(0)\|^2} w_2(0) \sqrt{\xi^2 \beta^2 - \|w(0)\|^2} & 0 \\ w_1(0) \sqrt{\xi^2 \beta^2 - \|w(0)\|^2} & w_2(0) \sqrt{\xi^2 \beta^2 - \|w(0)\|^2} & \|\dot{w}(0)\| \end{array} \right).$$

(3.26)

where $\dot{w}_1(0)$ and $\dot{w}_2(0)$ are the lifts of the stationary curves of $P_{\text{curve}}$ for appropriate boundary conditions and which coincide with the solutions of $P_{\text{mec}}$ for the same boundary conditions.

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$^6$which are the lifts of the stationary curves of $P_{\text{curve}}$ for appropriate boundary conditions and which coincide with the solutions of $P_{\text{mec}}$ for the same boundary conditions.
Thus after having \( \dot{w}(0) \neq 0 \), we have
\[
\begin{pmatrix}
\dot{y}(s) \\
\dot{z}(s)
\end{pmatrix} = \frac{w(s) \cdot \dot{w}(0)}{c\beta ||w(0)||} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (3.27)

In case \( W \neq 0 \) along with \( \dot{w}(0) \neq 0 \), we have
\[
\begin{pmatrix}
\dot{y}(s) \\
\dot{z}(s)
\end{pmatrix} = \frac{\int A(s') \, ds'}{c^2\beta^2 ||w(0)||} \begin{pmatrix} W\sqrt{c^2\beta^2 - ||w(0)||^2} \\ c\beta w(0) \cdot \dot{w}(0) \end{pmatrix},
\] (3.28)
where \( A(s) \in \mathbb{R}^{2x2} \) is given by
\[
A(s) = \frac{1}{||w(s)||^2 - \frac{W^2}{c^2\beta^2}} \begin{pmatrix}
w(s) \cdot \dot{w}(s) & -\frac{W}{c\beta} \sqrt{c^2\beta^2 - ||w(s)||^2} -\omega^3 w(s) \cdot \dot{w}(s)
\end{pmatrix}.
\] (3.29)

Proof. Observe that if one of the components of \( w(0) \) are \( \pm 1 \) along with \( \dot{w}(0) = 0 \) then \( s_{\text{max}} = 0 \). Hence the solution for a geodesic for such initial conditions is just a point.

We use the relations \( R^T \, dx = (\omega_1, \omega_2, \omega_3)^T \) and
\[
R^T \, dR = \begin{pmatrix} 0 & -\omega_6 & \omega_5 \\
\omega_6 & 0 & -\omega_4 \\
-\omega_5 & \omega_4 & 0
\end{pmatrix}
\] (3.30)
which hold at the points \((x, R) \in SE(3)\), to write the last six equations in (3.13) as \( d\lambda = \lambda g^{-1} \, dg \) where for any element \((x, R) \in SE(3)\), we have the \( 6 \times 6 \) matrix representation \( g \) given by
\[
g = \begin{pmatrix} R & \sigma_x R \\
0 & R
\end{pmatrix}
\] with the \( 3 \times 3 \) matrix \( \sigma_x \) such that \( \sigma_x b = a \times b \) for all \( a, b \in \mathbb{R}^3 \). Thus we have
\[
d\lambda - \lambda g^{-1} \, dg = 0 \iff (d\lambda)g^{-1} - \lambda g^{-1}(dg)g^{-1} = 0 \iff d(\lambda g^{-1}) = 0
\] (3.31)
with \( \lambda = (\lambda_1, \ldots, \lambda_6) \), the Lagrange multipliers which are already known.

Hence, as \( g(0) = e \), the geodesic must satisfy \( \Lambda(s)g^{-1}(s) = \lambda(0)g^{-1}(0) \) for all \( s \in [0, L] \), and therefore
\[
\lambda(s) = \lambda(0)g(s).
\] (3.32)

To make calculations easier, we translate and rotate the curve and solve a slightly easier equation and transform it back to the original curve. With \( \tilde{g} = \begin{pmatrix} \tilde{R} & \sigma_x \tilde{R} \\
0 & \tilde{R}
\end{pmatrix} \), we solve the system
\[
\lambda(s) = (c\beta, 0, 0, -\frac{W}{c\beta}, 0, 0)\tilde{g}(s),
\] (3.33)
with
\[
\tilde{g}(s) = \tilde{g}(0)g(s),
\] (3.34)
such that
\[
\lambda(0) = (c\beta, 0, 0, -\frac{W}{c\beta}, 0, 0)\tilde{g}(0).
\] (3.35)

Thus after having \( \tilde{g}(s) \), we retrieve the geodesic by using the relation
\[
g(s) = \tilde{g}(0)^{-1}\tilde{g}(s).
\] (3.36)
For the most general case, assuming non-vanishing denominators throughout, we see that choosing \(3.26\) and

\[
\begin{bmatrix}
\frac{1}{c_\beta^2 \| \mathbf{w}(0) \|^2} \\
W \sqrt{c_\beta^2 - \| \mathbf{w}(0) \|^2} \\
\mathbf{c}^\beta \mathbf{w}(0) \cdot \mathbf{w}(0)
\end{bmatrix},
\]

Equation \(3.35\) is satisfied. Then solving Equation \((3.33)\) for \(\tilde{x}, \tilde{y}\) and \(\tilde{z}\) and using \((3.34)\), we obtain \(\tilde{x}\) and the following system.

\[
\begin{bmatrix}
\tilde{y}(s) \\
\tilde{z}(s)
\end{bmatrix} = A(s) \begin{bmatrix}
\tilde{y}(s) \\
\tilde{z}(s)
\end{bmatrix}.
\]

Noting that \(A(s)\) and \(A(t)\) commute for all pairs \(s\) and \(t\), and hence using Wilcox formula \([32]\), we get the desired results.

Clearly, the formulas are not valid as the denominators in some of the expressions become zero. Hence we do the whole procedure keeping in mind the special cases right from the start and get the required results. The details of these computations can be found in \([10]\). \(\square\)

This results in the following corollary which states that the sub-Riemannian geodesics do not self intersect or roll up despite the fact that the absolute curvature \(\kappa(s) \to \infty\) as \(s \uparrow s_{\text{max}}\).

**Corollary 3.4.4.** Sub-Riemannian geodesics with initial conditions \(\mathbf{w}(0)\) and \(\dot{\mathbf{w}}(0)\), have a monotonically increasing component along the direction \(-\frac{w_1(0)}{c_\beta^2} \mathbf{e}_x - \frac{w_2(0)}{c_\beta^2} \mathbf{e}_y + \frac{\sqrt{c_\beta^2 - \| \mathbf{w}(0) \|^2}}{c_\beta^2} \mathbf{e}_z\). Hence they do not self intersect or roll up.

**Proof.** From Equation \(3.22\) we see that the tilde frame is given by \(R(0)(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)^T\). Hence the \(\tilde{x}\)-axis corresponds to \(-\frac{w_1(0)}{c_\beta^2} \mathbf{e}_x - \frac{w_2(0)}{c_\beta^2} \mathbf{e}_y + \frac{\sqrt{c_\beta^2 - \| \mathbf{w}(0) \|^2}}{c_\beta^2} \mathbf{e}_z\). Now, by Theorem \(3.4.3\), the component of the sub-Riemannian geodesic along this direction is

\[
\tilde{x}(s) = \frac{1}{c} \int_0^s \sqrt{1 - \| \mathbf{w}(\tau) \|^2} \, d\tau \Rightarrow \dot{\tilde{x}}(s) = \frac{1}{c} \sqrt{1 - \| \mathbf{w}(s) \|^2} > 0 \text{ for all } s \in (0, s_{\text{max}}).
\]

Thus we get the monotonicity along this direction. \(\square\)

We have already seen that \(W = 0\) gives us the planar curves. We expect the reverse also to be true. That is, taking the end conditions to be co-planar, we should get \(W = 0\) implying planar curves as the cuspsless sub-Riemannian geodesic of problem \(P_{\text{nec}}\) connecting them. The following corollary confirms that this is indeed the case.

**Theorem 3.4.5.** Let \(\mathbf{x} : [0, s_{\text{max}}] \to \mathbb{R}^3\) be the spatial part of a cuspsless sub-Riemannian geodesic in \((SE(3), H, G_3)\) given by Theorem \(3.4.3\) i.e. let \(\mathbf{x}\) be a local minimizer of \(P_{\text{curve}}\). Then for any \(s \in [0, s_{\text{max}}]\), one has that \(\mathbf{e}_z, \mathbf{x}(s)\) and \(\dot{\mathbf{x}}(s)\) are co-planar if and only if \(\mathbf{x}\) is a planar curve.

That is

\[
\mathbf{e}_z \cdot (\mathbf{x}(s) \times \dot{\mathbf{x}}(s)) = 0 \iff W = 0.
\]

**Proof.** First we show \(W = 0 \Rightarrow \mathbf{e}_z \cdot (\mathbf{x}(s) \times \dot{\mathbf{x}}(s)) = 0\). From Theorem \(3.4.3\) we get that \(W = 0 \Rightarrow \dot{\mathbf{y}} \equiv 0\). Which implies that the curve is planar. Also, \(\ddot{y}(s) \equiv 0 \Rightarrow \dot{\mathbf{x}}(0) \cdot (\mathbf{x}(s) \times \dot{\mathbf{x}}(s)) = 0 \Rightarrow \mathbf{e}_z \cdot (\mathbf{x}(s) \times \dot{\mathbf{x}}(s)) = 0\).

