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Flame dominated thermoacoustic instabilities in a system with high acoustic losses

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\textbf{A B S T R A C T}

The thermoacoustic oscillatory behaviour of a flame is experimentally investigated in the presence of large acoustic losses. Recently it has become clear that under such conditions an instability can occur due to an intrinsic local feedback loop at the heat source. The experimental results confirm that despite significant acoustic losses, thermoacoustic instabilities can still be present. These findings imply that the effectiveness of passive thermoacoustic damping devices is limited by the intrinsic stability properties of the flame.

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1. Introduction

The performance of combustion devices is often limited by the occurrence of thermoacoustic instabilities. Such instabilities are induced by a strong coupling between the acoustic field and the flames in a combustor, and may generate unacceptable noise levels or even lead to structural damage [1].

It is generally accepted that thermoacoustic instabilities occur due to the coupling between heat release fluctuations and the acoustic response of the combustion chamber. When the heat release and pressure fluctuations are in phase [2], the flame acts as an amplifier of acoustic waves, which are mostly reflected back towards the flame at the boundaries of the system. Therefore, an unstable feedbackloop can appear where the frequency of the oscillation is typically determined by one of the acoustic modes of the overall system.

Based on the foregoing interpretation, it should always be possible to prevent the unstable interaction by introducing anechoic, i.e. non-reflective, boundary conditions. In this case, from an energy viewpoint, the flame cannot overcome the damping induced by the acoustic losses.

In contrast, it was recently shown theoretically that the flame may still be unstable even under anechoic conditions [3,4]. The cause for this is that the flame can possess ‘intrinsic’ thermoacoustic modes. Within this theoretical framework, the flame element contains its own localized feedbackloop, which dominates the flame behaviour under anechoic boundary conditions. It was also shown [3], that for particular systems with only one sided anechoic conditions an unstable system mode can directly arise from such an intrinsic unstable mode of the flame. Therefore, the study of systems with low, but not zero, acoustic reflections allows to reveal the presence of the intrinsic flame instability.

Here, it should be stressed that the found intrinsic instability is of a completely different nature than the extensively studied hydrodynamic and diffusive-thermal instabilities reported in for example [5].

Recently, the intrinsic thermoacoustic instability has also been shown in the setting of direct numerical simulations of a laminar premixed flame [6,7]. On the other hand, in case the flame is intrinsically stable, it is possible to guarantee general thermoacoustic stability when a minimum level of acoustic losses can be achieved [8]. Nevertheless, there have only been few reported experiments of thermoacoustic behaviour under fully or partially anechoic conditions. In some studies using unconfined flames, i.e. flames freely radiating to the environment, the downstream conditions can be considered as anechoic [9–11]. Another notable exception is the studies reported in [12] where the reflection coefficient was actively controlled by means of an active control loop. However, in these studies the experimentally measured instabilities could be explained in the classic sense; the positive feedback of the flame causes destabilization of the acoustic mode. For example, the instability frequencies reported in [9] match the theoretical first, second

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and third, quarter wave modes of the closed-open ended duct configuration.

The goal of this work is to experimentally evaluate the thermoacoustic stability of a flame under partially anechoic conditions, and highlight the link to the intrinsic flame modes. To this end, a dedicated experiment is designed and conducted, where the flame/burner is embedded in an anechoic configuration with upstream anechoic conditions. Because the downstream boundary conditions are not anechoic, the experimental results alone are not sufficient to unequivocally prove the presence of the flame mode from the experiment alone. Nevertheless, careful consideration of both the theoretical predictions as well as the theoretical findings provides convincing clues that the experimental instabilities are indeed caused by the presence of the intrinsic flame mode.

