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Citation for published version (APA):
Cao, X., & Pop, I. S. (2014). *Two-phase porous media flows with dynamic capillary effects and hysteresis : uniqueness of weak solutions*. (CASA-report; Vol. 1428). Technische Universiteit Eindhoven.

Document status and date:
Published: 01/01/2014

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

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Download date: 31. Mar. 2023

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computer Science

CASA-Report 14-28
September 2014

Two-phase porous media flows with dynamic capillary effects and hysteresis: uniqueness of weak solutions

by

X. Cao, I.S. Pop

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

Two-phase porous media flows with dynamic capillary effects and hysteresis: uniqueness of weak solutions

X. Cao[1], **I.S. Pop**[1,2]

[1] Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, The Netherlands

[2] Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Bergen, Norway

Abstract. In this paper, we obtain the uniqueness of weak solutions for a two phase flow model in a porous medium. A particularity of the model is that the dynamic effects and hysteresis are included in the capillary pressure.
Keywords: Dynamic capillary pressure, two-phase flow, hysteresis, weak solution, uniqueness.

1 Introduction

We consider a mathematical model for two-phase flow in a porous medium. Two immiscible fluid phases are flowing through a porous medium occupying a bounded, connected domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$. Using $\bar{\Omega}$ and $\partial\Omega$ denote the closure and boundary of Ω. Let $T > 0$ be a given time. The phase pressures are denoted by p_w, p_n. The non-wetting phase saturation is s. We assume the porous medium is saturated by the two phases. Then from the Darcy law and mass conservation for each fluid give the system (see [1, 16])

$$\partial_t s - \nabla \cdot (k_n(s)\nabla p_n) - \nabla \cdot (k_n(s)\overrightarrow{g}) = 0, \tag{1}$$

$$-\partial_t s - \nabla \cdot (k_w(s)\nabla p_w) - \nabla \cdot (k_w(s)\overrightarrow{g}) = 0. \tag{2}$$

Here $\overrightarrow{g} \in \mathbb{R}^d$ is the gravity vector in direction $-\overrightarrow{e_d} = (0, ..., 0, -1) \in \mathbb{R}^d$. $k_n(s), k_w(s)$ are the permeabilities - two nonlinear functions depending on s. The system is closed by the relation between the phase pressures and saturation. Standardly, equilibrium models assume $p_w - p_n = p_c(s)$, p_c - decreasing with respect to s (see [12]). While experiments [3, 7, 17] have proved the limitation of this approach. Alternatively, models involving non-equilibrium effects are proposed in [2]:

$$p_n - p_w \in p_c(s) + \gamma(x)\text{sign}(\partial_t s) + \tau\partial_t s. \tag{3}$$

Here $\gamma \geq 0$, $\tau > 0$ are given and sign denotes the multi-valued function

$$\operatorname{sign}(\xi) = \begin{cases} 1 & \text{if} \quad \xi > 0, \\ -1 & \text{if} \quad \xi < 0, \\ \text{[-1,1]} & \text{if} \quad \xi = 0. \end{cases} \tag{4}$$

The second term on the right in (3) models a play-type hysteresis (see [2, 18]), while the last one accounts for dynamic effects in the phase pressure difference (see [15]). Following [11], for $\tau > 0$, the multi-valued function $\Phi : \xi \longmapsto \tau\xi + \gamma \operatorname{sign}(\xi)$ can be inverted. Its inverse $\Psi : \Phi^{-1} : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function satisfying

$$0 \leq \Psi'(\xi, x) \leq 1/\tau. \tag{5}$$

With this notation, (3) transforms into

$$\partial_t s = \Psi(p_n - p_w - p_c(s), x). \tag{6}$$

The model (1), (2), (6) is complemented by initial and boundary conditions

$$s(0, \cdot) = s_0, \tag{7}$$

$$p_n = p_w = 0 \quad \text{at } \partial\Omega, \quad \text{for all} \quad \text{t} \geq 0. \tag{8}$$

