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Heavy-traffic analysis through uniform acceleration of transitory queues with diminishing populations

Gianmarco Bet, Remco van der Hofstad, and Johan S.H. van Leeuwen

Abstract

We consider the $\Delta_{(i)}/GI/1$ queue, in which the arrival times of a fixed population of n customers are sampled independently from an identical distribution. This model recently emerged as the canonical model for so-called transitory queues that are non-stationary, time-varying and might operate only over finite time. The model assumes a finite population of customers entering the queue only once. This paper presents a method for analyzing heavy-traffic behavior by using uniform acceleration, which simultaneously lets the population n and the service rate grow large, while the initial resource utilization approaches one. A key feature of the model is that, as time progresses, more customers have joined the queue, and fewer customers can potentially join. This *diminishing population* gives rise to a class of reflected stochastic processes that vanish over time, and hence do not have a stationary distribution. We establish that, by suitably rescaling space and time, the queue length process converges to a Brownian motion on a parabola, a stochastic-process limit that captures the effect of a diminishing population by a negative quadratic drift. The stochastic-process limit provides insight into the macroscopic behavior (for n large) of the transitory queueing process, and the different phenomena occurring at different space-time scales.

1 Introduction

The queueing literature is to a large extent built on the assumption that arrivals are governed by some renewal process, a mathematically convenient assumption as it allows the full use of probabilistic tools based on regenerative processes and ergodic theory. This paper presents results for what is becoming the canonical model of the class of *transitory* queueing models that are non-stationary, time-varying and might operate only over finite time. The model assumes a finite population of customers entering the queue only once. As time progresses, more customers have joined the queue, and fewer customers can potentially join. This modelling assumption of a *diminishing population of customers* gives rise to a class of reflected stochastic processes that vanish over time, and hence do not have a stationary distribution. The relevant behavior of the processes only arises during a *finite window of time*, and therefore only the time-dependent behavior is of interest.

When we assume the arrival times of customers to be sampled independently from an identical distribution, the arrival times are ordered statistics, and the interarrival times are differences of ordered statistics. Further assuming a single server, and generally distributed independent service times, this model was coined the $\Delta_{(i)}/GI/1$ queue in [14], in which fluid and diffusion limits for the queue length process were established. In [15] a wider class of transitory queues was introduced, with the $\Delta_{(i)}/GI/1$ queue still as the prime example, and stochastic-process limits were established for large population sizes.

We will introduce a heavy-traffic regime for the *critically-loaded* $\Delta_{(i)}/GI/1$ queue, leading to stochastic-process limits and heavy-traffic approximations. Considering queueing processes in their critical regimes typically leads to a reduction in complexity, since the complicated processes can often be shown to converge to much simpler stochastic processes. Stochastic-process limits have been studied for single-server queues that have a time-varying arrival rate. Newell [26, 27] pioneered this direction by deriving diffusion approximations, see also [19, 21]. Rigorous results in terms of stochastic-process limits were obtained in [22] (building on [24, 25]). Here, stochastic-process limits were established as refinements to deterministic ODE limits for the time-dependent $M/M/1$ queue, known as the $M_t/M_t/1$ queue. See also [23] for state-dependent Markovian queues. However, these processes and their limits are not transitory. The technique used in [22] to develop Functional Law of Large Numbers (FLLN) and Functional Central Limit Theorem (FCLT) results uses strong approximation and what is known as the *uniform acceleration* (UA) technique. UA relies on the assumption that the relevant time scales for changes in the queue length process is of the order $1/\epsilon$ for some $\epsilon > 0$. Accelerating the process in a uniform manner by scaling the arrival and service rates by ϵ then reveals the dominant model behavior as $\epsilon \rightarrow 0$. While in [14, 15] the arrival and service rates are scaled in a similar manner, the time of the process itself is not scaled. The UA technique typically requires time scaling, which explains why the process limits for $\Delta_{(i)}/GI/1$ queue in [14] are for all time, and the process limits for the $M_t/M_t/1$ queue in [22] are for regimes in which both space and time are scaled. Due to additional ergodicity assumptions required, the UA technique has been extensively applied to non-stationary queueing systems with non-homogeneous Poisson input, but it remained unclear whether it is also useful for transitory queueing models of the kind introduced in [15]. In this paper we show how the key idea behind the UA technique can be applied to the $\Delta_{(i)}/GI/1$ queue, but due to the non-Markovian setting we choose a different mathematical approach that starts from the queue length process embedded at service completions. This embedded process resembles the so-called exploration processes that are used to study the structure of random graphs, and in fact this was what inspired us to take this route. We shall now explain this approach in terms of the easiest setting in which the identical distribution that generates the arrival times is exponential.

Assume a finite population of n customers, with n large, and where each customer has an independent exponential clock with mean $1/\lambda$. Customers join the queue when their clocks ring. The initial arrival process is then roughly Poisson with rate $n\lambda$. However, as time progresses, the arrival intensity decreases, due to those customers that have left the system. It is as if the arrival process is a Poisson process that is thinned over time. In order to create heavy-traffic conditions we let the population size n grow to infinity, while at the same time making sure that the (initial) traffic intensity ρ_n is close to one. Denote the mean service time by $1/\mu$. The system can initially be underloaded (when $n\lambda < \mu$), overloaded (when $n\lambda > \mu$), or critically-loaded (when $n\lambda = \mu$). In case $n\lambda > \mu$, the queue grows linearly with n and therefore, the correct scaling of the queue length to obtain meaningful limits is n (FLLN) for a first order approximation, and $n^{1/2}$ (FCLT) for second order approximations. This is the most relevant regime in [14].

We focus primarily on the critically-loaded regime, and we combine this with uniform acceleration through the population size n . Now clearly the arrival rate increases with n , and in order to let also the service rate increase we assume it to scale with n as well. In that case $n\lambda \approx n\mu$, in which case the queue length tends to infinity as the number of customers in the population n grows, but in a non-trivial manner. Therefore, the queue length process lies out-

side the reach of standard limit theorems such as FLLN and FCLT and a different space-time scaling is required. We create the critically-loaded regime by assuming that the server works at rate n , so that when the population size grows to infinity, so do both the arrival rate and the rate at which customers depart from the queue (similar as with UA). More generally, denote the density of the identical distribution as $f_T(t)$ (with $f_T(t) = \lambda e^{-\lambda t}$, $t \geq 0$ as a special case), and denote the i.i.d. service requirements of consecutive customers by S_1, S_2, \dots with generic random variable S . Assuming a service rate of n , the service times of consecutive customers are then $D_1 = S_1/n, D_2 = S_2/n, \dots$ with generic random variable D . The heavy-traffic regime we then consider is given by the condition

$$\rho_n := \frac{nf_T(0)\mathbb{E}[S]}{n} = f_T(0)\mathbb{E}[S] = 1, \quad \text{for } n \text{ large.} \quad (1.1)$$

Hence, returning to the exponential case $f_T(0) = \lambda$, this heavy-traffic regime simply says that the expected number of newly arriving customer during one service time is roughly one. For general service times, the condition can be understood by the fact that when $n \rightarrow \infty$ and time is accelerated, only the mass in zero $f_T(0)$ of the density matters for describing the new arrivals. In what follows we actually consider a slightly different definition of the random variables $(D_i)_{i=1}^\infty$ and this leads to

$$\rho_n = 1 + \beta n^{-1/3}. \quad (1.2)$$

The additional term $\beta n^{-1/3}$ arises from detailed calculations, but can be interpreted as the factor that describes the onset of the heavy-traffic period: when $\beta > 0$ (< 0) the queue is initially slightly overloaded (underloaded).

We assume n customers in a population, each one possessing a clock distributed as T . Whenever a clock rings, that customer joins the queue. Customers are served in a First Come First Served order. After service completion, the customer is removed from the system. Let $Q_n(k)$ denote the number of customers in the queue just after the service completion of the k th customer. The process $(Q_n(k))_{k \geq 0}$, embedded at service completions, is then given by $Q_n(0) = 0$ and

$$Q_n(k) = (Q_n(k-1) + A_n(k) - 1)^+, \quad k = 1, 2, \dots \quad (1.3)$$

with $x^+ = \max\{0, x\}$ and $A_n(k)$ the number of arrivals during the service time of the k th customer. Here $A_n(k)$ is given by

$$A_n(k) = \sum_{i=0}^n \mathbb{1}_{\{\sum_{j=1}^{k-1} D_k \leq T_{(i)} \leq \sum_{j=1}^k D_k\}} \quad (1.4)$$

with $T_{(1)} \leq T_{(2)} \leq \dots \leq T_{(n)}$ the order statistics of the random variables $(T_i)_{i=1}^n$. The queueing system defined in (1.3) and (1.4) neglects idle times, which is a simplification of the $\Delta_{(i)}/G/1$ model. However, due to the criticality assumption, the idle times vanish in the limit and our result thus also holds for the $\Delta_{(i)}/G/1$ model. Neglecting the idle periods from the start will greatly simplify the analysis, and we return to this issue in Section 2 and 7.2.

It is clear that the resulting queueing process is strongly influenced by the service time distribution. In particular, the heavy-traffic behavior is crucially different depending on whether the second moment of the service time distribution is finite or not. In this paper we assume throughout that $\mathbb{E}[S^2] < \infty$, in which case the queueing process is in the domain of attraction of Brownian motion. Indeed, for the queueing process embedded at times of service completions,

we scale space and time so that the stochastic-process limit turns out to be a Brownian motion on a parabola. The latter process is defined as $(W_t)_{t \geq 0} = (at + bt^2/2 + cB_t)_{t \geq 0}$ with $(B_t)_{t \geq 0}$ a standard Brownian motion, and a, b, c constants. The scaling limits that we obtain in this paper are reflected versions of $(W_t)_{t \geq 0}$, with $b < 0$ so that eventually the free process $(W_t)_{t \geq 0}$ drifts to minus infinity as bt^2 , causing the reflected process to be essentially stuck to zero. This effect is due to the *diminishing population* effect. One could interpret the quadratic term in the limit as the (cumulative) effect of the customers already served not being able to join the queue again. The stochastic-process limit provides insight into the macroscopic behavior (for n large) of the transitory queueing process, and the different phenomena occurring at different space-time scales. It also gives insight into the orders of the average queue lengths, the probability of large queue lengths occurring, and the time scales of busy periods.

1.1 Comparison with known results

To create the right circumstances for non-degenerate limiting behavior in heavy traffic, we work not only under the critical-load assumption (1.1), but also under the additional assumption that the maximum of the density $f_T(t)$, $t \geq 0$ is assumed in $t = 0$. The reason is that our asymptotic regime lets the population grow large, leading to many possible events in an extremely short time interval (acceleration), hence making the behavior of the arrival distribution around zero the dominant factor. If the density $f_T(t)$ would be maximal at some point $t_{\max} > 0$, then we should have changed our critical-load assumption accordingly, and hence accelerate time around the point t_{\max} instead of time zero. Let us explain this subtle point better below, and at the same time draw a comparison with existing lines of related research.

The $\Delta_{(i)}/GI/1$ queue has already been considered in [14] and [21]. Both papers allow for $t_{\max} > 0$, and consider a wide range of possible system behaviors related to the maximum $f_T(t_{\max})$. In our notation, system behavior will differ depending on whether $f_T(t_{\max})\mathbb{E}[S]$ is smaller, equal, or larger than 1, and in fact, even when $f_T(t_{\max})\mathbb{E}[S] \approx 1$, additional regimes can be considered by changing the rate of convergence (in our case this has to be $1 + \beta/n^{-1/3}$). Indeed, Louchard [21] studies a wide range of system behaviors, by dividing time into intervals that are associated with specific assumptions on $f_T(t_{\max})\mathbb{E}[S]$, and then deriving convergence results within an interval. In that way Louchard [21] identifies using mostly intuitive arguments the possible behaviors of the model. Our setting would then correspond to Cases 2 and 3 in [21], with $\alpha = 1/3$. Hence, while Louchard [21] identifies many possible stochastic-process limits, over all time while the system possibly changes between temporarily underloaded, overloaded and critical conditions, we exclusively focus on the unique set of conditions that lead to a critical system behavior, and it is that one specific heavy-traffic regime for which we formally derive the stochastic-process limits. We also identify further conditions on the density f_T that influence limiting behavior. We for instance show that different stochastic-process limits arise depending on how many derivatives of the density $f_T(t)$ in t_{\max} are zero. In the paper we focus on the case that only the first derivative is zero, and all higher derivatives are not, but our methodology can be used to obtain process limits for all such scenarios, as discussed in Section 6.

Hence, to force non-degenerate limiting behavior in the critical $\Delta_{(i)}/GI/1$ queue, uniform acceleration is required, in combination with additional assumptions on the arrival times density in its maximum. This approach was inspired by earlier work of Louchard [21] for the same queue, as previously discussed, but also by the work of Mandelbaum and Massey [22] for the $M_t/M_t/1$ queue. Mandelbaum and Massey derive a fluid approximation through a FLLN and use this approximation to classify various operating regimes of the queue. In this setting, our result

corresponds to the ‘Onset of Critical Loading’ regime [22, Theorem 3.4] and the results of [14] correspond to the FLLN and the FCLT [22, Theorems 2.1 and 2.2], and indeed the $M_t/M_t/1$ queue and the $\Delta_{(i)}/GI/1$ queue are intimately related. To see this, consider an inhomogeneous Poisson process $N(\int_0^t \lambda(s)ds)$ (with $N(\cdot)$ a Poisson process of rate one). Then, conditioned on $\{N(\int_0^t \lambda(s)ds) = k\}$, the k points themselves are a family of i.i.d. random variables with density given by $\lambda(s)/\int_0^t \lambda(z)dz$. In particular, if $\int_0^\infty \lambda(z)dz < \infty$, conditioned on the total number of Poisson points in $[0, \infty)$ (say, n), the points are a family of n i.i.d. random variables. In [15], the authors explore this connection in relation to the $\Delta_{(i)}/G/1$ model in great detail.

2 Model assumptions and main results

We consider a finite population of n customers. At time 0, a first customer joins the system, leaving a population of $n - 1$ customers that will eventually join the system in the future. The arrival times of the $n - 1$ remaining customers are drawn independently from a common distribution. Let the random variable T have distribution function F_T . The arrival times are then a collection of i.i.d. random variables $(T_i)_{i \geq 1}$ such that $T_i \sim T$. Assume that T admits a positive density function $f_T(\cdot)$, with $f_T(0) \in (0, \infty)$. Equivalently, the distribution function of T can be (properly) expanded near every point of its domain up to the first order term. In addition, as a technical assumption, we assume that the sublinear terms of F_T decay as quickly as

$$F_T(x) - F_T(\bar{x}) = f_T(\bar{x}) \cdot (x - \bar{x}) + o(|\bar{x} - x|^{4/3}) \quad \forall \bar{x} \in (0, \infty). \quad (2.1)$$

This is, for example, the case when $F_T \in \mathcal{C}^2([0, \infty))$. As an additional technical assumption, this error should be uniform over \bar{x} , for small values of \bar{x} , as in

$$\sup_{\bar{x} \leq Cy^{1/3}} |F_T(\bar{x} + y) - F_T(\bar{x}) - f_T(\bar{x})y| = o(y), \quad (2.2)$$

where we set $x - \bar{x} = y$ for convenience. This is, for example, the case when f'_T and f''_T exist in a neighborhood of zero and are different from zero in zero (which is the case when T is exponentially distributed). Indeed, if $f''_T(0) < 0$, $F_T(\bar{x} + y) - F_T(\bar{x}) - f_T(\bar{x})y$ is decreasing because its derivative (with respect to \bar{x}) is $f_T(\bar{x} + y) - f_T(\bar{x}) - f'_T(\bar{x})y = f''(\zeta)y < 0$, where $\zeta \in (\bar{x}, \bar{x} + y)$ and \bar{x} is sufficiently small. Then

$$\sup_{\bar{x} \leq Cy^{1/3}} |F_T(\bar{x} + y) - F_T(\bar{x}) - f_T(\bar{x})y| = |F_T(y) - F_T(0) - f_T(0)y| = o(y). \quad (2.3)$$

Assume moreover that $f'_T(\cdot)$ exists and is continuous in a neighborhood of 0. In particular it holds that

$$f_T(x) = c_T + f'_T(0) \cdot x + o(x), \quad (2.4)$$

where we denote $c_T = f_T(0)$. Since $\lim_{x \rightarrow \infty} f_T(x) = 0$ and f_T is continuous on $[0, \infty)$, it admits a maximum in $[0, \infty)$. A crucial additional assumption is that this maximum is attained in 0, that is

$$c_T = \sup_{x \geq 0} f_T(x). \quad (2.5)$$

The service requirements of consecutive customers are given by the i.i.d. random variables $(S_i)_{i=1}^n$, independent from n . We assume that $\mathbb{E}[S_1^2] < \infty$. We assume that the service capacity per time unit scales as $\frac{1+\beta n^{-1/3}}{n}$, so that the service times are given by

$$D_i = \frac{S_i}{n} \left(1 + \beta n^{-1/3}\right), \quad i = 1, \dots, n. \quad (2.6)$$

The critical or heavy-traffic regime for this queueing system turns out to obey the condition

$$c_T \cdot \mathbb{E}[S_1] = 1, \quad (2.7)$$

which can be rewritten as $c_T = \mu + o(n^{-1/3})$ with $\mu := 1/\mathbb{E}[S_1]$. To better understand the role of β rewrite (2.7) as

$$c_T \cdot \mathbb{E}[S_1(1 + \beta n^{-1/3})] = 1 + \beta n^{-1/3} + o(n^{-1/3}). \quad (2.8)$$

Thus the parameter β in the scaling factor $1 + \beta n^{-1/3}$ measures the position of the system inside the so-called *critical window*. Condition (2.7) implies that $c_T \cdot \mathbb{E}[S_1] \approx 1$, so that roughly one customer joins the system during one service time. Indeed, if we consider n independent potential customers, each with a (very small) probability of joining the queue given by $c_T \cdot \mathbb{E}[D_1]$, then the average number of arrivals during one service will be $(n-1) \cdot (c_T \cdot \mathbb{E}[S_1]/n) \approx 1$. The queue length process $(Q_n(k))_{k \geq 0}$ embedded at service completions is then given by (1.3) and (1.4). Note the crucial difference from the $\Delta_{(i)}/G/1$ model, that is: *if the queue is empty after a service completion, a customer is chosen uniformly at random from the population of remaining customers and is immediately placed into service*. Therefore, our model does not correspond precisely to the standard $\Delta_{(i)}/G/1$ model. It *does* coincide in distribution when stopped at the end of the first busy period, and the approximation is reasonable for situations of heavy traffic, in which systems are hardly ever empty, and if empty a new customer arrives quickly. Indeed, for the heavy-traffic regimes that we will consider, the idle periods will turn out to be (asymptotically) negligible. We expand on this issue in the exponential case in (2.13)-(2.16) below.

