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The two-scale approach to hydrodynamic limits for non-reversible dynamics

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Abstract

In [2], a new method to study hydrodynamic limits was developed for reversible dynamics. In this work, we generalize this method to a family of non-reversible dynamics. As an application, we obtain quantitative rates of convergence to the hydrodynamic limit for a weakly asymmetric version of the Ginzburg-Landau model endowed with Kawasaki dynamics. These results also imply local Gibbs behavior, following a method of [1].

1 Introduction

In this work, we are interested in generalizing the results of [2] on hydrodynamic limits to the case of weakly asymmetric interacting spin systems. We obtain quantitative rates of convergence to the hydrodynamic limit for such dynamics. Our main contribution is a method of controlling the effects of the antisymmetric component of the dynamic.

A typical result of convergence to the hydrodynamic limit consists in proving that, under a suitable time-space scaling and for nice initial conditions, a random systems with a large number of particles behaves like a deterministic object, given as the solution of a partial differential equation.

In [2], a new method to study such problems was developed. It consists in establishing estimates in Wasserstein distance between the distribution of the system and a well-chosen macroscopic state, given as the solution of a differential equation. The main elements are a coarse-graining argument and a logarithmic Sobolev inequality. It was applied to dynamics of the form

$$dX_t = -A\nabla H(X_t)dt + \sqrt{2A}dW_t$$

on some Euclidean space, where A is a positive definite matrix, H is the Hamiltonian and W is a Wiener process. In the case where A and H correspond to the Ginzburg-Landau model endowed with Kawasaki dynamics, they obtained scaling limits of the form

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2}{\partial \theta^2} \varphi'(\rho).$$

In this work, we add an extra term to the previous dynamic, and study

$$dX_t = -A\nabla H(X_t)dt - J\nabla H(X_t)dt + \sqrt{2A}dW_t$$

where J is an antisymmetric matrix. This extra term makes the dynamic non-reversible, but does not modify the invariant measure. For a particular choice of J , we obtain a scaling limit of the form

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2}{\partial \theta^2} \varphi'(\rho) + \frac{\partial}{\partial \theta} \varphi'(\rho).$$

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Our method is restricted to the case where the square of the antisymmetric part $-J^2$ is controlled by A (in the sense of symmetric matrices). This is because if the antisymmetric component becomes dominant in the scaling limit, we would expect the limiting PDE to be hyperbolic (rather than parabolic), and estimates in Wasserstein distances would not be adapted.

These estimates in Wasserstein distance also allow us to study local Gibbs behavior (which is stronger form of convergence) by using an interpolation inequality, following a method developed in [1]. We also obtain quantitative rates of convergence for the microscopic free energy to its scaling limit.

The plan of the paper is as follows: in Section 2, we present the framework and our main results. Section 3 contains the proofs of our results in the abstract setting. In section 4, we give the proofs of convergence to the hydrodynamic limit for the Ginzburg-Landau model endowed with a weakly asymmetric version of Kawasaki dynamics.

Notations

- C denotes a positive constant, which may vary from line to line, or even within a line;
- ∇ is the gradient, Hess stands for Hessian, $|\cdot|$ is the norm and $\langle \cdot, \cdot \rangle$ is an inner product. If necessary, a subscript will indicate the space on which these are taken.
- A^t is the adjoint of the operator A .
- Φ is the function defined by $\Phi(x) := x \log x$ on \mathbf{R}_+ .
- $\text{Ent}_\mu(f) = \int f(\log f)\mu - (\int f\mu) \log(\int f\mu)$ is the entropy of the positive function f with respect to the measure μ .
- Z is a constant enforcing unit mass for a probability measure.

2 Framework and main results

2.1 Abstract setting

Let X, Y be two Euclidean spaces with $X \subset \mathbf{R}^N, Y \subset \mathbf{R}^M$. We think of X as the microscopic space and Y as the macroscopic space. N and M can then be thought of as the size of the microscopic and macroscopic data respectively. Let A and J be respectively a positive definite symmetric and an anti-symmetric linear operators on X . Let $H: X \rightarrow \mathbf{R}$ be a given function. We consider the stochastic dynamic on X that is given by the following stochastic differential equation

$$dX_t = -A\nabla H(X_t) dt - J\nabla H(X_t) dt + \sqrt{2A} dW_t, \quad (1)$$

where W_t is a Wiener process, and \sqrt{A} is the square root of the matrix A . When $J \neq 0$, this is a non-reversible process, and the Fokker-Planck equation associated to this SDE is

$$\partial_t(f\mu) = \text{div}[\mu(A + J)\nabla f], \quad (2)$$

where μ is the invariant measure of the dynamic, which is

$$\mu(dx) := \frac{1}{Z} \exp(-H(x)) dx.$$

In the application we have in mind, which we shall present in the next section, A will be the discrete Laplacian, and J the discrete derivation.

We now introduce an abstract framework for the notion of coarse-graining operator. Let $P: X \rightarrow Y$ be a linear operator such that

$$NPP^t = \text{id}_Y, \quad (3)$$

where P^t is the adjoint operator of P . We think of $y = Px$ as the macroscopic state associated to the microscopic state x . This operator induces a decomposition of the invariant measure into a macroscopic component and a fluctuation component. Let $\bar{\mu}(dy) = P_{\#}\mu$ be the push-forward of μ under the operator P and $\mu(dx|y)$ be the conditional measure of μ given $Px = y$, i.e., for each y , $\mu(dx|y)$ is a probability measure on X and satisfies that for any test function φ

$$\int_X \varphi(x) d\mu(x) = \int_Y \left(\int_{Px=y} \varphi(x) \mu(dx|y) \right) \bar{\mu}(dy). \quad (4)$$

Applying the technique in [2], we show that under certain conditions, the macroscopic profile $y = Px$, with law given by $\bar{f}(t, y) = \int_{Px=y} f(t, x) \mu(dx)$, is close to the solution of the following differential equation

$$\frac{d\eta}{dt} = -(\bar{A} + \bar{J})\nabla\bar{H}(\eta(t)). \quad (5)$$

In this equation, \bar{A} is a symmetric, positive definite operator and \bar{J} is another operator on Y defined by

$$\bar{A}^{-1} = PA^{-1}NP^t, \quad \bar{J} = \bar{A}PA^{-1}NJP^t, \quad (6)$$

and $\bar{H}: Y \rightarrow \mathbf{R}$ is the macroscopic Hamiltonian that satisfies

$$\bar{\mu}(dy) = \exp(-N\bar{H}(y))dy. \quad (7)$$

In order to state the assumptions, we need to recall the definition of the Logarithmic Sobolev inequality. A probability measure $\nu \in \mathcal{P}(X)$ is said to satisfy an LSI with constant $\rho > 0$ (abbreviation LSI(ρ)) if, for any locally Lipschitz, nonnegative function $f \in L^1(\nu)$,

$$\int \Phi(f) d\nu - \Phi\left(\int f d\nu\right) \leq \frac{1}{2\rho} \int \frac{|\nabla f|^2}{f} d\nu.$$

Assumptions: Throughout the paper, we assume that

- (i) $\kappa := \max_{x \in X} \{ \langle \text{Hess } H(x) \cdot u, v \rangle, u \in \text{Ran}(NP^tP), v \in \text{Ran}(\text{id}_X - NP^tP), |u| = |v| = 1 \} < \infty$;
- (ii) There is $\rho > 0$ such that $\mu(dx|y)$ satisfies LSI(ρ) for all y ;
- (iii) There exist $\lambda, \Lambda > 0$ such that $\lambda \text{Id} \leq \text{Hess}\bar{H} \leq \Lambda \text{Id}$;
- (iv) There is $\alpha > 0$ such that $\int_X |x|^2 f \mu(dx) \leq \alpha N$;
- (v) There is $\beta > 0$ such that $\inf_{y \in Y} \bar{H}(y) \geq -\beta$;
- (vi) There is $\gamma > 0$ such that for all $x \in X$,

$$|(\text{id}_X - NP^tP)x|^2 \leq \gamma M^{-2} \langle x, Ax \rangle_X;$$

- (vii) There are constants C_1 and C_2 such that the initial datum satisfy

$$\int \Phi(f(0, x)) \mu(dx) \leq C_1 N \quad \text{and} \quad \bar{H}(\eta_0) \leq C_2;$$

- (viii) There is a $\tau > 0$ such that $A \geq \tau \text{Id}$;
- (ix) $-J^2 \leq cA$;
- (x) J and A commute.

Under these assumptions, we have the following bound on the Wasserstein distances between $f\mu$ and δ_η .

Theorem 2.1. *Let $\mu(dx) = \exp(-H(x)) dx$ be a probability measure on X , and let $P: X \rightarrow Y$ satisfy (3). Let $A: X \rightarrow X$ be a symmetric, definite positive operator, and $f(t, x)$ and $\eta(t)$ be the solutions of (2) and (5), with initial data $f(t, \cdot)$ and η_0 respectively. Suppose that the assumptions above hold. Define*

$$\Theta(t) := \frac{1}{2N} \int_X (x - NP^t \eta(t)) \cdot A^{-1}(x - NP^t \eta(t)) f(t, x) \mu(dx).$$

Then for any $T > 0$, we have

$$\max \left\{ \sup_{0 \leq t \leq T} \Theta(t), \frac{\lambda}{8} \int_0^T \left(\int_Y |y - \eta(t)|_Y^2 \bar{f}(t, y) \bar{\mu}(dy) \right) dt \right\} \leq e^{\frac{8c\Lambda^2}{\lambda} T} [\Theta(0) + E(T, M, N)],$$

where $E(T, M, N) \rightarrow 0$ as $N \uparrow \infty, M \uparrow \infty, \frac{N}{M} \uparrow \infty$. More precisely,

$$\begin{aligned} E(T, M, N) = & T \left(\frac{M}{N} \right) + \frac{4c\gamma\Lambda^2 T}{\lambda} \left(\alpha + \frac{2C_1}{\hat{\rho}} \right) \frac{1}{M} + C_1 \left(\frac{\gamma\kappa^2}{2\lambda\rho^2} + \frac{2c\gamma\kappa^2}{\tau\lambda\rho^2} + \frac{4\gamma c}{\lambda\tau} \right) \frac{1}{M^2} \\ & + \sqrt{2T\gamma} \left(\alpha + \frac{2C_1}{\hat{\rho}} \right)^{\frac{1}{2}} \left\{ \left(1 + \sqrt{\frac{c}{\tau}} + \frac{\sqrt{2c\gamma}}{M} \right) \sqrt{C_1} \right. \\ & \left. + \sqrt{2} \left(1 + \sqrt{\frac{c}{\tau}} \right) (H(\eta_0) - H(\eta_T)) + CT(1 + e^{CT\bar{H}(\eta_0)})^{\frac{1}{2}} \right\} \frac{1}{M}, \end{aligned}$$

where

$$\hat{\rho} := \frac{1}{2} \left(\rho + \lambda + \frac{\kappa^2}{\rho} - \sqrt{\left(\rho + \lambda + \frac{\kappa^2}{\rho} \right)^2 - 4\rho\lambda} \right).$$

Remark 2.2 (Remarks on the assumptions). Assumptions (i) to (viii) are collected from [2] and [1]. Assumption (ix) means that the asymmetric effect is controlled by the symmetric one. Its main use is to rule out situations where the scaling limit is a hyperbolic equation (this would be the case for a continuous analog of the asymmetric exclusion process), which the two-scale approach doesn't seem to handle. Assumption (x) is natural if we think of J and A are finite approximations of first and second derivatives operators, which is the application we have in mind. It could be replaced by an appropriate bound on the symmetric part of $PA^{-1}JNP^t$ (which is the macroscopic component of the commutator between A^{-1} and J), and an additional bound of the form $|\text{Tr}(PJA^{-1}NP^t)| \leq CM$. But since our proof is already fairly technical, and we do not have an application in mind that would warrant the greater generality, we decided to just assume that A and J commute, and simplify the proof. All these assumptions will be used in Lemma 3.4 to estimate the time derivative of $\Theta(t)$. In particular, (ii) and (vi) are used to handle the covariance and fluctuations terms respectively.