Remark 1.1: Other boundary conditions are possible, but for clarity, we restrict the presentation to (8).
The following assumptions are made:

- **A1:** The functions k_w, $k_n : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous. Further, $\delta, M_k > 0$ exist such that $\delta \leq k_w(s), k_n(s) \leq M_k < \infty$, for all $s \in \mathbb{R}$.
- **A2:** $p_c(\cdot) \in C^1(\mathbb{R})$ is increasing and Lipschitz continuous, there exist $m_p, M_p > 0$ such that $m_p \leq p_c'(s) \leq M_p$, for all $s \in \mathbb{R}$.
- **A3:** Ω is a $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ domain.
- **A4:** $\gamma(x) \in C^{0,1}(\bar{\Omega})$.
- **A5:** $s_0 \in C^{0,\alpha}(\bar{\Omega})$.

Remark 1.2: Commonly, the permeabilities encountered in the literatures ([4]) are

$$k_w(s) = (1 - s)^p, \quad k_n(s) = s^q, \quad \text{with} \quad p, q > 1,$$

and

$$p_c(s) = (1 - s)^{-\frac{1}{\lambda}}, \qquad \lambda > 1, \text{ for } s \in [0, 1].$$

Then **A1** is not satisfied when s approaches to 0 or 1. We consider here a regularized approximation of these functions.

2 Uniqueness

Existence results for the model considered here are proved in [11]. In this section, we provide a rigorous proof of the uniqueness of weak solutions to (1), (2), (6). We use common notations for function spaces, namely L^2, $W^{1,2}$, $W_0^{1,2}$, and Bochner space $L^2(0,T;X)$. Further, by $C>0$, we have a generic constant. We follow [11] and consider weak solutions solving

Problem $\mathrm{P_e}$**:** Given s_0 satisfying **A5**, find $p_n \in L^2(0,T;W_0^{1,2}(\Omega))$, $p_w \in L^2(0,T;W_0^{1,2}(\Omega))$ and $s \in W^{1,2}(0,T;L^2(\Omega))$, such that $s(\cdot,0)=s_0$ in Ω, and

$$(\partial_t s,\phi)+(k_n(s)\nabla p_n,\nabla\phi)+(k_n(s)\overrightarrow{g},\nabla\phi)=0, \tag{9}$$

$$(-\partial_t s,\psi)+(k_w(s)\nabla p_w,\nabla\psi)+(k_w(s)\overrightarrow{g},\nabla\psi)=0, \tag{10}$$

$$(\partial_t s,\rho)=(\Psi(p_n-p_w-p_c(s),x),\rho), \tag{11}$$

for any $\phi,\psi \in L^2(0,T;W_0^{1,2}(\Omega))$ and $\rho \in L^2(0,T;L^2(\Omega))$.

In [11], the hysteresis is modeled by considering (3) valid a.e.. This immediately implies that (6) holds a.e. and further (11). In this respect, the weak solution of Problem $\mathrm{P_e}$ is also a solution in [11]. The existence of weak solutions for Problem $\mathrm{P_e}$ has been proved in [11]. Here we show that weak solution is unique. Unique results for a similar model but without hysteresis are obtained in [5]. To this aim, some intermediate results are needed. We start with essential bounds for the gradients of p_n and p_w.

Theorem 2.1. *Let (p_n,p_w,s) be a weak solution to Problem P_e. Then one has $\nabla p_n, \nabla p_w \in L^\infty((0,T]\times\bar{\Omega})$.*

Proof. First we show that $\|\nabla p_n\|_{L^2(\Omega)} \in L^\infty(0,T)$ and $\|\nabla p_w\|_{L^2(\Omega)} \in L^\infty(0,T)$. Taking $\phi=p_n$ in (9), $\psi=p_w$ in (10) and adding the resulting equations give

$$\begin{aligned}(\partial_t s,p_n-p_w)+\|\sqrt{k_n(s)}\nabla p_n\|^2_{L^2(\Omega)}+\|\sqrt{k_w(s)}\nabla p_w\|^2_{L^2(\Omega)}\\ +(k_n(s)\overrightarrow{g},\nabla p_n)+(k_w(s)\overrightarrow{g},\nabla p_w)=0.\end{aligned} \tag{12}$$