Now we focus on the other direction of the implication. Let us consider the curve \(\dot{\mathbf{x}} : [0, s_{\text{max}}] \to S^2\). We know that it can have a minimum curvature of 1 and only when it aligns with a great circle on \(S^2\). From Equation \(3.30\), we have that

\[
\mathbf{R} = R \begin{pmatrix}
0 & 0 & \kappa_1 \\
0 & 0 & \kappa_2 \\
-\kappa_1 & -\kappa_2 & 0
\end{pmatrix}.
\]
Using this, we have

$$\dot{x}(s) = R(s)e_z \Rightarrow \ddot{x}(s) = R(s) \begin{pmatrix} \kappa_1(s) \\ \kappa_2(s) \\ 0 \end{pmatrix} \Rightarrow \dddot{x}(s) = R(s) \begin{pmatrix} \dot{\kappa}_1(s) \\ \dot{\kappa}_2(s) \\ -\kappa(s)^2 \end{pmatrix}.$$ 

So we have $\dddot{x}(s) \times \dddot{x}(s) = R(s) \begin{pmatrix} -\kappa(s)^2 \kappa_2(s) \\ \kappa(s)^2 \kappa_1(s) \\ (\kappa(s)^2 + \beta^2)W \end{pmatrix}$. Hence the curvature $K_{\text{tan}}$ of $\dddot{x}$ is given as

$$K_{\text{tan}}(s) = \frac{\|\dddot{x}(s) \times \dddot{x}(s)\|}{\|\dddot{x}(s)\|^3} = \frac{\sqrt{\kappa(s)^6 + (\kappa(s)^2 + \beta^2)^2W^2}}{\kappa(s)^3} = \frac{1 + (\kappa(s)^2 + \beta^2)^2W^2}{\kappa(s)^6}.$$ 

Thus we see that if we let $W \neq 0$, then $K_{\text{tan}}(s) > 1$ for any $s \in (0,s_{\text{max}})$. The curve $\dddot{x}$ gets aligned with great circles only at cusp points where $\kappa(s) = \infty$ which never occurs in an interior point. Thus we can conclude that the curve $\dddot{x}$ can intersect a great circle on $S^2$ at most at two points. As otherwise it must have geodesic curvature equal to zero (and hence total curvature = normal curvature = 1) at some interior point and this is not possible. Therefore, any three points along this curve can never lie simultaneously on a plane passing through the origin. In particular,

$$\dddot{x}(0) \cdot (\dddot{x}(\tau) \times \dddot{x}(s)) \neq 0 \Rightarrow \int_0^s (\dddot{x}(0) \cdot (\dddot{x}(\tau) \times \dddot{x}(s))) d\tau \neq 0 \Rightarrow \dddot{x}(0) \cdot (\dddot{x}(s) \times \dddot{x}(s)) \neq 0.$$

So $W \neq 0 \Rightarrow e_z \cdot (\dddot{x}(s) \times \dddot{x}(s)) = 0 \Rightarrow (S^2 \times \mathbb{R}^2 \times S^1$ (see [4,9]).

**Corollary 3.4.6.** Given admissible coplanar end conditions for $P_{\text{curve}}$, the unique cuspleless stationary curve connecting them is planar.

**Proof.** We apply Theorem 3.4.5 with $s = L$ and note that the planar solutions with $W = 0$ of $P_{\text{curve}}$ coincide with the unique two dimensional globally minimizing sub-Riemannian geodesics connecting the corresponding points in $\mathbb{R}^2 \times S^1$ (see [4,9]).

As a result we have the following corollary.

**Corollary 3.4.7.** Let $R$ denote the range of the exponential map of cuspleless geodesics in $SE(2)$, which coincides with the set of admissible end conditions of $P_{\text{curve}}$ in $SE(2)$. Then the set

$$\{ \left( z_1, \frac{x_1}{\sqrt{y_1^2 + y_1^2}} \sin \theta_1, \frac{y_1}{\sqrt{y_1^2 + y_1^2}} \sin \theta_1, \cos \theta_1 \right)^T | (x_1, \theta_1) \in R \}$$

is a set of end conditions admitting a unique global cuspleless minimizer of $P_{\text{curve}}$.

**3.5 Definition of the exponential map of $P_{\text{curve}}$**

Now that we have the end point of a geodesic as a function of the initial conditions $w(0), \dot{w}(0)$, and the length $L$, we can define the exponential map of the control problem $P_{\text{curve}}$ on the extended manifold $Z$. Note that so far we have expressed $s_{\text{max}}$ in terms of $w(0)$ and $\dot{w}(0)$ (or equivalently $e$ and $W$) in Theorem 3.4.1. However, in the definition of the exponential map below we stress the dependence of $s_{\text{max}} = s_{\text{max}}(\lambda(0))$ on the initial momentum given by

$$\lambda(0) = \sum_{i=1}^5 \lambda_i(0) \omega^i \big|_e = -\dot{w}_1(0) \omega^1 \big|_e - \dot{w}_2(0) \omega^2 \big|_e + \sqrt{c^2 \beta^2} \|\dot{w}(0)\|^2 \omega^3 \big|_e - \dot{w}_2(0) \omega^4 \big|_e + w_1(0) \omega^5 \big|_e.$$ 

(3.37)
Definition 3.5.1. Using the arc-length parametrization, we consider the canonical ODE system for $\Gamma(s) = (g(s), \kappa(s), \lambda(s))$ on $Z$ given by

$$\dot{\Gamma}(s)|_{\Gamma(s)} = 0,$$

which gives rise to an ODE system

$$\dot{\Gamma}(s) = F(\Gamma(s)), \quad s \in [0, L],$$

$$\Gamma(0) = (e, \kappa(0), \lambda(0)),$$

with unity element $e = (0, I) \in SE(3)$ and with $\kappa = (\kappa_1, \kappa_2)^T$ where $\kappa_1(0) = \frac{\beta \lambda_5(0)}{\sqrt{1 - (\lambda_4(0))^2 + \lambda_5(0)^2}}$ and $\kappa_2(0) = \frac{-\beta \lambda_4(0)}{\sqrt{1 - (\lambda_4(0))^2 + \lambda_5(0)^2}}$ where $F$ denotes the corresponding flow field on the manifold $Z$ given as

$$F:\left( \begin{array}{ccc} R & \sigma_2 R & \kappa(s), \lambda(s) \\ 0 & R & \kappa(s), \lambda(s) \\ \end{array} \right) = \left( \begin{array}{ccc} -\beta \lambda_4(0) & \beta \lambda_4(0) \lambda_5 - \lambda_1 \lambda_5 & \lambda_5 \\ \beta \lambda_4(0) - \lambda_1 \lambda_5 & -\lambda_4 & \lambda_5 \\ \end{array} \right) - \frac{\beta^2}{\lambda_3} \left( \begin{array}{ccc} \lambda_1 & 0 \\ 0 & \lambda_2 \\ \end{array} \right) \lambda \left( \begin{array}{ccc} K & \sigma_2 \\ 0 & K \\ \end{array} \right)$$

where

$$K = \frac{\beta^2}{\lambda_3} \left( \begin{array}{ccc} 0 & 0 & \lambda_5 \\ 0 & 0 & -\lambda_4 \\ -\lambda_5 & \lambda_4 & 0 \\ \end{array} \right).$$

This ODE has a unique solution

$$\Gamma(s) = \Gamma(0) e^{sF} \quad s \in [0, L].$$

Define

$$C = \{\lambda(0) \in T^*|_{e}(SE(3))|\beta^{-2}\lambda_3(0)^2 + \lambda_4(0)^2 + \lambda_5(0)^2 = 1\},$$

$$D = \{(\lambda(0), l) \in C \times \mathbb{R}^+|l \leq s_{\text{max}}(\lambda(0)) \neq 0\}$$

and on this set, we define $\tilde{\text{Exp}} : D \to SE(3)$ by

$$\tilde{\text{Exp}}(\lambda(0), l) = \pi \circ e^{lF}(e, \kappa(0), \lambda(0))$$

with $\pi : Z \to SE(3)$ being the natural projection. Note that this exponential map is different from the Lie group valued exponential map defined on the Lie algebra.

3.6 Symmetries of the exponential map

We now look at the linear symmetries of the above given exponential map of $P_{\text{curve}}$. In comparison to [19], we look at the symmetries which retain curvature and torsion along the curve and not at the symmetries involving time inversion, i.e. $s \mapsto L - s$. The conservation law

$$\lambda_1(0)^2 + \lambda_2(0)^2 - \beta^2(\lambda_4(0)^2 + \lambda_5(0)^2) = \beta^2(c^2 - 1) \quad (3.39)$$

along the curves suggest $O(2,2)$ to be the symmetry group for the vectors $(\lambda_1(0), \lambda_2(0), \lambda_4(0), \lambda_5(0))^T$. As we need the pointwise absolute curvature and the torsion to be identical after transformation, we need $\lambda_4(0)^2 + \lambda_5(0)^2$ to be preserved and hence $\lambda_1(0)^2 + \lambda_2(0)^2$ and hence $\lambda_3(0)$ is preserved as well by the transformation. Hence, the transformations are of the form

$$P = \left( \begin{array}{cccc} O_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & O_2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) \in O(6)$$
with $O_1, O_2 \in O(2)$.

Next we note that $P$ has to commute with the ODE system satisfied by the momentum variables, recall Equation (3.17). Hence,

$$
\begin{pmatrix}
O_1 & 0 \\
0 & O_2
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & -\beta^2 \\
0 & 0 & \beta^2 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 & -\beta^2 \\
0 & 0 & \beta^2 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
O_1 & 0 \\
0 & O_2
\end{pmatrix}
$$

$$
\Rightarrow O_2 = det(O_1)O_1^{-T}.
$$

Thus we get that the required symmetry group is $O(2)$ with the transformation given by

$$
P = \begin{pmatrix}
O_1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & det(O_1)O_1^{-1} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

The following corollary gives us the rotational and reflectional symmetries of the exponential map given above.

**Corollary 3.6.1.** Let $P \in \mathbb{R}^{6\times 6}$ be given by

$$
P = \begin{pmatrix}
Q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & det(Q)O_1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

with $Q \in O(2)$ arbitrary. Then we have the following symmetry property of the exponential map

$$
\bar{Exp}_e(P\lambda(0), l) = \left( o, \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot \bar{Exp}_e(\lambda(0), l) \cdot \left( o, \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix} \right).
$$

Here the group product $\cdot$ is on the Euclidean group $E(3)$.