2. Theoretical background

In the one-dimensional modelling of thermoacoustic instabilities, the velocity sensitivity flame is usually modelled by a two-step approach. First, the thermal response of the flame is characterized by a flame transfer function, relating the heat release rate \( Q \) [W] to acoustic velocity excitation \( u \) [m/s],

\[
\mathcal{F}(\omega) = \frac{Q}{\frac{\partial Q}{\partial u}}, \tag{1}
\]

where \( \mathcal{F} \) and \( \mathcal{R} \) denote the fluctuating and mean parts respectively and \( \omega = \omega_0 + i\omega_1 \) the complex frequency variable. In the second step, the coupling between heat release and acoustic variables is modelled by the Rankine-Hugoniot jump conditions \([13,14]\). Overall, this leads to the definition of a flame scattering matrix, modelling the acoustic response of the flame. It was recently revealed that under certain conditions on the flame transfer function the flame scattering matrix may be intrinsically unstable due to the local feedback between acoustic velocity and heat release. In particular, the intrinsic poles are given by the solution of,

\[
\mathcal{F}(\omega) = -\frac{\epsilon + 1}{\theta}, \tag{2}
\]

where \( \epsilon = \frac{p_{ac}}{\rho \alpha_c} \) is the ratio of specific acoustic impedances, and \( \theta = \frac{\rho}{\rho_v} - 1 \) is proportional to the temperature ratio. Here, the term \( \epsilon \) and \( \theta \) refers to the complex frequency values at which the response of the scattering matrix elements tend to infinity, as is often encountered in control and system theory, see for example \([15]\). If any of the solutions, the intrinsic poles, of Eq. (2) have an imaginary part \( \omega_1 < 0 \) the flame is intrinsically unstable. And, as a consequence, thermoacoustic instabilities may still occur even in anechoic conditions \([3,4]\).

Thus, in case of an experiment in (nearly) anechoic conditions, one should examine Eq. (2) for clues about the expected behaviour. Since the right hand side is for a fixed temperature ratio merely a constant, a variation in frequency and stability of the poles will typically be caused by variations in the flame transfer function. Figure 1 depicts the general relation between intrinsic mode frequency, stability and the flame frequency response \( \mathcal{F}(\omega) \). Because any solution will have to satisfy \( \mathcal{F}(\omega) = -\pi \), the real part of the pole is, in approximation, determined by the \( -\pi \) crossing of the phase of the flame frequency response \( \mathcal{F}(\omega_0) \). The stability on the other hand, is determined by the gain at the frequency of the crossing. In general, the pole will be unstable in case the gain is larger than \( \frac{\epsilon + 1}{\theta} \).

Therefore, given a sufficiently high gain, a variation in the phase characteristic of the flame transfer function will lead to a variation of the frequency of the expected intrinsic poles, and by extension the frequency of thermoacoustic oscillations in anechoic conditions. This expectation forms the target of our experiments. It is well known that for many flames the phase decays linearly due to the time delay nature of the response. As a consequence, any parameter which affects the time delay can be considered an effective method to vary the flame intrinsic poles.

2.1. Partly anechoic conditions

In the current case of a partial anechoic setup, any unstable mode is invariably a result of the complete system, i.e. the coupling between acoustics and the flame, and not only of the flame. In essence, adding acoustic reflection, either up- or downstream, will start to move the system poles from the flame-intrinsic modes (open-loop poles) towards a new location \([3,16]\). As such, both the reflection coefficient and the duct length on the non-anaechoic side of the flame become relevant parameters. The extend of the perturbation then depends on the exact choice of length of the duct and reflection coefficient. As such, some downstream lengths are better suited to closely match the behaviour of the intrinsic flame modes than others. An optimal duct-length will hardly shift the stability and frequency of the intrinsic mode, and cause the system mode to be very closely matched to the intrinsic mode. It is important to realize however, that in the absence of an (unstable) flame-intrinsic mode around this frequency, there would be no instability observed at that particular parameter combination.