For the first term of (12), we note that (3) holds almost everywhere. Then since $\mathrm{sign}(\xi)\xi \geq 0$ for any $\xi\in\mathbb{R}$, one has

$$\int_\Omega \partial_t s(p_n-p_w) \geq \int_\Omega \tau|\partial_t s|^2dx+\int_\Omega p_c(s)\partial_t s dx \geq \frac{\tau}{2}\|\partial_t s\|^2_{L^2(\Omega)}-\frac{1}{2\tau}\int_\Omega |p_c(s)|^2dx. \tag{13}$$

Further, since $s\in L^\infty(0,T;L^2(\Omega))$ (see [11, 14]), by using the Cauchy-Schwarz inequality, **A1** and **A2**, (12) gives

$$\|\nabla p_n\|^2_{L^2(\Omega)}+\|\nabla p_w\|^2_{L^2(\Omega)} \leq C, \quad \text{for almost every } t. \tag{14}$$

Then substituting (6) into (1) and (2) respectively, one has

$$-\nabla\cdot(k_n(s)\nabla p_n)=-\Psi(p_n-p_w-p_c(s),x)+\nabla\cdot(k_n(s)\overrightarrow{g}), \tag{15}$$

$$-\nabla \cdot (k_w(s)\nabla p_w) = \Psi(p_n - p_w - p_c(s), x) + \nabla \cdot (k_w(s)\overrightarrow{g}). \tag{16}$$

Using Theorem 14.1 in [13] gives for almost every t,

$$\|p_n\|_{C^{0,\alpha}(\bar{\Omega})} + \|p_w\|_{C^{0,\alpha}(\bar{\Omega})} \leq C. \tag{17}$$

Further, from (6), for almost every $x, y \in \Omega$ $(x \neq y)$ and $t > 0$, ζ and $\tilde{\zeta}$ depending on x, y, t exist, such that

$$\begin{aligned}
&\partial_t \frac{s(t,x) - s(t,y)}{|x-y|^\alpha} \\
=&\frac{\Psi\big((p_n - p_w - p_c(s))(t,x), x\big) - \Psi\big((p_n - p_w - p_c(s))(t,y), y\big)}{|x-y|^\alpha} \\
=&\frac{\Psi\big(p_n(t,x) - p_w(t,x) - p_c(s(t,x)), x\big) - \Psi\big(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), x\big)}{|x-y|^\alpha} \\
&+ \frac{\Psi\big(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), x\big) - \Psi\big(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), y\big)}{|x-y|^\alpha} \\
=&\Psi'(\zeta, x)\left(\frac{p_n(t,x) - p_n(t,y)}{|x-y|^\alpha} - \frac{p_w(t,x) - p_w(t,y)}{|x-y|^\alpha} - \frac{p_c(s)(t,x) - p_c(s)(t,y)}{|x-y|^\alpha}\right) \\
&+ \frac{\Psi(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), x) - \Psi(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), y)}{|x-y|^\alpha} \\
=&\Psi'(\zeta, x)\left(\frac{p_n(t,x) - p_n(t,y)}{|x-y|^\alpha} - \frac{p_w(t,x) - p_w(t,y)}{|x-y|^\alpha} - p_c'(\tilde{\zeta}) \cdot \frac{s(t,x) - s(t,y)}{|x-y|^\alpha}\right) \\
&+ \frac{\Psi\big(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), x\big) - \Psi\big(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), y\big)}{|x-y|^\alpha}.
\end{aligned} \tag{18}$$

