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Unconstrained and constrained stabilisation of bilinear discrete-time systems using polyhedral Lyapunov functions

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The constrained and unconstrained stabilisation problem of discrete-time bilinear systems is investigated. Using polyhedral Lyapunov functions, conditions for a polyhedral set to be both positively invariant and domain of attraction for systems with second-order polynomial nonlinearities are first established. Then, systematic methods for the determination of stabilising linear feedback for both constrained and unconstrained bilinear systems are presented. Attention is drawn to the case where no linear control law rendering the pre-specified desired domain of attraction positively invariant exists. For this case, an approach guaranteeing the existence of a possibly suboptimal solution is established.

Keywords: positively invariant sets; discrete-time bilinear systems; polyhedral Lyapunov functions; input/state constraints

1. Introduction

Bilinear systems are a special class of nonlinear systems, where the nonlinear part involves products of the state and input variables. Such systems deserve attention for a few reasons (Bruni, Pillo, and Koch 1974; Mohler and Kolodziej 1980; Favoreel, De Moor, and van Overschee 1999): firstly, many engineering applications (e.g. thermal, chemical, nuclear processes, transmission and power systems) as well as models in biology, socioeconomic, ecology etc. are naturally described by bilinear systems. Secondly, many nonlinear systems can be adequately approximated by bilinear systems. Lastly, bilinear models are suitable candidates for black-box modelling of nonlinear systems because of the availability of well-established identification algorithms. There is a plethora of articles published in the last 40 years concerning the analysis and design of stabilising controllers for bilinear systems.

For continuous-time systems, one of the proposed approaches to the stabilisation problem is to select a quadratic Lyapunov function (Gutman 1981, Amato, Cosention, Fiorillo, and Merola 2009; Tarbouriech, Queinnec, Calliero, and Peres 2009). Specifically, in Amato et al. (2009), the solution of an LMI problem leads to the computation of linear state-feedback control laws rendering a pre-specified set of states domain of attraction of the corresponding closed-loop system. In order to find a Lyapunov

function, the dynamics of the nonlinear part of the system are modelled by linear differential inclusions. Similar results are established in Tarbouriech et al. (2009), where an ellipsoidal domain of attraction is iteratively enlarged. For open-loop stable systems in Gutman (1981), a nonlinear ‘quadratic’ control law is preferred. In a more recent work, Chen (1998), normalised ‘quadratic’ control was used to achieve exponential asymptotic stability. In Chen, Chang, and Lai (2000), the authors present a bang–bang sliding mode control technique for single-input systems where the stability region strongly depends on the sliding function designed via a pole assignment based method. Piecewise-constant feedback laws are computed in Khapalov and Mohler (1998) by studying the behaviour of an auxiliary bilinear system which has an additional input in the drift term. An application of this method is in Mohler and Khapalov (2000). Using a quadratic cost function which is modified by non-negative penalising functions, global asymptotic stability was achieved for open-loop stable systems in Benallou, Mellichamp, and Seborg (1988), while in Ekman (2005) a suboptimal control law was computed for the infinite bilinear quadratic regulator when approximating the derived Riccati equation by a power series. In a more recent work (Athanasopoulos, Bitsoris, and Vassilaki 2010), polyhedral Lyapunov functions were utilised to compute a stabilising linear

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state-feedback control law, whereas in Amato, Calabrese, Cosentino, and Merola (2008) the same type of Lyapunov functions was used for the analysis of quadratic systems.