**Proof.** Let us denote the new quantities obtained by the transformation of initial condition by adding the subscript $Q$ to the symbols. We also use the identification of $\lambda$ and $w, \dot{w}$ as given by Theorem 3.3.2, Equation (3.10). So we have

$$
e_\lambda = \sqrt{1 - ||Qw(0)||^2 + \beta^{-2}||Q\dot{w}(0)||^2} = \sqrt{1 - ||w(0)||^2 + \beta^{-2}||\dot{w}(0)||^2} = \epsilon.
$$

And also, if $det(Q) = 1$, we have

$$
W_Q = Det(Qw(0)|Q\dot{w}(0)) = Det(w(0)|\dot{w}(0)) = W.
$$

If $det(Q) = -1$, we have

$$
W_Q = Det(Qw(0)|Q\dot{w}(0)) = -Det(w(0)|\dot{w}(0)) = -W.
$$

Hence it is clear that $s_{\text{max}} = s_{\text{max}}$. Now we see that $w_Q(s) = Qw(s)$ and $\dot{w}_Q(s) = Q\dot{w}(s)$ for all $s \in [0, s_{\text{max}}]$. Hence we have that $||w_Q(s)|| = ||w(s)||$, $||\dot{w}_Q(s)|| = ||\dot{w}(s)||$ and $w_Q(s) \cdot \dot{w}_Q(s) = w(s) \cdot \dot{w}(s)$ for all $s \in [0, s_{\text{max}}]$. Therefore we conclude that $\bar{x}_Q \equiv \bar{x}$.

Now let us consider the case $\dot{w}(0) = 0$. We have that for $s \in [0, s_{\text{max}}]$,

$$
\begin{pmatrix}
x_Q(s) \\
y_Q(s) \\
z_Q(s)
\end{pmatrix} = \begin{pmatrix}
Q \frac{w(s)}{c^3} \\
\dot{Q} \frac{w(s)}{c^3} \\
\bar{x}(s)
\end{pmatrix} = \begin{pmatrix}
Q & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
x(s) \\
y(s) \\
z(s)
\end{pmatrix}.
$$

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Figure 3: Illustration of rotational symmetries in case of several planar cuspless sub-Riemannian geodesics of problem $P_{\text{mec}}$ (i.e. the cuspless sub-Riemannian geodesics of problem $P_{\text{mec}}$ that satisfy $W = 0$) with the initial position $(0, 0, 0)$ and initial tangent $e_z$.

We now consider the case $\dot{w}(0) \neq 0$. We readily have that $\tilde{y}_Q \equiv \det(Q)\tilde{y}$ and $\tilde{z}_Q \equiv \tilde{z}$.

We get
\[
\tilde{R}_Q(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \det(Q) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{R}(0) \begin{pmatrix} 0 & 0 \\ Q^T & 0 \\ 0 & 1 \end{pmatrix}.
\]

Hence for $s \in [0, s_{\text{max}}]$,
\[
x_Q(s) = \tilde{R}_Q(0)^T (\tilde{x}_Q(s) - \tilde{x}_Q(0)) = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \tilde{R}(0)^T (\tilde{x}(s) - \tilde{x}(0)) = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} x(s),
\]

\[\Rightarrow \dot{x}_Q(s) = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \dot{x}(s).
\]

Similar calculations hold for the case $\det(Q) = -1$. Using Equation (3.30), we have that
\[
R^T \dot{R} = \begin{pmatrix} 0 & 0 & \kappa_1 \\ 0 & 0 & \kappa_2 \\ -\kappa_1 & -\kappa_2 & 0 \end{pmatrix} \Rightarrow R(s) = I + \int_0^s (-\kappa_1(\tau)\dot{x}(\tau), -\kappa_2(\tau)\dot{x}(\tau), \dot{x}(\tau)) d\tau
\]
\[
\Rightarrow R_Q(s) = I + \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} (R(s) - I) \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} R(s) \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix}.
\]

Thus we have
\[
(x(s), R_Q(s)) = \left(0, \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot (x(s), R(s)) \cdot \left(0, \begin{pmatrix} Q^T & 0 \\ 0 & 1 \end{pmatrix} \right),
\]

and hence we get the result. $\square$

Figures 3 and 4 depict both the rotational and the reflectional symmetries of the cuspless sub-Riemannian geodesics of problem $P_{\text{mec}}$. To generate the figures we have set
\[
w(0) = \|w(0)\|(\cos\theta, \sin\theta)^T \quad \text{and} \quad \dot{w}(0) = \|\dot{w}(0)\|(\cos(\theta - \Theta), \sin(\theta - \Theta))^T.
\]

Here, $\Theta$ denotes the angle between $w(0)$ and $\dot{w}(0)$. For both these figures, we fixed $\|w(0)\|$ and $\|\dot{w}(0)\|$. For Figure 3, we took $\Theta = 0$ and varied $\theta$. For Figure 4, we fixed $\theta$ and varied $\Theta$. The plane of reflection corresponds to $\Theta = 0$. See Figure 5 for an intuitive explanation of the relation of $\Theta$ with respect to the momentum variables.
Figure 4: Illustration of reflectional symmetry of certain cuspless sub-Riemannian geodesics of problem $P_{mec}$ with initial position $(0, 0, 0)$ and initial tangent $e_z$. Here, these curves are produced by rotating $\dot{w}(0)$ by certain angles while keeping $w(0)$ fixed. The plane of reflection contains the middle curve and corresponds to the case when $\dot{w}(0)$ is parallel to $w(0)$. 
Figure 5: Illustration of momentum and its preservation laws. As momentum is a dual vector we visualize basis vectors $\omega^k|_e$, with $e$ being the unit element $(0, I)$ in $SE(3)$, as equidistantly spaced hyper-planes (in grey) orthogonal to $A_k$, $k = 1, \ldots, 5$. The angle $\Theta$ is the angle between $\mathbf{w} = \|\mathbf{w}\|(\cos \theta, \sin \theta)$ and $\mathbf{\dot{w}} = \|\mathbf{\dot{w}}\|(\cos(\theta - \Theta), \sin(\theta - \Theta))$, so that by Equation (3.10), $\Theta =\Theta + \pi/2$ (mod $2\pi$) denotes the angle between the transversal part of the spatial momentum, and the angular momentum. For co-planar sub-Riemannian geodesics one has $W = \Theta = 0 = W(s) = \Theta(s)$. Indeed, for such curves angular momentum stays orthogonal to spatial momentum.

Spatial momentum

$$\lambda_1 \omega^1 + \lambda_2 \omega^2 + \lambda_3 \omega^3$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \epsilon^2 \beta^2$$

$$\lambda_1 = -\sqrt{\epsilon^2 \beta^2 - \lambda_3^2 \cos \theta}$$

$$\lambda_2 = -\sqrt{\epsilon^2 \beta^2 - \lambda_3^2 \sin \theta}$$

Angular momentum

$$\lambda_4 \omega^4 + \lambda_5 \omega^5$$

$$\lambda_4^2 + \lambda_5^2 = 1 - \beta^{-2} \lambda_3^2, \quad \lambda_6 = 0$$

$$\lambda_4 = -\beta^{-1} \sqrt{\beta^2 - \lambda_3^2 \sin(\theta - \Theta)}$$

$$\lambda_5 = \beta^{-1} \sqrt{\beta^2 - \lambda_3^2 \cos(\theta - \Theta)}$$
4 Numerics

In the previous sections, the space of cuspless sub-Riemannian geodesics was parameterized, effectively solving the Initial Value Problem (IVP) (where one specifies initial curvature vector and its derivative) associated to \( P_{\text{curve}} \). In this section we will discuss how to sample this solution space robustly and effectively, and describe a basic approach to numerically solve the corresponding Boundary Value Problem (BVP) (where one also specifies the final position and tangent).

4.1 Solving the initial value problem associated to \( P_{\text{curve}} \)

Two approaches to solve the IVP have been discussed in the previous sections. In Section 2 Mumford’s variational approach was used to derive Equation (2.6), which combined with the Frenet-Serret formulas, given in Equation (2.1), describes the possible cuspless sub-Riemannian geodesics of problem \( P_{\text{mec}} \) in terms of \( w(0), \dot{w}(0), \theta \), and \( C \). Here \( \theta \) describes the angle between the normal and the \( x \)-axis, giving the following initial conditions of the Frenet-Serret equations:

\[
T(0) = e_z, \quad N(0) = \cos(\theta)e_x + \sin(\theta)e_y, \quad B(0) = -\sin(\theta)e_x + \cos(\theta)e_y. \quad (4.1)
\]

Section 3 shows how Lagrange multipliers can be used to obtain an analytical expression for the same cuspless sub-Riemannian geodesics of problem \( P_{\text{mec}} \), parameterizing the solution space in terms of \( w(0) \) and \( \dot{w}(0) \). Recall that these parameterizations are one-to-one related by means of Corollary 3.3.7.

Of course either method can be used to sample the solution space of \( P_{\text{curve}} \):

- **Integration of the Frenet-Serret Formulae** For given \( w(0) \) and \( \dot{w}(0) \), Equation (2.6) can be solved numerically, after which the cuspless sub-Riemannian geodesic of problem \( P_{\text{mec}} \) \( s \mapsto x(s) \) is found straightforwardly after (numerical) integration of the Frenet-Serret formulas.

- **Integration of the Pfaffian equations** The analytical solutions obtained in Section 3, recall Theorem 3.4.3, can be implemented directly. Apart from the computationally heavy elliptical integrals, the bulk of the running time is due to the matrix exponentiation in Equation (3.28). However, because of the structure of \( A(s) \), this exponentiation can be simplified as follows.

**Corollary 4.1.1.** Given \( A(s) \) as defined in Equality (3.29), we may write

\[
\int e^A(s') ds' = \sqrt{\frac{\|w(0)\|^2 - \frac{c^2}{c^2+\beta^2}}{\|w(0)\|^2 - \frac{c^2}{c^2+\beta^2}}} \begin{pmatrix} \cos \phi(s) & -\sin \phi(s) \\ \sin \phi(s) & \cos \phi(s) \end{pmatrix}, \quad (4.2)
\]

with \( \phi(s) = \int_0^s \frac{c^2}{\|w(s')\|^2 - \frac{c^2}{c^2+\beta^2}} ds' \).
Proof. From Theorem 3.4.3, we have that
\[ A(s) = \frac{w(s) \cdot w(s)}{\|w(s)\|^2 - \frac{W^2}{c^2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{W}{\sqrt{c^2 \beta^2 - \|w(s)\|^2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
\[ \Rightarrow \int_0^s A(s') ds' = \ln \left( \begin{pmatrix} \|w(s)\|^2 - \frac{W^2}{c^2} & W^2 \beta^2 \\ \|w(0)\|^2 - \frac{W^2}{c^2} & W^2 \beta^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \phi(s) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \]
\[ \Rightarrow e^{\int_0^s A(s') ds'} = \begin{pmatrix} \|w(s)\|^2 - \frac{W^2}{c^2} & W^2 \beta^2 \\ \|w(0)\|^2 - \frac{W^2}{c^2} & W^2 \beta^2 \end{pmatrix}^{\frac{\|w(s)\|^2 - \frac{W^2}{c^2}}{\|w(0)\|^2 - \frac{W^2}{c^2}}} \begin{pmatrix} \cos \phi(s) & -\sin \phi(s) \\ \sin \phi(s) & \cos \phi(s) \end{pmatrix}. \]

4.1.1 Implementation

The two implementations have been compared extensively and both consistently produce the same results, though at the end, the analytical approach is superior in terms of speed, cf. Figure 6. The implementation of the solution to the boundary value problem of $P_{\text{curve}}$ is based on this method.