Define

$$\Gamma(t,x,y) = \frac{\Psi(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), x) - \Psi(p_n(t,y) - p_w(t,y) - p_c(s(t,y)), y)}{|x-y|^\alpha}. \tag{19}$$

By **A2** - **A4**, and since $p_n, p_w \in C^{0,\alpha}(\bar{\Omega})$, for almost every t, we have

$$|\Gamma(t,x,y)| + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|p_n(t,x) - p_n(t,y)|}{|x-y|^\alpha} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|p_w(t,x) - p_w(t,y)|}{|x-y|^\alpha} \leq C. \tag{20}$$

Defining $w : (0,T] \times \Omega^2 \to \mathbb{R}$ as

$$w = \frac{s(t,x) - s(t,y)}{|x-y|^\alpha}, \tag{21}$$

w satisfies

$$\partial_t w = fw + g, \tag{22}$$

where $f(t,x) = -\Psi'(\zeta,x)\cdot p_c'(\tilde{\zeta})$, and $g(t,x) = \Psi'(\zeta,x)(\frac{p_n(t,x)-p_n(t,y)}{|x-y|^\alpha} - \frac{p_w(t,x)-p_w(t,y)}{|x-y|^\alpha}) + \Gamma(t,x,y)$. Note that, (5), (20) and **A2** give $f,g \in L^\infty((0,T]\times\bar{\Omega})$.
Multiplying (22) by w and integrating from 0 to t leads to

$$\frac{1}{2}w^2(t) = \int_0^t fw^2(z)dz + \int_0^t gw(z)dz + \frac{1}{2}\left(\frac{s_0(x)-s_0(y)}{|x-y|^\alpha}\right)^2. \tag{23}$$

Since $f,g \in L^\infty((0,T]\times\bar{\Omega})$ and $s_0 \in C^{0,\alpha}(\bar{\Omega})$ from **A5**, we have

$$w^2(t) \le C(1+\int_0^t w^2 dz), \quad \text{for every } t. \tag{24}$$

Using Gronwall's inequality yields $w \le C$, implying that

$$\frac{|s(t,x)-s(t,y)|}{|x-y|^\alpha} \le C, \quad \text{for almost every } x,y\in\Omega,\ \text{for every } t. \tag{25}$$

Let Ω_c be the subset of Ω, where (25) holds everywhere. Clearly, $\Omega\backslash\Omega_c$ is zero measured. For any $x\in\Omega\backslash\Omega_c$, we consider a sequence $\{x_n\}_{n\in\mathbb{N}}\in\Omega_c$ converging to x, and define

$$s(t,x) = \lim_{\substack{n\to+\infty \\ x_n\in\Omega_c}} s(t,x_n). \tag{26}$$

In the view of (25), $s(x)$ does not depend on the choice of $\{x_n\}_{n\in\mathbb{N}}$. With this choice, $s\in C^{0,\alpha}(\bar{\Omega})$ (see [8]).
Finally, by Theorem 8.33 and Corollary 8.35 in [10] (see also [6]), we get

$$|p_n|_{1,\alpha} \le C(|p_n|_0 + |\Psi|_0 + |k_n(s)|_{0,\alpha}), \tag{27}$$

$$|p_w|_{1,\alpha} \le C(|p_w|_0 + |\Psi|_0 + |k_w(s)|_{0,\alpha}), \tag{28}$$

implying $\nabla p_n, \nabla p_w \in L^\infty((0,T]\times\bar{\Omega})$. □

Theorem 2.2. *Problem P_e has at most one solution.*

Proof. Let (u,p_n^u,p_w^u) and (v,p_n^v,p_w^v) be the two solutions of Problem $\mathrm{P_e}$, then one has

$$\begin{aligned}&\big(\partial_t(u-v),\phi\big) + (k_n(v)\nabla(p_n^u-p_n^v),\nabla\phi)\\ &+((k_n(u)-k_n(v))\nabla p_n^u,\nabla\phi) + ((k_n(u)-k_n(v))\overrightarrow{g},\nabla\phi) = 0,\end{aligned} \tag{29}$$