3. Experiment design

In a practical setting it is clear that completely anechoic conditions are extremely difficult, if not impossible, to realize. On the downstream side of the flame for example, the hot combustion gases introduce temperature gradients, which can lead to additional acoustic reflection. In the current contribution we therefore focus on a configuration where only the upstream reflection coefficient is minimized by adding an acoustic horn, see Fig. 2. In the following, it is shown that even under such partly anechoic conditions, Eq. (2) still captures the most important trends.

The overall geometry then is made out of the upstream horn, an inlet for methane, a fan, the burner, and a downstream quartz tube.

In general, one can then recognize three important considerations for the current experimental setup (i) choice of the downstream length, (ii) the horn design, (iii) the choice of the flame and burner.
However, in practice the risk for locking of the frequency to a tube mode increases for increasing length. Indeed, in experiments with longer tubes, we found that the system tended to lock-on to a certain frequency of oscillation more often. The chosen length of $L_2 = 0.27$ m and diameter of 6 cm provided a good comprise, minimizing the influence of the reflection on stability and frequency compared to the intrinsic flame mode. For more results regarding different downstream lengths see [17].

4.1. Horn design

An appropriately designed acoustic horn is a very efficient way to obtain a low in-duct reflection coefficient. The complete derivation of the reflection coefficient of the throat of the horn is beyond the scope of this paper. Good introductions to this subject can be found in [18–20]. For the current purpose a horn with a sufficiently low cut-off frequency, the frequency below which the horn is not effective, is required. In this case, the cut-off frequency is chosen to be $f_c = 150$ Hz which leads to a horn length of 1.23 m and a horn mouth radius of 0.36 m. The frequency dependent reflection coefficient is measured by means of a multi-microphone setup, see for example [21]. More specifically, the setup consists out of an aluminium tube of 1 m long, 50 mm inner diameter, and 2 cm wall thickness. The tube is equipped with six BSWA MPA416 microphones at an equal distance of 175 mm. The measured reflection coefficient of the horn along with the theoretical prediction is depicted in Fig. 3. Note that the cut-off frequency $f_c$ is easily recognized. The question if the measured horn reflection is low enough to uncover the scattering matrix poles is answered in Appendix B.

4.2. Choice of burner

Note that the overall goal of the current work is to extend the existing knowledge about thermoacoustic instabilities in reflecting conditions towards that of the non-reflecting regime. As such, it is preferable to choose a burner and flame configuration whose flame transfer function is well known. A perforated burnerdeck on which small premixed Bunsen type flames are stabilized is therefore a good candidate [22–25]. For this type of burner and flames, the amplitude starts at 1 and decreases to zero after a higher gain in the mid-frequency range (100–300 Hz). The phase on the other hand tends to decrease linearly, as the result of the typical time

4. Downstream length

The purpose of quartz tube in the downstream section is twofold: (1) to choose a length which limits the influence of the downstream reflection on the perturbation of the intrinsic flame mode, see Section 2.1, and (2) make the acoustic behaviour of this part well defined. Thus, the final selection of the downstream length is born out of compromise. On the one hand, a shorter tube allows eliminating the possibility of any frequency matching of the tube modes and the expected frequency of intrinsic instability. On the other hand, a longer tube guarantees plane acoustic waves, and validity of the theory, and the establishment of a more uniform temperature profile.
delay between the acoustic velocity excitation and the heat release variation. The time delay is mainly determined by the ratio of flame height to gas velocity \( \bar{V} \). As a consequence, the time delay increases with hole diameter, and for lean flames decreases with increasing equivalence ratio. The latter property is crucial in the design of the measurement. In addition, the transfer function also depends on burnerdeck temperature and pitch, amongst other parameters [25].

With the foregoing behaviour in mind, one can expect that for a fixed burner geometry the frequency of the intrinsic modes will be determined by the equivalence ratio, i.e. time delay, whereas the stability will be mainly influenced by the gain. Thus, in order to assess the frequency dependence of the modes with a variation of the time delay, the gain should be sufficiently high for a wide range of equivalence ratios. When the considered burner has the desired behaviour, the stability/instability and the frequency variation with changing equivalence ratio can serve as the main parameters of interest.