Given a process $X(t)$ with trajectories in $\mathcal{D}([0, \infty), \mathbb{R})$, the space of càdlàg functions (endowed with the Skorohod J_1 topology), we denote by $\phi(X(\cdot))(t)$ its reflected version, obtained through the map

$$\phi(f)(x) := f(x) - \inf_{y \leq x} (f \wedge 0)(y). \quad (2.9)$$

It is possible to give an equivalent definition of the queue length process $Q_n(k)$ through the reflection map. Define the process $(N_n(k))_{k \geq 0}$ by $N_n(0) = 1$ and

$$N_n(k) = N_n(k-1) + A_n(k) - 1. \quad (2.10)$$

Then it is easy to see that

$$(Q_n(k))_{k \geq 0} = (\phi(N_n(\lfloor \cdot \rfloor))(k))_{k \geq 0} \quad \text{w.p. 1.}$$

To avoid unnecessary notation, for all discrete-time processes $(X(k))_{k \geq 0}$, we write $X(t)$, with $t \in \mathbb{R}^+$ instead of $X(\lfloor t \rfloor)$. Note that the process defined in (2.10) may take negative values. We first prove a limit theorem for $N_n(k)$ and then apply the reflection map to obtain a limit for the queue length $Q_n(k)$ defined in (1.3). Given a sequence of random variables $(X_n)_{n \geq 1}$, we denote by $X_n \xrightarrow{d} X$ convergence in distribution (or weak convergence).

We remark that the criticality condition (2.7) is highly nonstandard. The following observations provides further insight into it. It is possible to prove that the queue length process in the $\Delta_{(i)}/G/1$ model, when rescaled by n , converges to a (deterministic) function $q(t)$ (see [14, Theorem 1]). By exploiting the explicit expression of the limit function and standard continuity arguments, it is possible to prove that $q(t)$ is identically zero if and only if $\sup_{t \geq 0} f_T(t) \leq \mu$. Therefore, the queue length grows linearly with time if and only if $\sup_{t \geq 0} f_T(t) > \mu$ and the condition $\sup_{t \geq 0} f_T(t) = \mu$ is the threshold between supercritical and subcritical scaling. Hence, (2.7) defines the critical regime in which the queue grows like a non-trivial power of n , but not linearly.

The critical behaviour is determined in the following theorem:

Theorem 1. *Let $Q_n(t)$ be the process defined in (1.3) and (1.4), associated with the arrival random variables $(T_i)_{i \geq 1}$ and service times $(D_i)_{i \geq 1}$. Then*

$$n^{-1/3}Q_n(\cdot \cdot n^{2/3}) \xrightarrow{d} \phi(W)(\cdot), \quad (2.11)$$

where W is the diffusion process

$$W(t) = \beta \cdot t + \frac{f'_T(0)}{c_T^2} \cdot \frac{t^2}{2} + \sigma B(t) \quad (2.12)$$

with $\sigma^2 = c_T^2 \cdot \mathbb{E}[S_1^2]$ and $t \mapsto B(t)$ a standard Brownian motion.

Recall that $\phi(W)$ denotes the reflected version of W .

Exponential arrivals. A case of particular interest is when $f_T(x) = \lambda e^{-\lambda x}$, so that $c_T = \sup_{x \geq 0} f_T(x) = \lambda$ and

$$\frac{f'_T(0)}{c_T^2} = -1.$$

Moreover, the conditional law of (1.4) (conditioned on $\{A_n(k-1), \sum_{j=1}^{k-1} D_k\}$) is distributed as

$$\sum_{i=0}^{P_n(k)} \mathbb{1}_{\{T_{i,k} \leq D_k\}}, \quad (2.13)$$

where $T_{i,k} \stackrel{d}{=} T_i$, which means that the clocks are re-drawn after each service, and $P_n(k) := n - Q_n(k-1) - (k-1)$ is the number of customers still in the population. With a slight abuse of notation, we will write T_i instead of $T_{i,k}$. By directly defining (2.13) as the random variable that describes the new arrivals during the n th service time, this case becomes significantly lighter on the technicalities and provides useful insights. For example, the memoryless property of the exponential clocks allows an explicit and clarifying description of the difference between the $\Delta_{(i)}/G/1$ model and our queue. Recall that, whenever the queue is empty at the end of a service, we draw a customer uniformly at random from the queue and place it in service, effectively eliminating idle times. On the contrary, idle periods *do* occur in the $\Delta_{(i)}/G/1$ model and, conditioned on the process up to time t , the idle period starting in t is distributed as

$$I \sim \frac{\text{Exp}(1)}{n - |v_t|}, \quad (2.14)$$

where $\text{Exp}(1)$ represents an exponential random variable with rate λ and $|\nu_t|$ represents the number of customers who have been served, or are already in the queue, by time t . We will see that $|\nu_t| \approx t$, for $t = O(n^{2/3})$, and therefore its contribution in (2.14) is negligible and we can think of the idle periods up to time $t = O(n^{2/3})$ as being independent exponential random variables with rate n . We can easily couple our queue and the $\Delta_{(i)}/G/1$ model in the following way: The arrival times of the two queues coincide until the end of the first busy period. The first customer to enter the queue after the idle period in the $\Delta_{(i)}/G/1$ queue is also the customer that is selected to be served immediately in our queue. Then, the clocks are reset and are the same for both queues, until the beginning of the next busy period, and so on. Hence, the two queues are only time-shifted by a random time change. If I_1, I_2, \dots indicate the length of the idle times, and $\beta_n(t)$ indicates the number of busy periods before t , this time change takes the form

$$t \mapsto t' := t - \sum_{i=1}^{\beta_n(t)} I_i. \quad (2.15)$$

Since an explicit expression for the idle period is available in (2.14), we can estimate the average time-shift between the two queues at time t to be

$$\sum_{i=1}^{\beta_n(t)} I_i \approx \beta_n(t) \cdot \frac{1}{n} \quad (2.16)$$

In Section 7 it will be shown that $n^{-1/3}\beta_n(tn^{2/3})$ converges in distribution to a proper random variable, hence (2.16) is negligible in the limit for times of the order $O(n^{2/3})$.

We will treat the exponential arrivals case first and afterwards we will elaborate on how to generalize it. Note that the exponential distribution satisfies assumptions (2.1) and (2.4). However, the theorem we prove for the arrivals defined in (2.13) holds for all random variables $(T_i)_{i \geq 1}$ satisfying

$$F_T(x) = c_T \cdot x + o(x^{4/3}), \quad (2.17)$$

which is an assumption strictly weaker than (2.1). We will refer to the model with arrivals as in (2.13) and assumption (2.17) as the *Resetting Clocks* (RC) model. We will refer to the model with arrivals as in (1.4) and assumptions (2.1), (2.4) as the *General Arrivals* (GA) model. For generally (non-exponentially) distributed T_i 's, the RC model cannot be interpreted as a queue, while the GA model remains to have a natural interpretation as a queueing system. Therefore, in most cases, the RC model and the GA model do not coincide.

The rest of the paper is devoted to proving Theorem 1. In Section 3 we give some auxiliary results used in the proof, which is done first for the RC model (Section 4) and then for the GA model (Section 5). Many results and ideas from Sections 3 and 4 carry over to the GA model, thus significantly more details are given for the RC model. In Section 3 we state the FCLT which will be used to prove convergence of a sequence of martingales to a Brownian motion. In Section 4 we carry out the core of the proof for the RC model, which consists of verifying four conditions: three for the FCLT plus the convergence of the drift to the parabolic term. In Section 5, we show how to adapt the proof to the GA model. In Section 6 we discuss extensions of the GA model by changing the assumptions and the density $f_T(t)$ in $t = 0$. In particular, we investigate what happens when $f_T'(0) = 0$. In Section 7 we discuss various connections with the literature in queueing, random graphs and statistics. There, we also discuss several future research directions.

3 Preliminaries

In this section, we present some auxiliary results used in the proof of Theorem 1. All random variables defined from now on are defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When needed, elements of Ω will always be denoted by ω . Given two real-valued random variables X, Y we say that X *stochastically dominates* Y , and we denote it by $Y \preceq X$, if

$$\mathbb{P}(X \leq x) \leq \mathbb{P}(Y \leq x), \quad \forall x \in \mathbb{R},$$

so that for every nondecreasing function $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\mathbb{E}[f(Y)] \leq \mathbb{E}[f(X)]. \quad (3.1)$$

We write $f(n) = O(g(n))$ for functions $f, g \geq 0$ and $n \rightarrow \infty$ if there exists a constant $C > 0$ such that $\lim_{n \rightarrow \infty} f(n)/g(n) \leq C$, and $f(n) = o(g(n))$ if $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$. Furthermore, we write $O_{\mathbb{P}}(a_n)$ for a sequence of random variables X_n for which $|X_n|/a_n$ is tight as $n \rightarrow \infty$. Moreover, we write $o_{\mathbb{P}}(a_n)$ for a sequence of random variables X_n for which $|X_n|/a_n \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$.

All the processes we deal with are elements of the space $\mathcal{D}([0, \infty), \mathbb{R})$ (or \mathcal{D} , for short) of càdlàg functions, defined as the space of all functions $f : [0, \infty) \rightarrow \mathbb{R}$ such that $\forall \bar{x} \in [0, \infty)$

- $\lim_{x \rightarrow \bar{x}^+} f(x) = f(\bar{x})$;
- $\lim_{x \rightarrow \bar{x}^-} f(x) < \infty$.

Hence, $N_n(\cdot)$ is a \mathcal{D} -valued random variable. Following [5], we say that X_n *converges in distribution* (or *weakly converges*) to X (and denote it by $X_n \xrightarrow{d} X$) if $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X)]$ as n tends to infinity for every $f(\cdot)$ that is real-valued, bounded and continuous. In particular, if X is \mathcal{D} -valued, $f(\cdot)$ can be any (almost everywhere) continuous function from \mathcal{D} to \mathbb{R} . Thus, to formally establish convergence in distribution in the space of \mathcal{D} -valued random variables, a metric, or at least a topology, on \mathcal{D} is needed (in order to define continuity of the functions $f(\cdot)$). Several topologies on the space \mathcal{D} have been defined (all by Skorokhod, in his famous paper [29]). For our purposes we will consider the J_1 topology, which can be described as being generated by some metric d_∞ on $\mathcal{D}([0, \infty), \mathbb{R})$ defined as an extension of some metric d_t on $\mathcal{D}([0, t], \mathbb{R})$. The latter is defined as follows. Let $\|\cdot\|$ indicate the supremum norm, $\text{id}(\cdot)$ the identity function on $[0, t]$ and Λ_t the space of nondecreasing homeomorphisms on $[0, t]$. Define for any $x_1, x_2 \in \mathcal{D}$

$$d_t(x_1, x_2) := \inf_{\lambda \in \Lambda_t} \{\max\{\|\lambda(\cdot) - \text{id}(\cdot)\|, \|x_1(\cdot) - x_2(\lambda(\cdot))\|\}\} \quad (3.2)$$

and

$$d_\infty(x_1, x_2) := \int_0^\infty e^{-t} [d_t(x_1, x_2) \wedge 1] dt. \quad (3.3)$$

It is possible to show (see, e.g., [31]) that this is the correct way of extending the metric, and thus the topology, from $\mathcal{D}([0, t], \mathbb{R})$ to $\mathcal{D}([0, \infty), \mathbb{R})$ in the sense that convergence with respect to d_∞ is equivalent to convergence with respect to d_t on any compact subset of the form $[0, t]$.

The general idea to prove Theorem 1 is to first show the weak convergence of the (rescaled) process $(N_n(k))_{k \geq 0}$ defined in (2.10) and then to deduce the convergence of the reflected process

$(\phi(N_n)(k))_{k \geq 0}$ exploiting the following continuous-mapping theorem. In fact, this is a special case of a general technique known as the *continuous-mapping approach* (in [31] and [32] a detailed description of this technique is given). Through the continuous-mapping approach one reduces the problem of establishing convergence of random objects to one of continuity of suitable functions.

Theorem 2 (Continuous Mapping Theorem). *If $X_n \xrightarrow{d} X$ and f is continuous almost surely with respect to the distribution of X , then $f(X_n) \xrightarrow{d} f(X)$.*

Suppose we have shown that $n^{-1/3}N_n(\cdot n^{2/3}) \xrightarrow{d} W(\cdot)$. To prove Theorem 1 we are left to prove that the function $\phi : \mathcal{D} \rightarrow \mathcal{D}$ given by

$$\phi : f(\cdot) \mapsto f(\cdot) - \inf_{y \leq \cdot} f^-(y) \quad (3.4)$$

is continuous almost surely with respect to the distribution of $W(\cdot)$. For this note that $\mathbb{P}(W(\cdot) \in \mathcal{C}) = 1$, where $\mathcal{C} = \mathcal{C}([0, \infty), \mathbb{R}) \subset \mathcal{D}$ denotes the space of continuous functions from $[0, \infty)$ to \mathbb{R} . Then by [31, Theorem 4.1] and [31, Theorem 6.1], ϕ is continuous almost surely with respect to the distribution of $W(\cdot)$.

To prove the convergence of the rescaled queue length process we make use of a well-known martingale functional central limit theorem (see [10], §7) in the special case in which the limit process is a standard Brownian motion. For a thorough overview of this important case, see [33]. For completeness we include the theorem that we shall apply, which is the one-dimensional version of the one contained in [33]. Recall that if $M(t)$ is a square-integrable martingale with respect to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, the *predictable quadratic variation process* associated with $M(t)$ is the unique nondecreasing, nonnegative, predictable, integrable process $B(t)$ such that $M^2(t) - B(t)$ is a martingale with respect to $\{\mathcal{F}_t\}_{t \geq 0}$.

Theorem 3 (MFCLT). *Let $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ be an increasing filtration and $\{\bar{M}_n\}_{n \in \mathbb{N}}$ be a sequence of continuous-time, real-valued, square-integrable martingales, each with respect to \mathcal{F}_n , such that $\bar{M}_n(0) = 0$. Assume that $(\bar{B}_n(t))_{t \geq 0}$, the predictable quadratic variation process associated with $(\bar{M}_n(t))_{t \geq 0}$, and $M_n(t)$ satisfy the following conditions:*

- (a) $\bar{B}_n(t) \xrightarrow{\mathbb{P}} \sigma^2 t, \quad \forall t \in [0, \infty]$;
- (b) $\lim_{n \rightarrow \infty} \mathbb{E}[\sup_{t \leq \bar{t}} |\bar{B}_n(t) - \bar{B}_n(t^-)|] = 0, \quad \forall \bar{t} \in \mathbb{R}^+$;
- (c) $\lim_{n \rightarrow \infty} \mathbb{E}[\sup_{t \leq \bar{t}} |\bar{M}_n(t) - \bar{M}_n(t^-)|^2] = 0, \quad \forall \bar{t} \in \mathbb{R}^+$.

Then, as $n \rightarrow \infty$, \bar{M}_n converges in distribution in $\mathcal{D}([0, \infty), \mathbb{R})$ to a centered Brownian motion with variance $\sigma^2 t$.

In order to apply the MFCLT, we apply a Doob-type decomposition to the process $N_n(k)$ to write it as the sum of a martingale—which will converge to the Brownian motion—and an appropriate drift term:

$$\begin{aligned} N_n(k) &= \sum_{i=1}^k (A_n(i) - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}]) + \sum_{i=1}^k (\mathbb{E}[A_n(i)|\mathcal{F}_{i-1}] - 1) \\ &=: M_n(k) + C_n(k), \end{aligned} \quad (3.5)$$

with $\{\mathcal{F}_i\}_{i \geq 1}$ the filtration generated by $(A_n(k))_{k \geq 1}$, i.e. $\mathcal{F}_i = \sigma(\{A_n(k)\}_{k=1}^i)$. Another Doob decomposition of interest is

$$M_n^2(k) = Z_n(k) + B_n(k) \quad (3.6)$$

with $Z_n(k)$ a martingale and $B_n(k)$ the discrete-time predictable quadratic variation of the process $M_n(k)$. Note that for every fixed n and k , $|M_n(k)|$ is bounded and thus has a finite second moment. Therefore $B_n(k)$ exists and is given by

$$\begin{aligned} B_n(k) &= \sum_{i=1}^k \mathbb{E}[(A_n(i) - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}])^2 | \mathcal{F}_{i-1}] \\ &= \sum_{i=1}^k (\mathbb{E}[A_n(i)^2 | \mathcal{F}_{i-1}] - \mathbb{E}[A_n(i) | \mathcal{F}_{i-1}]^2). \end{aligned} \quad (3.7)$$

To see this, we rewrite

$$\begin{aligned} M_n^2(k) &= \sum_{i=1}^k (A_n(i) - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}])^2 + \sum_{\substack{i,j \leq k \\ i \neq j}} (A_n(i) - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}])(A_n(j) - \mathbb{E}[A_n(j)|\mathcal{F}_{j-1}]) \\ &=: \sum_{i=1}^k (A_n(i) - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}])^2 + L_n(k). \end{aligned} \quad (3.8)$$

By developing $L_n(k)$ one can easily see that it is a martingale. The decomposition (3.6) then follows from:

$$\begin{aligned} Z_n(k) &:= \sum_{i=1}^k (A_n(i) - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}])^2 - \sum_{i=1}^k \mathbb{E}[(A_n(i) - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}])^2 | \mathcal{F}_{i-1}] + L_n(k) \\ B_n(k) &:= \sum_{i=1}^k \mathbb{E}[(A_n(i) - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}])^2 | \mathcal{F}_{i-1}]. \end{aligned} \quad (3.9)$$

Note that $Z_n(k)$ is the sum of two martingales and thus is itself a martingale.

We now turn to the case $T_i \sim \text{Exp}(\lambda)$. We begin with an important lemma, which should be thought as stating that, for the small-o term in the Taylor expansions we exploit, it holds

$$\mathbb{E}[o_{\mathbb{P}}(x)] = o(x).$$

Lemma 1. *Let S, T be positive random variables. Define $D := S/n$. If (2.17) holds for T and $\mathbb{E}[S^2] < \infty$, then*

$$\mathbb{E}[|\mathbb{P}(T \leq D|D) - c_T \cdot D|] = o(n^{-4/3}), \quad (3.10)$$

$$\mathbb{E}[|D(\mathbb{P}(T \leq D|D) - c_T \cdot D)|] = o(n^{-2}), \quad (3.11)$$

$$\mathbb{E}[|\mathbb{P}(T \leq D|D) - c_T \cdot D|^2] = o(n^{-2}). \quad (3.12)$$

Proof. Since

$$\mathbb{P}(T \leq D|D) = F_T(D) = c_T \cdot D + o(S^{4/3}n^{-4/3}), \quad (3.13)$$

pointwise convergence trivially holds:

$$n^{4/3}|F_T(D) - c_T \cdot D| \xrightarrow{\text{a.s.}} 0, \quad \text{as } n \rightarrow \infty. \quad (3.14)$$

Moreover, there exists a constant $C > 0$ such that

$$|F_T(x) - c_T \cdot x| \leq Cx^{4/3}, \quad (3.15)$$

which implies

$$n^{4/3}|F_T(D) - c_T \cdot D| \stackrel{\text{a.s.}}{\leq} n^{4/3} \cdot C(S/n)^{4/3}. \quad (3.16)$$

Since $\mathbb{E}[S^{4/3}] < \infty$, the random variable $n^{4/3}|F_T(D) - c_T \cdot D|$ is bounded by an integrable random variable not depending on n . We can then apply the Dominated Convergence Theorem to obtain (3.10). Using an analogous argument we can also prove (3.11) and (3.12). Pointwise convergence is again trivial. Dominance is obtained by noting that there exists constant $C_1, C_2 > 0$ such that

$$x|F_T(x) - c_T \cdot x| \leq C_1x^2 \quad \text{and} \quad |F_T(x) - c_T \cdot x|^2 \leq C_2x^2. \quad (3.17)$$

Indeed, for $x \ll 1$, $|F_T(x) - c_T \cdot x|^2 \leq C_2x^{8/3} \leq C_2x^2$ for some $N_2 > 0$, and for $x \gg 1$ it is enough to notice that $F_T(x)$ is bounded. The first bound in (3.17) is obtained in the same way. Since $\mathbb{E}[S^2] < \infty$ by assumption, (3.11) and (3.12) again follow by dominated convergence. \square

A random variable that plays a major role in the proof is the one associated with the arrivals when the queue would not deplete,

$$A'_n := \sum_{i=1}^n \mathbf{1}_{\{T_i \leq D\}}. \quad (3.18)$$

Note that

$$A_n(k) \preceq A'_n, \quad \forall k \geq 1. \quad (3.19)$$

Before turning to the proof of Theorem 1 in Section 4, we establish some lemmas that will prove to be useful both in the proof and in providing some insight into the behavior of the process $N_n(\cdot)$.