The hydrodynamic limit is obtained as a consequence.

Corollary 2.3. *Consider a sequence $\{X_\ell, Y_\ell, P_\ell, A_\ell, J_\ell, \mu_\ell, f_{0,\ell}, \eta_{0,\ell}\}_\ell$ satisfying the assumptions (i) to (x) with uniform constants $\kappa, \rho, \lambda, \Lambda, \alpha, \beta, \gamma, C_1, C_2$ and c . Suppose that*

$$N_\ell \xrightarrow{\ell \uparrow \infty} \infty; \quad M_\ell \xrightarrow{\ell \uparrow \infty} \infty; \quad \frac{M_\ell}{N_\ell} \xrightarrow{\ell \uparrow \infty} 0.$$

Further assume that

$$\lim_{\ell \uparrow \infty} \frac{1}{N_\ell} \int (x - N_\ell P^t \eta_{0,\ell}) \cdot A_\ell^{-1}(x - N_\ell P^t \eta_{0,\ell}) f_{0,\ell}(x) \mu_\ell(dx) = 0.$$

Then, for any $T > 0$:

(a) The microscopic variables are close to the solution of (5) in the penalized norm induced by A_ℓ^{-1} , uniformly in $t \in [0, T]$:

$$\lim_{\ell \uparrow \infty} \sup_{0 \leq t \leq T} \frac{1}{N_\ell} \int (x - N_\ell P^t \eta_\ell) \cdot A_\ell^{-1} (x - N_\ell P^t \eta_\ell) f_\ell(t, x) \mu_\ell(dx) = 0;$$

(b) The macroscopic variables are close to the solution of (5) in the strong $L^2(Y)$ norm, in a time-integrated sense:

$$\lim_{\ell \uparrow \infty} \int_0^T \int |y - \eta_\ell|_Y^2 \bar{f}(t, y) \bar{\mu}(dy) dt = 0.$$

Another topic of interest is whether the data behaves like a local Gibbs state.

Definition 2.4. The local Gibbs state with macroscopic profile $\eta \in Y$ is the probability measure on X whose density with respect to μ is given by

$$G(x) \mu(dx) := \frac{1}{Z} \exp(NP^t \nabla \bar{H}(\eta) \cdot x) \mu(dx).$$

Such a probability measure is close (in Wasserstein distance) to the associated macroscopic profile η .

In [6], it is shown that, if the initial data is close (in the sense of relative entropy) to a local Gibbs state, then this also holds at any positive time, for a time-dependent local Gibbs state. Since closeness in relative entropy is stronger (in the current setting) than closeness in Wasserstein distance, the kind of results obtained with Yau's method are stronger than those of the previous Corollary, but require a stronger assumption on the initial data.

In [4], it was shown that convergence in relative entropy actually holds at positive times, even if the initial data converges only in a weaker sense. In [1], the second author obtained a new proof of this fact in the reversible setting, using the two-scale approach. This method also yields quantitative rates of convergence in relative entropy. Now that we have generalized the two-scale approach to the non-reversible setting, the extension of the results of [1] follows.

Theorem 2.5. Let $G(t, x)$ be the time-dependent local Gibbs state associated to the solution η of (5). Under our assumptions, the following holds :

(a) The relative entropy with respect to the local Gibbs state is controlled as follows:

$$\int_0^T \frac{1}{N} \int \Phi \left(\frac{f(t, x)}{G(t, x)} \right) G(t, x) \mu(dx) dt = O \left(\sqrt{\Theta(0) + \frac{M}{N} + \frac{1}{M}} \right) \quad (8)$$

where the actual constants in the bound (which can be made explicit) depend on $T, \lambda, \Lambda, \alpha, \gamma, \rho, \kappa, \tau, c, C_1$ and C_2 , but not on M and N ;

(b) The difference between the microscopic free energy and the free energy associated with the macroscopic profile η is bounded as follows:

$$\begin{aligned} & \int_0^T \left| \frac{1}{N} \int \Phi(f(t, x)) \mu(dx) - \bar{H}(\eta(t)) \right| dt \\ &= O \left(\sqrt{\Theta(0) + \frac{M}{N} + \frac{1}{M}} \right) \\ &+ O \left(\frac{M}{N} \right) \times \max \left(\left| \log \left(\frac{\Gamma(Y, |\cdot|_Y)^{2/(M-1)}}{\Lambda N} \right) \right|, \left| \log \left(\frac{\Gamma(Y, |\cdot|_Y)^{2/(M-1)}}{\lambda N} \right) \right| \right), \quad (9) \end{aligned}$$

where $\Gamma(Y, |\cdot|_Y)$ is the Gaussian integral on the space Y with respect to the norm $|\cdot|_Y$.

2.2 Application to spin systems

We now give an application of Theorem 2.1 to a system of interacting continuous spins. The application we have in mind is when the matrices A and J are given by

$$A = N^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \quad (10)$$

and

$$J = \frac{N}{2} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix} \quad (11)$$

As in [2], let

$$H(x) := \sum_{i=1}^N \psi(x_i) \quad (12)$$

where $\psi : \mathbf{R} \rightarrow \mathbf{R}$ satisfies the following assumptions:

$$\psi(x) = \frac{1}{2}x^2 + \delta\psi(x); \quad \|\delta\psi\|_{C^2} < \infty \quad (13)$$

We consider the dynamic where A and J are given by (10) and (11) respectively. This corresponds to the system of stochastic differential equations

$$dX_i(t) = -N^2(2\psi(X_i) - \psi(X_{i+1}) - \psi(X_{i-1}))dt - \frac{N}{2}(\psi(X_{i+1}) - \psi(X_{i-1}))dt + N\sqrt{2}(dB_t^{i+1} - dB_t^i).$$

This is the dynamic studied in [3] and [2], to which we have added a weak asymmetric perturbation. This model is to the symmetric dynamic what the weakly asymmetric exclusion process is to the simple symmetric exclusion process, i.e., we have added an extra asymmetric term which has a scaling of lower order in N .

Since this dynamic conserves the mean spin $m = N^{-1} \sum X_i$, the natural space on which to work is

$$X_{N,m} := \left\{ x \in \mathbf{R}^N; \quad \frac{1}{N} \sum_{i=1}^N x_i = m \right\},$$

which we endow with the usual ℓ^2 scalar product. Following [2], the macroscopic space is

$$Y_{M,m} := \left\{ y \in \mathbf{R}^M; \quad \frac{1}{N} \sum_{i=1}^M y_i = m \right\},$$

which we endow with the L^2 scalar product

$$\langle y, \tilde{y} \rangle_Y := \frac{1}{M} \sum_{i=1}^M y_i \tilde{y}_i.$$

The coarse graining operator P is defined as

$$(Px)_i := \frac{1}{K} \sum_{j=(i-1)K+1}^{iK} x_j$$

where K is an integer such that $N = KM$. We can think of this coarse-graining operator as taking local averages of the microscopic profile over boxes of size K . This operator does satisfy the relation $PNP^t = \text{id}_Y$.

When K is large enough, it has been shown that the coarse-grained Hamiltonian \bar{H} is uniformly convex, so we will be able to apply the previous abstract Theorem.

Without loss of generality, we shall assume in the sequel that $m = 0$, since it does not play a role in our estimates.

To study the scaling limit, we need to embed our spaces $X_{N,m}$ into a single functional space. To a macroscopic profile $x \in X_{N,0}$, we associate the step function on the torus \bar{x} , defined by

$$\bar{x}(\theta) := x_i \quad \forall \theta \in \left[\frac{i-1}{N}, \frac{i}{N} \right).$$

We endow the space $L^2(\mathbb{T})$ with the H^{-1} norm, defined by

$$\|w\|_{H^{-1}}^2 = \int g^2 d\theta, \quad g' = w, \quad \int g d\theta = 0.$$

The closure of the spaces $X_{N,0}$ for this norm is the usual H^{-1} space of functions of average 0, which is the dual of the Sobolev space H^1 for the L^2 norm.

We can now state the hydrodynamic limit result we obtain for this model :

Theorem 2.6. *Let A_ℓ and J_ℓ be given by (10) and (11) respectively. Assume that ψ satisfies (13). Let $f(t, x)$ be a time-dependent probability density on $(X_{N,0}, \mu_{N,0})$ solving (2), with $f(0, \cdot) = f_0$ such that*

$$\int f_0 \log f_0 d\mu_{N,0} \leq CN$$

for some $C > 0$ and

$$\lim_{N \uparrow \infty} \int \|\bar{x} - \zeta_0\|_{H^{-1}}^2 f_0(x) \mu_{N,0}(dx) = 0$$

for some initial macroscopic profile $\zeta_0 \in L^2(\mathbb{T})$. The, for any $T > 0$, we have

$$\lim_{N \uparrow \infty} \sup_{0 \leq t \leq T} \int \|\bar{x} - \zeta(t, \cdot)\|_{H^{-1}}^2 f(t, x) \mu_{N,0}(dx) = 0$$

where ζ is the unique solution of

$$\begin{cases} \frac{\partial \zeta}{\partial t} = \frac{\partial^2}{\partial \theta^2} \varphi'(\zeta) + \frac{\partial}{\partial \theta} \varphi'(\zeta), \\ \zeta(0, \cdot) = \zeta_0, \end{cases} \quad (14)$$

where φ is the Cramér transform of ψ , i.e.

$$\varphi(m) = \sup_{\sigma \in \mathbf{R}} \left\{ \sigma m - \log \int_{\mathbf{R}} \exp(\sigma x - \psi(x)) dx \right\}. \quad (15)$$

We can also use [1] to study local Gibbs behavior, and convergence of the relative entropy.

Theorem 2.7. *Under the same assumptions as in Theorem 2.6, the following holds :*

$$\int_0^T \int_{X_N} \Phi \left(\frac{f_N(t, x)}{G_N(t, x)} \right) G_N(t, x) \mu_N(dx) dt \longrightarrow 0, \quad (16)$$

where $G_N(t, \cdot)$ is the local Gibbs state given by $\eta_N(t)$. As a consequence, we have convergence of the microscopic entropy to the hydrodynamic entropy, in a time-integrated sense :

$$\int_0^T \left| \frac{1}{N} \int \Phi(f_N(t, x)) \mu_N(dx) - \left(\int_{\mathbb{T}} \varphi(\zeta(\theta, t)) d\theta - \varphi \left(\int_{\mathbb{T}} \zeta(t, \theta) d\theta \right) \right) \right| dt \xrightarrow{N \rightarrow \infty} 0. \quad (17)$$

Moreover, convergence of $\frac{1}{N} \int \Phi(f_N(t, x)) \mu_N(dx)$ to $\int_{\mathbb{T}} \varphi(\zeta(\theta, t)) d\theta - \varphi \left(\int_{\mathbb{T}} \zeta(t, \theta) d\theta \right)$ holds uniformly on any time-interval $[\epsilon, T]$, for any $0 < \epsilon < T$.