$$\begin{aligned}&-\big(\partial_t(u-v),\psi\big) + \big(k_w(v)\nabla(p_w^u-p_w^v),\nabla\psi\big)\\ &+\big((k_w(u)-k_w(v))\nabla p_w^u,\nabla\psi) + ((k_w(u)-k_w(v))\overrightarrow{g},\nabla\psi) = 0,\end{aligned} \tag{30}$$

and

$$(\partial_t(u-v),\rho) = (\Psi(p_n^u-p_w^u-p_c(u),x) - \Psi(p_n^v-p_w^v-p_c(v),x),\rho), \tag{31}$$

for any $\phi, \psi \in L^2(0,T;W_0^{1,2}(\Omega))$, $\rho \in L^2(0,T;L^2(\Omega))$.
Since Ψ is Lipschitz, for almost every $(x,t) \in \Omega_T$, a ξ exists, such that

$$\big(\partial_t(u-v),\rho\big) = \big(\Psi^{'}(\xi,x)((p_n^u - p_n^v) - (p_w^u - p_w^v) - (p_c(u) - p_c(v))),\rho\big). \tag{32}$$

Further, let $(G_{u-v}, \tilde{G}_{u-v})$ be the weak solution pair of the elliptic system (see [5, 9]),

$$-\nabla\cdot(k_n(v)\nabla G_{u-v}) + \Psi^{'}(\xi,x)(\tilde{G}_{u-v} + G_{u-v}) = \Psi^{'}(\xi,x)(u-v), \tag{33}$$

$$-\nabla\cdot(k_w(v)\nabla \tilde{G}_{u-v}) + \Psi^{'}(\xi,x)(G_{u-v} + \tilde{G}_{u-v}) = \Psi^{'}(\xi,x)(u-v), \tag{34}$$

$$G_{u-v},\ \tilde{G}_{u-v} = 0, \quad \text{at } \partial\Omega. \tag{35}$$

The existence and uniqueness follow the Lax-Milgram lemma. Further, one has

$$(\Psi^{'}(\xi,x)\tilde{G}_{u-v},\lambda) + (\Psi^{'}(\xi,x)G_{u-v},\lambda) + (k_n(v)\nabla G_{u-v},\nabla\lambda) = (\Psi^{'}(\xi,x)(u-v),\lambda), \tag{36}$$

$$(\Psi^{'}(\xi,x)G_{u-v},\tilde{\lambda}) + (\Psi^{'}(\xi,x)\tilde{G}_{u-v},\tilde{\lambda}) + (k_w(v)\nabla \tilde{G}_{u-v},\nabla\tilde{\lambda}) = (\Psi^{'}(\xi,x)(u-v),\tilde{\lambda}), \tag{37}$$

for any $\lambda, \tilde{\lambda} \in W_0^{1,2}(\Omega)$.
Using the properties of Ψ, k_w, k_n, one immediately gets

$$\|G_{u-v}\|^2_{W^{1,2}(\Omega)} \le C\|u-v\|^2_{L^2(\Omega)}, \quad \text{and} \quad \|\tilde{G}_{u-v}\|^2_{W^{1,2}(\Omega)} \le C\|u-v\|^2_{L^2(\Omega)}. \tag{38}$$

Taking $\phi = G_{u-v}$ in (29), and $\psi = \tilde{G}_{u-v}$ in (30), one has

$$\begin{aligned}&(\partial_t(u-v),G_{u-v}) + (k_n(v)\nabla(p_n^u - p_n^v),\nabla G_{u-v})\\ &+((k_n(u)-k_n(v))\nabla p_n^u,\nabla G_{u-v}) + ((k_n(u)-k_n(v))\overrightarrow{g},\nabla G_{u-v}) = 0,\end{aligned} \tag{39}$$

$$\begin{aligned}&-(\partial_t(u-v),\tilde{G}_{u-v}) + (k_w(v)\nabla(p_w^u - p_w^v),\nabla \tilde{G}_{u-v})\\ &+((k_w(u)-k_w(v))\nabla p_w^u,\nabla \tilde{G}_{u-v}) + ((k_w(u)-k_w(v))\overrightarrow{g},\nabla \tilde{G}_{u-v}) = 0.\end{aligned} \tag{40}$$