Lemma 2. *Let $N_n(\cdot)$ be the process defined in (2.10). Then, for $k = O(n^{2/3})$, $n^{-2/3}N_n(k) \preceq G_n(k)$, where $G_n(k)$ is a random variable such that $G_n(k) \xrightarrow{\mathbb{P}} 0$.*

Proof. First note that

$$N_n(k-1) \preceq \sum_{j=1}^{k-1} \left(\sum_{l=1}^n \mathbf{1}_{\{T_l \leq D_j\}} - 1 \right) = \sum_{j=1}^{k-1} (A'_n(j) - 1). \quad (3.20)$$

By the Weak LLN for uncorrelated random variables (see, e.g., [20]) it is enough to prove that $\sup_{n \in \mathbb{N}} \text{Var}(A'_n) < \infty$. Write

$$\begin{aligned} \text{Var}(A'_n)^2 &= \mathbb{E}[(A'_n)^2] - \mathbb{E}[A'_n]^2 \\ &= \mathbb{E}[A'_n] + \mathbb{E}\left[\sum_{\substack{i,j \leq n \\ i \neq j}} \mathbf{1}_{\{T_i \leq D\}} \mathbf{1}_{\{T_j \leq D\}} \right] - \mathbb{E}[A'_n]^2. \end{aligned} \quad (3.21)$$

The terms $\mathbb{E}[A'_n]$ and $\mathbb{E}[A'_n]^2$ are uniformly bounded since

$$\mathbb{E}[A'_n] = 1 + \beta n^{-1/3} + o(n^{-1/3}). \quad (3.22)$$

Moreover,

$$\mathbb{E}\left[\sum_{\substack{i,j \leq n \\ i \neq j}} \mathbb{1}_{\{T_i \leq D\}} \mathbb{1}_{\{T_j \leq D\}}\right] = \sum_{\substack{i,j \leq n \\ i \neq j}} \mathbb{E}[F_T(D)F_T(D)] \leq C + o(1), \quad (3.23)$$

for some $C > 1$, where we have used Lemma 1 to bound the terms of lower order than D^2 appearing when Taylor expanding $F_T(D)F_T(D)$. Both error terms in (3.22) and (3.23) can be bounded from above by a constant independent from n . Therefore $\sup_{n \in \mathbb{N}} \text{Var}(A'_n) < \infty$ and the Weak LLN applies. \square

Intuitively Lemma 2 states that the queue length process is of order smaller than $n^{2/3}$. Note however that the convergence we established is not uniform in $j \leq k$, with $k = O(n^{2/3})$. We now work towards this result.

We will make use of the following well-known lemma for the order statistics of exponential random variables. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the order statistics of random variables X_1, \dots, X_n .

Lemma 3. *Let E_1, \dots, E_n be independent exponentially distributed random variables with mean one. Then,*

$$(E_{(j)})_{j=1}^n \stackrel{d}{=} \left(\sum_{s=1}^j \frac{E_s}{n-s+1} \right)_{j=1}^n.$$

In particular we can couple $(E_{(j)})_{j=1}^n$ with $(E_j)_{j=1}^n$ in such a way that for all $j \leq n$,

$$E_{(j)} \geq \frac{\text{a.s.}}{n} E_1 + \dots + E_j. \quad (3.24)$$

We next investigate the random variable A'_n . The following lemma states that on average, in the limit, the contribution to the queue length of arrivals of order $n^{1/3}$ or greater is negligible.

Lemma 4. *Define A'_n as in (3.18). Then $A_n'^2$ is stochastically dominated by a family of uniformly integrable (with respect to n) random variables. In particular,*

$$\mathbb{E}\left[(A'_n)^2 \mathbb{1}_{\{(A'_n)^2 > \varepsilon n^{2/3}\}}\right] \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.25)$$

Proof. Observe that, if $E_i \sim \text{Exp}(1)$, $u_i = 1 - \exp(-E_i)$ is a uniform random variable on $[0, 1]$ so that

$$A'_n = \sum_{i=1}^n \mathbb{1}_{\{T_i \leq D\}} = \sum_{i=1}^n \mathbb{1}_{\{T_{(i)} \leq D\}} \stackrel{d}{=} \sum_{i=1}^n \mathbb{1}_{\{F_T^{-1}(1 - \exp(-E_{(i)})) \leq D\}}. \quad (3.26)$$

Since the function $x \mapsto F_T^{-1}(1 - \exp(-x))$ is monotone, by using the coupling in Lemma 3,

$$\begin{aligned} \sum_{i=1}^n \mathbb{1}_{\{F_T^{-1}(1 - \exp(-E_{(i)})) \leq D\}} &\preceq \sum_{i=1}^n \mathbb{1}_{\{F_T^{-1}(1 - \exp(-\sum_{j=1}^i E_j/n)) \leq D\}} \\ &\stackrel{\text{a.s.}}{=} \sum_{i=1}^n \mathbb{1}_{\{\sum_{j=1}^i E_j \leq -n \log(1 - F_T(D))\}}. \end{aligned} \quad (3.27)$$

By (2.17), $F_T(x)/x$ is bounded from above by a positive constant $K \in \mathbb{R}^+$, so that

$$\sum_{i=1}^n \mathbb{1}_{\{\sum_{j=1}^i E_j \leq -n \log(1 - F_T(D))\}} \preceq \sum_{i=1}^n \mathbb{1}_{\{\sum_{j=1}^i E_j \leq -n \log(1 - K \cdot D)\}}. \quad (3.28)$$

Fix ε and find the respective C such that $-\log(1 - x) \leq Cx$ for all $0 \leq x \leq 1 - \varepsilon$. We do this in order to remove the dependencies from n . We then obtain that

$$\begin{aligned} A'_n &\preceq \left(\sum_{i=1}^{\infty} \mathbb{1}_{\{\sum_{j=1}^i E_j \leq CK \cdot nD\}} \right) \mathbb{1}_{\{K \cdot D \leq 1 - \varepsilon\}} + A'_n \mathbb{1}_{\{K \cdot D > 1 - \varepsilon\}} \\ &\preceq N(CK \cdot nD) + A'_n \mathbb{1}_{\{K \cdot D > 1 - \varepsilon\}}, \end{aligned} \quad (3.29)$$

where

$$N(t, \omega) := \sum_{i=1}^{\infty} \mathbb{1}_{\{\sum_{j=1}^i E_j \leq t\}}(\omega) \quad (3.30)$$

is a Poisson process with rate one. We now prove that each of the two terms in (3.29) is a family of uniformly integrable random variables, and thus also their sum is. Since by assumption S_k has a finite second moment, also $N(CK \cdot nD)$ has. Since the latter does not depend on n , it is uniformly integrable with respect to n . Moving to the second term, note that, since $A'_n \leq n$ a.s.,

$$\begin{aligned} \mathbb{E}[A_n'^2 \mathbb{1}_{\{K \cdot D > 1 - \varepsilon\}}] &\leq n^2 \mathbb{P}\left(S_k > \frac{(1 - \varepsilon)n}{K(1 + \beta n^{-1/3})}\right) \\ &\leq n^2 K^2 (1 + \beta n^{-1/3})^2 \frac{\mathbb{E}[S_k^2 \mathbb{1}_{\{S_k > \frac{(1 - \varepsilon)n}{K(1 + \beta n^{-1/3})}\}}]}{(1 - \varepsilon)^2 n^2}. \end{aligned} \quad (3.31)$$

Since $\mathbb{E}[S^2] < \infty$,

$$\mathbb{E}[S_k^2 \mathbb{1}_{\{S_k > \frac{(1 - \varepsilon)n}{K(1 + \beta n^{-1/3})}\}}] \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

The second moments of the second term in (3.29) converge to zero as n goes to infinity, and thus $\{A_n'^2 \mathbb{1}_{\{K \cdot D > 1 - \varepsilon\}}\}_{n \geq 1}$ is a uniformly integrable family. Therefore, $(N(CK \cdot nD) + A_n' \mathbb{1}_{\{K \cdot D > 1 - \varepsilon\}})^2$ is uniformly integrable. We have then shown that $(A_n'^2)_{n \geq 1}$ is stochastically dominated by a random variable with uniformly integrable second moments. The second claim then follows by the stochastic domination result in (3.1). \square

4 Proof of Theorem 1 for the RC model

Recall that $N_n(k)$ can be decomposed as $N_n(k) = 1 + M_n(k) + C_n(k)$, where $M_n(k)$ is a martingale and $C_n(k)$ is the drift term. Moreover, $M_n^2(k)$ was also written as $M_n^2(k) = Z_n(k) + B_n(k)$ with $Z_n(k)$ the Doob martingale and $B_n(k)$ its drift. The proof then consists of verifying the following conditions:

$$(i) \sup_{t \leq \bar{t}} |n^{-1/3} C_n(tn^{2/3}) - \beta t + \frac{1}{2} t^2| \xrightarrow{\mathbb{P}} 0, \quad \forall \bar{t} \in \mathbb{R}^+;$$

- (ii) $n^{-2/3}B_n(tn^{2/3}) \xrightarrow{\mathbb{P}} \sigma^2 t, \quad \forall t \in \mathbb{R}^+;$
- (iii) $\lim_{n \rightarrow \infty} n^{-2/3} \mathbb{E}[\sup_{t \leq \bar{t}} |B_n(tn^{2/3}) - B_n(tn^{2/3}-)|] = 0, \quad \forall \bar{t} \in \mathbb{R}^+;$
- (iv) $\lim_{n \rightarrow \infty} n^{-2/3} \mathbb{E}[\sup_{t \leq \bar{t}} |M_n(tn^{2/3}) - M_n(tn^{2/3}-)|^2] = 0, \quad \forall \bar{t} \in \mathbb{R}^+.$

Recall that $\sigma^2 := c_T^2 \mathbb{E}[S_1^2]$.

Condition (i) implies the convergence of the drift term, while conditions (ii)-(iv) imply the convergence of the (rescaled) process $M_n(k)$ to a centered Brownian motion, by Theorem 3. By standard convergence results one can then conclude that the rescaled version of the sum $C_n(k) + M_n(k)$ converges in distribution to the sum of the respective limits.

4.1 Proof of (i)

We first prove (i) and to that end, we expand the term $\mathbb{E}[A_n(i)|\mathcal{F}_{i-1}]$. Define ν_i as the set of the customers that are no longer in the population at the beginning of the service of the i -th customer. Then,

$$\begin{aligned}
\mathbb{E}[A_n(i)|\mathcal{F}_{i-1}] &= \sum_{s \notin \nu_{i-1}} \mathbb{E}[\mathbf{1}_{\{T_s \leq D_i\}}|\mathcal{F}_{i-1}] \\
&= \sum_{s \notin \nu_{i-1}} \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{T_s \leq D_i\}}|\mathcal{F}_{i-1}, D_i]|\mathcal{F}_{i-1}] \\
&= \sum_{s \notin \nu_{i-1}} \left(\mathbb{E}[c_T D_i|\mathcal{F}_{i-1}] + o(n^{-4/3}) \right), \tag{4.1}
\end{aligned}$$

where, in the last equality, we have used Lemma 1. Since D_i is independent from \mathcal{F}_{i-1} , we obtain

$$\mathbb{E}[A_n(i)|\mathcal{F}_{i-1}] = \sum_{s \notin \nu_{i-1}} \left(\mathbb{E}[c_T D_i] + o(n^{-4/3}) \right). \tag{4.2}$$

In what follows, we denote by $|A|$ the cardinality of a set A . The summation can then be simplified to

$$\begin{aligned}
\mathbb{E}[A_n(i)|\mathcal{F}_{i-1}] &= (n - |\nu_{i-1}|) \left(c_T \mathbb{E}[D_i] + o(n^{-4/3}) \right) \\
&= c_T \cdot \mathbb{E}[S_i] (1 + \beta n^{-1/3}) - c_T \mathbb{E}[S_i] (1 + \beta n^{-1/3}) \frac{|\nu_{i-1}|}{n} \\
&\quad + (n - |\nu_{i-1}|) o(n^{-4/3}), \tag{4.3}
\end{aligned}$$

where we note that $|\nu_i|$ is an integer-valued random variable. Then,

$$\mathbb{E}[A_n(i) - 1|\mathcal{F}_{i-1}] = c_T (1 + \beta n^{-1/3}) \mathbb{E}[S_i] - 1 - c_T (1 + \beta n^{-1/3}) \mathbb{E}[S_i] \frac{|\nu_{i-1}|}{n} + o(n^{-1/3}), \tag{4.4}$$

By (2.8), $c_T (1 + \beta n^{-1/3}) \mathbb{E}[S_i] = 1 + \beta n^{-1/3} + o(n^{-1/3})$, so that (4.4) yields

$$\mathbb{E}[A_n(i) - 1|\mathcal{F}_{i-1}] = \beta n^{-1/3} - \frac{|\nu_{i-1}|}{n} (1 + O(n^{-1/3})) + o(n^{-1/3}). \tag{4.5}$$

Note that the service times are independent from the history of the system, thus the conditioning has no effect. Since $|\nu_{i-1}| = N_n(i-1) + i - 1$, the drift term in the decomposition of $N_n(k)$ can now be written as

$$C_n(k) = k\beta n^{-1/3} - \left(\frac{k^2 - k}{2} + \sum_{i=1}^k N_n(i-1) \right) \left(\frac{1}{n} + O(n^{-4/3}) \right) + ko(n^{-1/3}). \quad (4.6)$$

The term $-\sum_{i=1}^k N_n(i-1)$ in (4.6) accounts for the fact that the customers already in the queue cannot rejoin it. This term converges to zero as n tends to infinity (after appropriate scaling) by the following result. In fact, Lemma 5 proves that the process $n^{-2/3}N_n(j)$ tends to zero in probability, uniformly on $j \leq an^{2/3}$, a significantly stronger statement than Lemma 2:

Lemma 5. *Let $(N_n(k))_{k \geq 0}$ be the process defined in (2.10) and $a \in \mathbb{R}^+$. Then*

$$n^{-2/3} \sup_{j \leq an^{2/3}} |N_n(j)| \xrightarrow{\mathbb{P}} 0. \quad (4.7)$$

Proof. Recall that $N_n(j) = M_n(j) + C_n(j)$. Then

$$\begin{aligned} \mathbb{P}\left(n^{-2/3} \sup_{j \leq an^{2/3}} |N_n(j)| \geq 2\varepsilon\right) &\leq \mathbb{P}\left(n^{-2/3} \sup_{j \leq an^{2/3}} |M_n(j)| \geq \varepsilon\right) \\ &\quad + \mathbb{P}\left(n^{-2/3} \sup_{j \leq an^{2/3}} |C_n(j)| \geq \varepsilon\right). \end{aligned} \quad (4.8)$$

We can separately bound the first and second terms. Applying Doob's inequality to the martingale $M_n(\cdot)$ gives

$$\mathbb{P}\left(n^{-2/3} \sup_{j \leq an^{2/3}} |M_n(j)| \geq \varepsilon\right) \leq \frac{\mathbb{E}[M_n^2(an^{2/3})]}{(\varepsilon n^{2/3})^2}. \quad (4.9)$$

By (3.6), $\mathbb{E}[M_n^2(k)] = \mathbb{E}[B_n(k)]$. Expanding this term gives

$$\begin{aligned} \mathbb{E}[B_n(k)] &= \mathbb{E}\left[\sum_{i=1}^k (\mathbb{E}[A_n(i)^2 | \mathcal{F}_{i-1}] - \mathbb{E}[A_n(i) | \mathcal{F}_{i-1}]^2)\right] \\ &\leq \mathbb{E}\left[\sum_{i=1}^k \mathbb{E}[A_n(i)^2 | \mathcal{F}_{i-1}]\right] \leq k \cdot \mathbb{E}[A_n'^2], \end{aligned} \quad (4.10)$$

where A_n' is defined as in (3.18). By rescaling we get

$$n^{-4/3} \mathbb{E}[B_n(an^{2/3})] \leq an^{-2/3} \mathbb{E}[A_n'^2]. \quad (4.11)$$

Since A_n' has a finite second moment as was shown in (3.23), the right-most term in (4.11) tends to zero as n tends to infinity.

For the second term in (4.8) we make use of the decomposition of the drift term in (4.6). From there we obtain

$$-(n^{-1} + O(n^{-4/3})) \sum_{i=1}^k \left((i-1) + N_n(i-1) \right) + ko(n^{-1/3}) \leq C_n(k) \leq k\beta n^{-1/3} + ko(n^{-1/3}), \quad (4.12)$$

and thus,

$$\begin{aligned} \sup_{j \leq an^{2/3}} |C_n(j)| &\stackrel{\text{a.s.}}{\leq} (an^{2/3}\beta n^{-1/3} + an^{2/3}o(n^{-1/3})) \\ &\vee \left((n^{-1} + O(n^{-4/3})) \sum_{i=1}^{an^{2/3}} \left((i-1) + N_n(i-1) \right) + an^{2/3}o(n^{-1/3}) \right), \end{aligned} \quad (4.13)$$

since the two sides of the bound (4.12) are monotone functions of k . By rescaling the first term by $n^{-2/3}$ we obtain $a\beta n^{-1/3} + a \cdot o(n^{-1/3})$, which tends to zero almost surely as n goes to infinity. The second term in (4.13) needs more attention. Notice that the function $i \mapsto i + N_n(i)$ is nonnegative and nondecreasing. Thus, isolating the sum, we can bound it through its final term:

$$\sum_{i=1}^{an^{2/3}} \left((i-1) + N_n(i-1) \right) \stackrel{\text{a.s.}}{\leq} an^{2/3}(an^{2/3} - 1 + N_n(an^{2/3} - 1)) \quad (4.14)$$

Rescaling by $n^{-2/3}$ we get for the second term in (4.13)

$$\begin{aligned} n^{-2/3} \left((n^{-1} + O(n^{-4/3})) \sum_{i=1}^{an^{2/3}} \left((i-1) + N_n(i-1) \right) + an^{2/3}o(n^{-1/3}) \right) \\ \stackrel{\text{a.s.}}{\leq} a(n^{-1} + O(n^{-4/3}))(an^{2/3} - 1 + N_n(an^{2/3} - 1)) + a \cdot o(n^{-1/3}) \end{aligned} \quad (4.15)$$

The $O(n^{-4/3})$ term is of lower order than n^{-1} and can be ignored. By Lemma 2, the right-hand side of (4.15) tends to zero in probability as n tends to infinity and this concludes the proof that the second term in (4.8) converges in probability to zero. \square

Substituting $k = tn^{2/3}$ into (4.6) and multiplying by $n^{-1/3}$ yields

$$n^{-1/3}C_n(tn^{2/3}) = \beta t - \left(\frac{t^2 - tn^{-2/3}}{2} + \sum_{i=1}^k N_n(i-1) \right) \left(1 + O(n^{-1/3}) \right) + o(1). \quad (4.16)$$

Both the small- o and the big- O terms in (4.16) are independent of t (or, equivalently, of k). Indeed, the small- o term originates from Lemma 1 (and is therefore independent of k) and the big- O term was introduced in (4.5) and depends only on n and β . Therefore, the convergence of $n^{-1/3}C_n(tn^{2/3})$ is uniform in $t \leq \bar{t}$ for fixed \bar{t} as required, and this concludes the proof of (i).