4.1.2 The range of the exponential map

There are a number of necessary restrictions on the possible boundary conditions for which we can get a cuspless sub-Riemannian geodesic of problem $P_{\text{curve}}$. We present some special cases which help us to get an idea about the range of the exponential map of $P_{\text{curve}}$. Note that this set coincides with the set of end conditions for which $P_{\text{curve}}$ is expected to be well defined. The first corollary gives us the possible final positions when the final direction is anti parallel to the initial direction.

**Corollary 4.1.2.** Let $(x_1, n_1)$ be the end condition of $P_{\text{curve}}$ with the initial condition being $(0, e_z)$. Then, given that $n_1 = -e_z$, a cuspless sub-Riemannian geodesic of problem $P_{\text{curve}}$ exists only for $x_1 \cdot e_z = 0$. Moreover, this condition is only possible for curves departing from a cusp and ending in a cusp.

**Proof.** Let $x$ be a cuspless sub-Riemannian geodesic of problem $P_{\text{curve}}$ with $\dot{x}(0) = -\dot{x}(L)$ for some $L \leq s_{\text{max}}$. This means that going to the tilde coordinates, we have $\tilde{x}(0) = -\tilde{x}(L)$, which...
implies \( \dot{z}(0) = -\dot{z}(L) \). But this is possible only if \( \dot{z}(0) = 0 = \dot{z}(L) \), which is possible only if \( \|w(0)\| = 1 \) and \( L = s_{max} \), i.e., if the geodesic both starts and ends in cusp. Then it is easy to see that \( z(s_{max}) = 0 \) (see corollary 4.1.4).

The following lemma helps us to get an important result on the range of the exponential map. It also explains the limited torsion seen in the plots of the cuspless sub-Riemannian geodesics of problem \( P_{mec} \).

**Lemma 4.1.3.** With \( \phi(s) \) given as in Corollary 4.1.1, \( |\phi(s)| < \pi \) for any \( s \in [0, s_{max}] \) for all \( w(0) \) and \( w(0) \) such that \( W \neq 0 \).

**Proof.** Clearly, \( \phi(s) \) is monotonically increasing or decreasing according to the sign of \( W \). Hence it is enough to prove the theorem for the case \( W > 0 \). The other case follows by a similar argument. So we assume \( W > 0 \). Now consider the quantities \( a(s) := \langle w(s), \dot{w}(s) \rangle \) and \( b(s) := \frac{W}{2c} \sqrt{c^2 \beta^2 - \|w(s)\|^2} \). Then we can define an angle \( \psi(s) \) such that \( \cos \psi(s) = \frac{a(s)}{\sqrt{a(s)^2 + b(s)^2}} \) and \( \sin \psi(s) = \frac{b(s)}{\sqrt{a(s)^2 + b(s)^2}} \). Since \(-1 \leq \cos \psi \leq 1 \) and \(-1 \leq \sin \psi \leq 1 \), we have that \( 0 \leq \psi(s) \leq \pi \) for all \( s \in [0, s_{max}] \). Also note that \( \psi(s_{max}) = 0 \). Now for all \( s \in (0, s_{max}) \), we have

\[
\frac{d}{ds}(\pi - \psi(s)) = -\frac{d}{ds} \arccos \left( \frac{a(s)}{\sqrt{a(s)^2 + b(s)^2}} \right) = \frac{b(s)a(s) - a(s)b(s)}{a(s)^2 + b(s)^2} = \frac{W}{2c} \sqrt{c^2 \beta^2 - \|w(s)\|^2} \left( 1 + \frac{\beta^2 \|w(s)\|^2}{\|w(s)\|^2} + \frac{\beta^2 (\langle w(s), \dot{w}(s) \rangle)^2}{\|w(s)\|^2 (c^2 \beta^2 - \|w(s)\|^2)} \right) > \frac{W}{2c} \sqrt{c^2 \beta^2 - \|w(s)\|^2} \|w(s)\|^2 - \frac{W}{2c} \beta^2 \|
\]

\[
\Rightarrow \int_0^{s_{max}} \frac{d}{ds}(\pi - \psi(s))ds > \int_0^{s_{max}} \phi(s)ds
\]

\[
\Rightarrow \phi(s_{max}) < \psi(0) \leq \pi.
\]

The monotonicity of \( \phi \) gives \( \phi(s) < \phi(s_{max}) < \psi(0) \leq \pi \) for all \( s \in [0, s_{max}] \).

The following corollaries about cuspless sub-Riemannian geodesics of problem \( P_{mec} \) departing from cusp point (i.e. \( \|w(0)\| = 1 \)) help us to speculate about the extremes of the range of the exponential map.

**Corollary 4.1.4.** Given that \( \|w(0)\| = 1 \), a cuspless sub-Riemannian geodesic can never have a negative component along the \( z \)-axis. Moreover, it can meet the \( z = 0 \) plane at non zero time only if \( s = s_{max} \) and \( W = 0 \).

**Proof.** With the assumptions \( \|w(0)\| = 1 \) and \( \langle w(0), \dot{w}(0) \rangle < 0 \) (which ensures \( s_{max} \neq 0 \)), we have that \( \|\dot{w}(0)\| = c \beta \), \( \gamma(0) = 0 \) and \( \ddot{z}(0) < 0 \). In this case, with the notation of Corollary 4.1.1 and again assuming \( W \neq 0 \), and for some \( s \in (0, s_{max}] \),

\[
z(s) = \ddot{z}(s) - \ddot{z}(0) = \left( \frac{\|w(s)\|^2 - \frac{W}{2c} \beta^2}{1 - \frac{W}{2c} \beta^2} \cos \phi(s) - 1 \right) \ddot{z}(0) > 0 \text{ (as by Lemma 4.1.3, } \phi < \pi) \text{.}
\]

Now for \( W = 0 \), because of the rotational symmetries depicted in Figure 3 (recall Corollary 3.6.1), it is enough to consider \( w_2(0) = 0 = \ddot{w}_2(0) \) and hence \( |w_1(0)| = 1 \) and \( |\ddot{w}_1(0)| = c \beta \). Then

\[
z(s) = \ddot{z}(s) - \ddot{z}(0) = \frac{\langle w(s) - w(0), \ddot{w}(0) \rangle}{c \beta \|w(0)\|}.
\]
If $\dot{w}_1(0) = c\beta$, we have $w_1(0) = -1$ which implies $(w(s) - w(0)) \cdot \dot{w}(0) = c\beta(1 + w_1(s)) \geq 0$. In case $w_1(0) = -c\beta$, we have $w_1(0) = 1$ which implies $(w(s) - w(0)) \cdot \dot{w}(0) = c\beta(1 - w_1(s)) \geq 0$. Moreover, $z = 0 \Rightarrow |w_1(s)| = 1 \Leftrightarrow s = s_{max}$. 

Hence, we conclude that as with the cusplless sub-Riemannian geodesics in $$(\tilde{M}, \tilde{H}, \tilde{G}_{\beta}) := (SE(2), ker \{- \sin \theta dx + \cos \theta dy\}, \beta^2(\cos \theta dx + \sin \theta dy) \otimes (\cos \theta dx + \sin \theta dy) + d\theta \otimes d\theta)$$ (4.4)
which are contained in the half space $x \geq 0$ [9], the extreme cases of the cusplless sub-Riemannian geodesics in $(M, H, G_{\beta})$ (recall Equation (3.3)) departing from a cusp are contained in the half space $z \geq 0$. The plane $z = 0$ is reached only by planar sub-Riemannian geodesics both departing from a cusp and ending in a cusp as can be observed in Figure 7 and Figure 8.

Based on our numerical experiments, we pose the following conjecture which is analogous to result in the two dimensional case of finding cusplless sub-Riemannian geodesics in $(\tilde{M}, \tilde{H}, \tilde{G}_{\beta})$ [10].

**Conjecture 4.1.5.** Let the range of the exponential map defined in Definition 3.5.1. be denoted by $\mathcal{R}$ and let $\mathcal{D}$ be as defined in Definition 3.5.1.

- $\text{Exp}: \mathcal{D} \to \mathcal{R}$ is a homeomorphism when $\mathcal{D}$ and $\mathcal{R}$ are equipped with the subspace topology.
- $\widetilde{\text{Exp}}: \text{int}(\mathcal{D}) \to \text{int}(\mathcal{R})$ is a diffeomorphism. Here int$(S)$ denotes the interior of the set $S$.

The boundary of the range is given as

$$\partial \mathcal{R} = S_B \cup S_R \cup S_L \text{ with }$$

$$S_B = \{ \widetilde{\text{Exp}}(\lambda(0), s_{max}(\lambda(0)))|\lambda(0) \in \mathcal{C} \} \text{ and }$$

$$S_R = \{ \widetilde{\text{Exp}}(\lambda(0), s)|\lambda(0) \in \mathcal{C} \text{ and } \lambda_4(0)^2 + \lambda_5(0)^2 = 1 \text{ and } s > 0 \}$$

$$S_L = \{ \{0, R| R \in SO(3) \} \}.$$ (4.5)

This conjecture would imply that no conjugate points, recall Remark 1.0.2, arise within $\mathcal{R}$ and the problem $P_{\text{curve}}$ (1.3) is well posed for all end conditions in $\mathcal{R}$.

The proof of this conjecture would be on similar lines as in Appendix F of [9]. If the conjecture is true, we have a reasonably limited set of possible directions per given final positions for which a cusplless sub-Riemannian geodesic of problem $P_{\text{curve}}$ exists. Then the cones (of admissible end conditions for $P_{\text{curve}}$) in Figure 7 which are the image of the boundary of the phase space of $(w(0), \dot{w}(0))$ under $\widetilde{\text{Exp}}$, represent the boundary of the possible reachable angles by cusplless sub-Riemannian geodesics of problem $P_{\text{curve}}$. Figure 8 shows the special case of the end conditions being on a unit circle containing the $z$-axis. The final tangents are always contained within the cones at each position. Numerical computations indeed seem to confirm that this is the case (see Figure 8). The blue points on the boundary of the cones correspond to $S_B$ while the red points correspond to $S_R$ given in Equality (4.5).