Choosing $\lambda = p_n^u - p_n^v$ in (36) and $\tilde{\lambda} = p_w^u - p_w^v$ in (37) gives

$$\begin{aligned}(k_n(v)\nabla G_{u-v},\nabla(p_n^u - p_n^v)) = (\Psi^{'}(\xi,x)(u-v),p_n^u - p_n^n) &- (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_n^u - p_n^n)\\ &- (\Psi^{'}(\xi,x)G_{u-v},p_n^u - p_n^n),\end{aligned} \tag{41}$$

$$(k_w(v)\nabla\tilde{G}_{u-v},\nabla(p_w^u-p_w^v)) = (\Psi^{'}(\xi,x)(u-v),p_w^u-p_w^n) - (\Psi^{'}(\xi,x)G_{u-v},p_w^u-p_w^n) - (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_w^u-p_w^n). \tag{42}$$

Substitute (41) into (39) and (42) into (40), we find that

$$\begin{aligned}(\partial_t(u-v),G_{u-v}) &- (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_n^u-p_n^n) - (\Psi^{'}(\xi,x)G_{u-v},p_n^u-p_n^n)\\ &+(\Psi^{'}(\xi,x)(u-v),p_n^u-p_n^n) + ((k_n(u)-k_n(v))\nabla p_n^u,\nabla G_{u-v})\\ &+((k_n(u)-k_n(v))\overrightarrow{g},\nabla G_{u-v}) = 0,\end{aligned} \tag{43}$$

$$\begin{aligned}-(\partial_t(u-v),\tilde{G}_{u-v}) &- (\Psi^{'}(\xi,x)G_{u-v},p_w^u-p_w^n) - (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_w^u-p_w^n)\\ &+(\Psi^{'}(\xi,x)(u-v),p_w^u-p_w^n) + ((k_w(u)-k_w(v))\nabla p_w^u,\nabla\tilde{G}_{u-v})\\ &+((k_w(u)-k_w(v))\overrightarrow{g},\nabla\tilde{G}_{u-v}) = 0.\end{aligned} \tag{44}$$

Taking $\rho = u - v$ into (32) yields

$$\begin{aligned}(\Psi^{'}(\xi,x)(p_n^u-p_n^v),u-v) &= (\partial_t(u-v),u-v) + (\Psi^{'}(\xi,x)(p_w^u-p_w^v),u-v)\\ &+ (\Psi^{'}(\xi,x)(p_c(u)-p_c(v)),u-v).\end{aligned} \tag{45}$$

Using this into (43) and subtracting (44), the resulting equation gives

$$\begin{aligned}&(\partial_t(u-v),G_{u-v}) + (\Psi^{'}(\xi,x)G_{u-v},p_w^u-p_w^n) - (\Psi^{'}(\xi,x)G_{u-v},p_n^u-p_n^n)\\ &+(\partial_t(u-v),\tilde{G}_{u-v}) + (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_w^u-p_w^n) - (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_n^u-p_n^n)\\ &+(\partial_t(u-v),u-v) + (\Psi^{'}(\xi,x)(p_c(u)-p_c(v)),u-v)\\ &+((k_n(u)-k_n(v))\nabla p_w^u,\nabla G_{u-v}) - ((k_w(u)-k_w(v))\nabla p_n^u,\nabla\tilde{G}_{u-v})\\ &+((k_n(u)-k_n(v))\overrightarrow{g},\nabla G_{u-v}) - ((k_w(u)-k_w(v))\overrightarrow{g},\nabla\tilde{G}_{u-v}) = 0.\end{aligned} \tag{46}$$