4.2 Proof of (ii)

In order to prove (ii) we first compute

$$\begin{aligned} \mathbb{E}[A_n(i)^2 | \mathcal{F}_{i-1}] &= \mathbb{E} \left[\sum_{j \notin \nu_{i-1}} \mathbf{1}_{\{T_j \leq D_i\}}^2 + \sum_{\substack{l \neq m \\ l, m \notin \nu_{i-1}}} \mathbf{1}_{\{T_l \leq D_i\}} \mathbf{1}_{\{T_m \leq D_i\}} \middle| \mathcal{F}_{i-1} \right] \\ &= \mathbb{E}[A_n(i) | \mathcal{F}_{i-1}] + \mathbb{E} \left[\sum_{\substack{l \neq m \\ l, m \notin \nu_{i-1}}} \mathbf{1}_{\{T_l \leq D_i\}} \mathbf{1}_{\{T_m \leq D_i\}} \middle| \mathcal{F}_{i-1} \right], \end{aligned} \quad (4.17)$$

which yields

$$\begin{aligned}
\mathbb{E}[A_n(i)^2|\mathcal{F}_{i-1}] - \mathbb{E}[A_n(i)|\mathcal{F}_{i-1}] &= \sum_{\substack{l \neq m \\ l, m \notin \nu_{i-1}}} \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{T_l \leq D_i\}} \mathbb{1}_{\{T_m \leq D_i\}} | \mathcal{F}_{i-1}, D_i] | \mathcal{F}_{i-1}] \\
&= \sum_{\substack{l \neq m \\ l, m \notin \nu_{i-1}}} \mathbb{E}[\mathbb{P}(T_l \leq D_i | D_i) \mathbb{P}(T_m \leq D_i | D_i) | \mathcal{F}_{i-1}] \\
&= \sum_{\substack{l \neq m \\ l, m \notin \nu_{i-1}}} \mathbb{E}[F_T(D_i)^2 | \mathcal{F}_{i-1}]. \tag{4.18}
\end{aligned}$$

Using (2.17) we can rewrite the summation term as

$$\mathbb{E}[F_T(D_i)^2 | \mathcal{F}_{i-1}] = c_T^2 \mathbb{E}[D_i^2] + 2c_T \mathbb{E}[D_i(F_T(D_i) - c_T D_i)] + \mathbb{E}[(F_T(D_i) - c_T D_i)^2]. \tag{4.19}$$

The second and third terms are $o(n^{-2})$ by Lemma 1. Therefore, we rewrite (4.18) as

$$\sum_{\substack{l \neq m \\ l, m \notin \nu_{i-1}}} \left(\frac{c_T^2}{n^2} (1 + \beta n^{-1/3})^2 \mathbb{E}[S^2] + o(n^{-2}) \right) = \frac{|\Xi_{i-1}| c_T^2}{n^2} (1 + \beta n^{-1/3})^2 \mathbb{E}[S^2] + o(1). \tag{4.20}$$

Note that the cardinality of the set $\Xi_{i-1} := \{(l, m) : l \neq m, l, m \notin \nu_{i-1}\}$ is

$$|\Xi_{i-1}| = (n - N_n(i-1) - i + 1)^2 - (n - N_n(i-1) - i + 1), \tag{4.21}$$

thus of the order n^2 . Then, for $k = O(n^{2/3})$,

$$\begin{aligned}
B_n(k) &= \sum_{i=1}^k (\mathbb{E}[A_n(i)^2 | \mathcal{F}_{i-1}] - \mathbb{E}[A_n(i) | \mathcal{F}_{i-1}])^2 \\
&= \sum_{i=1}^k \left(\frac{|\Xi_{i-1}| c_T^2}{n^2} (1 + \beta n^{-1/3})^2 \mathbb{E}[S^2] + \mathbb{E}[A_n(i) | \mathcal{F}_{i-1}] - \mathbb{E}[A_n(i) | \mathcal{F}_{i-1}]^2 + o(1) \right). \tag{4.22}
\end{aligned}$$

Using (4.5), together with the observation that $|\nu_i|/n = O_{\mathbb{P}}(n^{-1/3})$ uniformly for $i = O(n^{2/3})$,

$$\begin{aligned}
B_n(k) &= \sum_{i=1}^k \left(\frac{|\Xi_{i-1}| c_T^2}{n^2} (1 + \beta n^{-1/3})^2 \mathbb{E}[S^2] + O_{\mathbb{P}}(n^{-1/3}) + o(1) \right) \\
&= \sum_{i=1}^k \frac{(n - N_n(i-1) - i + 1)^2 - (n - N_n(i-1) - i + 1)}{n^2} c_T^2 (1 + \beta n^{-1/3})^2 \mathbb{E}[S^2] \\
&\quad + O_{\mathbb{P}}(kn^{-1/3}) + o(k).
\end{aligned}$$

We then split the term inside the summation to isolate the contribution of the process N_n , and write

$$\begin{aligned}
B_n(k) &= \sum_{i=1}^k \frac{(n - i + 1)^2 - (n - i + 1)}{n^2} c_T^2 (1 + \beta n^{-1/3})^2 \mathbb{E}[S^2] \\
&\quad + \sum_{i=1}^k \frac{N_n(i-1)(N_n(i-1) - 2(n - i + 1) + 1)}{n^2} c_T^2 (1 + \beta n^{-1/3})^2 \mathbb{E}[S^2] \\
&\quad + O_{\mathbb{P}}(kn^{-1/3}) + o(k). \tag{4.23}
\end{aligned}$$

By Lemma 5 the second term (accounting for the process history) tends to zero in probability when rescaled appropriately. An elementary computation shows that

$$\begin{aligned} \frac{c_T^2 \mathbb{E}[S^2]}{n^2} \sum_{l=n-k}^{n-1} (l^2 - l) &= \frac{\sigma^2}{n^2} (2/3k + k^2 - 1/3k^3 - 2kn - k^2n + kn^2) \\ &= \sigma^2 k + O(k^2 n^{-1}). \end{aligned} \quad (4.24)$$

The remaining terms were omitted because they are of order smaller than $O(k^2 n^{-1})$ when $k = sn^{2/3}$. When rescaling space and time appropriately in (4.23) we finally obtain that

$$n^{-2/3} B_n(tn^{2/3}) \xrightarrow{\mathbb{P}} \sigma^2 t,$$

as required.

4.3 Proof of (iii)

Note that by basic Doob decomposition theory, $(B_n(k))_{k \in \mathbb{N}}$ in (3.6) is almost surely increasing. For (iii), we need to estimate the largest possible jump

$$n^{-2/3} |B_n(k+1) - B_n(k)| = n^{-2/3} |\mathbb{E}[A_n(k+1)^2 | \mathcal{F}_k] - \mathbb{E}[A_n(k+1) | \mathcal{F}_k]^2|, \quad (4.25)$$

with $k = O(n^{2/3})$. In order to apply the Dominated Convergence Theorem, we first prove that it converges pointwise to zero, and then that it is (almost surely) bounded. The jump (4.25) has already been implicitly computed as the term in the summation in (4.23) and it takes the form

$$\begin{aligned} &n^{-2/3} |\mathbb{E}[A_n(k+1)^2 | \mathcal{F}_k] - \mathbb{E}[A_n(k+1) | \mathcal{F}_k]^2| \\ &= n^{-2/3} \left| c_T^2 \mathbb{E}[S_{k+1}^2] \left(\frac{(n-k)^2 - (n-k)}{n^2} + \frac{N_n(k)(N_n(k) - 2(n-k) + 1)}{n^2} \right) + O_{\mathbb{P}}(n^{-1/3}) \right| \end{aligned} \quad (4.26)$$

The $O_{\mathbb{P}}(n^{-1/3})$ term is a byproduct of $\mathbb{E}[A_n(k) | \mathcal{F}_{k-1}] - \mathbb{E}[A_n(k) | \mathcal{F}_{k-1}]^2$ (as computed in (4.22) and following calculations). We now compute it precisely by using the exact expression for $\mathbb{E}[A_n(k) | \mathcal{F}_{k-1}]$ found in (4.5), as follows

$$|\mathbb{E}[A_n(k) | \mathcal{F}_{k-1}] - \mathbb{E}[A_n(k) | \mathcal{F}_{k-1}]^2| \stackrel{\text{a.s.}}{\leq} \beta n^{-1/3} + 2|\nu_{k-1}|n^{-1} + 2\beta|\nu_{k-1}|n^{-4/3} + o(n^{-1/3}).$$

The right-most term is bounded by 3 for all sufficiently large values of n , uniformly in $k \leq tn^{2/3}$ (recall that $|\nu_k| \leq n$). Then, by inserting this into (4.26),

$$\begin{aligned} &n^{-2/3} |\mathbb{E}[A_n(k+1)^2 | \mathcal{F}_k] - \mathbb{E}[A_n(k+1) | \mathcal{F}_k]^2| \\ &\stackrel{\text{a.s.}}{\leq} n^{-2/3} \left| \sigma^2 \left(\frac{(n-k)^2 - (n-k)}{n^2} + \frac{N_n(k)(N_n(k) - 2(n-k) + 1)}{n^2} \right) + 3 \right|. \end{aligned} \quad (4.27)$$

Since $|N_n(k)| \leq n$, we can find a constant C such that, uniformly in $k \leq tn^{2/3}$,

$$n^{-2/3} |\mathbb{E}[A_n(k+1)^2 | \mathcal{F}_k] - \mathbb{E}[A_n(k+1) | \mathcal{F}_k]^2| \stackrel{\text{a.s.}}{\leq} n^{-2/3} (3 + \sigma^2 C). \quad (4.28)$$

This gives both the assumptions in the Dominated Convergence Theorem (almost sure convergence and dominance with an integrable random variable) and therefore concludes the proof of (iii).

4.4 Proof of (iv)

We tackle (iv) through a coupling argument. First observe that

$$\begin{aligned} n^{-2/3} \mathbb{E}[\sup_{t \leq \bar{t}} |M_n(tn^{2/3}) - M_n(tn^{2/3}-)|^2] &= n^{-2/3} \mathbb{E}[\sup_{k \leq \bar{t}n^{2/3}} |A_n(k) - \mathbb{E}[A_n(k)|\mathcal{F}_{k-1}]|^2] \\ &\leq n^{-2/3} \mathbb{E}[\sup_{k \leq \bar{t}n^{2/3}} |A_n(k)|^2] + n^{-2/3} \mathbb{E}[\sup_{k \leq \bar{t}n^{2/3}} |\mathbb{E}[A_n(k)|\mathcal{F}_{k-1}]|^2]. \end{aligned} \quad (4.29)$$

The second term in (4.29) is easily estimated. Using the calculations on $\mathbb{E}[A_n(k)|\mathcal{F}_{k-1}]$ done in (4.5), and that the second term there is negative, yields

$$0 \stackrel{\text{a.s.}}{\leq} \mathbb{E}[A_n(k)|\mathcal{F}_{k-1}] \stackrel{\text{a.s.}}{\leq} 1 + O(n^{-1/3}).$$

For the first term, we will use a coupling argument. For $\varepsilon > 0$, we split

$$\begin{aligned} \mathbb{E}\left[\sup_{k \leq \bar{t}n^{2/3}} |A_n(k)|^2\right] &= \mathbb{E}\left[\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 \mathbf{1}_{\{\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 \leq \varepsilon n^{2/3}\}}\right] \\ &\quad + \mathbb{E}\left[\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 \mathbf{1}_{\{\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 > \varepsilon n^{2/3}\}}\right]. \end{aligned} \quad (4.30)$$

After multiplying (4.30) by $n^{-2/3}$, the first term can trivially be bounded by ε , while for the second term we estimate

$$\begin{aligned} \mathbb{E}\left[\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 \mathbf{1}_{\{\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 > \varepsilon n^{2/3}\}}\right] &\leq \sum_{k=1}^{\bar{t}n^{2/3}} \mathbb{E}[A_n(k)^2 \mathbf{1}_{\{A_n(k)^2 > \varepsilon n^{2/3}\}}] \\ &\leq \sum_{k=1}^{\bar{t}n^{2/3}} \mathbb{E}[A'_n(k)^2 \mathbf{1}_{\{A'_n(k)^2 > \varepsilon n^{2/3}\}}] \\ &= \bar{t}n^{2/3} \mathbb{E}[A'_n(1)^2 \mathbf{1}_{\{A'_n(1)^2 > \varepsilon n^{2/3}\}}], \end{aligned} \quad (4.31)$$

where we have used the stochastic domination in (3.18). By Lemma 4, $\mathbb{E}[A'_n(1)^2 \mathbf{1}_{\{A'_n(1)^2 > \varepsilon n^{2/3}\}}] \rightarrow 0$ and thus

$$n^{-2/3} \mathbb{E}\left[\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 \mathbf{1}_{\{\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 > \varepsilon n^{2/3}\}}\right] \rightarrow 0.$$

This concludes the proof of (iv).

5 Proof of Theorem 1 for the GA model

We now generalize the result obtained in the direction of allowing for non-exponential arrival times. Consider a family of arrival times $(T_i)_{i=1}^n$ and denote their order statistics by $T_{(1)} \leq T_{(2)} \leq \dots \leq T_{(n)}$. As before, let $(D_j)_{j \geq 1}$ be the rescaled service times with $D_j := S_j(1 + \beta n^{-1/3})/n$. Then, at time k , the number of arrivals during every service is defined as

$$A_n(k) := \sum_{i=1}^n \mathbf{1}_{\{\sum_{j=1}^{k-1} D_j < T_{(i)} \leq \sum_{j=1}^k D_j\}}. \quad (5.1)$$

Note that, unlike in the RC model, the arrival clocks are not reset at every service time, but rather ring once and for all. Therefore, in this setting, slightly stronger assumptions are needed. We assume that $F_T(\cdot)$ can be Taylor expanded in a neighborhood of every point, as in (2.1) and that the density $f_T(\cdot)$ can be Taylor expanded in a neighborhood of zero, as in (2.4).

5.1 Supporting lemmas

For readability, throughout this section we will use the notation $\Sigma_j := \sum_{i=1}^j D_i$. In this section we present several useful lemmas that are appropriate adaptations of lemmas in Section 3. We provide the proof only when it is substantially different from their counterparts in Section 3.

Lemma 6. *In the present setting and for $k = O(n^{2/3})$, the following holds:*

$$\mathbb{E}[|F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - f_T(\Sigma_{k-1}) \cdot D_k| | \Sigma_{k-1}] = o_{\mathbb{P}}(n^{-4/3}). \quad (5.2)$$

$$\mathbb{E}[|D_k(F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - f_T(\Sigma_{k-1}) \cdot D_k)| | \Sigma_{k-1}] = o_{\mathbb{P}}(n^{-2}). \quad (5.3)$$

$$\mathbb{E}[|F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - f_T(\Sigma_{k-1}) \cdot D_k|^2 | \Sigma_{k-1}] = o_{\mathbb{P}}(n^{-2}). \quad (5.4)$$

Moreover, all the statements of convergence hold uniformly for $k = O(n^{2/3})$.

Proof. We give the proof for (5.3), the rest can be shown in an analogous way. Note that, by our assumptions on F_T ,

$$\begin{aligned} & \mathbb{E}[n^2 |D_k \cdot (F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - f_T(\Sigma_{k-1})D_k)| | \Sigma_{k-1}] \\ & \stackrel{\text{a.s.}}{\leq} \sup_{y \leq Cn^{-1/3}} n^2 \mathbb{E}[|F_T(y + D_k) - F_T(y) - f_T(y)D_k| \cdot D_k] \\ & \leq n^2 \mathbb{E}[\sup_{y \leq Cn^{-1/3}} |F_T(y + D_k) - F_T(y) - f_T(y)D_k| \cdot D_k] \end{aligned} \quad (5.5)$$

since with high probability $\Sigma_{k-1} \leq Cn^{-1/3}$ for some $C > 0$. The right term tends to zero by the Dominated Convergence Theorem and assumption (2.2), and this immediately implies (5.3). \square

As we have already seen in the RC model, the queue length is roughly of the order $n^{1/3}$ around time $tn^{2/3}$. The following lemma is the counterpart for the GA model of Lemma 2 and we prove it in a similar fashion.

Lemma 7. *Let $N_n(k)$ be the queue length process defined in (2.10). Then $n^{-2/3}N_n(k) \preceq G_n(k)$, where $G_n(k)$ is a random variable such that $G_n(k) \xrightarrow{\mathbb{P}} 0$ for $k = O(n^{2/3})$.*

Proof. For simplicity we only consider $k = tn^{2/3}$. Fix an arbitrary $\varepsilon > 0$. Then,

$$\begin{aligned} & \mathbb{P}\left(n^{-2/3} \sum_{j=1}^{tn^{2/3}} \left(\sum_{l=1}^n \mathbb{1}_{\{\Sigma_{j-1} \leq T_{(l)} \leq \Sigma_j\}} \right) - n^{-2/3}t \geq \varepsilon\right) \\ & \leq \mathbb{P}\left(n^{-2/3} \left| \sum_{j=1}^{tn^{2/3}} \left(\sum_{l=1}^n \mathbb{1}_{\{\Sigma_{j-1} \leq T_{(l)} \leq \Sigma_j\}} \right) - n^{-2/3}t \right| \geq \varepsilon\right) \\ & = \mathbb{P}\left(n^{-2/3} \left| \sum_{l=1}^n \mathbb{1}_{\{T_{(l)} \leq \Sigma_{tn^{2/3}}\}} - n^{-2/3}t \right| \geq \varepsilon\right) \\ & \leq \frac{\mathbb{E}\left[\left(\sum_{l=1}^n \mathbb{1}_{\{T_{(l)} \leq \Sigma_{tn^{2/3}}\}}\right)^2 \mathbb{1}_{\{\left|\sum_{l=1}^n \mathbb{1}_{\{T_{(l)} \leq \Sigma_{tn^{2/3}}\}} - n^{-2/3}t\right| \geq \varepsilon n^{2/3}\}}\right]}{n^{4/3}\varepsilon^2}. \end{aligned} \quad (5.6)$$

We are left to bound the expected value in (5.6). To do so, we define the event $\mathcal{E}_n := \{|\sum_{l=1}^n \mathbf{1}_{\{T_{(l)} \leq \Sigma_{tn^{2/3}}\}} - n^{-2/3}t| \geq \varepsilon n^{2/3}\}$ and write

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{l=1}^n \mathbf{1}_{\{T_l \leq \Sigma_{tn^{2/3}}\}}\right)^2 \mathbf{1}_{\mathcal{E}_n}\right] &\leq n + \mathbb{E}\left[\sum_{h \neq k} \mathbf{1}_{\{T_h \leq \Sigma_{tn^{2/3}}\}} \mathbf{1}_{\{T_k \leq \Sigma_{tn^{2/3}}\}} \mathbf{1}_{\mathcal{E}_n}\right] \\ &\leq n + n^2 \mathbb{E}[F_T(\Sigma_{tn^{2/3}})^2 \mathbf{1}_{\mathcal{E}_n}] \leq n + Cn^2 \mathbb{E}[(\Sigma_{tn^{2/3}})^2 \mathbf{1}_{\mathcal{E}_n}] \\ &\leq n + Cn^{4/3} \mathbb{E}[S^2 \mathbf{1}_{\mathcal{E}_n}], \end{aligned} \tag{5.7}$$

for a large constant C . The last inequality in (5.7) was obtained by an application of the Cauchy-Schwarz inequality. Since $\mathbb{P}(\mathcal{E}_n) \rightarrow 0$ and $\mathbb{E}[S^2] < \infty$, by inserting (5.7) into (5.6) and the Dominated Convergence Theorem, we get the desired convergence. \square

The following lemma reduces the task of estimating quantities involving order statistics of exponentials, which are possibly complicated objects, to the one of dealing with a Poisson process, which is much simpler. It will be used later for proving Lemma 9, which is the equivalent in this setting of Lemma 4.