For the discrete-time case, most approaches are related to the optimal and model-based predictive control theory. In Basic, Cannon, and Kouvaritakis (2003), the computation of polytopic invariant sets of low complexity for constrained single-input bilinear systems was studied. In particular, switching between feedback linearisation and state feedback takes place and renders a region of the state space, both invariant and feasible. Stability is ensured by introducing the new concept of partially invariant and feasible polytopes. The extension of this work in Liao, Cannon, and Kouvaritakis (2005) also takes into account the behaviour of the dynamics of the system when input-output feedback linearisation is applied and it is shown that in some cases much larger invariant sets can be produced. The construction of terminal invariant sets along with the corresponding control law is a common task when using model predictive control (MPC). In Cannon, Deshmukh, and Kouvaritakis (2003), this is done for general input-affine nonlinear systems, a category in which bilinear systems belong. The dynamics of the system are modelled by polytopic linear difference inclusions. The invariant sets are produced by solving a nonlinear or a sequence of linear programming problems. Larger and more complex polytopes are computed in Cannon, Kouvaritakis, and Deshmukh (2004). In a generalised predictive control approach, Fontes, Maitelli, and Salazar (2002), the bilinear model is approximated by a quasi-linear model with an extra term that compensates the prediction error. In a similar MPC formulation of the stabilisation problem for constrained bilinear systems (Fontes, Dorea, and Garcia 2008), conditions for the convergence of the prediction error for the case of one-step prediction and for single-input systems are given. The terminal invariant set is computed by treating the system as a linear parameter varying one. Using quadratic Lyapunov functions, conditions for a globally stabilising nonlinear control law for open-loop stable linear systems are given in Kim, Kim, Lin, and Kim (2002).

The problem studied in this article is formulated as follows: given a set of initial states, determine a linear state-feedback control law such that a subset or the whole set of initial states is a domain of attraction for the resulting closed-loop system and state and/or input constraints are satisfied. The resulting closed-loop system is a nonlinear system with second-order polynomial nonlinearities. In contrast to all well-known approaches based on choosing quadratic Lyapunov functions, the stability analysis of this

class of systems is carried out by using polyhedral Lyapunov functions which are implicitly given by the problem specifications. Thus, as a first step, algebraic conditions guaranteeing the positive invariance of polyhedral sets and the stability for this class of nonlinear systems are established. Then, these conditions are used to develop systematic design techniques for the constrained and unconstrained control problems of bilinear systems. The appropriate control laws are obtained by solving linear programming problems. The proposed design techniques can be applied to stable or unstable multiple-input multiple-output bilinear systems with linear input and/or state constraints.

This article is organised as follows: in Section 2, necessary notations as well as the problem statement are given. In Section 3, algebraic conditions guaranteeing the positive invariance of polyhedral sets and the asymptotic stability of systems with second-order polynomial nonlinearities are established. Then, in Section 4, design techniques for the unconstrained and constrained stabilisation problems are developed. Finally, in Section 5, two numerical examples illustrating the effectiveness of the proposed methods are given, while in Section 6 conclusions are drawn.

2. Problem statement

Throughout this article, capital letters denote real matrices and lower case letters denote column vectors or scalars. \mathbb{R}^n denotes the real n -space and $\mathbb{R}^{n \times m}$ denotes the set of real $n \times m$ matrices. Given a real $n \times m$ matrix, $A = (a_{ij})$, $A^+ = (a_{ij}^+)$ and $A^- = (a_{ij}^-)$ are $n \times m$ matrices with entries defined by the relations $a_{ij}^+ = \max\{a_{ij}, 0\}$ and $a_{ij}^- = -\min\{a_{ij}, 0\}$. Thus, $A = A^+ - A^-$. Given a square matrix $D = (d_{ij})$, $D^\delta = (d_{ij}^\delta)$ denotes the diagonal matrix with $d_{ii}^\delta = d_{ii}$ and $D^\mu = (d_{ij}^\mu)$ denotes the square matrix with $d_{ii}^\mu = 0$ and $d_{ij}^\mu = d_{ij}$ for $i \neq j$. Thus $D = D^\delta + D^\mu$. For two $n \times m$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \odot B = \sum_{i=1}^n \sum_{j=1}^m a_{ij} b_{ij}$ denotes their component-wise inner product called the Frobenius inner product. The inequality $A \leq B$ ($A < B$) with $A, B \in \mathbb{R}^{n \times m}$ is equivalent to $a_{ij} \leq b_{ij}$ ($a_{ij} < b_{ij}$). Similar notation holds for vectors. Finally, T denotes the time set $T = \{0, 1, 2, \dots\}$.