Since deducing this result from 2.5 is nearly the same as in [1], we omit the proof. The only significant difference is proving that the solution of the hydrodynamic equation ζ is smooth on $[\epsilon, T]$, which is a known result, that can be proven by a straightforward adaptation of the proof of Proposition 3.22 in [1].

3 Proof of the abstract results

In this section, we prove Theorem 2.1 and provide a sketch of proof of Theorem 2.5.

3.1 Proof of Theorem 2.1

Following the approach of [2], we prove Theorem 2.5 in three steps : first we differentiate with respect to time the Wasserstein distance between $f(t)\mu$ and the macroscopic profile $\eta(t)$, then we derive an upper bound for the quantity we obtain, before integrating in time and applying Gronwall's Lemma to obtain the result.

Lemma 3.1. *The time-derivative of $\Theta(t)$ is given by the following formula*

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2N} \int_X (x - NP^t \eta(t)) \cdot A^{-1}(x - NP^t \eta(t)) f(t, x) \mu(dx) \\ &= \frac{M}{N} - \int_Y (y - \eta) \cdot (\nabla_Y \bar{H}(y) - \nabla_Y \bar{H}(\eta)) \bar{f}(t, y) \bar{\mu} dy \\ & \quad - \int_Y PJA^{-1} NP^t (y - \eta) \cdot (\nabla_Y \bar{H}(y) - \nabla_Y \bar{H}(\eta)) \bar{f}(t, y) \bar{\mu} dy \\ & \quad - \int_Y (y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu} dy \\ & \quad - \frac{1}{N} \int_X (\text{id}_X - NP^t P) x \cdot \nabla f(t, x) \mu(dx) \\ & \quad + \int \bar{A} \nabla_Y \bar{H}(\eta) \cdot PA^{-1} (\text{id}_X - NP^t P) x f \mu(dx) \\ & \quad + \int_Y PJA^{-1} NP^t (y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{d}y \\ & \quad + \int_Y PJA^{-1} (\text{id}_X - NP^t P) x \cdot P \nabla f(t, x) \mu(dx) \\ & \quad + \frac{1}{N} \int_X (\text{id}_X - NP^t P) JA^{-1} (x - NP^t \eta) \cdot \nabla f(t, x) \mu(dx) \\ & \quad + \int_X PA^{-1} (\text{id}_X - NP^t P) x \cdot \bar{J} \nabla_Y \bar{H}(\eta) f(t, x) \mu(dx). \end{aligned} \quad (18)$$

Proof. We have

$$\begin{aligned}
& \frac{d}{dt} \frac{1}{2N} \int_X (x - NP^t \eta(t)) \cdot A^{-1}(x - NP^t \eta(t)) f(t, x) \mu(dx) \\
& \stackrel{(2)}{=} -\frac{1}{N} \int_X A^{-1}(x - NP^t \eta) \cdot (A + J) \nabla f \mu(dx) - \int P^t \frac{d\eta}{dt} \cdot A^{-1}(x - NP^t \eta) f \mu(dx) \\
& \stackrel{(5)}{=} -\frac{1}{N} \int_X A^{-1}(x - NP^t \eta) \cdot A \nabla f \mu(dx) + \int \bar{A} \nabla_Y \bar{H}(\eta) \cdot P A^{-1}(x - NP^t \eta) f \mu(dx) \\
& \quad - \frac{1}{N} \int A^{-1}(x - NP^t \eta) \cdot J \nabla f \mu(dx) + \int A^{-1}(x - NP^t \eta) \cdot P^t \bar{J} \nabla \bar{H}(\eta) f \mu(dx) \\
& = (I) + (II) + (III) + (IV). \tag{19}
\end{aligned}$$

We now use the decomposition $x = NP^t P x + (\text{id}_X - NP^t P)x$ to transform each term on the right hand side of (19). We need the following definition of the μ -covariance of two functions $f, g \in L^2(\mu)$

$$\text{cov}_\mu(f, g) = \int fg \, d\mu - \left(\int f \, d\mu \right) \left(\int g \, d\mu \right). \tag{20}$$

The first two terms, (I) and (II), are already done in [2]. We repeat here for the sake of completeness.

$$\begin{aligned}
(I) &= -\frac{1}{N} \int_X (x - NP^t \eta) \cdot \nabla f \mu(dx) \\
&= -\int_X P^t (Px - \eta) \cdot \nabla f \mu(dx) - \frac{1}{N} \int (\text{id}_X - NP^t P)x \cdot \nabla f \mu(dx). \tag{21}
\end{aligned}$$

We now transform the first term in (21) using (4) and Lemma 21 in [2].

$$\begin{aligned}
& -\int_X P^t (Px - \eta) \cdot \nabla f \mu(dx) = -\int (Px - \eta) \cdot P \nabla f \mu(dx) \\
& \quad - \stackrel{(4)}{=} \int_Y (y - \eta) \cdot P \int_{Px=y} \nabla f \mu(dx|y) \bar{\mu} dy \\
& \quad \stackrel{[2, (36)]}{=} -\frac{1}{N} \int (y - \eta) \cdot \nabla_Y \bar{f} \bar{\mu} dy - \int (y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu} dy \\
& \quad \stackrel{(7)}{=} \frac{1}{N} \int \nabla_Y \cdot y \bar{f} \bar{\mu} dy - \int (y - \eta) \cdot \nabla_Y \bar{H}(y) \bar{f} \bar{\mu} dy - \int (y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu} dy \\
& = \frac{\dim Y}{N} - \int (y - \eta) \cdot \nabla_Y \bar{H}(y) \bar{f} \bar{\mu} dy - \int (y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu} dy.
\end{aligned}$$

We obtain

$$(I) = \frac{\dim Y}{N} - \int (y - \eta) \cdot \nabla_Y \bar{H}(y) \bar{f} \bar{\mu} dy - \int (y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu} dy - \frac{1}{N} \int (\text{id}_X - NP^t P)x \cdot \nabla f \mu(dx). \tag{22}$$

Now we proceed with (II).

$$\begin{aligned}
(II) &= \int \bar{A} \nabla_Y \bar{H}(\eta) \cdot P A^{-1} NP^t (Px - \eta) f \mu(dx) + \int \bar{A} \nabla_Y \bar{H}(\eta) \cdot P A^{-1} (\text{id}_X - NP^t P)x f \mu(dx) \\
& \stackrel{(6)}{=} \int \nabla_Y \bar{H}(\eta) \cdot (Px - \eta) f \mu(dx) + \int \bar{A} \nabla_Y \bar{H}(\eta) \cdot P A^{-1} (\text{id}_X - NP^t P)x f \mu(dx) \\
& \stackrel{(4)}{=} \int_Y (y - \eta) \cdot \nabla_Y \bar{H}(\eta) \bar{f} \bar{\mu}(dy) + \int \bar{A} \nabla_Y \bar{H}(\eta) \cdot P A^{-1} (\text{id}_X - NP^t P)x f \mu(dx). \tag{23}
\end{aligned}$$

Next, we continue with (III).

$$\begin{aligned}
(III) &= \frac{1}{N} \int JA^{-1}(x - NP^t\eta) \cdot \nabla f \mu(dx) \\
&= \frac{1}{N} \int PJA^{-1}(x - NP^t\eta) \cdot NP \nabla f \mu(dx) + \frac{1}{N} \int (\text{id}_X - NP^tP)JA^{-1}(x - NP^t\eta) \cdot \nabla f \mu(dx) \\
&= \frac{1}{N} \int PJA^{-1}NP^t(Px - \eta) \cdot NP \nabla f \mu(dx) + \frac{1}{N} \int PJA^{-1}(\text{id}_X - NP^tP)x \cdot NP \nabla f \mu(dx) \\
&\quad + \frac{1}{N} \int (\text{id}_X - NP^tP)JA^{-1}(x - NP^t\eta) \cdot \nabla f \mu(dx).
\end{aligned}$$

The first term on the right hand side of the expression above can be transformed further using Lemma 21 in [2] as done for (I).

$$\begin{aligned}
\frac{1}{N} \int PJA^{-1}(x - NP^t\eta) \cdot NP \nabla f \mu(dx) &= \frac{1}{N} \int_Y \left(\int_{Px=y} PJA^{-1}NP^t(y - \eta) \cdot NP \nabla f \mu(dx|y) \right) \bar{\mu}(dy) \\
&= \frac{1}{N} \int_Y PJA^{-1}NP^t(y - \eta) \cdot \left[\nabla_Y \bar{f}(y) + NP \text{cov}_{\mu(dx|y)}(f, \nabla H) \right] \bar{\mu}(dy) \\
&= \frac{1}{N} \int_Y PJA^{-1}NP^t(y - \eta) \cdot \nabla_Y \bar{f}(y) \bar{\mu}(dy) + \int_Y PJA^{-1}NP^t(y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu}(dy) \\
&= -\frac{\text{Tr}(PJA^{-1}NP^t)}{N} + \int_Y PJA^{-1}NP^t(y - \eta) \cdot \nabla_Y \bar{H}(y) \bar{f} \bar{\mu}(dy) \\
&\quad + \int_Y PJA^{-1}NP^t(y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu}(dy).
\end{aligned}$$

Since $PJA^{-1}NP^t$ is anti-symmetric, $\text{Tr}(PJA^{-1}NP^t)=0$, and we obtain

$$\begin{aligned}
(III) &= \int_Y PJA^{-1}NP^t(y - \eta) \cdot \nabla_Y \bar{H}(y) \bar{f} \bar{\mu}(dy) \\
&\quad + \int_Y PJA^{-1}NP^t(y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu}(dy) \\
&\quad + \frac{1}{N} \int PJA^{-1}(\text{id}_X - NP^tP)x \cdot NP \nabla f \mu(dx) \\
&\quad + \frac{1}{N} \int (\text{id}_X - NP^tP)JA^{-1}(x - NP^t\eta) \cdot \nabla f \mu(dx). \tag{24}
\end{aligned}$$

Finally, we now transform (IV).

$$\begin{aligned}
(IV) &= \int PA^{-1}NP^t(Px - \eta) \cdot \bar{J} \nabla_Y \bar{H}(\eta) f \mu(dx) + \int PA^{-1}(\text{id}_X - NP^tP) \cdot \bar{J} \nabla_Y \bar{H}(\eta) f \mu(dx) \\
&\stackrel{(4)}{=} \int_Y PA^{-1}NP^t(y - \eta) \cdot \bar{J} \nabla_Y \bar{H}(\eta) \bar{f} \bar{\mu}(dy) + \int PA^{-1}(\text{id}_X - NP^tP) \cdot \bar{J} \nabla_Y \bar{H}(\eta) f \mu(dx) \\
&\stackrel{(6)}{=} - \int PJA^{-1}NP^t(y - \eta) \cdot \nabla_Y \bar{H}(\eta) \bar{f} \bar{\mu}(dy) + \int PA^{-1}(\text{id}_X - NP^tP) \cdot \bar{J} \nabla_Y \bar{H}(\eta) f \mu(dx). \tag{25}
\end{aligned}$$

Substituting (22)-(25) into (19), we obtain (18) and the lemma is proven. \square

The following auxiliary lemma will be helpful in the sequel. The second and the third parts are respectively (54) and (52) in [2]; we put them here for readers' convenience.