Given the requirements described above, the particular burner of choice is made of a 1 mm thick brass plate of which the central 5 cm diameter contains a hexagonal pattern of holes. The holes have a diameter of 2 mm placed at a pitch of 4.5 mm (D2-P4.5) and the total open area of the burner is 342.43 mm². The burnerdeck is mounted in a cooled deck holder, which keeps the outer perimeter of the burnerdeck at a fixed temperature of 30 °C. The flame transfer function for two different equivalence ratios is shown in Fig. 4. Figure 5 provides an impression of the burner under operation. Note the quartz ring placed on the burner deck. The purpose of this ring is to properly condition the flames on the circumference of the burner which, if this ring would not be present, would detach from the burner deck.

4.3 Measurement equipment

In order to supply the methane air mixture to the system, a fan was installed 0.055 m below the burnerdeck, thereby also ensuring proper mixing. From an acoustical viewpoint, the area change as a result of the central motor of the fan is about 33%, but the effective length is small, only 10 mm. Note that the blades are generally too thin to contribute to the area change (0.5 mm). Therefore the fan does not introduce any significant reflections in the frequency range of interest.

The methane flow rate is controlled by a von Bronckhorst mass flow controller. Under some operating conditions, a certain amount of air is premixed into the methane supply line such that the fan could be operated in the correct voltage range. Note that a variation of fan voltage leads to a smooth variation of equivalence ratio, since the air flow is varied for a fixed fuel flow rate. The equivalence ratio is calculated from the CO₂ and O₂ concentration in the flue gas as measured by a gas analyser connected to the downstream tube.

In order to perform further detailed measurements, a constant temperature anemometer (CTA), measuring (acoustic) velocity fluctuations \( u' \), and a pressure transducer for pressure \( p' \) are mounted 0.025 and 0.115 m upstream of the burnerdeck. In addition, a photomultiplier tube (PMT) with OH\(^*\) filter is used as a measure for the varying heat release signal \( Q' \). This instrumentation is added to assess the phase differences between the velocity, pressure and heat release signals. Furthermore, the temperature of the burnerdeck is measured by an N-Type thermocouple, while the flue gas temperature at the duct end is measured by an S-type thermocouple. All relevant data is digitized at a sampling rate of \( f_s = 10 \) kHz by a set of National instruments DAQ systems connected to a PC running Labview.

Note that flame transfer functions are needed to compare the modelling expectations to the experimental results. For this purpose, the upstream horn was substituted by a fully premixed supply tube, connected to a loudspeaker, and a set of mass flow controllers. The CTA and PMT are used to provide the necessary signals to calculate the flame transfer function. The transfer function measurements are performed at the same burner operating points as the instabilities are measured. In order to determine the solutions of Eq. (2), one needs to apply a system identification procedure on the measured data, see Appendix A. In short, the procedure consists out of fitting a rational transfer function, i.e. a model, on the measured frequency response. Because such model can be evaluated in the complete complex domain, a Nelder–Mead root finding algorithm can be used to solve Eq. (2) [26]. Note however, that the validity of the fits can be expected to decrease with increasing distance to the real \(\omega_r\) axis.

4.4 Measurement procedure

As noted before, for the selected burner type, the equivalence ratio is one of the most effective parameters to control the frequency of the intrinsic mode. Therefore, the basic experiment consists of fixing the thermal power, by fixing the flow rate of methane, and varying the fan voltage. In case any instability is observed the \( Q', u' \) and \( p' \) signals are then recorded.

5. Results

Figure 6a shows the measured instability frequencies (○) and the predicted intrinsic mode frequencies (×) as function of the equivalence ratio for the D2-P4.5 case at a methane flow of
2.1 ln/min (1.39 kW). The frequency range varies smoothly between 200 and 380 Hz as the equivalence ratio increases. This is remarkable, as it indicates that the instability does not lock on to a fixed acoustic mode.