Bilinear discrete-time systems are described by difference equations of the form

$$x(t+1) = Ax(t) + Bu(t) + \begin{bmatrix} x^T(t)C_1 \\ x^T(t)C_2 \\ \vdots \\ x^T(t)C_n \end{bmatrix} u(t), \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $t \in T$ is the time variable and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C_i \in \mathbb{R}^{n \times m}$, $i = 1, 2, \dots, n$.

For linear state-feedback control laws $u(t) = Kx(t)$ with $K \in \mathbb{R}^{m \times n}$, the resulting closed-loop system is described by the difference equation

$$x(t+1) = (A + BK)x(t) + \begin{bmatrix} x^T(t)C_1Kx(t) \\ x^T(t)C_2Kx(t) \\ \vdots \\ x^T(t)C_nKx(t) \end{bmatrix}. \quad (2)$$

This equation describes a nonlinear system with second-order polynomial nonlinearities.

The unconstrained stabilisation problem to be investigated is formulated as follows: given system (1) and a bounded subset of the state space defined by the inequalities

$$-w_2 \leq Gx \leq w_1 \quad (3)$$

with $G \in \mathbb{R}^{p \times n}$, $w_1 \in \mathbb{R}^p$, $w_1 > 0$, $w_2 \in \mathbb{R}^p$, $w_2 > 0$, determine a linear state-feedback control law $u(t) = Kx(t)$ making this set a domain of attraction of the resulting closed-loop system (2). Due to the presence of nonlinearities in the resulting closed-loop system, this problem may not possess any solution even if pair (A, B) is stabilisable. In this case, the problem is the determination of a linear state-feedback control law $u(t) = Kx(t)$ making a subset of the set defined by inequalities (3) domain of attraction of the resulting closed-loop system.

In the constrained stabilisation problem, control constraints of the form

$$-\rho_2 \leq u(t) \leq \rho_1 \quad (4)$$

with $\rho_1 \in \mathbb{R}^m$, $\rho_1 > 0$, $\rho_2 \in \mathbb{R}^m$, $\rho_2 > 0$ are also imposed. The problem is the determination of a linear state-feedback control law $u(t) = Kx(t)$ such that all initial states belonging to the set defined by inequalities (3) are transferred asymptotically to the origin while the control constraints (4) are satisfied. If, due to the presence of nonlinearities in the resulting closed-loop system or/and control constraints there does not exist any control law making the set defined by inequalities (3) domain of attraction, then the problem is the determination of a linear state-feedback control law $u(t) = Kx(t)$ rendering a subset of the set defined by inequalities (3) domain of attraction of the resulting closed-loop system.

3. Stability and polyhedral positively invariant sets

Given a dynamical system, a subset of its state space is said to be *positively invariant* if all trajectories starting

from this set remain in it for all future instances. This property is very important for control problems with state constraints or/and input constraints when using state-feedback control laws. Thus, if the state constraints define an admissible subset of the state space, a solution to the control problem under state constraints is a stabilising linear control law making this admissible set positively invariant with respect to the resulting closed-loop system. Since in practical control problems the state constraints are usually expressed by linear inequalities, the admissible set is a polyhedron. Therefore, it is very important to establish conditions guaranteeing positive invariance of polyhedral sets of the form (3) with respect to nonlinear systems of the form (2).

The following lemma which provides necessary and sufficient conditions for a set defined by a nonlinear vector inequality of the form $v(x) \leq w$ to be positively invariant with respect to a nonlinear discrete-time system is very important for the development of the results of this article.