Lemma 3.2. *We have the following estimate*

1. For every $y \in Y$

$$|PJA^{-1}NP^t y|^2 \leq c \langle \bar{A}^{-1} y, y \rangle \leq \frac{c}{\tau} |y|_Y^2, \quad (26)$$

$$\langle \bar{A}PJA^{-1}NP^t y, PJA^{-1}NP^t y \rangle \leq c|y|^2. \quad (27)$$

2. For every $x \in X$

$$(\text{id}_X - NP^t P)x \cdot A^{-1}(\text{id}_X - NP^t P)x \leq \frac{\gamma}{M^2} |x|^2. \quad (28)$$

3. It holds that

$$|NP^t P \text{cov}_{\mu(dx|y)}(f, \nabla H)|^2 \leq \gamma \frac{\kappa^2}{\rho^2} \frac{1}{M^2} \bar{f} \int \frac{1}{f} \nabla f \cdot A \nabla f \mu(dx|y). \quad (29)$$

Proof. We only need to prove the first part.

We start with (26). The first inequality is obtained using the assumption (2) and the fact that $NP^t P$ is an orthogonal projection as follows.

$$\begin{aligned} \langle PJA^{-1}NP^t y, PJA^{-1}NP^t y \rangle &= \frac{1}{N} \langle NP^t PJA^{-1}NP^t y, JA^{-1}NP^t y \rangle \\ &\leq \frac{1}{N} \langle JA^{-1}NP^t y, JA^{-1}NP^t y \rangle \\ &= -\frac{1}{N} \langle J^2 A^{-1}NP^t y, A^{-1}NP^t y \rangle \\ &\leq \frac{c}{N} \langle A^{-1}NP^t y, NP^t y \rangle \quad (\text{used assumption (ix) here}) \\ &= c \langle NPA^{-1}P^t y, y \rangle \\ &= c \langle \bar{A}^{-1} y, y \rangle. \end{aligned}$$

Now we prove the second one. Since τ is a lower bound on the spectral value of A , $\frac{1}{\tau}$ is an upper bound on that of A^{-1} . Hence

$$\langle \bar{A}^{-1} y, y \rangle = \langle PA^{-1}NP^t y, y \rangle = \frac{1}{N} \langle A^{-1}NP^t y, NP^t y \rangle \leq \frac{1}{N\tau} \langle NP^t y, NP^t y \rangle = \frac{1}{\tau} |y|_Y^2.$$

Next, we prove (27). By duality, we have

$$\begin{aligned} \langle \bar{A}PJA^{-1}NP^t y, PJA^{-1}NP^t y \rangle &= \sup_z \{2\langle PJA^{-1}NP^t y, z \rangle - \langle \bar{A}^{-1} z, z \rangle\} \\ &\stackrel{(26)}{\leq} \sup_z \{2\langle y, PJA^{-1}NP^t z \rangle - c^{-1}|PJA^{-1}NP^t z|^2\} \\ &\leq \sup_z \{2\langle y, z \rangle - c^{-1}|z|^2\} \\ &\leq c|y|^2. \end{aligned}$$

□

Lemma 3.3. *If $f(t, x)$ and $\eta(t)$ satisfy the assumptions of theorem 2.1, then for any $T < \infty$ we have*

$$\int_0^T \int \frac{1}{f} \nabla f \cdot A \nabla f(t, x) \mu(dx) dt = \int \Phi(f(0, x)) \mu(dx) - \int \Phi(f(T, x)) \mu(dx); \quad (30)$$

$$\int_0^T \langle \bar{A} \nabla_Y \bar{H}(\eta), \nabla_Y \bar{H}(\eta) \rangle dt \leq 2(H(\eta_0) - H(\eta_T)) + CT(1 + e^{CT} \bar{H}(\eta_0)), \quad (31)$$

where $C > 0$ is a constant;

$$\left(\int |x|^2 f(t, x) \mu(dx) \right)^{\frac{1}{2}} \leq \left(\int |x|^2 \mu(dx) \right)^{\frac{1}{2}} + \left(\frac{2}{\bar{\rho}} \int \Phi(f(0, x)) \mu(dx) \right)^{\frac{1}{2}}. \quad (32)$$

Proof. The proof of this lemma is similar to that of proposition 24 in [2]. We prove (30) first. We have

$$\begin{aligned}
\frac{d}{dt} \int \Phi(f(t, x)) \mu(dx) &= \int (\log f + 1) \partial_t (f \mu) \\
&= \int (\log f + 1) \operatorname{div}(\mu(A + J) \nabla f) \\
&= - \int (A + J) \nabla f \cdot \frac{\nabla f}{f} \mu(dx) \\
&= - \int \frac{1}{f} A \nabla f \cdot \nabla f \mu(dx) \quad (\text{since } J \text{ is anti-symmetric}). \quad (33)
\end{aligned}$$

Thus (30) follows. Next we prove (31). We have

$$\begin{aligned}
\frac{d}{dt} \bar{H}(\eta(t)) &= \langle \dot{\eta}(t), \nabla_Y \bar{H}(\eta) \rangle \\
&\stackrel{(5)}{=} - \langle \bar{A} \nabla_Y \bar{H}(\eta), \nabla_Y \bar{H}(\eta) \rangle - \langle \bar{J} \nabla_Y \bar{H}(\eta), \nabla_Y \bar{H}(\eta) \rangle \\
&= - \langle \bar{A} \nabla \bar{H}(\eta), \nabla \bar{H}(\eta) \rangle - \langle \bar{A} P J A^{-1} N P^t \nabla \bar{H}(\eta), \nabla \bar{H}(\eta) \rangle \\
&\leq \frac{1}{2} \langle \bar{A} P J A^{-1} N P^t \nabla \bar{H}(\eta), P J A^{-1} N P^t \nabla \bar{H}(\eta) \rangle - \frac{1}{2} \langle \bar{A} \nabla_Y \bar{H}(\eta), \nabla_Y \bar{H}(\eta) \rangle \\
&\stackrel{(27)}{\leq} \frac{c}{2} |\nabla \bar{H}(\eta)|^2 - \frac{1}{2} \langle \bar{A} \nabla_Y \bar{H}(\eta), \nabla_Y \bar{H}(\eta) \rangle
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{d}{dt} \bar{H}(\eta(t)) + \frac{1}{2} \langle \bar{A} \nabla_Y \bar{H}(\eta), \nabla_Y \bar{H}(\eta) \rangle \\
\leq \frac{c}{2} |\nabla \bar{H}(\eta)|^2 \\
\leq C(|\eta|^2 + 1) \\
\leq C(\bar{H}(\eta) + 1).
\end{aligned}$$

In the above estimate, $C > 0$ is a general constant. Note that we have used the assumption (iii). The above Gronwall-type inequality implies that for every $t \geq 0$, we have $\bar{H}(\eta(t)) \leq e^{C(T+1)} \bar{H}(\eta_0)$, and

$$\int_0^T \langle \bar{A} \nabla_Y \bar{H}(\eta), \nabla_Y \bar{H}(\eta) \rangle dt \leq 2(H(\eta_0) - H(\eta_T)) + CT(1 + e^{CT} \bar{H}(\eta_0)).$$

By (33), $\int \Phi(f(t, x)) \mu(dx)$ is non-increasing in t ; hence the proof of (32) is the same as that of (46) in [2]. \square

Lemma 3.4. *We have the following estimate*

$$\begin{aligned}
\frac{d}{dt} \Theta(t) - \frac{8c\Lambda^2}{\lambda} \Theta(t) + \frac{\lambda}{8} \int |y - \eta|^2 \bar{f} \bar{\mu}(dy) \\
\leq \frac{M}{N} + \frac{4c\gamma\Lambda^2}{2\lambda N M^2} \int |x|^2 f \mu(dx) \\
+ \left(\frac{\gamma\kappa^2}{2\lambda\rho^2 M^2} + \frac{2c\gamma\kappa^2}{\tau\lambda\rho^2 M^2} + \frac{4\gamma c}{\lambda\tau M^2} \right) \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \\
+ \frac{\sqrt{\gamma}}{M} \left(\int \frac{1}{N} |x|^2 f \mu(dx) \right)^{\frac{1}{2}} \left[\left(1 + \sqrt{\frac{c}{\tau}} + \frac{\sqrt{2c\gamma}}{M} \right) \left(\int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}} \right. \quad (34) \\
\left. + \left(1 + \sqrt{\frac{c}{\tau}} \right) (\bar{A} \nabla_Y \bar{H}(\eta) \cdot \nabla_Y \bar{H}(\eta))^{\frac{1}{2}} \right]. \quad (35)
\end{aligned}$$

Proof. We estimate each term in (18). The 2nd, 4th and 5th terms are already done in [2]. We get

$$- \int_Y (y - \eta) \cdot (\nabla_Y \bar{H}(y) - \nabla_Y \bar{H}(\eta)) \bar{f} \bar{\mu} dy \leq -\lambda \int |y - \eta|_Y^2 \bar{f} \bar{\mu} dy, \quad (36)$$

$$\left| \int (y - \eta) \cdot P \text{cov}_{\mu(dx|y)}(f, \nabla H) \bar{\mu} dy \right| \leq \frac{\gamma \kappa^2}{2\lambda \rho^2 M^2} \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) + \frac{\lambda}{2} \int |y - \eta|_Y^2 \bar{f} \bar{\mu}(dy), \quad (37)$$

$$\left| \frac{1}{N} \int (\text{id}_X - NP^t P)x \cdot \nabla f \mu(dx) \right| \leq \left(\frac{\gamma}{M^2} \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \cdot \int \frac{1}{N} |x|^2 f \mu(dx) \right)^{\frac{1}{2}}. \quad (38)$$

We estimate the 3rd term. Since

$$\begin{aligned} |PJA^{-1}NP^t(y - \eta)| \cdot |\nabla_Y \bar{H}(y) - \nabla_Y \bar{H}(\eta)| &\leq \Lambda |y - \eta| \cdot |PJA^{-1}NP^t(y - \eta)| \\ &\stackrel{(26)}{\leq} \Lambda |y - \eta| \sqrt{c\langle \bar{A}^{-1}(y - \eta), y - \eta \rangle} \\ &\leq \frac{\lambda}{8} |y - \eta|^2 + \frac{2c\Lambda^2}{\lambda} \langle \bar{A}^{-1}(y - \eta), y - \eta \rangle, \end{aligned}$$

we have

$$\begin{aligned} &\left| \int_Y PJA^{-1}NP^t(y - \eta) \cdot (\nabla_Y \bar{H}(y) - \nabla_Y \bar{H}(\eta)) \bar{f} \bar{\mu}(dy) \right| \\ &\leq \int_Y |PJA^{-1}NP^t(y - \eta)| |\nabla_Y \bar{H}(y) - \nabla_Y \bar{H}(\eta)| \bar{f} \bar{\mu}(dy) \\ &\leq \frac{\lambda}{8} \int_Y |y - \eta|^2 \bar{f} \bar{\mu}(dy) + \frac{2c\Lambda^2}{\lambda} \int_Y \langle \bar{A}^{-1}(y - \eta), y - \eta \rangle \bar{f} \bar{\mu}(dy) \\ &= \frac{\lambda}{8} \int_Y |y - \eta|^2 \bar{f} \bar{\mu}(dy) + \frac{2c\Lambda^2}{\lambda} \frac{1}{N} \int_X \langle A^{-1}NP^t(Px - \eta), NP^t(Px - \eta) \rangle f \mu(dx) \\ &\leq \frac{\lambda}{8} \int_Y |y - \eta|^2 \bar{f} \bar{\mu}(dy) + \frac{2c\Lambda^2}{\lambda} \frac{2}{N} \int_X \langle A^{-1}(x - NP^t\eta), (x - NP^t\eta) \rangle f \mu(dx) \\ &\quad + \frac{2c\Lambda^2}{\lambda} \frac{2}{N} \int_X \langle A^{-1}(\text{id}_X - NP^t P)x, (\text{id}_X - NP^t P)x \rangle f \mu(dx) \\ &\leq \frac{\lambda}{8} \int_Y |y - \eta|^2 \bar{f} \bar{\mu}(dy) + \frac{8c\Lambda^2}{\lambda} \Theta(t) + \frac{4c\gamma\Lambda^2}{\lambda NM^2} \int |x|^2 f \mu(dx). \end{aligned} \quad (39)$$