Lemma 8. *Consider the order statistics $(E_{(i)})_{i=1}^{n-v}$ ($v \leq n$) of n exponential unit mean random variables. Define $|\Upsilon_{(0,c)}^{(n-v)}|$ as the cardinality of the set $\Upsilon_{(0,c)}^{(n-v)} := \{j \in \{1, \dots, n-v\} : E_{(j)} \in (0, c)/n\}$. Then,*

$$|\Upsilon_{(0,c)}^{(n-v)}| \preceq N\left(\frac{n-v}{n}c\right), \tag{5.8}$$

where $N(t)$ is a Poisson process with unit rate.

Proof. The statement is a consequence of Lemma 3. Fix $j \in \{1, \dots, n-v\}$. By definition of stochastic domination

$$\mathbb{P}(E_{(j)} \leq c/n) \leq \mathbb{P}\left(\frac{\sum_{i=1}^j E_i}{n-v} \leq \frac{c}{n}\right) \leq \mathbb{P}\left(\Pi_j \leq \frac{n-v}{n}c\right), \tag{5.9}$$

where Π_j is the j -th point of a Poisson process with rate one. The computation in (5.9) intuitively means that there are more Poisson points in an interval of length $(n-v)\frac{c}{n}$ than order statistics in an interval of length $\frac{c}{n}$. This implies (5.8). \square

Since

$$N\left(\frac{n-v}{n} \cdot c\right) \preceq N(c), \quad \forall v \leq n,$$

it follows from (5.8) that

$$|\Upsilon_{(0,c)}^{(n-v)}| \preceq N(c), \quad \forall v \leq n. \tag{5.10}$$

Corollary 1. *Under the same assumptions as in Lemma 8,*

$$\Upsilon_{(a,b)}^{(n)} \preceq N(b-a). \tag{5.11}$$

Proof. By Lemma 8,

$$\mathbb{P}(N(b-a) \leq x) \leq \mathbb{P}(|\Upsilon_{(0,b-a)}^{(n-v)}| \leq x). \quad (5.12)$$

Note that, by the memoryless property,

$$\mathbb{P}(|\Upsilon_{(0,b-a)}^{(n-v)}| \leq x) \stackrel{\text{a.s.}}{=} \mathbb{P}(|\Upsilon_{(a,b)}^{(n)}| \leq x \mid |\Upsilon_{(0,a)}^{(n)}| = v). \quad (5.13)$$

Since the left side of (5.12) does not depend on v , by combining (5.12) and (5.13) and taking the expectation on both sides in order to remove the conditioning, we get

$$\mathbb{P}(N(b-a) \leq x) \leq \mathbb{P}(|\Upsilon_{(a,b)}^{(n)}| \leq x), \quad (5.14)$$

which is (5.11). \square

One of the cornerstones of the analysis in Section 4 was Lemma 4. An analogous version holds in this general setting.

Lemma 9. $A_n(k)$ is stochastically bounded by a random variable with uniformly integrable (with respect to n) second moment, uniformly in $k \leq tn^{2/3}$.

Proof. Note that $T_{(i)} \stackrel{d}{=} F_T^{-1}(1 - \exp(-E_{(i)}))$, where $(E_{(i)})_{i=1}^n$ are the order statistics of unit mean exponential random variables. Then,

$$\begin{aligned} A_n(k) &= \sum_{i=1}^n \mathbb{1}_{\{\Sigma_{k-1} \leq T_{(i)} \leq \Sigma_k\}} \stackrel{d}{=} \sum_{i=1}^n \mathbb{1}_{\{\Sigma_{k-1} \leq F_T^{-1}(1 - \exp(-E_{(i)})) \leq \Sigma_k\}} \\ &\stackrel{d}{=} \sum_{i=1}^n \mathbb{1}_{\{F_T(\Sigma_{k-1}) - 1 \leq -\exp(-E_{(i)}) \leq F_T(\Sigma_k) - 1\}} \\ &\stackrel{d}{=} \sum_{i=1}^n \mathbb{1}_{\{-\log(1 - F_T(\Sigma_{k-1})) \leq E_{(i)} \leq -\log(1 - F_T(\Sigma_k))\}}. \end{aligned} \quad (5.15)$$

By Corollary 1,

$$A_n(k) \leq N\left(n \log\left(\frac{1 - F_T(\Sigma_{k-1})}{1 - F_T(\Sigma_k)}\right)\right). \quad (5.16)$$

By splitting the event space Ω we can write (5.16) as

$$A_n(k) \leq N\left(n \log\left(\frac{1 - F_T(\Sigma_{k-1})}{1 - F_T(\Sigma_k)}\right)\right) \mathbb{1}_{\{\Sigma_k \leq \bar{x}\}} + A_n(k) \mathbb{1}_{\{\Sigma_k > \bar{x}\}}, \quad (5.17)$$

where \bar{x} is independent of n and determined later on. The plan now is to show that the first term in (5.17) is bounded by a random variable independent of n and with finite second moment, and that the second term has second moments converging to zero, as n tends to infinity. These two facts together imply that the right-hand side of (5.17) has uniformly integrable second moments.

The first term is easily bounded. We now choose \bar{x} in such a way that $1 - F_T(\bar{x}) > 0$. By Taylor expanding the function $x \mapsto \log\left(\frac{1 - F_T(\Sigma_{k-1})}{1 - F_T(\Sigma_{k-1} + x)}\right)$, we get

$$\begin{aligned} N\left(n \log\left(\frac{1 - F_T(\Sigma_{k-1})}{1 - F_T(\Sigma_k)}\right)\right) \mathbf{1}_{\{\Sigma_k \leq \bar{x}\}} &= N\left(n \frac{f_T(\Sigma_{k-1})}{1 - F_T(\Sigma_{k-1})} \frac{S_k}{n} + o_{\mathbb{P}}(1)\right) \mathbf{1}_{\{\Sigma_k \leq \bar{x}\}} \\ &\leq N\left(\frac{M}{1 - F_T(\bar{x})} S_k + C S_k\right), \end{aligned} \quad (5.18)$$

where we have used that the density $f_T(\cdot)$ admits a maximum value M since it is continuous, $0 \leq f_T(0) < \infty$ and $\lim_{t \rightarrow \infty} f_T(t) = 0$, and we bounded the error term by $C S_k$, $C > 0$. The right-most term in (5.18) has finite second moment, since $\mathbb{E}[S^2] < \infty$. For the second term we proceed by noting that

$$A_n(k)^2 \mathbf{1}_{\{\Sigma_k \geq \bar{x}\}} \stackrel{\text{a.s.}}{\leq} A_n(k)^2 \mathbf{1}_{\{D_k \geq \bar{x}/2\}} + A_n(k)^2 \mathbf{1}_{\{\Sigma_k \geq \bar{x}, D_k < \bar{x}/2\}}, \quad (5.19)$$

The mean of the first term can be bounded by

$$\mathbb{E}[A_n(k)^2 \mathbf{1}_{\{D_k \geq \bar{x}/2\}}] \leq n^2 \mathbb{P}(S_k \geq n\bar{x}/2) \leq n^2 \frac{\mathbb{E}[S_k^2 \mathbf{1}_{\{S_k \geq n\bar{x}/2\}}]}{(n\bar{x}/2)^2}, \quad (5.20)$$

and the right-hand side tends to zero as n tends to infinity since $\mathbb{E}[S_k^2] < \infty$. For the second term, some more work is needed. First observe that $\mathbf{1}_{\{\Sigma_k \geq \bar{x}, D_k < \bar{x}/2\}} \leq \mathbf{1}_{\{\Sigma_{k-1} \geq \bar{x}/2\}}$. After dominating as usual $A_n(k)^2$ by n^2 , we compute

$$\begin{aligned} \mathbb{E}[A_n(k)^2 \mathbf{1}_{\{\Sigma_k \geq \bar{x}, D_k < \bar{x}/2\}}] &\leq n^2 \mathbb{E}[\mathbb{E}[\mathbf{1}_{\{\Sigma_{k-1} \leq T_i \leq \Sigma_k\}} \mathbf{1}_{\{\Sigma_{k-1} \geq \bar{x}/2\}} | \Sigma_k]] \\ &= n^2 \mathbb{E}[\mathbf{1}_{\{\Sigma_{k-1} \geq \bar{x}/2\}} \mathbb{E}[\mathbf{1}_{\{\Sigma_{k-1} \leq T_i \leq \Sigma_k\}} | \Sigma_k]] \\ &= n^2 \mathbb{E}[\mathbf{1}_{\{\Sigma_{k-1} \geq \bar{x}/2\}} (F_T(\Sigma_k) - F_T(\Sigma_{k-1}))]. \end{aligned} \quad (5.21)$$

By applying the Mean Value Theorem to $F_T(\cdot)$, we obtain

$$|F_T(\Sigma_k) - F_T(\Sigma_{k-1})| \stackrel{\text{a.s.}}{\leq} M \cdot D_k, \quad (5.22)$$

where we recall that $M = \max_{t \in [0, \infty)} f_T(t)$. Inserting this into (5.21),

$$\mathbb{E}[A_n(k)^2 \mathbf{1}_{\{\Sigma_k \geq \bar{x}, D_k < \bar{x}/2\}}] \leq n M \mathbb{E}[S_k \mathbf{1}_{\{\Sigma_{k-1} \geq \bar{x}/2\}}] = n M \mathbb{E}[S_k] \mathbb{P}(\Sigma_{k-1} \geq \bar{x}/2).$$

It is easy to see that the right-hand side converges to zero by using Chebyshev's inequality. Taking n so large that $n^{2/3} \mathbb{E}[S] \leq \frac{n\bar{x}}{4}$,

$$\begin{aligned} \mathbb{P}(\Sigma_{k-1} \geq \bar{x}/2) &\leq \mathbb{P}\left(\left|\sum_{i=1}^{tn^{2/3}} S_i - n^{2/3} \mathbb{E}[S_i]\right| \geq n\bar{x}/4\right) \\ &\leq 16 \frac{tn^{2/3} \text{Var}(S_i)}{n^2 \bar{x}^2} = o(n^{-1}). \end{aligned} \quad (5.23)$$

This concludes the proof that the second moment of the second term in (5.17) tends to zero as n tends to infinity. \square

We conclude with a useful application of Doob's inequality:

Lemma 10. *Assume $(S_i)_{i \geq 0}$ is a sequence of i.i.d. random variables with finite second moment. Then for any $\alpha, \beta > 0$ such that $\alpha < 2\beta$,*

$$\frac{\sup_{k \leq tn^\alpha} |\sum_{i=1}^k S_i - k \cdot \mathbb{E}[S]|}{n^\beta} \xrightarrow{\mathbb{P}} 0. \quad (5.24)$$

Proof. Define $M_k := \sum_{j=1}^k (S_j - \mathbb{E}[S])$. Then $(M_k)_{k \geq 1}$ is a martingale. Therefore, by Doob's inequality applied to the sub-martingale $(|M_k|)_{k \geq 1}$, we have

$$\begin{aligned} \mathbb{P}\left(\frac{\sup_{k \leq tn^\alpha} |M_k|}{n^\beta} > \varepsilon\right) &\leq \frac{\mathbb{E}[M_{tn^\alpha}^2]}{\varepsilon^2 n^{2\beta}} \\ &= \frac{tn^\alpha \mathbb{E}[(S - \mathbb{E}[S])^2]}{\varepsilon^2 n^{2\beta}}. \end{aligned}$$

This converges to zero since $\alpha < 2\beta$. Note that ε can depend on n , for example by defining $\varepsilon := n^{-\delta}$ and choosing δ such that $\delta < \beta - \alpha/2$. \square

5.2 Proof of Theorem 1

The proof consists, again, in verifying four conditions in order to apply Theorem 3. The first is the convergence of the drift

$$(i) \sup_{t \leq \bar{t}} |n^{-1/3} C_n(tn^{2/3}) - \beta t - f'_T(0)/c_T^2 \cdot \frac{t^2}{2}| \xrightarrow{\mathbb{P}} 0, \quad \forall \bar{t} \in \mathbb{R}^+;$$

while (ii), (iii) and (iv) are technical conditions and analogous to the ones given in Section 4. The filtration we consider henceforth is defined as

$$\mathcal{F}_i := \sigma(\{A_n(j), D_j\}_{j \leq i}). \quad (5.25)$$

5.2.1 Proof of (i)

Computing the asymptotic drift essentially amounts to computing the discrete drift, which in turn depends heavily on

$$\begin{aligned} \mathbb{E}[A_n(k) | \mathcal{F}_{k-1}] &= \sum_{i=1}^n \mathbb{E} \left[\mathbb{1}_{\{\Sigma_{k-1} \leq T_{(i)} \leq \Sigma_k\}} \middle| \mathcal{F}_{k-1} \right] \\ &= \sum_{i \notin \nu_{k-1}} \mathbb{E} \left[\mathbb{1}_{\{\Sigma_{k-1} \leq T_{(i)} \leq \Sigma_k\}} \middle| \mathcal{F}_{k-1} \right] \\ &= \sum_{i \notin \nu_{k-1}} \mathbb{E} \left[\mathbb{1}_{\{T_{(i)} \leq \Sigma_k\}} \middle| \Sigma_{k-1}, \{T_{(i)} \geq \Sigma_{k-1}\} \right], \end{aligned} \quad (5.26)$$

where, as before, ν_i denotes the set of the customers that no longer remain in the population at the beginning of the service of the i -th customer. The second (almost sure) equality is due to the fact that the σ -algebra \mathcal{F}_{k-1} contains information on how many arrivals have occurred up to time $k-1$ (formally, ν_{k-1} is \mathcal{F}_{k-1} -measurable). Adding the conditioning on $\{T_{(i)} \geq \Sigma_{k-1}\}$ does not influence the conditional expectation, since $T_{(i)}$ is such that $i \notin \nu_{k-1}$. Note also that

$i \notin \nu_{k-1}$ implies $T_{(i)} \geq \Sigma_{k-1}$. Then, defining for simplicity $\mathcal{E}_{k-1} := \{\Sigma_{k-1}, \{T_i \geq \Sigma_{k-1}\}\}$,

$$\begin{aligned} \mathbb{E}[A_n(k)|\mathcal{F}_{k-1}] &= \sum_{i \notin \nu_{k-1}} \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\Sigma_{k-1} \leq T_{(i)} \leq \Sigma_k\}} | D_k, \mathcal{E}_{k-1}] \middle| \mathcal{E}_{k-1}\right] \\ &= (n - |\nu_{k-1}|) \mathbb{E}\left[\frac{F_T(\Sigma_k) - F_T(\Sigma_{k-1})}{1 - F_T(\Sigma_{k-1})} \middle| \mathcal{E}_{k-1}\right] \\ &= \frac{(n - |\nu_{k-1}|)}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[F_T(\Sigma_k) - F_T(\Sigma_{k-1}) | \mathcal{E}_{k-1}]. \end{aligned} \quad (5.27)$$

We now rearrange the terms in order to distinguish between the ones contributing to the limit and those vanishing, as follows

$$\begin{aligned} \mathbb{E}[A_n(k)|\mathcal{F}_{k-1}] - 1 &= \frac{(n - |\nu_{k-1}|)}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[F_T(\Sigma_k) - F_T(\Sigma_{k-1}) | \mathcal{E}_{k-1}] - \frac{1 - F_T(\Sigma_{k-1})}{1 - F_T(\Sigma_{k-1})} \\ &= \frac{n}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - 1/n | \mathcal{E}_{k-1}] \\ &\quad - \frac{1}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[|\nu_{k-1}|(F_T(\Sigma_k) - F_T(\Sigma_{k-1})) - F_T(\Sigma_{k-1}) | \mathcal{E}_{k-1}] \\ &=: A_n^{(1)}(k) - A_n^{(2)}(k). \end{aligned} \quad (5.28)$$

$A_n^{(1)}(k)$ groups all the terms appearing in the limit, while $A_n^{(2)}(k)$ groups all the terms of lower order, vanishing in the limit. We treat them separately, starting with $A_n^{(1)}(k)$. The term $F_T(\Sigma_k) - F_T(\Sigma_{k-1})$ can be simplified by our assumptions. By (2.1) and Lemma 6,

$$A_n^{(1)}(k) = \frac{n}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[f_T(\Sigma_{k-1})D_k - n^{-1} + o_{\mathbb{P}}(n^{-4/3}) | \mathcal{E}_{k-1}], \quad (5.29)$$

and by (2.4),

$$\begin{aligned} \mathbb{E}[f_T(\Sigma_{k-1})D_k | \mathcal{E}_{k-1}] &= (c_T + f'_T(0)\Sigma_{k-1} + o(\Sigma_{k-1})) \mathbb{E}[D] \\ &= c_T \mathbb{E}[D_k] + f'_T(0)\Sigma_{k-1} \mathbb{E}[D_k] + o(\Sigma_{k-1}) \mathbb{E}[D_k], \end{aligned} \quad (5.30)$$

where, with a slight abuse of notation, we denoted $|f_T(\Sigma_{k-1}) - c_T - f'_T(0)\Sigma_{k-1}|$ by $o(\Sigma_{k-1})$. Since, for $k = O(n^{2/3})$,

$$n^{1/3}|f_T(\Sigma_{k-1}) - c_T - f'_T(0)\Sigma_{k-1}| \xrightarrow{\text{a.s.}} 0, \quad (5.31)$$

also convergence in probability holds, that is, $o(\Sigma_{k-1}) = o_{\mathbb{P}}(n^{-1/3})$. In particular $o(\Sigma_{k-1}) \cdot \mathbb{E}[D_k] = o_{\mathbb{P}}(n^{-4/3})$. Inserting (5.30) into (5.29) yields

$$\begin{aligned} A_n^{(1)}(k) &= \frac{n}{1 - F_T(\Sigma_{k-1})} \left(c_T \mathbb{E}[D_k] - 1/n + f'_T(0) \mathbb{E}[D_k] \Sigma_{k-1} + o_{\mathbb{P}}(n^{-4/3}) \right) \\ &= \frac{1}{1 - F_T(\Sigma_{k-1})} \left(c_T \mathbb{E}[S_k] - 1 + \frac{f'_T(0)}{n} \sum_{j=1}^{k-1} S_j \mathbb{E}[S_k] + o_{\mathbb{P}}(n^{-1/3}) \right). \end{aligned} \quad (5.32)$$