Lemma 3.1 (Bitsoris and Gravalou 1995; Bitsoris and Truffet 2006): *The set*

$$P(v, w) \triangleq \{x \in \mathbb{R}^n : v(x) \leq w\} \quad (5)$$

with $v(x)$, $v: \mathbb{R}^n \rightarrow \mathbb{R}^q$ and $w \in \mathbb{R}^q$ is a positively invariant set of system

$$x(t+1) = f(x(t)) \quad (6)$$

with $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$, if and only if there exists a nondecreasing function $h(y)$, $h: \mathbb{R}^q \rightarrow \mathbb{R}^q$ such that

$$v(f(x)) \leq h(v(x))$$

and

$$h(w) \leq w.$$

We shall use this result to establish conditions guaranteeing that a polyhedral set defined by linear inequalities (3) is positively invariant with respect to the closed-loop system (2).

Let

$$y_1 = \begin{bmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1p} \end{bmatrix} = Gx, \quad y_2 = \begin{bmatrix} y_{21} \\ y_{21} \\ \vdots \\ y_{2p} \end{bmatrix} = -Gx \quad (7)$$

and $Y^M = (y_{ij}^M)$, $Y^m = (y_{ij}^m)$ be $p \times p$ matrices whose elements are defined by the relations

$$y_{ij}^M \triangleq \max(y_{1i}y_{1j}, y_{2i}y_{2j}), \quad (8)$$

$$y_{ij}^m \triangleq \max(y_{1i}y_{2j}, y_{2i}y_{1j}). \quad (9)$$

Theorem 3.2: *The polyhedral set*

$$Q(G, w_1, w_2) \triangleq \{x \in \mathbb{R}^n : -w_2 \leq Gx \leq w_1\} \quad (10)$$

with $G \in \mathbb{R}^{p \times n}$, $w_1 \in \mathbb{R}^p$, $w_1 > 0$, $w_2 \in \mathbb{R}^p$, $w_2 > 0$ is positively invariant with respect to the nonlinear closed-loop system (2) if there exist matrices $H \in \mathbb{R}^{p \times p}$ and $D_j \in \mathbb{R}^{p \times p}$ $j=1, 2, \dots, p$ such that

$$G(A + BK) = HG, \quad (11)$$

$$\sum_{i=1}^n g_{ji} C_i K = G^T D_j G, \quad j = 1, 2, \dots, p \quad (12)$$

and

$$h(w) \leq w, \quad (13)$$

where

$$h(y) = \begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} D_1^+ \odot Y^M + D_1^{\mu^-} \odot Y^m \\ \vdots \\ D_p^+ \odot Y^M + D_p^{\mu^-} \odot Y^m \\ D_1^- \odot Y^M + D_1^{\mu^+} \odot Y^m \\ \vdots \\ D_p^- \odot Y^M + D_p^{\mu^+} \odot Y^m \end{bmatrix}, \quad (14)$$

and

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Proof: See Appendix A. □

Let us now assume that the origin $x=0$ is an equilibrium state of a nonlinear system. Then, the following important lemma holds:

Lemma 3.3 (Bitsoris 1984; Bitsoris and Gravalou 1995; Bitsoris and Truffet 2006): *Let $v(x)$, $v: \mathbb{R}^n \rightarrow \mathbb{R}^q$ be a vector-valued function such that the scalar function $\max_{i=1, 2, \dots, q} \{v_i(x)\}$ is positive definite. If there exist a nondecreasing function $h(y)$, $h: \mathbb{R}^q \rightarrow \mathbb{R}^q$, $h(0)=0$ and a vector $w \in \mathbb{R}^q$, $w > 0$ satisfying inequalities*

$$v(f(x)) \leq h(v(x)) \quad (15)$$

and

$$h(rw) < rw \quad \forall r \in (0, a] \quad (16)$$

where a is a positive real number, then the equilibrium $x=0$ of system (6) is asymptotically stable,

$$v^*(x) = \max_{i=1, 2, \dots, q} \left\{ \frac{v_i(x)}{w_i} \right\} \quad (17)$$

is a Lyapunov function, and $P(v, rw)$, $\forall r \in (0, a]$ are domains of attraction of the equilibrium $x=0$.