Next we estimate the 6th term.

$$\begin{aligned} &\int \bar{A} \nabla_Y \bar{H}(\eta) \cdot PA^{-1}(\text{id}_X - NP^t P)x f \mu(dx) \\ &= \int P^t \bar{A} \nabla_Y \bar{H}(\eta) \cdot A^{-1}(\text{id}_X - NP^t P)x f \mu(dx) \\ &\leq \left(\int P^t \bar{A} \nabla_Y \bar{H}(\eta) \cdot A^{-1}NP^t \bar{A} \nabla_Y \bar{H}(\eta) f \mu(dx) \right)^{\frac{1}{2}} \left(\frac{1}{N} \int (\text{id}_X - NP^t P)x \cdot A^{-1}(\text{id}_X - NP^t P)x f \mu(dx) \right)^{\frac{1}{2}} \end{aligned}$$

Since

$$P^t \bar{A} \nabla_Y \bar{H}(\eta) \cdot A^{-1}NP^t \bar{A} \nabla_Y \bar{H}(\eta) = \bar{A} \nabla_Y \bar{H}(\eta) \cdot PA^{-1}NP^t \bar{A} \nabla_Y \bar{H}(\eta) = \bar{A} \nabla_Y \bar{H}(\eta) \cdot \nabla_Y \bar{H},$$

and from (28), we have

$$\int \bar{A} \nabla_Y \bar{H}(\eta) \cdot PA^{-1}(\text{id}_X - NP^t P)x f \mu(dx) \leq (\bar{A} \nabla_Y \bar{H}(\eta) \cdot \nabla_Y \bar{H}(\eta))^{\frac{1}{2}} \left(\frac{\gamma}{NM^2} \int |x|^2 f \mu(dx) \right)^{\frac{1}{2}}. \quad (40)$$

Next, we estimate the 7th term.

$$\begin{aligned}
& \left| \int PJA^{-1}NP^t(y-\eta) \cdot P\text{cov}_{\mu(dx|y)}(f, \nabla H)\bar{\mu}(dy) \right| \\
& \leq \left(\int |PJA^{-1}NP^t(y-\eta)|^2 \bar{f}\bar{\mu}(dy) \cdot \int \frac{1}{f} |P\text{cov}_{\mu(dx|y)}|_Y^2 \bar{\mu}(dy) \right)^{\frac{1}{2}} \\
& \stackrel{(26),(29)}{\leq} \left(\frac{2c}{\tau} \gamma \frac{\kappa^2}{\rho^2} \frac{1}{M^2} \int |y-\eta|^2 \bar{f}\bar{\mu}(dy) \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}} \\
& \leq \frac{2c\gamma\kappa^2}{\tau\lambda\rho^2 M^2} \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) + \frac{\lambda}{8} \int |y-\eta|^2 \bar{f}\bar{\mu}(dy). \tag{41}
\end{aligned}$$

For the 8th term, we have

$$\begin{aligned}
& \int PJA^{-1}(\text{id}_X - NP^tP)x \cdot P\nabla f \mu(dx) = \frac{1}{N} \int NP^tPJA^{-1}(\text{id}_X - NP^tP)x \nabla f \mu(dx) \\
& \leq \left(\int \frac{1}{N} NP^tPJA^{-1}(\text{id}_X - NP^tP)x \cdot A^{-1}NP^tPJA^{-1}(\text{id}_X - NP^tP)x f \mu(dx) \right)^{\frac{1}{2}} \left(\int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\begin{aligned}
& \langle NP^tPJA^{-1}(\text{id}_X - NP^tP)x, A^{-1}NP^tPJA^{-1}(\text{id}_X - NP^tP)x \rangle \\
& \leq \frac{1}{\tau} \langle NP^tPJA^{-1}(\text{id}_X - NP^tP)x, NP^tPJA^{-1}(\text{id}_X - NP^tP)x \rangle \\
& \leq \frac{1}{\tau} \langle JA^{-1}(\text{id}_X - NP^tP)x, JA^{-1}(\text{id}_X - NP^tP)x \rangle \\
& = \frac{1}{\tau} \langle -J^2A^{-1}(\text{id}_X - NP^tP)x, A^{-1}(\text{id}_X - NP^tP)x \rangle \\
& \leq \frac{c}{\tau} \langle (\text{id}_X - NP^tP)x, A^{-1}(\text{id}_X - NP^tP)x \rangle \\
& \stackrel{(28)}{\leq} \frac{c\gamma}{\tau M^2} |x|^2,
\end{aligned}$$

we obtain

$$\int PJA^{-1}(\text{id}_X - NP^tP)x \cdot P\nabla f \mu(dx) \leq \left(\frac{c\gamma}{\tau M^2} \int \frac{1}{N} |x|^2 f \mu(dx) \cdot \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}}. \tag{42}$$

Next we estimate the 9th term. Set $z = JA^{-1}(x - NP^t\eta)$. We have

$$\begin{aligned}
& \frac{1}{N} \int (\text{id}_X - NP^tP)JA^{-1}(x - NP^t\eta) \cdot \nabla f \mu(dx) = \frac{1}{N} \int (\text{id}_X - NP^tP)z \cdot \nabla f \mu(dx) \\
& \leq \left(\int \frac{1}{N} (\text{id}_X - NP^tP)z \cdot A^{-1}(\text{id}_X - NP^tP)z f \mu(dx) \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}} \\
& \stackrel{(28)}{\leq} \left(\frac{\gamma}{M^2} \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \int \frac{1}{N} |z|^2 f \mu(dx) \right)^{\frac{1}{2}}.
\end{aligned}$$

We estimate the second integral inside the parentheses. It holds that

$$\begin{aligned}
|z|^2 &= \langle JA^{-1}(x - NP^t\eta), JA^{-1}(x - NP^t\eta) \rangle \\
&= \langle -J^2A^{-1}(x - NP^t\eta), A^{-1}(x - NP^t\eta) \rangle \\
&\stackrel{(ix)}{\leq} c \langle A^{-1}(x - NP^t\eta), x - NP^t\eta \rangle \\
&\leq 2c \left(\langle A^{-1}NP^t(Px - \eta), NP^t(Px - \eta) \rangle + \langle A^{-1}(\text{id}_X - NP^tP)x, (\text{id}_X - NP^tP)x \rangle \right) \\
&\stackrel{(28)}{\leq} 2c \left(\frac{1}{\tau} |NP^t(Px - \eta)|^2 + \frac{\gamma}{M^2} |x|^2 \right) \\
&= 2c \left(\frac{N}{\tau} |Px - \eta|^2 + \frac{\gamma}{M^2} |x|^2 \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{N} \int (\text{id}_X - NP^tP)JA^{-1}(x - NP^t\eta) \cdot \nabla f \mu(dx) \\
&\leq \left(\frac{\gamma}{M^2} \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}} \left(\frac{2c\gamma}{\tau} \int |y - \eta|^2 \bar{f} \bar{\mu}(dy) + \frac{2c\gamma}{M^2N} \int |x|^2 f \mu(dx) \right)^{\frac{1}{2}} \\
&\leq \frac{4\gamma c}{M^2\lambda\tau} \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) + \frac{\lambda}{8} \int |y - \eta|^2 \bar{f} \bar{\mu}(dy) \\
&\quad + \left(\frac{\gamma}{M^2} \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}} \left(\frac{2c\gamma}{M^2N} \int |x|^2 f \mu(dx) \right)^{\frac{1}{2}}. \tag{43}
\end{aligned}$$

Finally, we estimate the 10th term. Since

$$\begin{aligned}
&\langle PA^{-1}NP^t\bar{J}\nabla\bar{H}(\eta), \bar{J}\nabla\bar{H}(\eta) \rangle \\
&= \langle PJA^{-1}NP^t\nabla\bar{H}(\eta), \bar{A}PJA^{-1}NP^t\nabla\bar{H}(\eta) \rangle \\
&\leq c|\nabla\bar{H}(\eta)|^2 \\
&\leq \frac{c}{\tau} \langle \bar{A}\nabla_Y\bar{H}(\eta), \nabla_Y\bar{H}(\eta) \rangle,
\end{aligned}$$

we have

$$\begin{aligned}
&\left| \int A^{-1}(\text{id}_X - NP^tP)x \cdot P^t\bar{J}\nabla_Y\bar{H}(\eta) f \mu(dx) \right| \\
&\leq \left(\int P^t\bar{J}\nabla_Y\bar{H}(\eta) \cdot A^{-1}NP^t\bar{J}\nabla_Y\bar{H}(\eta) f \mu(dx) \right)^{\frac{1}{2}} \left(\int \frac{1}{N} (\text{id}_X - NP^tP)x \cdot A^{-1}(\text{id}_X - NP^tP)x f \mu(dx) \right)^{\frac{1}{2}} \\
&\leq \left(\frac{c}{\tau} \bar{A}\nabla_Y\bar{H}(\eta) \cdot \nabla_Y\bar{H}(\eta) \right)^{\frac{1}{2}} \left(\frac{\gamma}{NM^2} \int |x|^2 f \mu(dx) \right)^{\frac{1}{2}}. \tag{44}
\end{aligned}$$

Summing up from (36) to (44), we obtain

$$\begin{aligned}
& \frac{d}{dt}\Theta(t) - \frac{8c\Lambda^2}{\lambda}\Theta(t) + \frac{\lambda}{8} \int |y - \eta|^2 \bar{f}\bar{\mu}(dy) \\
& \leq \frac{M}{N} + \frac{4c\gamma\Lambda^2}{2\lambda NM^2} \int |x|^2 f\mu(dx) \\
& \quad + \left(\frac{\gamma\kappa^2}{2\lambda\rho^2 M^2} + \frac{2c\gamma\kappa^2}{\tau\lambda\rho^2 M^2} + \frac{4\gamma c}{\lambda\tau M^2} \right) \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \\
& \quad + \frac{\sqrt{\gamma}}{M} \left(\int \frac{1}{N} |x|^2 f\mu(dx) \right)^{\frac{1}{2}} \left[\left(1 + \sqrt{\frac{c}{\tau}} + \frac{\sqrt{2c\gamma}}{M} \right) \left(\int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}} \right. \\
& \quad \left. \left(1 + \sqrt{\frac{c}{\tau}} \right) (\bar{A} \nabla_Y \bar{H}(\eta) \cdot \nabla_Y \bar{H}(\eta))^{\frac{1}{2}} \right].
\end{aligned}$$