The criticality assumption $c_T \mathbb{E}[S_k] = 1 + \beta n^{-1/3} + o(n^{-1/3})$ leads to

$$A_n^{(1)}(k) = \frac{1}{1 - F_T(\Sigma_{k-1})} \left(\beta + f'_T(0) \mathbb{E}[S_k] \frac{\sum_{j=1}^{k-1} S_j}{n^{2/3}} + o_{\mathbb{P}}(1) \right) \cdot n^{-1/3}. \quad (5.33)$$

Recall that the drift term is defined as

$$C_n(s) = \sum_{k=1}^s (\mathbb{E}[A_n(k)|\mathcal{F}_{k-1}] - 1) = \sum_{k=1}^s (A_n^{(1)}(k) - A_n^{(2)}(k)) =: C_n^{(1)}(s) + C_n^{(2)}(s). \quad (5.34)$$

We now sum (5.33) over k , obtaining

$$C_n^{(1)}(s) = \sum_{k=1}^s \frac{\beta n^{-1/3}}{1 - F_T(\Sigma_{k-1})} + \frac{f'_T(0)\mathbb{E}[S_1]}{n} \sum_{k=1}^s \frac{\sum_{j=1}^{k-1} S_j}{1 - F_T(\Sigma_{k-1})} + \sum_{k=1}^s \frac{1}{1 - F_T(\Sigma_{k-1})} o_{\mathbb{P}}(n^{-1/3}). \quad (5.35)$$

By scaling time as $s = tn^{2/3}$ and space as $n^{-1/3}$, we obtain

$$n^{-1/3}C_n^{(1)}(tn^{2/3}) = \sum_{k=1}^{tn^{2/3}} \frac{\beta n^{-2/3}}{1 - F_T(\Sigma_{k-1})} + \frac{f'_T(0)\mathbb{E}[S_1]}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} \frac{\sum_{j=1}^{k-1} S_j}{1 - F_T(\Sigma_{k-1})} + \sum_{k=1}^{tn^{2/3}} \frac{n^{-1/3} o_{\mathbb{P}}(n^{-1/3})}{1 - F_T(\Sigma_{k-1})}. \quad (5.36)$$

Before continuing, we give two easy but useful remarks:

Lemma 11. *Let $(S_i)_{i \geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{E}[S_1^2] < \infty$. Then*

$$\left| \frac{\sum_{n=1}^N \sum_{i=1}^n S_i}{N^2} - \frac{\mathbb{E}[S_1]}{2} \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } N \rightarrow \infty. \quad (5.37)$$

Moreover,

$$\left| \frac{\sum_{n=1}^N (\sum_{i=1}^n S_i)^2}{N^3} - \frac{\mathbb{E}[S_1]^2}{3} \right| \xrightarrow{\mathbb{P}} 0, \quad \text{as } N \rightarrow \infty. \quad (5.38)$$

Proof. Both claims can be proven through Lemma 10. We omit the details. \square

Another useful ingredient is the following Taylor expansion:

$$\begin{aligned} \frac{1}{1 - F_T(\Sigma_{k-1})} &= 1 + F_T(\Sigma_{k-1}) + \left(\frac{1}{1 - F_T(\Sigma_{k-1})} - 1 - F_T(\Sigma_{k-1}) \right) \\ &= 1 + O_{\mathbb{P}}(\Sigma_{k-1}). \end{aligned} \quad (5.39)$$

In what follows, we compute the limits (in probability) for each term in (5.36).

First term in (5.36). By (5.39) and Lemma 11, we have for the first term

$$\begin{aligned} &\sup_{t \leq \bar{t}} \left| n^{-2/3} \sum_{k=1}^{tn^{2/3}} \frac{1}{1 - F_T(\Sigma_{k-1})} - t \right| \\ &= \sup_{t \leq \bar{t}} \left| n^{-2/3} \sum_{k=1}^{tn^{2/3}} \left(F_T(\Sigma_{k-1}) + \left(\frac{1}{1 - F_T(\Sigma_{k-1})} - 1 - F_T(\Sigma_{k-1}) \right) \right) \right| \\ &\leq n^{-2/3} \sum_{k=1}^{\bar{t}n^{2/3}} 2F_T(\Sigma_{k-1}), \end{aligned}$$

where we have dominated the error term in the Taylor expansion (5.39) by $F_T(\Sigma_{k-1})$ and used the fact that, as a function of t , the summation is an increasing function. We can uniformly dominate F_T as follows

$$\begin{aligned} n^{-2/3} \sum_{k=1}^{\bar{t}n^{2/3}} 2F_T(\Sigma_{k-1}) &\leq n^{-2/3} \sum_{k=1}^{\bar{t}n^{2/3}} 2 \sup_{k \leq \bar{t}n^{2/3}} F_T(\Sigma_{k-1}) \\ &= 2\bar{t}F_T(\Sigma_{\bar{t}n^{2/3}}). \end{aligned} \quad (5.40)$$

The right-hand side of (5.40) tends to zero almost surely, and thus also in probability.

Second term in (5.36). Again, by (5.39), the second term simplifies to

$$\frac{f'_T(0)\mathbb{E}[S]}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} \frac{\sum_{j=1}^{k-1} S_j}{1 - F_T(\Sigma_{k-1})} = \frac{f'_T(0)\mathbb{E}[S]}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} \sum_{j=1}^{k-1} S_j + \frac{f'_T(0)\mathbb{E}[S]c_T}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} \sum_{j=1}^{k-1} S_j O_{\mathbb{P}}(\Sigma_{k-1}).$$

By the first statement of Lemma 11 the first term converges to $\frac{t^2}{2} \cdot f'_T(0)\mathbb{E}[S_1]^2$ uniformly in t . Indeed,

$$\begin{aligned} \sup_{t \leq \bar{t}} \left| n^{-4/3} \sum_{k=1}^{tn^{2/3}} \sum_{j=1}^{k-1} S_j - \frac{t^2}{2} \mathbb{E}[S] \right| &= \sup_{t \leq \bar{t}} \left| \frac{\sum_{n=1}^{tn^{2/3}} n\mathbb{E}[S]}{n^{4/3}} - \frac{t^2}{2} \mathbb{E}[S] + \frac{\sum_{n=1}^{tn^{2/3}} (\sum_{i=1}^n S_i - n\mathbb{E}[S])}{n^{4/3}} \right| \\ &= \sup_{t \leq \bar{t}} \left| \frac{t\mathbb{E}[S]}{2n^{2/3}} + \frac{\sum_{n=1}^{tn^{2/3}} (\sum_{i=1}^n S_i - n\mathbb{E}[S])}{n^{4/3}} \right| \\ &\stackrel{\text{a.s.}}{\leq} \bar{t} \cdot \frac{\mathbb{E}[S]}{2n^{2/3}} + \bar{t} \cdot \frac{\sup_{k \leq tn^{2/3}} |(\sum_{i=1}^k S_i - k\mathbb{E}[S])|}{n^{2/3}}. \end{aligned} \quad (5.41)$$

The second term converges to zero in probability by Lemma 11. Indeed,

$$\begin{aligned} \sup_{t \leq \bar{t}} \left| \frac{C}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} \left(\sum_{j=1}^{k-1} S_j \right) O_{\mathbb{P}}(\Sigma_{k-1}) \right| &\stackrel{\text{a.s.}}{\leq} \frac{2f'_T(0)\mathbb{E}[S]c_T}{n^{4/3}} \sum_{k=1}^{\bar{t}n^{2/3}} \left| \left(\sum_{j=1}^{k-1} S_j \right) F_T(\Sigma_{k-1}) \right| \\ &\stackrel{\text{a.s.}}{\leq} \frac{2C}{n^{7/3}} \sum_{k=1}^{\bar{t}n^{2/3}} \left| \left(\sum_{j=1}^{k-1} S_j \right)^2 \right|, \end{aligned} \quad (5.42)$$

where, in the first inequality, we exploited the bound $1/(1 - F_T(\Sigma_{k-1})) \leq 2F_T(\Sigma_{k-1})$, and in the second inequality a similar one for the error term in the Taylor expansion of $F_T(\cdot)$. The right-most term in (5.42) then converges to zero in probability by Lemma 11.

Third term in (5.36). The error term originates from the Taylor expansion of $n(F_T(\Sigma_k) - F_T(\Sigma_{k-1}))$ done in (5.29). To see that it is uniform in $k \leq \bar{k}$, we write $F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - f_T(\Sigma_{k-1})D_k = \epsilon_k$ and, since by assumption f' exists and is continuous in a nbh of 0, we bound it as

$$|n\epsilon_k| \leq n \sup_{x \leq Cn^{-1/3}} |f'_T(x)| \frac{D_k^2}{2}, \quad (5.43)$$

similarly as in Lemma 6, when $k = O(n^{2/3})$. In particular,

$$\sup_{k \leq tn^{2/3}} \mathbb{E}[n\epsilon_k | \mathcal{E}_{k-1}] \leq \sup_{x \leq Cn^{-1/3}} |f'_T(x)| \frac{\mathbb{E}[S^2]}{2n} = o(n^{-1/3}), \quad (5.44)$$

and the right-hand side is independent from k .

To conclude, it remains to be proven that $n^{-1/3}C_n^{(2)}(s)$ vanishes in the limit, when appropriately rescaled. To do so, we develop the terms of $A_n^{(2)}(k)$ similarly as before, obtaining

$$\begin{aligned} A_n^{(2)}(k) &= \frac{1}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[\nu_{k-1} | (f_T(\Sigma_{k-1})D_k + o_{\mathbb{P}}(n^{-1})) - c_T \Sigma_{k-1} + o_{\mathbb{P}}(n^{-1/3}) | \mathcal{E}_{k-1}] \\ &= \frac{1}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[\nu_{k-1} | f_T(\Sigma_{k-1})D_k - c_T \Sigma_{k-1} + \nu_{k-1} | o_{\mathbb{P}}(n^{-1}) + o_{\mathbb{P}}(n^{-1/3}) | \mathcal{E}_{k-1}] \\ &= \frac{\mathbb{E}[\nu_{k-1} | c_T D_k - c_T \Sigma_{k-1} + f'_T(0) \nu_{k-1} | \Sigma_{k-1} D_k + \nu_{k-1} | o_{\mathbb{P}}(n^{-1}) + o_{\mathbb{P}}(n^{-1/3}) | \mathcal{E}_{k-1}]}{1 - F_T(\Sigma_{k-1})}, \end{aligned} \quad (5.45)$$

where $o_{\mathbb{P}}(n^{-1})$ is a convenient notation for $F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - f_T(\Sigma_{k-1})D_k$ and $o_{\mathbb{P}}(n^{-1/3})$ for $F_T(\Sigma_{k-1}) - c_T \Sigma_{k-1}$. We now sum (5.45) over k to obtain

$$\begin{aligned} C_n^{(2)}(s) &= \sum_{k=1}^s \frac{1}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[\nu_{k-1} | c_T D_k - c_T \Sigma_{k-1} + f'_T(0) \nu_{k-1} | \Sigma_{k-1} D_k \\ &\quad + \nu_{k-1} | o_{\mathbb{P}}(n^{-1}) + o_{\mathbb{P}}(n^{-1/3}) | \mathcal{E}_{k-1}]. \end{aligned} \quad (5.46)$$

Recall that $|\nu_k| = k + N_n(k)$. Intuitively, k is of a much larger order than $N_n(k)$, therefore at first approximation we ignore $N_n(k)$ and we will later prove convergence of the terms containing it. We rescale, and split $C_n^{(2)}(k)$ into

$$\begin{aligned} n^{-1/3}C_n^{(2)}(tn^{2/3}) &= n^{-1/3} \sum_{k=1}^{tn^{2/3}} \frac{1}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[(k-1)c_T D_k - c_T \Sigma_{k-1} | \mathcal{E}_{k-1}] \\ &\quad + n^{-1/3} \sum_{k=1}^{tn^{2/3}} \frac{1}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[f'_T(0)(k-1)\Sigma_{k-1} D_k + (k-1)o_{\mathbb{P}}(n^{-1}) + o_{\mathbb{P}}(n^{-1/3}) | \mathcal{E}_{k-1}] + \varepsilon_n, \end{aligned} \quad (5.47)$$

where ε_n represents the terms containing $N_n(k)$. Again we rescale and perform a term-by-term analysis.

First term in (5.47). Expanding $(1 - F_T(\Sigma_{k-1}))^{-1}$ gives

$$\frac{c_T}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} \mathbb{E}[(k-1)S_k - \sum_{j=1}^{k-1} S_j | \mathcal{E}_{k-1}] + \frac{c_T^2}{n^{1/3}} \sum_{k=1}^{tn^{2/3}} O_{\mathbb{P}}(\Sigma_{k-1}) \cdot \mathbb{E}[(k-1)D_k - \Sigma_{k-1} | \mathcal{E}_{k-1}], \quad (5.48)$$

The second term is almost surely dominated by the first for n sufficiently large, so that it is enough to show (uniform) convergence of the first term. By Lemma 11,

$$-\frac{1}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} \sum_{j=1}^{k-1} S_j \xrightarrow{\mathbb{P}} -\frac{t^2}{2} \mathbb{E}[S], \quad \frac{1}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} (k-1) \mathbb{E}[S_k] \xrightarrow{\mathbb{P}} \frac{t^2}{2} \mathbb{E}[S]. \quad (5.49)$$

Therefore, (5.48) converges to zero in probability. Moreover, the convergence is uniform in $t \leq \bar{t}$ by Lemma 10.

Second term in (5.47). Expanding $(1 - F_T(\Sigma_{k-1}))^{-1}$ and ignoring all but the highest order term, which can be almost surely dominated, we get for the second term

$$f'_T(0)\mathbb{E}[S]n^{-7/3} \sum_{k=1}^{tn^{2/3}} (k-1) \sum_{j=1}^{k-1} S_j + n^{-1/3} \sum_{k=1}^{tn^{2/3}} (k-1)o_{\mathbb{P}}(n^{-1}) + tn^{1/3}o_{\mathbb{P}}(n^{-1/3}). \quad (5.50)$$

One can check, similarly as in Lemma 11, that $N^{-3} \sum_{k=1}^N k \sum_{j=1}^k S_j$ converges in probability to a non-trivial limit. Since $7/3 > (2/3) \cdot 3$, the first term converges to zero in probability. In addition, it converges uniformly in $t \leq \bar{t}$ because of the monotonicity of the sum. The small- o terms can be uniformly dominated as has already been done in (5.43).

Third term in (5.47). The remaining term is

$$\varepsilon_n = \sum_{k=1}^{tn^{2/3}} \frac{n^{-1/3}}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[N_n(k-1)(c_T D_k + f'_T(0)\Sigma_{k-1} D_k + o_{\mathbb{P}}(n^{-1})) | \mathcal{E}_{k-1}].$$

Again it is sufficient to prove (uniform) convergence for the first term in the Taylor expansion of $(1 - F_T(\Sigma_{k-1}))^{-1}$. This simplifies the previous expression to

$$\frac{c_T \mathbb{E}[S]}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} N_n(k-1) + \frac{f'_T(0)\mathbb{E}[S]}{n^{4/3}} \sum_{k=1}^{tn^{2/3}} N_n(k-1)\Sigma_{k-1} + n^{-4/3} \sum_{k=1}^{tn^{2/3}} N_n(k-1)o_{\mathbb{P}}(n^{-1}). \quad (5.51)$$

The second and third terms are again almost surely dominated by the first for n large. Moreover, the first converges to zero uniformly in probability by Lemma 12 below:

Lemma 12. *Let $(N_n(k))_{k \geq 0}$ be the process (2.10) for the General Arrivals Model and $a \in \mathbb{R}^+$. Then,*

$$n^{-2/3} \sup_{j \leq an^{2/3}} |N_n(j)| \xrightarrow{\mathbb{P}} 0. \quad (5.52)$$

Proof. The proof proceeds as in Lemma 5. We split $N_n(j)$ as the sum of a martingale and a predictable process, $N_n(j) = M_n(j) + C_n(j)$, and bound each one separately. The term containing the martingale $M_n(j)$ can be bounded through Doob's inequality, leaving us to bound $\mathbb{E}[M_n^2(an^{2/3})]/(\varepsilon n^{2/3})^2$. As was noticed in Lemma 5, $\mathbb{E}[M_n(k)^2] = \mathbb{E}[B_n(k)]$, where $B_n(k)$ is the predictable quadratic variation of $M_n(k)$, and its expectation is given by

$$\mathbb{E}[B_n(k)] = \mathbb{E}\left[\sum_{i=1}^k (\mathbb{E}[A_n(i)^2 | \mathcal{F}_{i-1}] - \mathbb{E}^2[A_n(i) | \mathcal{F}_{i-1}])\right] \leq \sum_{i=1}^k \mathbb{E}[A_n(i)^2]. \quad (5.53)$$

This term can be easily bounded exploiting Lemma 9. If A'_n is the random variable provided by Lemma 9, we have

$$\frac{1}{(\varepsilon n^{2/3})^2} \mathbb{E}[B_n(an^{2/3})] \leq \frac{1}{(\varepsilon n^{2/3})^2} \sum_{i=1}^{an^{2/3}} \mathbb{E}[(A'_n(i))^2] = \frac{\mathbb{E}[A'_n{}^2]}{\varepsilon n^{2/3}}, \quad (5.54)$$

which tends to zero because A'_n has uniformly integrable (with respect to n) second moment. The $C_n(j)$ term (that was computed in (5.28) and (5.34)) is the difference of two increasing processes, and thus can be easily bounded from either side as was done in Lemma 4. \square

This concludes the proof that

$$\sup_{t \leq \bar{t}} |n^{-1/3} C_n^{(2)}(tn^{2/3})| \xrightarrow{\mathbb{P}} 0.$$

and thus we have proven that

$$\sup_{t \leq \bar{t}} |n^{-1/3} C_n(n^{2/3}t) - \beta t - t^2 f'_T(0) \mathbb{E}[S]^2/2| \xrightarrow{\mathbb{P}} 0.$$

5.2.2 Proof of (ii)

First we compute $\mathbb{E}[A_n(k)^2 | \mathcal{F}_{k-1}]$. By proceeding as in (4.18) we obtain

$$\begin{aligned} \mathbb{E}[A_n(k)^2 | \mathcal{F}_{k-1}] - \mathbb{E}[A_n(k) | \mathcal{F}_{k-1}] &= \mathbb{E} \left[\sum_{\substack{l \neq m \\ l, m \notin \nu_{k-1}}} \mathbb{1}_{\{\Sigma_{k-1} \leq T_{(m)} \leq \Sigma_k\}} \cdot \mathbb{1}_{\{\Sigma_{k-1} \leq T_{(l)} \leq \Sigma_k\}} | \mathcal{F}_{k-1} \right] \\ &= \sum_{\substack{l \neq m \\ l, m \notin \nu_{k-1}}} \mathbb{E} \left[\frac{F_T(\Sigma_k) - F_T(\Sigma_{k-1})}{1 - F_T(\Sigma_{k-1})} \cdot \frac{F_T(\Sigma_k) - F_T(\Sigma_{k-1})}{1 - F_T(\Sigma_{k-1})} | \mathcal{F}_{k-1} \right] \\ &= \frac{\sum_{l \neq m} \mathbb{E}[(f_T(\Sigma_{k-1})D_k + o_{\mathbb{P}}(D_k^{-4/3}))^2 | \mathcal{F}_{k-1}]}{(1 - F_T(\Sigma_{k-1}))^2}, \end{aligned} \quad (5.55)$$

where the sum is over the set $\{l, m : l \neq m, l, m \notin \nu_{k-1}\}$ wherever not specified. We also denoted, for convenience, $F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - f_T(\Sigma_{k-1})D_k$ as $o_{\mathbb{P}}(D_k^{-4/3})$. We proceed as in (4.19) and (4.20). By Lemma 6,

$$\begin{aligned} &\mathbb{E}[A_n(k)^2 | \mathcal{F}_{k-1}] - \mathbb{E}[A_n(k) | \mathcal{F}_{k-1}] \\ &= \frac{1}{(1 - F_T(\Sigma_{k-1}))^2} \sum_{l \neq m} \mathbb{E}[f_T^2(\Sigma_{k-1})D_k^2 + 2f_T(\Sigma_{k-1})D_k \cdot o_{\mathbb{P}}(D_k^{4/3}) + o_{\mathbb{P}}(D_k^2) | \mathcal{F}_{k-1}] \\ &= \frac{1}{(1 - F_T(\Sigma_{k-1}))^2} \sum_{l \neq m} \left((c_T + f'_T(0)\Sigma_{k-1} + o_{\mathbb{P}}(\Sigma_{k-1}))^2 \mathbb{E}[D_k^2] + o_{\mathbb{P}}(n^{-2}) \right). \end{aligned} \quad (5.56)$$

As usual, here $o_{\mathbb{P}}(\Sigma_{k-1})$ is a shorthand notation for $f_T(\Sigma_{k-1}) - c_T - f'_T(0)\Sigma_{k-1}$. Developing the coefficient of $\mathbb{E}[D_k^2]$ reveals that it has the form $c_T^2 + \alpha_k n^{-1/3} + \beta_k o(n^{-1/3})$, with α_k and β_k converging in probability to a constant, for $k = O(n^{2/3})$. We can ignore all the terms except the one with the leading order, c_T^2 . From this point onwards the computations are identical to (4.22).