For the sake of argument, let us make a first approximation of an acoustic resonance. First, assume that the flame acts as an closed-ended boundary condition, meaning that a quarter wave mode (closed-open) is the most likely acoustic mode. In addition, this will yield the lowest eigen frequency to compare to the experiment using simple boundary conditions. Assuming an average temperature half of the adiabatic flame temperature at the lowest $\Phi, T = 837$ K, one obtains a speed of sound of $c \approx 570$ m/s. This would yield an approximate first quarter wave mode at 526 Hz, which clearly is more than double than what is measured at that equivalence ratio: 215 Hz. Therefore it is unlikely that the results can be explained on the basis of an acoustic resonance only.

Returning to the results shown in Fig. 6, this is exactly the behaviour which can be expected from the presence of an intrinsic flame mode, for which the oscillating frequency is directly linked to the equivalence ratio through the time delay. In fact, the predicted intrinsic mode frequencies ($\times$) match extremely well to the measured frequencies.

Stability-wise, the flame intrinsic modes are expected to be unstable over the full $\Phi$ range. Indeed, the experimental results confirm this to be the case. Note that apparently the downstream reflections do not alter the system mode frequency significantly compared to the intrinsic mode frequency.

Figure 6b depicts the corresponding phase relations between heat release $Q'$, velocity $u'$, and pressure $p'$ versus equivalence ratio. An example of the underlying data for the measured frequency of 382 Hz at $\Phi = 0.92$ is shown in Fig. 7. Note that the $u'$ signal as measured by the CTA is not perfectly sinusoidal due to high acoustic velocity amplitude in the limit cycle oscillation. Furthermore, the indicated phases between pressure and velocity are corrected for the distance between two. As expected for an intrinsic mode, the velocity and heat release signals are almost perfectly out of phase. Of course, given the frequency match between the measured ($\circ$), and intrinsic flame frequencies ($\times$) shown in Fig. 6a, this should be the case. In addition, the velocity and pressure upstream of the flame are also out of phase, indicating a purely travelling wave. First of all, this indicates that the horn is indeed acting as a nearly anechoic termination. Second, note that acoustic pressure and heat release are also in phase, which, according to the Rayleigh criterion, indicates that the heat-release is acting as an efficient acoustic energy source. As Fig. 6b shows, the phase relations for the other measured unstable frequencies are nearly identical. These findings are completely in-line with the expectation of the flame as an intrinsically unstable, sound producing, element.

In summary, the presented results agree qualitatively as well as quantitatively with the theoretical predictions based on Eq. (2).

6. Discussion and conclusions

The current contribution considers a fundamental aspect of the flame acoustic coupling. In particular, we have shown that even in the presence of large acoustic losses, a thermoacoustic system may exhibit instabilities which cannot be explained as system acoustic modes. The reason for this is the fact that according to the theory, the flame-acoustic coupling itself can contain intrinsically unstable modes [3,14]. This is especially evident from the stability properties of the burner scattering matrix. As the acoustic reflections decrease, the eigenfrequencies of the system become fully determined by the flame intrinsic modes. From the viewpoint of the thermoacoustic feedback loop, the decrease of up- and downstream reflections from the flame effectively puts the flame in open-loop,
thereby revealing the intrinsic thermoacoustic stability properties of the flame.

Notwithstanding the encouraging experimental results, one should be aware of practical limitations of an experimental setting. In particular, it would be desirable to also reduce the downstream reflection coefficient in a verifiable manner. In practice, this represents a difficult task because of the hot downstream flue gas and associated temperature gradients. Therefore, in the light of showing the presence of intrinsic flame modes, direct numerical simulations may be better suited. Recently, such direct numerical simulations of a flame under anechoic conditions are indeed performed, and confirm that under vanishing acoustic reflections the flame may still oscillate. Furthermore, it is shown that even in the case of partial downstream reflections, as considered in this paper, the intrinsic mode may dominate the system oscillation frequency [6,7].