□

Proof of Theorem 2.1. Denote by $R(t)$ the right hand side of (35). Set $D = \frac{8c\Lambda^2}{\lambda}$. For any $0 < t \leq T$, we have

$$\begin{aligned}
\frac{d}{dt} (e^{-Dt}\Theta(t)) + e^{-Dt} \frac{\lambda}{8} \int |y - \eta|^2 \bar{f}\bar{\mu}(dy) & \leq \frac{d}{dt} (e^{-Dt}\Theta(t)) + e^{-Dt} \frac{\lambda}{8} \int |y - \eta|^2 \bar{f}\bar{\mu}(dy) \\
& \leq e^{-Dt} S(t) \leq S(t).
\end{aligned} \tag{45}$$

Integrating (35) with respect to time, for any $0 < t \leq T$, we have

$$\begin{aligned}
e^{-Dt}\Theta(t) + \frac{\lambda}{8} e^{-Dt} \int_0^T \int |y - \eta|^2 \bar{f}\bar{\mu}(dy) & \leq e^{-Dt}\Theta(t) + \frac{\lambda}{8} e^{-Dt} \int_0^T \int |y - \eta|^2 \bar{f}\bar{\mu}(dy) \\
& \leq \Theta(0) + \int_0^T S(t) dt.
\end{aligned} \tag{46}$$

It follows that for any $T > 0$

$$\max \left\{ \sup_{t \in (0, T)} \Theta(t), \frac{\lambda}{8} \int_0^T \int_Y |y - \eta|^2 \bar{f}\bar{\mu}(dy) \right\} \leq e^{DT} \left(\Theta(0) + \int_0^T S(t) dt \right). \tag{47}$$

□

It remains to take care of each term in the right hand side of (47). Let $a, b > 0$ be two constants.

$$\begin{aligned}
& \int_0^T \int \frac{1}{N} |x|^2 f(t, x) \mu(dx) dt \stackrel{(32)}{\leq} 2 \left(\alpha + \frac{2C_1}{\hat{\rho}} \right) T; \\
& \int_0^T \int \frac{1}{Nf} \nabla \cdot A \nabla f \mu(dx) dt \stackrel{(30)}{\leq} C_1; \\
& \int_0^T \left(\int \frac{1}{N} |x|^2 f\mu(dx) \right)^{\frac{1}{2}} \left(a \left(\int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) \right)^{\frac{1}{2}} + b (\bar{A} \nabla_Y \bar{H}(\eta) \cdot \nabla_Y \bar{H}(\eta))^{\frac{1}{2}} \right) dt \\
& \leq \left(\int_0^T \int \frac{1}{N} |x|^2 f\mu(dx) dt \right)^{\frac{1}{2}} \left(a \left(\int_0^T \int \frac{1}{Nf} \nabla f \cdot A \nabla f \mu(dx) dt \right)^{\frac{1}{2}} + b \left(\int_0^T \bar{A} \nabla_Y \bar{H}(\eta) \cdot \nabla_Y \bar{H}(\eta) dt \right)^{\frac{1}{2}} \right) \\
& \leq \sqrt{2T} \left(\alpha + \frac{2C_1}{\hat{\rho}} \right)^{\frac{1}{2}} \left(a\sqrt{C_1} + \sqrt{2b}(H(\eta_0) - H(\eta_T)) + CT(1 + e^{CT}\bar{H}(\eta_0))^{\frac{1}{2}} \right).
\end{aligned}$$

Substituting these estimate to (47) concludes the proof of Theorem 2.1.

3.2 Sketch of proof of Theorem 2.5

In this section, we give the main arguments of the proof of Theorem 2.5, which exactly follows the method of [1].

- First, we decompose the relative entropy with respect to the local Gibbs state into a macroscopic component and a fluctuations component. Since $G(x)$ only depends on the macroscopic profile $y = Px$, we have

$$\text{Ent}_{G\mu}(f\mu) = \text{Ent}_{\bar{G}\bar{\mu}}(\bar{f}\bar{\mu}) + \int_Y \text{Ent}_{\mu(dx|y)}(f\mu)\bar{G}(y)\bar{\mu}(dy).$$

- The fluctuations component $\int_0^T \int_Y \text{Ent}_{\mu(dx|y)}(f\mu)\bar{G}(y)\bar{\mu}(dy)dt$ can be bounded using the logarithmic Sobolev inequality for $\mu(dx|y)$, assumption (vi) and the bound on the microscopic entropy production of Lemma 3.3.
- For the macroscopic component, since $\bar{G}\bar{\mu}$ is log-concave, we can use the HWI inequality of [5], which states that

$$\text{Ent}_{\bar{G}\bar{\mu}}(\bar{f}\bar{\mu}) \leq W_2(\bar{f}\bar{\mu}, \bar{G}\bar{\mu})\sqrt{I_{\bar{G}\bar{\mu}}(\bar{f}\bar{\mu})},$$

where the Wasserstein distance W_2 is taken with respect to the norm $|\cdot|_Y$, and I is the Fisher information

$$I_{\bar{G}\bar{\mu}}(\bar{f}\bar{\mu}) := \int \frac{|\nabla(\bar{f}/\bar{G})|^2}{\bar{f}/\bar{G}} \bar{G}d\bar{\mu}.$$

As a consequence, to obtain convergence in relative entropy, we only require convergence in Wasserstein distance and a bound on the Fisher information.

- We already have a bound on $\int_0^T W_2(\bar{f}\bar{\mu}, \delta_{\eta(t)})^2 dt$ from Theorem 2.1. Moreover,

$$W_2(\bar{G}\bar{\mu}, \delta_\eta)^2 \leq \frac{M}{\lambda N}$$

by Proposition 4.1 of [1]. A bound on $\int_0^T W_2(\bar{f}\bar{\mu}, \bar{G}\bar{\mu})dt$ immediately follows from the triangle inequality.

- Finally, the time-integral of the Fisher information can be bounded using the bounds on the entropy production of Lemma 3.3. This concludes the proof of (a).
- (b) can be deduced from (a) using elementary inequalities and the bound

$$\begin{aligned} & \left| \frac{1}{N} \int \Phi(G^\eta)d\mu - \bar{H}(\eta) \right| \\ & \leq \frac{(M-1)}{2N} \max \left(\left| \log \left(\frac{\Gamma(Y, |\cdot|_Y)^{2/(M-1)}}{\Lambda N} \right) \right|, \left| \log \left(\frac{\Gamma(Y, |\cdot|_Y)^{2/(M-1)}}{\lambda N} \right) \right| \right) \\ & \quad + \sqrt{\frac{M}{\lambda N}} |\nabla \bar{H}(\eta)|_Y, \end{aligned}$$

which was proven in Proposition 4.1 of [1].

4 Application part

In this section, we prove Theorem 2.6. First, we give a precise definition of the notion of weak solution to the limiting equation (14).

Definition 4.1. $\zeta = \zeta(t, \theta)$ is called a weak solution of (14) on $[0, T] \times \mathbb{T}^1$ if

$$\zeta \in L^\infty(L^2), \quad \frac{\partial \zeta}{\partial t} \in L^2(H^{-1}), \quad \varphi'(\zeta) \in L^2(L^2), \quad (48)$$

and

$$\left\langle g, \frac{\partial \zeta}{\partial t} \right\rangle_{H^{-1}} = - \int_{\mathbb{T}^1} g \varphi'(\zeta) d\theta + \int_{\mathbb{T}^1} G \varphi'(\zeta) d\theta, \quad \text{for all } g \in L^2(\mathbb{T}^1), \quad \text{for almost every } t \in [0, T], \quad (49)$$

where G is the (unique up to a set of Lebesgue measure 0) function on the torus such that $\int_{\mathbb{T}^1} G d\theta = 0$ and $G' = g$.

As in Corollary 2.3, consider a sequence $\{M_\ell, N_\ell\}_{\ell=1}^\infty$ such that

$$M_\ell \uparrow \infty; \quad N_\ell \uparrow \infty; \quad \frac{N_\ell}{M_\ell} \uparrow \infty.$$

Let $\bar{\eta}_0^\ell$ be a step-function approximation of ζ_0 , such that

$$\|\bar{\eta}_0^\ell - \zeta_0\|_{L^2} \xrightarrow{\ell \uparrow \infty} 0. \quad (50)$$

Consider η^ℓ the solutions to

$$\frac{d\eta^\ell}{dt} = -(\bar{A} + \bar{J})\nabla_Y \bar{H}(\eta^\ell), \quad \eta^\ell(0) = \bar{\eta}_0^\ell.$$

To obtain Theorem 2.6 from Theorem 2.3, we shall need to study the convergence of the sequence η^ℓ . It is given by the following result.

Proposition 4.2. *With the notations above, the sequence of step functions $\bar{\eta}^\ell$ converge strongly in $L^\infty(H^{-1})$ to the unique weak solution of (14) with initial condition ζ_0 .*

The key estimate which will allow us to pass to the limit is the fact that, when N goes to infinity, the Euclidean product associated to A^{-1} behaves like the H^{-1} norm. This is the content of the following lemma :

Lemma 4.3. *There exists $C < +\infty$ such that, for any $x \in X$, if \bar{x} is the associated step function, then*

$$\frac{1}{C} \|\bar{x}\|_{H^{-1}}^2 \leq \frac{1}{N} \langle A^{-1}x, x \rangle \leq C \|\bar{x}\|_{H^{-1}}^2.$$

Moreover, if \bar{x} is bounded in L^2 , then

$$\left| \|\bar{x}\|_{H^{-1}}^2 - \frac{1}{N} \langle A^{-1}x, x \rangle \right| \leq \frac{C}{N}.$$

These estimates have been proven in section 6.3 of [2].

We delay the proof of Proposition 4.2, and first prove Theorem 2.6

Proof of Theorem 2.6. Our aim is to apply Corollary 2.3. To do this, we need to check that assumptions (i) to (x) hold with uniform constants. Assumptions (i) to (vii) have been checked in [2], and assumption (viii) in [1]. Assumption (x) can be immediately check by the direct

computation of JA and AJ . Finally, it is easy to see that for any $x \in X$, we have

$$\begin{aligned}
\langle -J^2x, x \rangle &= |Jx|^2 \\
&= \frac{N^2}{4} \sum_{i=1}^N (x_{i+1} - x_{i-1})^2 \\
&\leq \frac{N^2}{4} \sum_{i=1}^N 2(x_{i+1} - x_i)^2 + 2(x_i - x_{i-1})^2 \\
&= N^2 \sum_{i=1}^N (x_{i+1} - x_i)^2 \\
&= \langle Ax, x \rangle
\end{aligned} \tag{51}$$

and therefore assumption (ix) holds with $c = 1$.

Applying Corollary 2.3, we get

$$\lim_{\ell \uparrow \infty} \sup_{0 \leq t \leq T} \int \langle (x - NP^t \eta^\ell(t)), A^{-1}(x - NP^t \eta^\ell(t)) \rangle f(t, x) \mu(dx) = 0.$$

By Lemma 4.3, this implies

$$\lim_{\ell \uparrow \infty} \sup_{0 \leq t \leq T} \int \|\bar{x} - \bar{\eta}^\ell(t)\|_{H^{-1}}^2 f(t, x) \mu(dx) = 0.$$

Applying Proposition 4.2 and using the triangle inequality then concludes the proof. \square

We now turn to the proof of Proposition 4.2. It is based on the following six lemmas, and closely follows the method of [2], with additional arguments to take into account the extra first-order term.