### 4.2 Solving the boundary value problem associated to $P_{\text{curve}}$

With the work described in the previous sections, $P_{\text{curve}}$ has been reduced to finding values for $w(0), \dot{w}(0)$ and $L$ such that the corresponding cusplless sub-Riemannian geodesic of problem $P_{\text{mec}}$ $s \mapsto x(w(0), \dot{w}(0), s)$ satisfies

$$x(w(0), \dot{w}(0), L) = x_1,$$

$$\dot{x}(w(0), \dot{w}(0), L) = n_1.$$ (4.6)

We can solve this algorithmically by writing

$$(w(0), \dot{w}(0), l) = \arg \inf_{w_0 \in [-1, 1]^2, w_0 \in \mathbb{R}^2, L \geq 0} \|x(w_0, \dot{w}_0, L) - x_1\|_2^2 + c\|\dot{x}(w_0, \dot{w}_0, L) - n_1\|_2^2,$$ (4.7)
Figure 7: A comparison of the possible end conditions of $P_{\text{curve}}$ for the two dimensional and the three dimensional cases. On right, possible tangent directions are depicted of cuspless sub-Riemannian geodesics with initial position at the origin and the initial direction along $e_z$ and the final positions at unit distance from the origin. As, in the $SE(2)$ (shown on left) case within sub-Riemannian manifold given in Equation (4.4), this set of possible directions at each point is connected and the cones are the boundaries. These cones are obtained by considering the end conditions of sub-Riemannian geodesics that either begin with a cusp point (shown in red) or end at a cusp point (shown in blue). Figure 8 depicts the comparison in the special case when we set the end conditions on a unit circle containing the $z$-axis.
Figure 8: A comparison of the cones of reachable angles by the cuspless sub-Riemannian geodesics in the two dimensional case as in [4, 9] and those in the three dimensional case respectively. It represents the special case in Figure 7 of the end conditions being on a unit circle containing the $z$-axis. The intersection of the cones in Figure 8b with $x = 0$ coincides with the cones depicted in Figure 8a, which is in accordance with Theorem 3.4.5.
Figure 9: An illustration of the spatial part of arbitrary cuspless sub-Riemannian geodesics in 
\((SE(3), H, \mathcal{G}_\beta)\) and the cones of reachable angles as depicted in Figures 7 and 8. Note that the 
cuspless sub-Riemannian geodesics are always contained within the cones. We checked this for 
many more cases, which supports our Conjecture 4.1.5.
Figure 10: An illustration of a successful agent update step in an iteration of the differential evolution algorithm. $x = (w(0), \dot{w}(0), L)$ denotes the agent under consideration. From the three random agents $x_j$, $x_k$ and $x_l$ a new point $x_s$ is calculated, which is combined with the original agent $x$ to form $x_{\text{new}}$.

\[ \dot{x}(w_0, \dot{w}_0, L) = \left. \frac{\partial x(w_0, \dot{w}_0, s)}{\partial s} \right|_{s=L}, \quad \| \cdot \| : \mathbb{R}^3 \rightarrow \mathbb{R}^+, \quad \| \cdot \|_{S^2} : \mathbb{S}^2 \rightarrow \mathbb{R}^+ \]

is the Euclidean norm in $\mathbb{R}^3$, $\| \cdot \|_{S^2}$ is the Riemannian distance on the sphere, and $c$ is a user defined scaling parameter. Note that though the cuspless sub-Riemannian geodesic of problem $P_{\text{mec}}$ connecting $(0, I)$ to $(x_1, n_1)$ is clearly a global minimizer with respect to Equation (4.7), it need not necessarily be unique as we have not shown formally that the exponential map (see Definition 3.5.1) of the underlying geometric control problem is injective (i.e. we have not proven the absence of conjugate points in $\mathcal{R}$).

4.2.1 Direct-search optimization

We have considered two direct-search algorithms to determine the minimizers described in Equation (4.7), the simplex-based Nelder-Mead method \cite{21}, and differential evolution \cite{24, 30}:

- **Nelder-Mead** Let $D$ be the domain of the function $x$ in Equation (4.6), coinciding with the domain of the exponential map, recall Definition 3.5.1. The Nelder-Mead method is initialized by randomly picking six elements out of dom that form a simplex which approaches a minimum by iteratively transforming the element with the lowest value. The exact transformations are given in ref \cite{21}. The algorithm is assumed to have optimized and terminates when both the difference in position and function value of the two best vertices in two consecutive simplices are below the respective thresholds.

- **Differential Evolution** Differential evolution optimizes a function by maintaining a population of candidate solutions taken randomly from dom, called agents, that is refreshed at every iteration. The updated candidate solutions are obtained by randomly combining previously considered solutions \cite{24, 30}, and retained only if the function value of the updated agent is lower than that of the original one. The algorithm is assumed to have optimized and terminates when both the difference in position and function value of the two best agents in two consecutive populations are below the thresholds. An illustration of one (successful) update in a single iteration of the algorithm is shown in Figure 10.

4.2.2 Results

Figures 11a and 11b show the performance of the Nelder-Mead method and the differential evolution method respectively. The maximum number of iterations have been taken as 500 and 100 respectively. Increasing this number does not increase the accuracy of the methods by much as much of the problem is solved in a few number of steps. The average running time of the
Figure 11: The performance of the Nelder-Mead method (in (a)) and the differential evolution method (in (b)) in terms of the fraction of trials against the error $\Psi$, which equals the infimum attained as given in the right hand side of the Equation (4.7).

Nelder-Mead method is 16.00 seconds as opposed to 28.23 seconds for the differential evolution method.

5 Conclusion

We have explicitly derived the cuspless sub-Riemannian geodesics in the space of three dimensional positions and directions by minimizing the functional $\int_0^L \sqrt{\kappa(s)^2 + \beta^2} \, ds$. By the classical variational approach, we obtained a nonlinear ODE for the curvature of the geodesic. Then we used a Lagrangian reduction approach based on the works by Bryant and Griffiths [5] and succeeded in obtaining a couple of linear ODE’s for a function of the curvature which we used to obtain a relatively simple and analytical description of the cuspless sub-Riemannian geodesic. Application of Pontryagin’s Maximum Principle in a Hamiltonian approach yielded the same ODE’s. As a result, our stationary curves are locally optimal. Using the obtained analytical expressions for the cuspless sub-Riemannian geodesics, we derived a number of properties like reflectional and rotational symmetries of the geodesics. Furthermore, we showed that in case of planar boundary conditions, the cuspless sub-Riemannian geodesics in $\mathbb{R}^3 \times S^2$ coincide with the cuspless sub-Riemannian geodesics in $\mathbb{R}^2 \times S^1$. We also obtained bounds on the torsion of the curves. Moreover, we succeeded in formulating explicit simple bounds of the exponential map in some relevant special cases. The obtained formulas for the geodesics contribute to fast implementation of the geodesics. These implementations induced experiments that provided strong evidence supporting our conjecture that the restriction of the exponential map to the interior of the domain is a diffeomorphism as is proved to be the case in the two dimensional setup [3, 4, 19, 25, 26]. However, we could not prove the conjecture as injectivity of the exponential map is hard to prove, providing a challenging problem for future research.

A The Cartan connection and proof of Theorem 3.3.3

In this section, we generate the differential geometrical construction of the Cartan connection in $SE(2)$ [14] and its sub-Riemannian manifold ($SE(2), \ker \{- \sin \theta \, dx + \cos \theta \, dy\}, G_\beta := \beta^2 \{ \cos \theta \, dx + \sin \theta \, dy \} \otimes (\cos \theta \, dx + \sin \theta \, dy) + d\theta \otimes d\theta$) to $SE(3)$ and its sub-Riemannian manifold ($SE(3), H := \ker \{ \omega_1 \} \cap \ker \{ \omega_2 \} \cap \ker \{ \omega_5 \}, G_\beta := \beta^2 \omega_3 \otimes \omega_3 + \omega_4 \otimes \omega_4 + \omega_5 \otimes \omega_5$).

In both the cases, the connection can be used to structure the Pfaffian system for sub-Riemannian geodesics. We will show that along sub-Riemannian geodesics, momentum is parallel.
transported.

Furthermore, we will show that there is a discrepancy between auto-parallel curves and sub-Riemannian geodesics due to the torsion of the Cartan connection. This is in contrast to the flat $\mathbb{R}^d$ case where straight lines are the shortest paths. In our case, auto-parallel (horizontal) curves are (horizontal) exponential curves which do not coincide with the sub-Riemannian geodesics. Finally, we note that the phrase Cartan connection is used for connections on principal fibre bundles, on the associated vector bundle, on the frame bundle and on the tangent bundle. Although, they are related, for details see [29] [14].

In the Subsection A.1, for the sake of simplicity, we avoid the sub-Riemannian structure and consider the Cartan connection on $SE(3)$.

Subsequently in [A.2] we will structure the Pfaffian system dramatically and show that momentum is parallel transported w.r.t Cartan connection along sub-Riemannian geodesics.

Finally in [A.3] we do the same thing but now we include the sub-Riemannian structure given in (3.3) formally into the Cartan connection. In fact, we provide a principal fibre bundle viewpoint on the sub-Riemannian manifold $(SE(3), H, G_\beta)$. This provides a geometric understanding on the manifold itself (rather than its tangent bundle) and we exploit this understanding in the derivation of the sub-Riemannian geodesics.

The main results of this appendix are summarized in Theorem 3.3.3.