Another issue is created by the fact that the current theoretical predictions are based on a linear theory, while in practice the flame oscillation is nonlinearly saturated. Although a number of studies highlight that the transfer function phase and gain are highly amplitude dependent [27]. In our previous studies [25] we have established that for this type of burner and excitation level this is not really the case. Indeed, in the experiments the frequency of the instability onset and main frequency of limit cycle are found to be very close.

Here it is appropriate to mention that measurement data for other burners is described in [17]. The results agree qualitatively to the predictions of Eq. (2).

From a thermoacoustic modelling viewpoint, the intrinsic instability notion does not invalidate the common network analysis of thermo-acoustic instability, in fact the mentioned instability has always been hidden inside it and completes the interpretation of the system analysis [3,4]. As a result, it can shed new light on previously unexplainable or not fully understood results from network codes.

Overall, the notion of a possible flame intrinsic instability of thermoacoustic nature raises many new questions in the field of thermoacoustic instabilities.

Acknowledgments

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Appendix A

Given the measured frequency dependent reflection coefficient of the horn, it is interesting to evaluate its influence on the system modes. Figure 8 depicts the computed complex eigenvalues of the system with an upstream horn, \( L_2 = 0.27 \) m and the D2-P4.5 burner at \( \Phi = 0.99 \) and 1.39 kW. As shown, there is a general trend of increasingly positive \( \omega_r \) for increasing frequency, except the point at 453 Hz, which is a result of the intrinsic mode. This is exactly as expected, since the horn will cause an increase in damping due to the decrease of reflection coefficient with frequency. In addition, the multitude of eigenvalues is a result of the total effective length of the setup, which is the sum of the horn length, upstream duct, and downstream duct. Note that by coincidence this leads to a stable acoustic mode at the same frequency as the unstable intrinsic flame mode at 454 Hz. As the figure demonstrates, the horn is well suited to create sufficient acoustic losses in order to uncover the flame intrinsic modes.

Appendix B

In order to directly calculate the poles of the system and scattering matrices according to Eq. (2), information about the flame transfer function in the complete complex domain is required. However, the flame frequency response is usually only measured at a finite set \( \Omega' \), \( k = 1, \ldots, K \) of real frequencies, leading to \( \mathcal{F}_m(\Omega'_{k}) \). To solve this problem, a model based on the frequency response data should be identified first. Here, we consider models from the family of rational functions of the form,

\[
\mathcal{F}(\omega, a, b) = \frac{\sum_{n=0}^{N} a_n (i \omega)^n}{\sum_{n=0}^{Z} b_n (i \omega)^n},
\]

where \( N \) and \( Z \) are respectively the order of the denominator and numerator, and \( a_n \) and \( b_n \) are the unknown coefficients to be fitted. For a measured frequency response \( \mathcal{F}_m(\Omega'_{k}) \), the solution of the fitting procedure is then given by

\[
\arg \min_{a, b} \sum_{k=1}^{K} |\mathcal{F}_m(\Omega'_{k}) - \mathcal{F}(\Omega'_{k}, a, b)|^2.
\]

Note that given a certain measured frequency response \( \mathcal{F}_m(\Omega'_{k}) \) one first has to make a choice with respect to the orders \( N \) and \( Z \).
before one can proceed with the fitting procedure. Because measured flame transfer functions are always (semi-)proper, one normally considers only functions with $N \leq Z$. In addition, due to the time delay nature of the transfer function phase, the order of the fit is mainly determined by the magnitude of the delay. It was found that for typical flame transfer functions used here, $N = 7$, $Z = 8$ provides good results. Figure 9 depicts the resulting pole locations in the complex plane. Table 1 provides an overview of the exact fitting coefficients.

### Appendix C

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