Lemma 4.4. *Assume \bar{H} is convex. Then η satisfies (5) with initial condition $\eta(0) = \eta_0$ if and only if*

$$2 \int_0^T \bar{H}(\eta) \beta(t) dt \leq \int_0^T [\bar{H}(\eta + g) + \bar{H}(\eta - PA^{-1}NJP^t g)] \beta(t) dt - \int_0^T \langle g, (\bar{A})^{-1} \eta \rangle_Y \dot{\beta}(t) dt, \tag{52}$$

for all $g \in Y$ and smooth $\beta: [0, T] \rightarrow [0, \infty)$.

Similarly, assume that φ is convex. Then ζ satisfies (49) if and only if

$$\begin{aligned}
&2 \int_0^T \int_{\mathbb{T}^1} \varphi(\zeta(t, \theta)) \beta(t) d\theta dt \\
&\leq \int_0^T \int_{\mathbb{T}^1} [\varphi(\zeta(t, \theta) + g(\theta)) + \varphi(\zeta(t, \theta) - G(\theta))] \beta(t) d\theta dt - \int_0^T \langle g(\cdot), \zeta(t, \cdot) \rangle_{H^{-1}} \dot{\beta}(t) dt,
\end{aligned} \tag{53}$$

for all $g \in L^2 \mathbb{T}^1$ and smooth $\beta: [0, T] \rightarrow [0, \infty)$, where G is the (unique up to a set of Lebesgue measure 0) function on the torus such that $\int_{\mathbb{T}^1} G d\theta = 0$ and $G' = g$.

Proof. The proof of this Lemma is modified from that of Lemma 36 in [2]. We show that (5) is equivalent to (52). The equivalence of (49) and (53) follows analogously.

The weak form of (5) is given by

$$\int_0^T \langle g, (\bar{A})^{-1} \eta \rangle_Y \dot{\beta}(t) dt = \int_0^T [\langle g, \nabla_Y \bar{H}(\eta) \rangle_Y - \langle PA^{-1}NJP^t g, \nabla_Y \bar{H}(\eta) \rangle_Y] \beta(t) dt, \tag{54}$$

for all $g \in Y$ and smooth $\beta: [0, T] \rightarrow [0, \infty)$. We now show that (54) implies (52). Since \bar{H} is convex, we have

$$\begin{aligned} \langle g - PA^{-1}NJP^t g, \nabla_Y \bar{H}(\eta) \rangle_Y &\leq (\bar{H}(\eta + g) - \bar{H}(\eta)) + (\bar{H}(\eta - PA^{-1}NJP^t g) - \bar{H}(\eta)) \\ &= -2\bar{H}(\eta) + \bar{H}(\eta + g) + \bar{H}(\eta - PA^{-1}NJP^t g). \end{aligned} \quad (55)$$

Substituting (55) into (54), we obtain (52)

$$\begin{aligned} &\int_0^T \langle g, (\bar{A})^{-1}\eta \rangle_Y \dot{\beta}(t) dt \\ &\leq -2 \int_0^T \bar{H}(\eta) \beta(t) dt + \int_0^T [\bar{H}(\eta + g) + \bar{H}(\eta - PA^{-1}NJP^t g)] \beta(t) dt. \end{aligned} \quad (56)$$

Next we show (52) implies (54). Take $\tilde{g} = \varepsilon g$ in (54), for some $\varepsilon > 0$ and $g \in Y$, we get

$$\int_0^T \langle g, (\bar{A})^{-1}\eta \rangle_Y \dot{\beta}(t) dt \leq \int_0^T \left[\frac{\bar{H}(\eta + \varepsilon g) - \bar{H}(\eta)}{\varepsilon} + \frac{\bar{H}(\eta - \varepsilon PA^{-1}NJP^t g) - \bar{H}(\eta)}{\varepsilon} \right] \beta(t) dt.$$

By passing to the limit $\varepsilon \rightarrow 0$, we get

$$\int_0^T \langle g, (\bar{A})^{-1}\eta \rangle_Y \dot{\beta}(t) dt \leq \int_0^T \langle g - PA^{-1}NJP^t g, \nabla_Y \bar{H}(\eta) \rangle_Y \beta(t) dt.$$

Similarly now by taking $\tilde{g} = -\varepsilon g$, we obtain the opposite inequality

$$\int_0^T \langle g, (\bar{A})^{-1}\eta \rangle_Y \dot{\beta}(t) dt \geq \int_0^T \langle g - PA^{-1}NJP^t g, \nabla_Y \bar{H}(\eta) \rangle_Y \beta(t) dt.$$

Thus (54) is proven. \square

Lemma 4.5. *Let $\{\eta^\ell\}_{\ell=1}^\infty$ be a sequence of solutions of (5) with initial data η_0^ℓ satisfying $\|\bar{\eta}_0^\ell\|_{L^2} \leq C$. There exists a constant C independent of ℓ such that*

$$\int_0^T \left\langle \frac{d\eta^\ell}{dt}(t), (\bar{A})^{-1} \frac{d\eta^\ell}{dt}(t) \right\rangle dt \leq C, \quad (57)$$

$$\sup_{t \in [0, T]} \langle \eta^\ell(t), \eta^\ell(t) \rangle_Y \leq C. \quad (58)$$

As a consequence, there is a subsequence of the sequence of the associated step functions $\bar{\eta}^\ell$ and a function η_* such that

$$\bar{\eta}^\ell \rightharpoonup \eta_* \quad \text{weak-}^* \text{ in } L^\infty(L^2) = (L^1(L^2))^*.$$

Proof. According to proof of (31), we have

$$\bar{H}(\eta^\ell(t)) \leq e^{C(T+1)} \bar{H}(\eta_0^\ell) \quad \text{for all } t \in [0, T]. \quad (59)$$

Since \bar{H} is strictly convex, we obtain

$$\langle \eta^\ell(t), \eta^\ell(t) \rangle_Y \leq C(\bar{H}(\eta^\ell(t)) + 1) \leq Ce^{C(T+1)} \bar{H}(\eta_0^\ell) \leq C,$$

which is (58). Now we establish (57). From (5), we have

$$\begin{aligned} \langle \dot{\eta}^\ell(t), (\bar{A})^{-1}\dot{\eta}^\ell(t) \rangle &= \langle \bar{A}(I + PJA^{-1}NP^t) \nabla_Y \bar{H}(\eta^\ell(t)), (I + PJA^{-1}NP^t) \nabla_Y \bar{H}(\eta^\ell(t)) \rangle \\ &\leq 2(\langle \bar{A} \nabla_Y \bar{H}(\eta^\ell(t)), \nabla_Y \bar{H}(\eta^\ell(t)) \rangle + \langle \bar{A} PJA^{-1}NP^t \nabla_Y \bar{H}(\eta^\ell(t)), PJA^{-1}NP^t \nabla_Y \bar{H}(\eta^\ell(t)) \rangle) \\ &\stackrel{(27)}{\leq} 2(\langle \bar{A} \nabla_Y \bar{H}(\eta^\ell(t)), \nabla_Y \bar{H}(\eta^\ell(t)) \rangle + c|\nabla_Y \bar{H}(\eta^\ell(t))|^2) \\ &\stackrel{(iii)}{\leq} 2(\langle \bar{A} \nabla_Y \bar{H}(\eta^\ell(t)), \nabla_Y \bar{H}(\eta^\ell(t)) \rangle + C(\bar{H}(\eta^\ell(t)) + 1)) \end{aligned}$$

Therefore,

$$\int_0^T \langle \dot{\eta}^\ell(t), (\bar{A})^{-1} \dot{\eta}^\ell(t) \rangle dt \leq 2 \int_0^T (\langle \bar{A} \nabla_Y \bar{H}(\eta^\ell(t)), \nabla_Y \bar{H}(\eta^\ell(t)) \rangle + C(\bar{H}(\eta^\ell(t)) + 1)) dt$$

$$\stackrel{(31),(59)}{\leq} C,$$

which is (57). \square

Lemma 4.6. *Let $\{\eta^\ell\}_1^\infty$ be a sequence of solutions of (5) satisfying Lemma 4.5. We take any subsequence that the associated step functions weak-* convergence in $(L^1(L^2))^*$ to a limit η_* . Then on any bounded time interval, we have*

$$\eta_* \in L^\infty(L^2), \quad \frac{\partial \eta_*}{\partial t} \in L^2(H^{-1}), \quad \varphi'(\eta_*) \in L^2(L^2). \quad (60)$$

Proof. Having the estimate in Lemma 4.5, the proof of this Lemma is the same as that of Lemma 35 in [2]; hence we omit it here. \square

Lemma 4.7. *If $\bar{g}^\ell \rightarrow \bar{g}$ strongly in $H^{-1}(\mathbb{T})$, then $-\overline{PA^{-1}JNP^t g^\ell} \rightarrow G$ strongly in $L^2(\mathbb{T})$ where G is the primitive of \bar{g} .*

Proof. Set

$$D = N \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad (61)$$

then we can write

$$A = DD^T, \quad J = \frac{1}{2}(D^T - D).$$

Hence

$$JA^{-1} = \frac{1}{2}(D^T - D)(D^T D)^{-1} = \frac{1}{2}(D^T - D)D^{-1}(D^T)^{-1} = \frac{1}{2}(D^{-1} - (D^T)^{-1}). \quad (62)$$

The inverse of D and D^T can be computed explicitly

$$D^{-1} = \frac{1}{2N} \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & & & \\ & -1 & & & & \\ & & & & & 1 \end{pmatrix}, \quad (D^T)^{-1} = (D^{-1})^T.$$

So we obtain

$$D^{-1} - (D^T)^{-1} = \frac{1}{N} \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & & & \\ -1 & & & & & \\ & & & & & 0 \end{pmatrix}, \quad (63)$$

Let $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_M \end{pmatrix} \in Y = \mathbf{R}^M$ be given. We now compute $\overline{PA^{-1}JNP^t \xi}$ explicitly in three steps.