A.1

Consider the principal fibre bundle

$$P = (SE(3), SE(3)/SE(3) \equiv e, \pi, k)$$

with $R_{g_1}g_2 = g_2g_1$, $g_1, g_2 \in SE(3)$, with total space $SE(3)$, base manifold and structure group $SE(3)$ and projection $\pi$: $\pi(g) = e$,

which consists of only one fibre $\pi^{-1}(e) = SE(3)$. Then,

$$w_g(X_g) = \sum_{i=1}^{6} (\omega^i|_g, X_g) A_i$$

for all $X \in T(SE(3))$

defines a Cartan-Ehresman connection on $P$. (For definition, see [29]) Note that $w_g = (L_g^{-1})_*$ for all $g \in SE(3)$ since

$$w_g \left( \sum_{j=1}^{6} \alpha^j A_j|_g \right) = \sum_{i=1}^{6} (\omega^i|_g, \sum_{j=1}^{6} \alpha^j A_j|_g) A_i$$

$$= \sum_{i=1}^{6} \alpha^i A_i$$

and

$$(L_g)_* \left( \sum_{i=1}^{6} \alpha^i A_i \right) = \sum_{j=1}^{6} \alpha^j A_j|_g,$$

where subscript asterisk denotes the push forward and with $L_g h = gh$.

Via the group representation

$$g \mapsto Ad(g) := (L_g^{-1}R_g)_*,$$
we obtain the associated vector bundle \( SE(3) \times \text{Ad} \mathcal{L}(SE(3)) \) with corresponding connection form
\[
\tilde{\omega} = \sum_{i=1}^{6} (\text{Ad})_{*}(A_{j}) \otimes \omega^{j} = \sum_{i=1}^{6} \text{ad}(A_{j}) \otimes \omega^{j} = \sum_{j=1}^{6} \left( \sum_{i=1}^{6} \sum_{k=1}^{6} c_{ij}^{k} A_{k} \otimes \omega^{i} \right) \otimes \omega^{j} \quad (A.1)
\]
where \( (\text{Ad})_{*} = \text{ad} \) is given by \( \text{ad}(X) = [\cdot, X] \) [17] and where we identified this linear map \( \text{ad} \) in the usual way with a mixed \((1,1)\)-tensor in the second equality of Equation (A.1).

This in turn produces a \( 3 \times 3 \) matrix valued one form on the frame bundle
\[
\tilde{\omega}_{k}^{j} = -\tilde{\omega}(\omega^{j}, \cdot, A_{k})
\]
which yield the following Koszul connection on the tangent bundle \( T(SE(3)) \)
\[
\nabla_{X}A = \sum_{k=1}^{6} \left( \frac{\partial^{k}}{\partial \gamma^{k}} A_{k} + a^{k} \sum_{j=1}^{6} \tilde{\omega}_{k}^{j}(X)A_{j} \right)
= \sum_{k=1}^{6} \left( a^{k} A_{k} + a^{k} \sum_{i,j=1}^{6} c^{j}_{ki} \frac{\partial^{i}}{\partial \gamma^{i}} A_{j} \right) \quad (A.2)
\]
with \( X = \sum_{i=1}^{6} \frac{\partial^{i}}{\partial \gamma^{i}} A_{i} |_{\gamma}, A = \sum_{k=1}^{6} a^{k} A_{k} \) vector fields along curve \( \gamma : [0,1] \to SE(3) \) and
\[
\tilde{\omega}_{k}^{j}(\frac{\partial^{i}}{\partial \gamma^{i}} A_{i}) = \frac{\partial^{i}}{\partial \gamma^{i}} \tilde{\omega}_{k}^{j}(A_{i}) = -\frac{\partial^{i}}{\partial \gamma^{i}} \tilde{\omega}(\omega^{j}, A_{i}, A_{k}) = -\gamma^{i} c^{j}_{ki}
\]
so that the Christoffel symbols equal minus the structure constants and
\[
a^{k} = X(a^{k}) = \sum_{i=1}^{6} \frac{\partial^{i}}{\partial \gamma^{i}} A_{i}(a_{k}).
\]

**Lemma A.1.1.** The covariant derivative of covector field \( \omega^{k}|_{\gamma} \) w.r.t. \( \gamma \) equals
\[
\nabla_{\gamma} \omega^{k}|_{\gamma} = \sum_{i,j=1}^{6} c_{ij}^{k} \frac{\partial^{i}}{\partial \gamma^{i}} \omega^{j}|_{\gamma}.
\]

**Proof.** By Equation (A.2) we have
\[
\nabla_{\gamma} A_{j} = -\sum_{i,k=1}^{6} c_{ij}^{k} A_{k} \frac{\partial^{i}}{\partial \gamma^{i}}.
\quad (A.3)
\]

For all \( k, j = 1, \ldots, 6 \), we have
\[
d \langle \omega^{k}|_{\gamma}, A_{j}|_{\gamma} \rangle = 0
\quad \Rightarrow \langle \nabla_{\gamma} \omega^{k}|_{\gamma}, A_{j}|_{\gamma} \rangle + \langle \omega^{k}|_{\gamma}, \nabla_{\gamma} A_{j}|_{\gamma} \rangle = 0.
\]

So by (A.3), we have
\[
\langle \nabla_{\gamma} \omega^{k}|_{\gamma}, A_{j}|_{\gamma} \rangle = \langle \omega^{k}|_{\gamma}, \sum_{i} c_{ij}^{k} A_{k} \frac{\partial^{i}}{\partial \gamma^{i}} \rangle
= \sum_{i} c_{ij}^{k} \frac{\partial^{i}}{\partial \gamma^{i}} \quad \text{for all } j = 1, \ldots, 6
\quad \Rightarrow \nabla_{\gamma} \omega^{k}|_{\gamma} = \sum_{i,j} c_{ij}^{k} \frac{\partial^{i}}{\partial \gamma^{i}} \omega^{j}|_{\gamma},
\]
which proves the result. □
A.2 The role of the Cartan connection in the sub-Riemannian geodesics: Parallel transport of momentum

The Cartan connection helps in providing more intuition about the sub-Riemannian geodesics. As is seen by the following calculations, these curves actually describe parallel transport of the momentum vectors w.r.t. Cartan connections.

We start with the assumption of the momentum being parallely transported along \( \gamma \), that is,

\[
\nabla_{\dot{\gamma}} \lambda = 0 \quad (A.4)
\]

\[
\Leftrightarrow \nabla_{\dot{\gamma}} \left( \sum_{j=1}^{6} \lambda_j \omega^j |_{\gamma} \right) = 0
\]

\[
\Leftrightarrow \sum_{j=1}^{6} \lambda_j \omega^j |_{\gamma} + \sum_{k=1}^{6} \lambda_k \nabla_{\dot{\gamma}} \omega^k |_{\gamma} = 0
\]

by Lemma A.1.1

\[
\sum_{j=1}^{6} \lambda_j \omega^j |_{\gamma} + \sum_{k=1}^{6} \lambda_k \omega^k |_{\gamma} = 0
\]

\[
\Leftrightarrow \dot{\lambda}_j + \sum_{k=1}^{6} \lambda_k \epsilon^{k \lambda} \gamma^i = 0 \quad \text{for all } j = 1, \ldots, 6 \quad \text{(as } \omega^i \text{ are independent for } j = 1, \ldots, 6) \]

\[
\Leftrightarrow \langle d\lambda_j, \gamma \rangle + \sum_{k=1}^{6} \lambda_k \epsilon^{k \lambda} \gamma^i = 0 \quad \text{for all } j = 1, \ldots, 6 \quad \text{(as } d\lambda_j = \dot{\lambda}_j ds \text{ and } \dot{\gamma} = \sum_{i=1}^{6} \gamma^i A_i |_{\gamma})
\]

\[
\therefore \langle \omega^i, \dot{\gamma} \rangle = 0 \quad \forall i \in \{1,2,6\} \quad \Leftrightarrow \langle A_j | d\psi, \dot{\gamma} \rangle = 0
\]

So, we arrive at the Pfaffian system

\[
A_j | d\psi = 0 \quad \text{for all } j = 1, \ldots, 6,
\]

which is exactly as the last six equations in Equation (3.13) in Theorem 3.3.2 for the sub-Riemannian geodesics of \( P_{\text{curve}} \). This means that the cuspless sub-Riemannian geodesics given by Theorem 3.3.2 have the momentum parallel transported w.r.t the Cartan connection \( \nabla \).

A.3 The Cartan connection \( \vec{\nabla} \) on the sub-Riemannian manifold \( (SE(3), H, G_{\beta}) \)

In the sub-Riemannian manifold \( (SE(3), \ker \{\omega_1\} \cap \ker \{\omega_2\} \cap \ker \{\omega_6\}, G_{\beta}) \), the directions \( A_1, A_2 \) and \( A_6 \) are prohibited in the tangent space. To get a better grasp on what this means on the manifold level, we will consider principal fibre bundles.

To this end, we consider the subgroup isomorphic to \( SE(2) \) given by \( \widehat{SE(2)} = \{ \exp (c^1 A_1 + c^2 A_2 + c^6 A_6) | c^1, c^2, c^6 \in \mathbb{R} \} \) with \( A_k = A_k |_c \). It represents rotations and translations in the \( xy \)-plane. For simplicity, we denote this group by \( SE(2) \). So we consider the principal fibre bundle \( P = (SE(3), SE(3)/SE(2), \pi, \mathbb{R}) \) with \( R_h g = gh, h \in SE(2), \pi(g) = [g] = gSE(2) \). On \( P \), we consider the Maurer-Cartan form \( \bar{w} = (L_h^{-1})_* \), more precisely

\[
\bar{w}(A_g) = \sum_{i \in \{3,4,5\}} \langle \omega^i | g, X_g \rangle A_i
\]

It can be verified that \( \bar{w} \) is a Cartan-Ehresmann connection on \( P \).

Via the group representation \( SE(2) \ni h \mapsto Ad(h) := (L_{h^{-1}} R_h)_* \), we obtain the associated vector bundle (def. 3.7 in [14]) \( (SE(3) \times_{Ad|_{SE(2)}} L(SE(3))) \) with corresponding connection form

\[
\bar{w} = \sum_{i \in \{3,4,5\}} (Ad|_{SE(3)})(A_j) \otimes \omega^i = \sum_{i \in \{3,4,5\}} ad(A_j) \otimes \omega^i = \sum_{i,j,k} \epsilon^{ijk} A_k \otimes \omega^i \otimes \omega^j
\]

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where \( \text{ad}(X) = [\cdot, X] \).