5.2.3 Proof of (iii) and (iv)

The proof of (iii) in the RC model can be carried over to the GA model without any significant change, since it relies only on (ii) and Lemma 12.

For (iv), we split the quantity according to

$$\begin{aligned} n^{-2/3} \mathbb{E} \left[\sup_{t \leq \bar{t}} |M_n(tn^{2/3}) - M(tn^{2/3}-)|^2 \right] &\leq n^{-2/3} \mathbb{E} \left[\sup_{k \leq \bar{t}n^{2/3}} |A_n(k)|^2 \right] \\ &\quad + n^{-2/3} \mathbb{E} \left[\sup_{k \leq \bar{t}n^{2/3}} |\mathbb{E}[A_n(k) | \mathcal{F}_{k-1}]|^2 \right]. \end{aligned} \quad (5.57)$$

The second term is straightforward. Indeed, (5.28) and (5.33) give the crude bound

$$\mathbb{E}[A_n(k)|\mathcal{F}_{k-1}] \stackrel{\text{a.s.}}{\leq} A_n^{(1)}(k) \stackrel{\text{a.s.}}{\leq} C + o_{\mathbb{P}}(1),$$

for $k = O(n^{2/3})$ and some positive constant $C > 1$, uniform over $k \leq \bar{t}n^{2/3}$. The first term can also be estimated imitating (4.30). Indeed, fix $\varepsilon > 0$ and split it as

$$\begin{aligned} \mathbb{E}\left[\sup_{k \leq \bar{t}n^{2/3}} |A_n(k)|^2\right] &= \mathbb{E}\left[\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 \mathbf{1}_{\{\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 \leq \varepsilon n^{2/3}\}}\right] \\ &\quad + \mathbb{E}\left[\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 \mathbf{1}_{\{\sup_{k \leq \bar{t}n^{2/3}} A_n(k)^2 > \varepsilon n^{2/3}\}}\right]. \end{aligned} \quad (5.58)$$

The first term is trivially bounded. The bounding of the second term proceeds as in (4.31) and is concluded through Lemma 9.

6 The ℓ -th order contact case

The technique developed until here can be exploited to prove limit results for the more general case in which the function $t \mapsto f_T(t) - \mu$ has ℓ -th order contact in zero, defined as follows:

Definition 1 (*ℓ -th order contact point*). *Given a smooth, real-valued, function $f(t)$, we say it has ℓ -th order contact in \bar{t} if $f(\bar{t}) = 0$, $f^{(l)}(\bar{t}) = 0$ for $l = 1, \dots, \ell - 1$ and $f^{(\ell)}(\bar{t}) \neq 0$.*

If $f(\bar{t}) = o(1)$ and all the other assumptions on Definition 1 are kept, we still say that f has an ℓ -th order contact in \bar{t} . Indeed, our criticality assumption is $f_T(0) - \mu = o(1)$, where the error term is specified later. The assumption that the argmax of f_T is zero allows us to consider both odd and even order contacts. In this case, Theorem 1 can be generalized. In the following we consider the GA model, i.e., the arrivals are defined by (1.4).

Theorem 4 (*Asymptotics for the critical ℓ -th order contact queue*). *Assume that the function $f_T(t) - \mu$ has ℓ -th order contact in 0, where $\ell \geq 1$. Define*

$$\alpha = \frac{\ell}{\ell + 1/2}. \quad (6.1)$$

Then,

$$n^{-\alpha/2} Q_n(tn^\alpha) \xrightarrow{d} \phi(W)(t), \quad (6.2)$$

where $\phi(W)(t)$ is the reflected version of

$$W(t) = \beta t - ct^{\ell+1} + \sigma B(t), \quad c \in \mathbb{R}^+, \quad (6.3)$$

and $B(t)$ is a standard Brownian motion.

Note that $\ell = 1$ (corresponding to the case $f'_T(0) < 0$) returns the correct scaling used in Theorem 1. Moreover, the case $\ell = 2$ has already been known in the literature for quite some time, at least at a heuristic level. Newell derived the correct exponents ($\alpha = 4/5$) through an argument using the Fokker-Planck equation associated with the queue length process (see [26],

part III¹). Notice also that $\lim_{\ell \rightarrow \infty} \frac{\alpha}{2} = \lim_{\ell \rightarrow \infty} \frac{\ell}{2\ell+1} = \frac{1}{2}$, suggesting that the right scaling for the uniform arrivals case (∞ -order contact) is the diffusive one.

The following heuristics motivate the expressions for α and the limit process. We first consider a first-order approximation of the queue, as follows:

$$\begin{aligned} Q_n(tn^\alpha) &= \sum_{i=1}^{tn^\alpha} \left(\sum_{j=1}^n \mathbb{1}_{\{\sum_{l=1}^{i-1} D_l \leq T_{(j)} \leq \sum_{l=1}^i D_l\}} - 1 \right) \\ &\approx \sum_{i=1}^{tn^\alpha} (n(F_T(\Sigma_i) - F_T(\Sigma_{i-1})) - 1) \\ &\approx \sum_{i=1}^{tn^\alpha} (f_T(\Sigma_{i-1}) \cdot \mathbb{E}[S] - 1) \approx \sum_{i=1}^{tn^\alpha} \left(\sum_{l=1}^{i-1} S_l/n \right)^\ell f_T^{(\ell)}(0), \end{aligned} \quad (6.4)$$

where the last approximation comes from the criticality assumption. This computation gives us the leading order term (up to a multiplicative constant) of the queue length process, that is

$$Q_n(tn^\alpha) \approx \sum_{i=1}^{tn^\alpha} \frac{i^\ell}{n^\ell} = t^{\ell+1} n^{(\ell+1)\alpha - \ell}. \quad (6.5)$$

The queue is the sum of (the order of) n^α contributions, thus (ignoring dependencies) the correct spatial scaling in order to obtain Gaussian fluctuations is $n^{\alpha/2}$. Finally, in order to obtain both a deterministic drift and a Brownian contribution in the limit, the order of magnitude of the first order approximation (6.5) should equate that of the diffusion approximation, $n^{\alpha/2}$. Equating the two yields

$$(\ell + 1)\alpha - \ell = \alpha/2 \Rightarrow \alpha = \frac{\ell}{\ell + 1/2}. \quad (6.6)$$

The formal proof of Theorem 4 mimics what has been done for the case $\ell = 1$. One might expect that assumptions on the existence of higher moments of S_i are needed, but this is not the case. In what follows, we perform the key steps in the analysis in order to show how to proceed in the general case.

6.1 Discussion and proof of Theorem 4

Assumptions. Recall the definition of α in (6.1). In particular, $\alpha < 1$. Some simple relations hold between α and ℓ and we will use these throughout this section:

$$\ell - \frac{\alpha}{2} = \alpha\ell, \quad \ell + \frac{\alpha}{2} = \alpha(\ell + 1). \quad (6.7)$$

We need to adapt the model assumptions to the ℓ -th order contact case. Assume that the distribution function of the arrival times satisfies

$$F_T(x) - F_T(\bar{x}) = f_T(\bar{x}) \cdot (x - \bar{x}) + o(|x - \bar{x}|^{1+\alpha/2}). \quad (6.8)$$

¹Note that the scaling parameter there (unluckily named β) is related to our setting by $\beta = 1/n^2$.

This is, for example, the case when $F_T \in \mathcal{C}^2([0, \infty))$. The service times are given by

$$D_i := \frac{S_i}{n}(1 + \beta n^{-\alpha/2}), \quad i \geq 1. \quad (6.9)$$

We assume that the maximum of the density $f_T(\cdot)$ is obtained in zero, and the criticality assumption is then defined as

$$c_T \cdot \mathbb{E}[S] = 1 + o(n^{-\alpha/2}). \quad (6.10)$$

In particular, the criticality assumption implies $c_T \cdot n\mathbb{E}[D] = 1 + \beta n^{-\alpha/2} + o(n^{-\alpha/2})$.

Proof. We proceed by proving conditions (i)-(iv). We will treat condition (i) in great detail as this changes profoundly, since the limiting drift is significantly different. We will then discuss how (ii)-(iv) follow from the previous calculations for the GA model.

Proof of (i). The starting point is again (5.28),

$$\begin{aligned} \mathbb{E}[A_n(k)|\mathcal{F}_{k-1}] - 1 &= \frac{n}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[(F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - 1/n)|\mathcal{E}_{k-1}] \\ &\quad - \frac{1}{1 - F_T(\Sigma_{k-1})} \mathbb{E}[|\nu_{k-1}|(F_T(\Sigma_k) - F_T(\Sigma_{k-1})) - F_T(\Sigma_{k-1})|\mathcal{E}_{k-1}] \\ &=: A_n^{(1)}(k) - A_n^{(2)}(k), \end{aligned} \quad (6.11)$$

and the corresponding drift decomposition

$$C_n(s) = \sum_{k=1}^s (A_n^{(1)}(k) - A_n^{(2)}(k)) =: C_n^{(1)}(s) + C_n^{(2)}(s). \quad (6.12)$$

Recall that $A_n^{(1)}(k)$ contains the terms appearing in the limit, while $A_n^{(2)}(k)$ the ones that vanish. Expanding gives us the equivalent of (5.33), in the form

$$A_n^{(1)}(k) = \frac{1}{1 - F_T(\Sigma_{k-1})} \cdot \left(\beta + \frac{f_T^{(\ell)}(0)}{\ell!} \mathbb{E}[S_1] \frac{(\sum_{j=1}^{k-1} S_j)^\ell}{n^{\ell\alpha}} + o_{\mathbb{P}}(1) \right) \cdot n^{-\alpha/2} \quad (6.13)$$

It is easy to check that both the linear part of the drift and the error term converge uniformly by proceeding as in (5.40) and the following computations. Therefore, we focus on the second term of $A_n^{(1)}(k)$,

$$\left(\mathbb{E}[S] \frac{f_T^{(\ell)}(0)}{\ell!} \right) n^{-\alpha/2} \sum_{i=1}^{tn^\alpha} \frac{1}{1 - F_T(\Sigma_{i-1})} \left(\frac{\sum_{j=1}^{i-1} S_j}{n} \right)^\ell, \quad (6.14)$$

for which we prove (uniform) convergence in probability. We begin by computing

$$\begin{aligned} \left| \frac{1}{n^{\alpha(\ell+1)}} \sum_{i=1}^{tn^\alpha} \left(\sum_{j=1}^{i-1} S_j \right)^\ell - \frac{t^{\ell+1}}{\ell+1} \mathbb{E}[S]^\ell \right| &= \frac{1}{n^{\alpha(\ell+1)}} \left| \sum_{i=1}^{tn^\alpha} \left(\sum_{j=1}^{i-1} S_j \right)^\ell - \sum_{i=1}^{tn^\alpha} ((i-1)\mathbb{E}[S])^\ell + o(n^{\alpha(\ell+1)}) \right| \\ &\leq \frac{1}{n^{\alpha(\ell+1)}} \sum_{i=1}^{tn^\alpha} \left| \left(\sum_{j=1}^{i-1} S_j \right)^\ell - (i-1)^\ell \mathbb{E}[S]^\ell \right| + o(1) \end{aligned} \quad (6.15)$$

We now make use of the following generalization of Lemma 10.

Lemma 13. Assume $(S_i)_{i \geq 0}$ is a sequence of i.i.d. random variables with finite second moment. Then for any $\alpha > 0, \beta \in \mathbb{R}$ such that $-\alpha < 2\beta$,

$$\frac{\sup_{k \leq tn^\alpha} \left| \left(\sum_{i=1}^k S_i \right)^\ell - k^\ell \mathbb{E}[S]^\ell \right|}{n^{\alpha+\beta}} \xrightarrow{\mathbb{P}} 0. \quad (6.16)$$

Proof. The proof is an application of Lemma 10, hence we only sketch it. We have

$$\begin{aligned} \sup_{k \leq tn^\alpha} \left| \left(\sum_{i=1}^k S_i \right)^\ell - k^\ell \mathbb{E}[S]^\ell \right| &\leq \sup_{k \leq tn^\alpha} \left| \left(k \mathbb{E}[S] + \sup_{k \leq tn^\alpha} \left| \sum_{i=1}^k S_i - k \mathbb{E}[S] \right| \right)^\ell - k^\ell \mathbb{E}[S]^\ell \right| \\ &\vee \sup_{k \leq tn^\alpha} \left| \left(k \mathbb{E}[S] - \sup_{k \leq tn^\alpha} \left| \sum_{i=1}^k S_i - k \mathbb{E}[S] \right| \right)^\ell - k^\ell \mathbb{E}[S]^\ell \right|. \end{aligned} \quad (6.17)$$

Both terms on the right can be shown to converge to zero when appropriately rescaled. To do so, it is enough to study the leading order term, which is

$$\sup_{k \leq tn^\alpha} k^{\ell-1} \mathbb{E}[S]^{\ell-1} \sup_{k \leq tn^\alpha} \left| \sum_{i=1}^k S_i - k \mathbb{E}[S] \right| = (tn)^\alpha \mathbb{E}[S]^{\ell-1} \sup_{k \leq tn^\alpha} \left| \sum_{i=1}^k S_i - k \mathbb{E}[S] \right|, \quad (6.18)$$

and this converges to zero when divided by $n^{\alpha+\beta}$, with $\beta > -\alpha/2$. \square

Note that when $\ell = 1$ we do, in fact, recover Lemma 10, since in that case $\alpha + \beta > \alpha - \alpha/2 = \alpha/2$.

Then, by Lemma 13 the right side of (6.15) converges to zero. The convergence is uniform in $t \leq \bar{t}$ by monotonicity in t . We can similarly analyse $A_n^{(2)}(k)$ and $C_n^{(2)}(k)$. Equation (5.46) then becomes

$$\begin{aligned} C_n^{(2)}(s) &= \sum_{k=1}^s \frac{1}{1 - F_T(\Sigma_{k-1})} \\ &\cdot \mathbb{E} \left[|\nu_{k-1}| c_T D_k - c_T \Sigma_{k-1} + \frac{f^{(\ell)}(0)}{\ell!} |\nu_{k-1}| (\Sigma_{k-1})^\ell D_k + |\nu_{k-1}| o_{\mathbb{P}}(n^{-1}) + o_{\mathbb{P}}(n^{-\alpha/2}) \mid \mathcal{E}_{k-1} \right], \end{aligned} \quad (6.19)$$

with the usual convention that $o_{\mathbb{P}}(n^{-1}) =: |F_T(\Sigma_k) - F_T(\Sigma_{k-1}) - f_T(\Sigma_{k-1}) D_k|$ and $o_{\mathbb{P}}(n^{-\alpha/2}) =: |F_T(\Sigma_{k-1}) - c_T \Sigma_{k-1}|$. This gives a decomposition of the drift $C_n^{(2)}(k)$ similar to (5.47) as

$$\begin{aligned} n^{-\alpha/2} C_n^{(2)}(tn^\alpha) &= n^{-\alpha/2} \sum_{k=1}^{tn^\alpha} \frac{1}{1 - F_T(\Sigma_{k-1})} \mathbb{E} \left[(k-1) c_T D_k - c_T \Sigma_{k-1} \mid \mathcal{E}_{k-1} \right] \\ &+ \sum_{k=1}^{tn^\alpha} \frac{n^{-\alpha/2}}{1 - F_T(\Sigma_{k-1})} \mathbb{E} \left[f_T^{(\ell)}(0) (k-1) (\Sigma_{k-1})^\ell D_k + (k-1) o_{\mathbb{P}}(n^{-1}) + o_{\mathbb{P}}(n^{-\alpha/2}) \mid \mathcal{E}_{k-1} \right] + \varepsilon_n, \end{aligned} \quad (6.20)$$

where ε_n groups all the terms containing $N_n(k)$. The first term is less straightforward now, because the first order terms in its expansion do not cancel out immediately. Consider (again ignoring the higher order terms in the expansion of $(1 - F_T(\Sigma_{k-1}))^{-1}$)

$$\begin{aligned} n^{-\alpha/2} \left| \sum_{k=1}^{tn^\alpha} \left((k-1) c_T \mathbb{E}[D_1] - c_T \Sigma_{k-1} \right) \right| &= c_T n^{-1-\alpha/2} \left| \sum_{k=1}^{tn^\alpha} \left(\sum_{j=1}^{k-1} S_j - (k-1) \mathbb{E}[S] \right) \right| \\ &\leq c_T n^{-1-\alpha/2} \sum_{k=1}^{tn^\alpha} \left| \sum_{j=1}^{k-1} S_j - (k-1) \mathbb{E}[S] \right| \end{aligned} \quad (6.21)$$

It is now sufficient to apply Lemma 10 with $\beta = 1 - \alpha/2$ (note that $2\beta < \alpha$). Therefore, the first term in (6.20) tends to zero in probability, uniformly in $t \leq \bar{t}$. The remaining terms are treated, without additional complications, in a similar manner as in Section 4.

Proof of (ii). This was proved through an analysis of the leading-order term, in which the ℓ -th derivative of the density played no role.

Proof of (iii). The proof of (iii) relies heavily on Lemma 12 which, in turn, relies on the analysis of the order statistics done in Lemma 9. Since the latter does not depend on the derivatives of the density (but rather on its continuity), the proof carries over.

Proof of (iv). Again, the proof relies on the analysis carried out in Lemma 9.

Having proved conditions (i)-(iv), this concludes the proof of Theorem 4. \square

7 Extended discussion

7.1 On the number of busy periods

Let $\beta_n(k)$ denote the number of busy periods which have been concluded before step k , then it is easy to see that it admits the following representation.