First, by definition of P^t , we have

$$NP^t\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_1 \\ \xi_2 \\ \vdots \\ \xi_2 \\ \vdots \\ \xi_M \\ \vdots \\ \xi_M \end{pmatrix} \in \mathbf{R}^N = \mathbf{R}^{KM}. \quad (64)$$

Second, from (62), (63) and (64), we have

$$A^{-1}JNP^t\xi = \frac{1}{2N} \begin{pmatrix} K(\xi_1 + \cdots + \xi_M) \\ (K-1)\xi_1 + K(\xi_2 + \cdots + \xi_M) \\ \vdots \\ K(\xi_2 + \cdots + \xi_M) \\ (K-1)\xi_2 + K(\xi_3 + \cdots + \xi_M) \\ \vdots \\ K(\xi_3 + \cdots + \xi_M) \\ \vdots \\ \xi_M \end{pmatrix} - \frac{1}{2N} \begin{pmatrix} \xi_1 \\ 2\xi_1 \\ \vdots \\ K\xi_1 \\ K\xi_1 + \xi_2 \\ \vdots \\ K(\xi_1 + \cdots + \xi_M) \end{pmatrix}.$$

Therefore, by definition of P ,

$$\begin{aligned} PA^{-1}JNP^t\xi &= \frac{1}{2M} \begin{pmatrix} \xi_1 + \xi_2 + \cdots + \xi_M \\ \xi_2 + \cdots + \xi_M \\ \vdots \\ \xi_M \end{pmatrix} - \frac{1}{2M} \begin{pmatrix} \xi_1 \\ \xi_1 + \xi_2 \\ \vdots \\ \xi_1 + \xi_2 + \cdots + \xi_M \end{pmatrix} \\ &= -\frac{1}{2M} \begin{pmatrix} \xi_1 \\ 2\xi_1 + \xi_2 \\ \vdots \\ 2(\xi_1 + \cdots + \xi_{M-1}) + \xi_M \end{pmatrix}. \end{aligned}$$

This implies that $-\overline{PA^{-1}JNP^t\xi} = \Upsilon_\xi$, where Υ_ξ is the primitive of $\bar{\xi}$. The assertion then follows since

$$(\bar{g}^\ell \rightarrow \bar{g} \text{ strongly in } H^{-1}(\mathbb{T})) \iff (\Upsilon_{g^\ell} \rightarrow \Upsilon_g \equiv G \text{ strongly in } L^2(\mathbb{T})).$$

□

Lemma 4.8. *Suppose that the sequence η^ℓ satisfies (57), (58) and (52), and consider a subsequence such that*

$$\bar{\eta}^\ell \rightharpoonup \eta_* \text{ weak-}^* \text{ in } L^\infty(L^2) = (L^1(L^2))^*.$$

holds. Let $\xi^\ell = \pi_\ell(\xi + \eta^\ell) - \eta^\ell$, where ξ is an arbitrary L^2 function and π_ℓ is the L^2 -projection onto elements of Y . Let Ξ be the primitive with average 0 of ξ . Then we have

(i)

$$\liminf_\ell \int_0^T \bar{H}(\eta^\ell(t))\beta(t)dt \geq \int_0^T \int_{\mathbb{T}} \varphi(\eta_*(t, \theta))\beta(t)d\theta dt;$$

(ii)

$$\lim_{\ell} \int_0^T \bar{H}(\eta^\ell(t) + \xi^\ell(t))\beta(t)dt = \int_0^T \int_{\mathbb{T}} \varphi(\eta_*(t, \theta) + \xi(\theta))\beta(t)d\theta dt;$$

(iii)

$$\lim_{\ell} \int_0^T \bar{H}(\eta^\ell(t) - PA^{-1}JNP^t\xi^\ell(t))\beta(t)dt = \int_0^T \int_{\mathbb{T}} \varphi(\eta_*(t, \theta) - \Xi(\theta))\beta(t)d\theta dt;$$

(iv)

$$\lim_{\ell} \int_0^T \langle \xi^\ell(t), \bar{A}^{-1}\eta^\ell(t) \rangle_Y \dot{\beta}(t)dt = \int_0^T \langle \xi(\theta), \eta_*(t, \theta) \rangle_{H^{-1}} \dot{\beta}(t)dt.$$

Proof. (i), (ii) and (iv) have already been proven in Lemma 37 of [2], so we only have to prove (iii).

Since η^ℓ converges to η_* and $PA^{-1}JNP^t\xi^\ell(t)$ converges to Ξ , by weak lower-semi continuity and the uniform convergence of ψ_K to φ we immediately get

$$\liminf_{\ell} \int_0^T \bar{H}(\eta^\ell(t) - PA^{-1}JNP^t\xi^\ell(t))\beta(t)dt \geq \int_0^T \int_{\mathbb{T}} \varphi(\eta_*(t, \theta) - \Xi(\theta))\beta(t)d\theta dt$$

so we only need to prove the associated upper bound. Let $g^\ell(t)$ be a sequence of elements of Y such that \bar{g}^ℓ strongly converges in $L^\infty(L^2)$ to $\eta_* - \Xi$. Since we then have

$$\int_0^T \bar{H}(g^\ell(t))\beta(t)dt \longrightarrow \int_0^T \int_{\mathbb{T}} \varphi(\eta_*(t, \theta) - \Xi(\theta))\beta(t)d\theta dt$$

we only need to show that

$$\limsup_{\ell} \int_0^T \bar{H}(\eta^\ell(t) - PA^{-1}JNP^t\xi^\ell(t))\beta(t)dt - \int_0^T \bar{H}(g^\ell(t))\beta(t)dt \leq 0.$$

Let A_M be the discrete Laplacian with scaling factor M^2 on Y . Since ψ_K is convex, we have

$$\begin{aligned} \bar{H}(\eta^\ell(t) - PA^{-1}JNP^t\xi^\ell(t)) - \bar{H}(g^\ell(t)) &= \frac{1}{M} \sum_{i=1}^M \psi_K(\eta_i^\ell - (PA^{-1}JNP^t\xi^\ell)_i) - \psi_K(g_i^\ell) \\ &\leq \frac{1}{M} \sum_{i=1}^M \psi'_K(\eta_i^\ell - (PA^{-1}JNP^t\xi^\ell)_i)(\eta_i^\ell - (PA^{-1}JNP^t\xi^\ell)_i - g_i^\ell) \\ &= \langle \nabla \bar{H}(\eta^\ell - PA^{-1}JNP^t\xi^\ell), (\eta^\ell - PA^{-1}JNP^t\xi^\ell - g^\ell) \rangle_Y \\ &\leq \langle A_M \nabla \bar{H}(\eta^\ell - PA^{-1}NP^t\xi^\ell), \nabla \bar{H}(\eta^\ell - PA^{-1}NP^t\xi^\ell) \rangle_Y^{1/2} \\ &\quad \times \langle A_M^{-1}(\eta^\ell - PA^{-1}NP^t\xi^\ell - g^\ell), (\eta^\ell - PA^{-1}NP^t\xi^\ell - g^\ell) \rangle_Y^{1/2} \end{aligned}$$

Since $\langle A_M^{-1}\cdot, \cdot \rangle_Y$ behaves like the H^{-1} norm, the fact that $\eta^\ell - PA^{-1}NP^t\xi^\ell$ and g^ℓ converge to the same limit in $L^\infty(H^{-1})$ implies that

$$\langle A_M^{-1}(\eta^\ell - PA^{-1}NP^t\xi^\ell - g^\ell), (\eta^\ell - PA^{-1}NP^t\xi^\ell - g^\ell) \rangle_Y \longrightarrow 0,$$

and therefore it will be enough to show that

$$\int_0^T \langle A_M \nabla \bar{H}(\eta^\ell - PA^{-1}NP^t\xi^\ell), \nabla \bar{H}(\eta^\ell - PA^{-1}NP^t\xi^\ell) \rangle_Y dt < C.$$

Since under our assumptions ψ'_K is bi-Lipschitz, we have

$$\begin{aligned}
& \langle A_M \nabla \bar{H}(\eta^\ell - PA^{-1}NP^t\xi^\ell), \nabla \bar{H}(\eta^\ell - PA^{-1}NP^t\xi^\ell) \rangle_Y \\
&= M \sum_{i=1}^M (\psi'_K(\eta_{i+1}^\ell - (PA^{-1}NP^t\xi^\ell)_{i+1}) - \psi'_K(\eta_i^\ell - (PA^{-1}NP^t\xi^\ell)_i))^2 \\
&\leq CM \sum_{i=1}^M (\eta_{i+1}^\ell - (PA^{-1}NP^t\xi^\ell)_{i+1} - (\eta_i^\ell - (PA^{-1}NP^t\xi^\ell)_i))^2 \\
&\leq CM \sum_{i=1}^M (\eta_{i+1}^\ell - \eta_i^\ell)^2 + ((PA^{-1}NP^t\xi^\ell)_{i+1} - (PA^{-1}NP^t\xi^\ell)_i)^2 \\
&\leq CM \sum_{i=1}^M (\psi'_K(\eta_{i+1}^\ell) - \psi'_K(\eta_i^\ell))^2 + \frac{C}{M} \sum_{i=1}^M (\xi_{i+1}^\ell - \xi_i^\ell)^2 \\
&\leq C \langle A_M \nabla \bar{H}(\eta^\ell), \bar{H}(\eta^\ell) \rangle + C \|\bar{\xi}^\ell\|_{L^2}^2.
\end{aligned}$$

Since $\bar{\xi}^\ell$ converges in L^2 , $\|\bar{\xi}^\ell\|_{L^2}^2$ is bounded. To conclude, we then only require (57) and the fact that

$$\langle A_M y, y \rangle \leq C \langle \bar{A} y, y \rangle \quad \forall y \in Y. \quad (65)$$

This statement is equivalent to bounding from below A_M^{-1} by \bar{A}^{-1} . This does hold, since we have

$$\begin{aligned}
\langle \bar{A}^{-1} y, y \rangle_Y &= \frac{1}{N} \langle A^{-1} NP^t y, NP^t y \rangle_X \\
&\leq C \|NP^t y\|_{H^{-1}}^2 \\
&\leq C \|\bar{y}\|_{H^{-1}}^2 \\
&\leq C \langle A_M^{-1} y, y \rangle_Y
\end{aligned}$$

which concludes the proof. \square

Finally, we need to prove uniqueness of solutions to the limiting PDE :

Lemma 4.9. *Given an initial condition ζ_0 , there is at most one solution to (14).*

Proof. Let ζ_1 and ζ_2 be two solutions of (14) with same initial condition. Let $F(t) := 2^{-1} \|\zeta_1(t, \cdot) - \zeta_2(t, \cdot)\|_{H^{-1}}^2$, and let g_1 and g_2 be mean-zero primitives (in space) of ζ_1 and ζ_2 . Then, for any $\lambda > 0$,

$$\begin{aligned}
F'(t) &= - \int_{\mathbb{T}} (\varphi'(\zeta_1) - \varphi'(\zeta_2)) (\zeta_1 - \zeta_2) d\theta + \int_{\mathbb{T}} (\varphi'(\zeta_1) - \varphi'(\zeta_2)) (g_1 - g_2) d\theta \\
&\leq - \frac{\inf \varphi''}{2} \int_{\mathbb{T}} (\zeta_1 - \zeta_2)^2 d\theta + \frac{\lambda}{2} \int_{\mathbb{T}} (\varphi'(\zeta_1) - \varphi'(\zeta_2))^2 d\theta \\
&\quad + \frac{1}{2\lambda} \int_{\mathbb{T}} (g_1 - g_2)^2 d\theta \\
&\leq - \frac{\inf \varphi''}{2} \int_{\mathbb{T}} (\zeta_1 - \zeta_2)^2 d\theta + \frac{\lambda \sup \varphi''}{2} \int_{\mathbb{T}} (\zeta_1 - \zeta_2)^2 d\theta + \frac{1}{\lambda} F(t)
\end{aligned}$$

Taking $\lambda = \frac{\inf \varphi''}{\sup \varphi''}$, we obtain a differential inequality which, by Gronwall's lemma, implies that $\zeta_1 = \zeta_2$. \square

We can now prove Proposition 4.2:

Proof of Proposition 4.2. According to Lemma 4.5, we can consider a subsequence such that

$$\bar{\eta}^\ell \rightharpoonup \eta_* \quad \text{weak-}^* \text{ in } L^\infty(L^2) = (L^1(L^2))^*.$$

and strongly in $L^\infty(H^{-1})$. By Lemma 4.6, η_* satisfies (48). According to Lemma 4.4, η^ℓ satisfies (52). Passing to the limit using Lemma 4.8, we see that η_* satisfies (53), and therefore is a weak solution of (14).

Since Lemma 4.9 guarantees uniqueness of the weak solution, the full sequence $(\eta^\ell)_\ell$ converges to the unique weak solution of (14). □

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