Further, taking $\rho = G_{u-v}$ and $\rho = \tilde{G}_{u-v}$ in (32) respectively give

$$\begin{aligned}(\partial_t(u-v),G_{u-v}) + (\Psi^{'}(\xi,x)G_{u-v},p_w^u-p_w^n) &- (\Psi^{'}(\xi,x)G_{u-v},p_n^u-p_n^n)\\ &= -(\Psi^{'}(\xi,x)G_{u-v},p_c(u)-p_c(v)),\end{aligned} \tag{47}$$

and

$$\begin{aligned}(\partial_t(u-v),\tilde{G}_{u-v}) + (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_w^u-p_w^n) &- (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_n^u-p_n^n)\\ &= -(\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_c(u)-p_c(v)).\end{aligned} \tag{48}$$

Substituting the above two equations into (46) leads to

$$\begin{aligned}&(\partial_t(u-v),u-v) - (\Psi^{'}(\xi,x)G_{u-v},p_c(u)-p_c(v)) - (\Psi^{'}(\xi,x)\tilde{G}_{u-v},p_c(u)-p_c(v))\\ &+(\Psi^{'}(\xi,x)(u-v),p_c(u)-p_c(v)) + ((k_n(u)-k_n(v))\nabla p_n^u,\nabla G_{u-v})\\ &-((k_w(u)-k_w(v))\nabla p_w^u,\nabla\tilde{G}_{u-v}) + ((k_n(u)-k_n(v))\overrightarrow{g},\nabla G_{u-v})\\ &-((k_w(u)-k_w(v))\overrightarrow{g},\nabla\tilde{G}_{u-v}) = 0.\end{aligned} \tag{49}$$

Integrating (49) from 0 to $\tilde{t}$, for any $\tilde{t} \in (0, T]$. Since $\nabla p_n, \nabla p_w \in L^\infty((0,T] \times \bar{\Omega})$, by using (5), **A1**, **A2** and (38), we obtain

$$\|(u-v)(\cdot,\tilde{t})\|^2_{L^2(\Omega)} \leq C \int_0^{\tilde{t}} \|(u-v)(\cdot,t)\|^2_{L^2(\Omega)} dt. \tag{50}$$

By Gronwall's inequality, $\|(u-v)(\cdot,\tilde{t})\|^2_{L^2(\Omega)} = 0$. Since $\tilde{t}$ is arbitrary, this gives $u = v$ a.e. in Ω and for all $t \in (0, T]$.
To show that $p_n^u = p_n^v$, $p_w^u = p_w^v$, we use (29) and (30). Since $u = v$, one has

$$(k_n(u)\nabla(p_n^u - p_n^v), \nabla\phi) = 0, \tag{51}$$

$$(k_w(u)\nabla(p_w^u - p_w^v), \nabla\psi) = 0, \tag{52}$$

for any $\phi, \psi \in W_0^{1,2}(\Omega)$, for almost every t.
The rest of the proof follows straightforwardly by taking $\phi = p_n^u - p_n^v, \psi = p_w^u - p_w^v$, and recalling that $p_n^u, p_n^v, p_w^u, p_w^v$ have equal traces on $\partial\Omega$. □

3 Conclusion

In this paper, we have proved the uniqueness of weak solutions to a non-degenerate system which models two-phase flow in porous media including hysteresis and dynamic effects in the capillary pressure. In doing so, we use arguments based on Green's function.

Acknowledgment. We thank Prof. Dr. Ben Schweizer, Dr. Agnes Lamacz (both Dortmund) for the helpful discussions. X. Cao would like to thank CSC (China Scholarship Council) for the financial support. The authors are members of the International Research Training Group NUPUS funded by the German Research Foundation DFG (GRK 1398) and The Netherlands Organization for Scientific Research NWO (DN 81-754).

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