Lemma 14. *Consider the RC or the GA model. Under the same assumptions as in Theorem 1,*

$$\beta_n(k) = - \inf_{j \leq k} (N_n(j) \wedge 0), \quad k \geq 0, \text{ a.s.} \quad (7.1)$$

Proof. It is easy to see that this holds for the first busy period. By definition, this starts in 0 and is such that $N_n(0) = 0$. Assume \bar{k} is the smallest $k \geq 0$ such that $N_n(\bar{k}) = 0$ and $N_n(\bar{k} + 1) = -1$. Then, by definition of N_n the queue is empty in \bar{k} and one customer is being served. During his service time (that is, in the interval $[\bar{k}, \bar{k} + 1]$) no arrivals occur. Therefore, the first busy period has ended. The second busy period then starts in $\bar{k} + 1$ and is such that $N_n(\bar{k} + 1) = -1$. By reasoning analogously as for the first busy period one can prove this also for the second and successive ones. \square

As a simple application of Theorem 1 we prove an asymptotic result for $\beta_n(\cdot)$.

Corollary 2. *Consider the RC or the GA model. Under the same assumptions as in Theorem 1,*

$$n^{-1/3} \beta_n(tn^{2/3}) \xrightarrow{d} - \inf_{s \leq t} (W(s) \wedge 0). \quad (7.2)$$

Proof. The operator $\psi : f \mapsto \psi(f)(t) = - \inf_{s \leq t} (f(s) \wedge 0)$ acting from \mathcal{D} to itself is Lipschitz continuous with respect to the Skorohod topology by [31, Theorem 6.1]. Note that

$$n^{-1/3} \beta_n(tn^{2/3}) = \psi(n^{-1/3} N_n(\cdot n^{2/3}))(t). \quad (7.3)$$

Then, since $n^{-1/3} N_n(\cdot n^{2/3}) \xrightarrow{d} W$, by the Continuous Mapping Theorem their images through ψ converge,

$$\psi(n^{-1/3} N_n(\cdot n^{2/3})) \xrightarrow{d} \psi(W), \quad (7.4)$$

and this is (7.2). \square

7.2 On the relationship between the RC/GA models and the $\Delta_{(i)}/G/1$ model

In proving Theorem 1 and 4 we heavily used the fact that our model neglects possible idle times. This allowed for a very simple recursive description of the queue length embedded at departures, which would have not been possible otherwise. However, this raises the issue whether or not the RC and GA models serve as good approximations for the more natural $\Delta_{(i)}/G/1$ model, at least in the limit for $n \rightarrow \infty$. As was noted in [15], the $\Delta_{(i)}/G/1$ is ‘canonical’ in the sense that several seemingly different finite-population queueing models exhibit the same limiting behavior. As it turns out, this is also the case for our model. This is a direct consequence of the fact that, due to criticality, the idle times in the $\Delta_{(i)}/G/1$ queue are negligible compared to the time scale at which we observe the process. This is particularly easy to see for the RC model, where an explicit coupling between the two queues, as described in Section 3, is possible. Conditioned on the queue at time t' , we can reconstruct the coupled $\Delta_{(i)}/G/1$ queue through the time change

$$t = t' + \sum_{i=1}^{\beta_n(t')} I_i, \quad (7.5)$$

where I_i is the length of the i -th idle period and $\beta(t')$ is the number of (completed) busy periods. In the following lemma, we show that the time-shift in the limit is the identity map.

Lemma 15. *Consider the GA model and assume the arrival times $(T_i)_{i=1}^n$ are exponentially distributed. Fix $t \in (0, \infty)$. Then, conditioned on $\{Q_n(s), s \in (0, tn^{2/3})\}$,*

$$\frac{n}{\beta_n(tn^{2/3})} \sum_{i=1}^{\beta_n(tn^{2/3})} I_i \xrightarrow{\mathbb{P}} 1. \quad (7.6)$$

Proof. As was noted in (2.14), I_i is distributed as an exponential random variable with rate $n - |\nu_{t_i}|$ and I_i is independent from I_j for $i \neq j$. We rewrite the sum as

$$\begin{aligned} \frac{n}{\beta_n(tn^{2/3})} \sum_{i=1}^{\beta_n(tn^{2/3})} \frac{E_i}{n - |\nu_{t_i}|} &= \frac{1}{\beta_n(tn^{2/3})} \sum_{i=1}^{\beta_n(tn^{2/3})} \frac{E_i}{1 - \frac{|\nu_{t_i}|}{n}} \\ &= \frac{1}{\beta_n(tn^{2/3})} \sum_{i=1}^{\beta_n(tn^{2/3})} E_i + \frac{1}{\beta_n(tn^{2/3})} \sum_{i=1}^{\beta_n(tn^{2/3})} E_i \frac{|\nu_{t_i}|}{n} + \varepsilon_n, \end{aligned} \quad (7.7)$$

where $\varepsilon_n = o_{\mathbb{P}}(1)$ and $(E_i)_{i \geq 1}$ are exponential random variables with rate 1. By Corollary 2, $\beta_n(tn^{2/3}) \geq Mn^\alpha$ w.h.p. for a fixed $M > 0$ and $\alpha \in (0, 1/3)$. By the LLN, the first term in (7.7) converges in probability to 1, and by Lemma 5 the second term (and consequently the error term) converges to zero. \square

We believe that the result in Lemma 15 can be extended to more general arrival times, but we refrain from this. Lemma 15 intuitively says that the total idle *time* up to time $tn^{2/3}$ is of the same order of magnitude of the *number* of busy periods up to time $tn^{2/3}$ times the average interarrival time. In particular, $\sum_{i=1}^{\beta_n(tn^{2/3})} I_i = o_{\mathbb{P}}(1)$, that is, the idle times up to times of the order $n^{2/3}$ are negligible.

We now present another, possibly illuminating, connection between the present work and [14], from the perspective of the latter. In [14], the queue length process Q_n is such that $Q_n = \phi(X_n)$, where

$$X_n(t) = (A_n(t) - nF_T(t)) - (S^n(B_n(t)) - \mu nB^n(t)) + n(F_T(t) - \mu t). \quad (7.8)$$

Here $A_n(t) := \sum_{i=1}^n \mathbb{1}_{\{T_i \leq t\}}$, F_T is the distribution function of T , S^n is the renewal process associated with the (rescaled) service times $S_1/n, S_2/n, \dots$ and $B_n := \int_0^t \mathbb{1}_{\{Q_n(s) > 0\}} ds$ is the busy time process. Then, for example, [14, Theorem 1] is proved by applying the *continuous-mapping* approach to the fluid-scaled process X_n/n . Let us now scale the process X_n as

$$\begin{aligned} n^{-1/3} X^n(tn^{-1/3}) &= n^{2/3} \left(\frac{A_n(tn^{-1/3})}{n} - F_T(tn^{-1/3}) \right) - n^{2/3} \left(\frac{S^n(B_n(tn^{-1/3}))}{n} - \mu B_n(tn^{-1/3}) \right) \\ &\quad + n^{2/3} (F_T(tn^{-1/3}) - \mu tn^{-1/3}). \end{aligned} \quad (7.9)$$

It is immediate that, if $F_T(x) \approx f_T(0)x + \frac{f_T'(0)}{2}x^2$, the third term in (7.9) yields the quadratic drift. Moreover, it is reasonable to believe that the first and second term converge to (independent) Brownian motions, respectively through a martingale FCLT (or generalized Donsker's Theorem) and through a FCLT for renewal processes. Since the sum of two independent Brownian motions is a Brownian motion itself, the limit of $n^{-1/3} X^n(tn^{-1/3})$ coincides with our result.

These heuristics suggest that the connection between our results on the embedded process and other results in the literature (such as [14] and [22]) is simply a time scaling by a factor n . Indeed, $n^{2/3} = n \cdot n^{-1/3}$, $n^{\frac{\ell}{\ell+1/2}} = n \cdot n^{\frac{1}{2\ell+1}}$ (the latter being the time scaling appearing in [22, Theorem 3.4]), and so on. Intuitively, this is because in the definition of the embedded queue process, the service times and arrival times interact, and thus scaling the former influences the latter, while in the usual construction of the queueing process, the arrival and service processes are given separately as two renewal processes and only the service process is scaled.

7.3 On other regimes of criticality

Throughout this paper, we have focused on the critical regime, when $\nu := \sup_{t \geq 0} f_T(t)$ is such that $\nu = \mu$. This is by far the least studied case in the literature, but other cases may be of interest too. In this section, we discuss some known results also in relation to our own results. The parameters of interest are ν and the threshold $\mu = 1/\mathbb{E}[S]$. We denote by \bar{t} the argmax of $f_T(t)$.

The case $\nu > \mu$ has been studied extensively in [14]. When $\nu > \mu$, the queue length grows linearly in a neighborhood of \bar{t} as has been discussed in Section 2. In particular, it is possible to prove that there exists a constant $m > 0$ such that the maximum of the fluid-scaled queue length process converges to m , meaning that the queue during ‘peak hour’ grows linearly.

When $\nu = \mu$, the argmax of $f_T(t)$ can be either zero, or greater than zero. If it is zero, and if $f_T'(0) < 0$, our results apply, and thus the correct scaling is $n^{-1/3}$ for the space, $n^{2/3}$ for the time around the argmax. If the first derivative (or possibly all the derivatives up to a finite order) of f_T is zero in zero, by appropriately modifying the scalings our approach still yields the correct limit, as was shown in Section 6. Here ‘correct’ means that both a deterministic drift indicating a depletion-of-points effect and a Brownian contribution appear. If the argmax \bar{t} is greater than zero, and $f_T(t)$ is sufficiently smooth, then we must have that $f_T'(\bar{t}) = 0$, suggesting that one has to investigate higher order derivatives as was done in Section 6 (note that the first

non-zero derivative will necessarily be even, provided $f'_T(t)$ is smooth). If $f_T(t)$ is not smooth, i.e. $f'_T(t)$ is discontinuous in \bar{t} , an even wider range of behaviors will possibly be displayed, due to the fact that $f_T(t)$ and μ may have different contact orders in \bar{t}^- and \bar{t}^+ .

The uniform arrivals case (uniform in, say, $[0, 1]$) is special, since all the derivatives of $f_T(t)$ are identically zero. In some sense, it interpolates between our result and [14]. The queue is critical during its entire lifetime, and thus it never grows linearly, similarly to our model. On the other hand, the uniform arrivals model does not lead to a downward drift. Intuitively, this is because there is no ‘tipping point’ after which the intensity of the stream of customer diminishes and therefore the depletion-of-points effect does not occur. By observing the queue at a macroscopic time scale, it displays diffusion-type fluctuations (of the order $n^{1/2}$), which end abruptly in 1, when customers stop joining and thus the queue behaves deterministically, with the server flushing out the customers still in the queue. The correct (optimal) scaling in this case has been first identified in [16] (Example 3, p. 365), and can be treated using the general theory developed in [14]. Indeed, [14, Theorem 1] implies that $Q_n(t)/n \xrightarrow{d} 0$, while [14, Theorem 2] implies that $Q_n(t)/\sqrt{n}$ converges to $X^*(t)$, the reflected version of $X(t)$, where $X(t)$ is defined as

$$X(t) = \begin{cases} B^0(t) - \sigma B(t), & t \in [0, 1], \\ 0, & t > 1, \end{cases}$$

where $B^0(t)$ is a standard Brownian Bridge.

The case $\nu < \mu$ is, from the scaling perspective, the least interesting one. The service rate is so large, compared to the instantaneous arrival rate $f_T(t)$, that the queue is empty most of the time and in no instant it builds up as a power of n . One is then tempted to conjecture that, for fixed t , the queue length $Q_n(t)$ converges in distribution to a proper random variable $Q(t)$. Moreover, for $s < t$, $Q_n(\cdot)$ will be zero w.h.p. at some time $\zeta \in (s, t)$, suggesting that $Q(s)$ is independent from $Q(t)$. Since locally the arrival process is roughly a Poisson process with rate $f_T(t)$, it is reasonable to also conjecture that $Q(t)$ is the stationary distribution of an $M/G/1$ queue with arrival rate $f_T(t)$. In fact, this has already been proven for the $M_t/M_t/1$ queue in [22] and, using analogous arguments, it is also possible to prove it for the $\Delta_{(i)}/G/1$ model with exponential arrivals.

7.4 On the relationship between the queue and random graphs

To the queue that we study, we can associate a directed random graph by saying that customer i is connected to customer j when i arrives during the service period of j . When the customers have an exponential clock, and conditionally on the customer not having joined the queue yet and the service time D_j of the presently served customer j , each customer i will be connected to customer j with probability

$$p_{ij} = 1 - e^{-D_j} = 1 - e^{-S_j/n}. \quad (7.10)$$

In particular when the service times are constant, i.e., $S_j \equiv \lambda$, these probabilities are precisely equal to the edge probabilities of the Erdős-Rényi random graph (see [30, 6, 17] for monographs on random graphs) with parameter $1 - e^{-\lambda/n} \approx \lambda/n$. In this case, Aldous [2] has shown that the scaling limit in Theorem 1 appears precisely when $\lambda = 1$, and with the constant $f'_T(0)/c_T^2$ replaced by 1. This follows, intuitively, by noting that the depletion-of-points effects in the two problems is the same. Of course, the process described by (1.3) is not the graph construction

itself, but is instead equivalent to the exploration of the connected components of the graph. Edges in the graph that form cycles are not explored, and thus also not observed, so that we obtain a tree rather than a graph.

When S_j is not a.s. constant, then the edge probabilities in (7.10) are not symmetric. We can think of this as the cluster exploration of a *directed* random graph, where, due to the depletion-of-points effect, at most one of the two possible edges between two vertices will ever be found. The service time random variables $(S_j)_{j=1}^n$ create *inhomogeneity* in the graph, of a form that has not been investigated in the random graph community.

If, instead of our assumptions, we assume that the clock of customer i has rate proportional to D_i (i.e., when the job size is larger, then it will join the queue more quickly, inversely proportional to the job size), then the edge probabilities become

$$p_{ij} = 1 - e^{-\lambda D_i D_j} = 1 - e^{-\lambda S_i S_j / n^2}. \quad (7.11)$$

Taking $\lambda = an$, this becomes $p_{ij} = 1 - e^{-aS_i S_j / n}$. This is closely related to the so-called *generalized random graph* with i.i.d. vertex weights, as studied in [3, 4, 13, 18]. See also [7] for the most general setting, and [8, 9, 28] for related models. Interestingly, a *size-biasing* takes place in the form that the vertices that are found by the exploration process have weights with the size-biased distribution S_j^* given by

$$\mathbb{P}(S_j^* \leq x) = \frac{\mathbb{E}[S_j \mathbb{1}_{\{S_j \leq x\}}]}{\mathbb{E}[S]}. \quad (7.12)$$

In [7] it is shown that the cluster exploration obeys the scaling in Theorem 1 precisely when $\mathbb{E}[S^3] < \infty$, while different scalings can occur when $\mathbb{E}[S^3] = \infty$ and S has a power-law distribution. Because of the size-biasing that is absent in our work, this suggests that similar scaling limits might appear for queues with vanishing populations of customers as in [3, 18] when S has a power-law distribution with $\mathbb{E}[S^2] = \infty$. This is an interesting problem for future research. Another interesting question is whether there is an interpretation for the edges forming cycles in our queueing model. In the random graph context, these edges are called the *surplus* and these play an important role in the scaling limit of random graphs (see e.g., [1]).

7.5 On the relationship between the queue and the Grenander estimator

The arrival process in (3.18) can be seen as (n times) the empirical distribution function for n trials of the random variable T . This seemingly trivial observation reveals a connection between our result Theorem 1 and some results in the statistics literature. Let us briefly (and roughly) introduce the setting of the latter. Suppose we are given a random variable T whose law is supported on $[0, 1]$ and admits a positive density $f_T(t)$ such that $\inf_{t \in [0, 1]} |f_T(t)| > 0$ and $t \mapsto f_T$ is non-increasing. Given that the distribution of T is sampled n times as T_1, \dots, T_n , our task is to estimate f_T . In [11] the so-called Grenander estimator is introduced as an estimator of f_T . It is constructed as follows. Let $F_T^{(n)}(t) := (1/n) \sum_{i=1}^n \mathbb{1}_{\{T_i \leq t\}}$ be the empirical distribution function of the given sample and let $\hat{F}_T^{(n)}(t)$ be the concave conjugate of $F_T^{(n)}(t)$. Recall that the concave conjugate is the (pointwise) smallest concave function such that

$$\hat{F}_T^{(n)}(t) \geq F_T^{(n)}(t), \quad \forall t \in [0, 1]. \quad (7.13)$$

Then, the Grenander estimator $\hat{f}_T^{(n)}$ is defined as the left derivative of $\hat{F}_T^{(n)}(t)$. It turns out that studying the *inverse process* $U_n(a) = \sup\{t \in [0, 1] : F_T^{(n)}(t) - at \text{ is maximal}\}$ is equivalent

to studying $\hat{f}_T^{(n)}(t)$, but considerably easier. In particular, in [12] the task of estimating the (rescaled) L^1 error of the estimator, $n^{1/3}\|\hat{f}_T^{(n)} - f_T\|_{L^1([0,1])}$, is proven to be equivalent to studying $\int |V_n^E(a)|da$, with $V_n^E(a) := n^{1/3}(U_n(a) - g(a))$ and $g(a)$ the inverse of f_T . It is readily seen that V_n^E can be rewritten as

$$V_n^E(a) = \sup\{t \in \mathcal{T} : D_n^E(a, t) - n^{1/3}at \text{ is maximal}\}, \quad (7.14)$$

for some appropriate set $\mathcal{T} = \mathcal{T}_n(a)$. Here D_n^E is the process

$$D_n^E(a, t) := n^{1/6}(E_n(g(a) + n^{-1/3}t) - E_n(g(a))) + n^{2/3}(F_T(g(a) + n^{-1/3}t) - F_T(g(a))), \quad (7.15)$$

where $E_n(x) = \sqrt{n}(F_T^{(n)}(x) - F_T(x))$. The problem in [12] of computing the error of the Grenander estimator then reduces to studying a certain complicated functional of the process

$$t \mapsto D_n^E(a, t) - n^{1/3}at, \quad (7.16)$$

regarding a as a parameter for the sake of exposition. Notice that this can be interpreted as the queue length process associated with a system in which the service rate is deterministic, $1/S \equiv a$ and the arrival process is given by $t \mapsto F_T^{(n)}(g(a) + t) - F_T^{(n)}(g(a))$. Moreover, note that the derivative of F_T in $g(a)$ is

$$F_T'(g(a)) = f_T(g(a)) = a = 1/S, \quad (7.17)$$

since g is the inverse of f_T . Remarkably, (7.17) as seen from our perspective is nothing but the criticality assumption in the point $g(a)$. In [12, Theorem 3.2] the authors prove a convergence result for V_n^E of the type of Theorem 1, the most important difference being that the functional of interest in our case is the reflection mapping, while they are interested in the (supremum among all of the) argmax. Their technique is similar to ours, in the sense that in order to prove convergence of the functional they first prove convergence of the underlying process. This is a two-sided Brownian motion in \mathbb{R} , $(B(t))_{t \in \mathbb{R}}$ originating from zero, i.e. $B(0) = 0$. Whether there is an equivalent, in this setting, of our ℓ -th order contact is an interesting question, and one which remains open for future research.

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