This in turn produces a \( 3 \times 3 \) matrix valued one form on the frame bundle
\[
\tilde{w}_k^i = -\tilde{w}(\omega^j, \cdot, A_k)
\]
which yields a connection on the tangent bundle \((SE(3), \ker\{\omega_1\} \cap \ker\{\omega_2\} \cap \ker\{\omega_6\})\)
\[
\nabla_X A = \sum_{k \in \{(3,4,5)\}} \left( \tilde{a}_k^i A_k + a^k \sum_{j \in \{(3,4,5)\}} \tilde{w}_k^j (X) A_j \right)
\]
\[
= \sum_{k \in \{(3,4,5)\}} \left( a^k A_k + a^k \sum_{j \in \{(3,4,5)\}} \gamma^j c_{ki}(A_j) \right)
\]
(A.5)

with \( X = \sum_{i \in \{(3,4,5)\}} \dot{\gamma}^i A_i, \ A = \sum_{k \in \{(3,4,5)\}} a^k A_k \) and \( \tilde{w}_k^i (A_i) = -\tilde{w}(\omega^j, A_i, A_k) = -c_{ik}^j \) where again the Christoffels are equal to minus the structure constants of the Lie algebra and where \( a^k = \sum_{i \in \{(3,4,5)\}} \gamma^i (A_i) \gamma^k \).

### A.4 Structuring the Pfaffian system

**Lemma A.4.1.** Let \( w \) denote the Cartan-Maurer form on \((SE(3), e, \pi, R)\). Let \( \tilde{w} \) denote the Cartan-Maurer form on \((SE(3), SE(3)/SE(2), \pi, R)\). Let \( H = \ker\{\omega_1\} \cap \ker\{\omega_2\} \cap \ker\{\omega_6\} = \text{span}\{A_3, A_4, A_5\} \). Then the stationary curves are determined by
\[
\dot{\gamma} \in H \Leftrightarrow \langle w - \tilde{w}, \dot{\gamma} \rangle = 0 \tag{A.6}
\]
\[
\langle dL(P), \lambda, \dot{\gamma} \rangle = 0 \tag{A.7}
\]
\[
\nabla_{\dot{\gamma}} \lambda = 0 \tag{A.8}
\]
where \( P = (\sigma, -\kappa_2 \sigma, \kappa_1 \sigma) = -\kappa_2 \sigma A_4 + \kappa_1 \sigma A_5 + \sigma A_3. \)

**Proof.** Equation (A.8) is already derived in Equation (A.4). Regarding Equation (A.6), we note that
\[
w - \tilde{w} = \sum_{i \in \{(3,4,5)\}} \text{ad}(A_i) \otimes \omega^i = 0 \Leftrightarrow \omega^1 = \omega^2 = \omega^6 = 0
\]
Regarding Equation (A.7) we note
\[
dL = \sigma^{-1} \sum_{i=1}^2 \frac{\partial L}{\partial \kappa_i} d(\sigma \kappa_i) + \left( \frac{\partial L}{\partial \sigma} - \sigma^{-1} \sum_{i=1}^2 \kappa_i \frac{\partial L}{\partial \kappa_i} \right) d\sigma
\]
\[
= \sum_{i=1}^2 \frac{\kappa_i}{\sqrt{\kappa_1^2 + \kappa_2^2 + \beta^2}} d(\sigma \kappa_i) + \frac{\beta^2}{\sqrt{\kappa_1^2 + \kappa_2^2 + \beta^2}} d\sigma
\]
\[
= \lambda_3 d(\sigma \kappa_1) - \lambda_4 d(\sigma \kappa_2) + \lambda_3 d\sigma.
\]

From this the result follows. \( \Box \)

Thus, this lemma provides a compact formulation of the system (3.13).

Both Cartan connections \( \nabla \) and \( \nabla \) have constant non-vanishing torsion which can be expressed in the Lie algebra structure elements.

In fact by straightforward computations and application of the Cartan connections given in Equations (3) and (4), we see that for constants \( c^k, k \in \{1, \ldots, 6\} \) and \( d^k, k \in \{3, 4, 5\}, \)
\[
\nabla_{\gamma} \gamma = 0 \Leftrightarrow \langle \omega^k | \gamma, \gamma \rangle = c^k \Leftrightarrow \gamma(s) = \gamma(0) e^{\sum_{i=1}^5 c^i A_k} ,
\]
\[
\nabla_{\gamma} \gamma = 0 \Leftrightarrow \langle \omega^k | \gamma, \gamma \rangle = d^k \Leftrightarrow \gamma(s) = \gamma(0) e^{\sum_{i=1}^5 d^i A_k} ,
\]
(A.9)

with \( A_k = A_k |_c. \)

As a result, auto-parallel curves do not coincide with stationary curves w.r.t. energy \( \int \sqrt{\|\kappa \sigma^2 + \beta^2 \sigma^2 \|} dt. \)

Equation (A.9) and Lemma (A.4.1) provide the proof of Theorem (3.3.3).
Consider the sub-Riemannian manifold \((M, H, G_\beta)\) with \(M = \mathbb{R}^3 \times S^2\), \(H = \text{span}\{A_3, A_4, A_5\}\) and \(G_\beta = \beta^2 \omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4 + \omega^5 \otimes \omega^5\). Note that a curve \(\gamma(\cdot) = (x(\cdot), R(\cdot))\) in \(SE(3)\) induces the curve \((x(\cdot), R(\cdot))e_3)\) in \(\mathbb{R}^3 \times S^2\) which we again denote by \(\gamma\).

Now we define the geometric control problem on \(\mathbb{R}^3 \times S^2\):

\[
\int_0^T \sqrt{(u^3)^2 + (u^4)^2 + (u^5)^2} \, dt \to \text{minimize (with free } T) \tag{31}
\]

with

\[
\ddot{\gamma}(t) = \sum_{i=1}^5 \langle \omega^i |_{\gamma(t)} \rangle A_i |_{\gamma(t)} = \sum_{i=3}^5 \langle \omega^i |_{\gamma(t)} \rangle A_i |_{\gamma(t)}
\]

where for \(i = 3, 4, 5\), \(u^i \in L_1([0, T])\).


Hence, there are no abnormal extremals in this geometric control problem (see Remark 1.0.1).

Also, as \(G_\beta\) is isotropic in the \([A_4, A_5]\) plane, it is well defined on the quotient \(SE(3)/\{0\} \times SO(2)\).

Note that there exists standard PMP for \(L_\infty([0, T])\) controls and there exists a recently generalized PMP for \(L_1([0, T])\) controls \([31]\). Despite the fact that \(L_\infty([0, T]) \subset L_1([0, T])\), the generalized PMP in this case produces (by a reparameterization argument) solutions which are equivalent to the solutions of the control problem

\[
\int_0^T (\beta^2 (u^3)^2 + (u^4)^2 + (u^5)^2) \, dt \to \text{minimize (with fixed } T) \tag{B.1}
\]

with

\[
\ddot{\gamma}(t) = \sum_{i=3}^5 \langle \omega^i |_{\gamma(t)} \rangle A_i |_{\gamma(t)} = \sum_{i=3}^5 \langle \omega^i |_{\gamma(t)} \rangle A_i |_{\gamma(t)}
\]

where for \(i = 1, 2, 3\), \(u^i \in L_1([0, T])\).

Applying the standard PMP, we have that there exists a Lipschitzian curve in the cotangent bundle given by \([0, T] \ni t \mapsto \mu(t) = (\gamma(t), \lambda(t))\) with \(\lambda(t) \in T_{\gamma(t)}^*(M)\) such that

\[
\mu(t) \neq 0, \quad \dot{\mu} = \tilde{H}(\mu(t))
\]

\[
H(\mu) = \max_{u \in \mathbb{R}^3} \left( L(u^3, u^4, u^5) - \sum_{i=3}^5 \mu_i u^i \right)
\]

where \(\mu = (\gamma, \lambda)\) and where the Hamiltonian is given by

\[
H(\mu) \equiv H(\lambda) = \frac{1}{2} (\lambda_3^2 + \lambda_4^2 + \lambda_5^2)
\]

\[
= \frac{1}{2} \left( |\langle \lambda, A_3 \rangle|^2 + |\langle \lambda, A_4 \rangle|^2 + |\langle \lambda, A_5 \rangle|^2 \right).
\]

The Hamiltonian vector field

\[
\tilde{H} = \sum_{i=1}^6 \alpha_i \frac{\partial}{\partial \lambda_i} + \beta^i A_i \tag{B.1}
\]

is such that it preserves the canonical symplectic structure

\[
\sigma = \sum_{i=1}^6 d\lambda_i \wedge \omega^i = \sum_{i=3}^5 d\lambda_i \wedge \omega^i
\]
and hence, we have
\[ \sigma(\vec{H}, \cdot) = -\mathrm{d}H \] (B.2)
\[ \mathrm{d}H = \sum_{i=3}^{5} A_i H \omega^i + \frac{\partial H}{\partial \lambda_i} \mathrm{d}\lambda_i. \] (B.3)

From Equations (B.1), (B.2) and (B.3), we obtain for \( i \)
\[ \alpha^i = -A_i H \] and hence, we have \[ \lambda_{\gamma} \]
\[ \therefore \lambda_{\gamma} \Rightarrow \lambda_{\delta} \]
\[ \lambda_{\gamma} + 1 \lambda_{\delta} \]
\[ \alpha^i = -A_i H \] and \( \beta^i = \frac{\partial H}{\partial \lambda_i} = \lambda_i. \)

Consequently, (noting that \( A_0 H = A_1 H = A_2 H = A_3 H = 0 \)), we have the Hamiltonian vector field
\[ \vec{H}(\mu) = \lambda_3 A_3 + \lambda_4 A_4 + \lambda_5 A_5 - A_4 H \frac{\partial}{\partial \lambda_4} - A_5 H \frac{\partial}{\partial \lambda_5} \]
\[ = \lambda_3 A_3 + \lambda_4 A_4 + \lambda_5 A_5 + \lambda_2 \lambda_3 \omega^3 - \lambda_1 \lambda_3 \omega^5. \]

We have the canonical ODE in the PMP
\[ \dot{\mu} = H(\mu). \]

Then the horizontal part of PMP (where time derivatives are w.r.t. sub-Riemannian arclength \( t \)) is given as
\[ \dot{\gamma} = \lambda_3 |_{\gamma} + \lambda_4 |_{\gamma} + \lambda_5 |_{\gamma} \]
\[ \Rightarrow \lambda_3 (t) = \langle \omega^3 |_{\gamma(t)}, \gamma(t) \rangle; \lambda_4 (t) = \langle \omega^4 |_{\gamma(t)}, \gamma(t) \rangle; \lambda_5 (t) = \langle \omega^5 |_{\gamma(t)}, \gamma(t) \rangle. \]

Note that the results are consistent with the earlier \( SE(2) \) results [9].

Now the vertical part of PMP
\[ \dot{\lambda} = \frac{d}{dt} \left( \sum_{k=1}^{5} \lambda_k \omega^k \right) = \lambda_2 \lambda_3 \omega^4 - \lambda_1 \lambda_3 \omega^5 \]
\[ \Rightarrow \dot{\lambda}_1 \omega^1 + \dot{\lambda}_2 \omega^2 + \dot{\lambda}_3 \omega^3 + \dot{\lambda}_4 \omega^4 + \dot{\lambda}_5 \omega^5 = \lambda_2 \lambda_3 \omega^4 - \lambda_1 \lambda_3 \omega^5. \]

Now note that for \( k = 1, 2, 3; \)
\[ \dot{\omega}^k = -\sum_{j=1}^{5} \sum_{j=1}^{5} \epsilon_{ji}^j \dot{\gamma}^j \omega^j \]
\[ \Rightarrow \dot{\omega} = -\lambda_3 \omega^3, \dot{\omega}^2 = \lambda_4 \omega^3, \dot{\omega}^3 = \lambda_5 \omega^1 - \lambda_4 \omega^2. \]

Hence we get for the vertical part of PMP
\[ (\dot{\lambda}_1 + \lambda_3 \lambda_5) \omega^1 + (\dot{\lambda}_2 - \lambda_3 \lambda_4) \omega^2 + (\dot{\lambda}_3 - \lambda_5 \lambda_1 + \lambda_4 \lambda_2) \omega^3 + \dot{\lambda}_4 \omega^4 + \dot{\lambda}_5 \omega^5 = \lambda_2 \lambda_3 \omega^4 - \lambda_1 \lambda_3 \omega^5 \]
\[ \Leftrightarrow \frac{d}{dt} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (-\lambda_3 \lambda_5, \lambda_3 \lambda_4, \lambda_1 \lambda_5 - \lambda_2 \lambda_4, \lambda_3 \lambda_2, -\lambda_3 \lambda_1). \]

Note that for \( \lambda_3 \neq 0, \frac{d}{dt} = \frac{d}{ds} \frac{ds}{dt} = \lambda_3 \frac{d}{ds}. \) Hence we have
\[ \frac{d}{ds} (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (-\lambda_5, \lambda_4, \frac{\lambda_1 \lambda_5 - \lambda_2 \lambda_4}{\lambda_3}, \lambda_2, -\lambda_1). \]

Thus it coincides with the Equation (3.17). Hence we conclude that PMP gives the same Pfaffian system as the Bryant and Griffiths method.
C Proof of Theorem 2.0.1

Let us consider a small perturbation of this curve towards its normal. The perturbed curve parameterized by the arc-length of the cuspless sub-Riemannian geodesic of problem \( P_{\text{mec}} \) can be given as

\[
x_N(s) = x(s) + h\delta_N(s)n(s)
\]  

where \( \delta_N \in C^\infty_0([0,L],\mathbb{R}) \) and \( 0 < h \in \mathbb{R} \) is very small. Then for \( s \in [0,L] \) using (2.1)

\[
\dot{x}_N(s) = \dot{t}(s) + h\delta_N(s)n(s) + h\dot{\delta}_N(s)n(s)
\]

\[
= (1 - h\delta_N(s)\kappa(s)) \left( \dot{t}(s) + h\delta_N(s)n(s) + h\dot{\delta}_N(s)\tau(s)b(s) \right) + O(h^2).
\]

Neglecting all second and higher order terms of \( h \), the arc-length of the new curve can be expressed as

\[
ds_N = (1 - h\delta_N(s)\kappa(s))ds.
\]

Let \( \phi \) denote the transformation of the parameters such that \( \phi(s_N) = s \). Then the unit tangent to the new curve parameterized by the arc-length of the old curve is given by

\[
t_N(s) := \frac{d}{ds_N}x_N(s) = \frac{ds}{ds_N} \frac{dx_N(s)}{ds} = t(s) + h\delta_N(s)n(s) + h\dot{\delta}_N(s)\tau(s)b(s)
\]

\[
= \frac{ds}{ds_N} \left( \frac{d}{ds}x_N(s) \right) = \frac{ds}{ds_N} \left( \frac{d}{ds}x(s) + h\delta_N(s)n(s) + h\dot{\delta}_N(s)\tau(s)b(s) \right)
\]

\[
= \frac{ds}{ds_N} \left( \frac{d}{ds}x(s) \right) = \frac{ds}{ds_N} \left( \frac{d}{ds}x(s) \right)
\]

\[
= \frac{ds}{ds_N} \left( \frac{d}{ds}x(s) + h\delta_N(s)n(s) + h\dot{\delta}_N(s)\tau(s)b(s) \right)
\]

\[
= \frac{ds}{ds_N} \left( \frac{d}{ds}x(s) \right)
\]

\[
= \frac{ds}{ds_N} \left( \frac{d}{ds}x(s) \right)
\]

Comparing with the Frenet-Serret equations, we have for the curvature \( \kappa_N \) of the new curve parameterized by the arc-length of the old curve

\[
\kappa_N(s) = \kappa(s) + h \left( \delta_N(s)(\kappa(s))^2 + \ddot{\delta}_N(s) - \delta_N(s)(\tau(s))^2 \right).
\]

Then the total energy of the new curve with length \( L_N \) is given by

\[
\mathcal{E}(x_N) = \int_0^{L_N} \sqrt{(\kappa_N(s))^2 + \beta^2} ds = \int_0^{L_N} \sqrt{(\kappa_N(s))^2 + \beta^2(1 - h\delta_N(s)\kappa(s))} ds. \quad (C.2)
\]

As the curve is already optimal, the derivative of the energy functional at this curve along any direction is 0. Hence, we have

\[
0 = \lim_{h \downarrow 0} \frac{1}{h} (\mathcal{E}(x_N) - \mathcal{E}(x))
\]

\[
= \lim_{h \downarrow 0} \frac{1}{h} \int_0^L \left( \sqrt{(\kappa_N(s))^2 + \beta^2} - \sqrt{(\kappa(s))^2 + \beta^2} \right) ds \quad \text{(using (C.2))}
\]

\[
= \int_0^L \sqrt{(\kappa(s))^2 + \beta^2} \left( \frac{\kappa(s)}{(\kappa(s))^2 + \beta^2} \left( \delta_N(s)(\kappa(s))^2 + \ddot{\delta}_N(s) - \delta_N(s)(\tau(s))^2 \right) - \delta_N(s)\kappa(s) \right) ds
\]

\[
= \int_0^L \delta_N(s) \left( \frac{d^2}{ds^2} \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} + \frac{(\kappa(s))^3}{\sqrt{(\kappa(s))^2 + \beta^2}} - \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \right) (\tau(s))^2 ds
\]

\[
- \int_0^L \delta_N(s)\kappa(s) \sqrt{(\kappa(s))^2 + \beta^2} ds \quad \text{(Using integration by parts)}.
\]
Now since $\delta_N$ is an arbitrary test function, we obtain Equation \((2.2)\). In the rest of this appendix, we shall derive Equation \((2.3)\). Now let us consider a small perturbation of this curve towards its binormal. The perturbed curve parameterized by the arc-length of the cuspless sub-Riemannian geodesic of problem $P_{\text{mec}}$ can be given as

$$x_B(s) = x(s) + h\delta_B(s)b(s)$$ \hspace{1cm} (C.3)

where $\delta_B \in C^\infty_0([0,L],[R])$. Then for $s \in [0,L]$ using \((2.1)\)

$$\dot{x}_B(s) = t(s) + h\dot{\delta}_B(s)b(s) + h\delta_B(s)\dot{b}(s) = t(s) + h\dot{\delta}_B(s)b(s) - h\delta_B(s)\tau(s)n(s)$$

Neglecting all second and higher order terms of $h$, the arc-length of the new curve becomes the same as the old curve. Hence, the new tangent becomes

$$t_B(s) = t(s) + h\dot{\delta}_B(s)b(s) - h\delta_B(s)\tau(s)n(s)$$

and

$$\Rightarrow \dot{t}_B(s) = \dot{t}(s) + h\dot{\delta}_B(s)b(s) + h\ddot{\delta}_B(s)b(s) - h\delta_B(s)\tau(s)n(s)$$

Comparing again with the Frenet-Serret equations, we have for the curvature $\kappa_B$ of the new curve parameterized by the arc-length of the old curve

$$\kappa_B(s) = \kappa(s) - 2h\dot{\delta}_B(s)\tau(s) - h\delta_B(s)\dot{\tau}(s).$$ \hspace{1cm} (C.4)

Then the total energy of the new curve with length $L_B$ (say), which is equal to $L$, is given by

$$E(x_B) = \int_0^L \sqrt{(\kappa_B(\phi(s)))^2 + \beta^2} ds$$ \hspace{1cm} (C.5)

Following the same arguments as in the previous subsection, we have Hence, we have

$$0 = \lim_{h \to 0} \frac{1}{h} (E(x_B) - E(x))$$

$$= \lim_{h \to 0} \frac{1}{h} \int_0^L \left( \sqrt{(\kappa_B(s))^2 + \beta^2} - \sqrt{(\kappa(s))^2 + \beta^2} \right) ds \text{ (using \((2.5)\))}$$

$$= \lim_{h \to 0} \frac{1}{h} \int_0^L \frac{(\kappa(s)^2 + \beta^2)}{\sqrt{(\kappa(s))^2 + \beta^2}} \left( 1 - \frac{\kappa(s)}{\kappa(s)^2 + \beta^2} \left( 2h\dot{\delta}_B(s)\tau(s) + h\delta_B(s)\dot{\tau}(s) \right) - 1 \right) ds$$

$$= \int_0^L \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \left( -2\dot{\delta}_B(s)\tau(s) - \delta_B(s)\dot{\tau}(s) \right) ds$$

$$= \int_0^L \left( \frac{2\kappa(s)\tau(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} - \dot{\tau}(s) \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \right) \delta_B(s) ds$$

$$= \int_0^L \left( 2\tau(s) \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} + \dot{\tau}(s) \frac{\kappa(s)}{\sqrt{(\kappa(s))^2 + \beta^2}} \right) \delta_B(s) ds$$

Once again since $\delta_N$ is an arbitrary test function, we obtain Equation \((2.3)\).
References