It is clear that hypotheses of this lemma also guarantee the positive invariance of all polyhedral sets $P(v, aw)$ with $r \in (0, a]$.

By combining this lemma with Theorem 3.2 we shall establish conditions guaranteeing both the positive invariance of a polyhedral sets $Q(G, aw_1, aw_2) \triangleq \{x \in \mathbb{R}^n : -aw_2 \leq Gx \leq aw_1\}$ with $a > 0$ and the asymptotic stability of the equilibrium $x=0$ of the closed-loop system.

Let H^*y and $g^*(y)$ be the linear and the nonlinear parts, respectively, of function $h(y)$, namely

$$H^*y = \begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} y, \quad (18)$$

and

$$g^*(y) = \begin{bmatrix} D_1^+ \odot Y^M + D_1^{\mu^-} \odot Y^m \\ \vdots \\ D_p^+ \odot Y^M + D_p^{\mu^-} \odot Y^m \\ D_1^- \odot Y^M + D_1^{\mu^+} \odot Y^m \\ \vdots \\ D_p^- \odot Y^M + D_p^{\mu^+} \odot Y^m \end{bmatrix}. \quad (19)$$

Theorem 3.4: *If there exist matrices $G \in \mathbb{R}^{p \times n}$ with rank $G = n$, $H \in \mathbb{R}^{p \times p}$ and $D_j \in \mathbb{R}^{p \times p}$ $j=1, 2, \dots, p$ satisfying relations (11), (12) and a scalar $a > 0$ such that*

$$H^*w + ag^*(w) < w, \quad (20)$$

then the equilibrium $x=0$ of the closed-loop system (2) is asymptotically stable,

$$v^*(x) = \max \left\{ \frac{(Gx)_1}{w_{11}}, \dots, \frac{(Gx)_p}{w_{1p}}, \frac{(-Gx)_1}{w_{21}}, \dots, \frac{(-Gx)_p}{w_{2p}} \right\} \quad (21)$$

is a Lyapunov function and the sets $Q(G, rw_1, rw_2)$ are both positively invariant and domains of attraction for all $r \in [0, a]$.

Proof: Following the same steps as in the proof of Theorem 3.2 we establish the relation $v(f(x)) \leq h(v(x))$ with $h(y) = H^*y + g^*(y)$ where H^*y and $g^*(y)$ are given by (18) and (19), respectively. By construction, function $h(y)$ is nondecreasing and the scalar function

$$\max \{(Gx)_1, \dots, (Gx)_p, -(Gx)_1, \dots, -(Gx)_p\}$$

is positive definite because, by hypotheses, rank $G = n$. In addition, from (20) it follows that

$$\begin{aligned} h(rw) &= H^*rw + g^*(rw) \\ &= rH^*w + r^2g^*(w) \\ &= r(H^*w + rg^*(w)) \\ &\leq r(H^*w + ag^*(w)) \quad \forall r \in (0, a) \\ &< rw \quad \forall r \in (0, a). \end{aligned}$$

Thus, all hypotheses of Lemma 3.3 are satisfied for $v(x) = [(Gx)_1 \ (Gx)_2 \ \dots \ (Gx)_p \ -(Gx)_1 \ -(Gx)_2 \ \dots \ -(Gx)_p]^T$. Therefore $v^*(x)$ defined by (21) is a Lyapunov function and, as a result, all sets $Q(G, rw_1, rw_2)$ for $r \in [0, a)$ are both positively invariant and domains of attraction. \square

Now, we are in a position to establish conditions for a bounded polyhedral set $Q(G, w_1, w_2)$ to be both positively invariant and domain of attraction of the equilibrium $x = 0$ of the closed-loop system (2). These conditions are obtained by applying the result stated in Theorem 3.4 for $a = 1$.

Corollary 3.5: *If there exist matrices $H \in \mathbb{R}^{p \times p}$ and $D_j \in \mathbb{R}^{p \times p}$ $j = 1, 2, \dots, p$ satisfying relations (11), (12) and*

$$H^*w + g^*(w) < w, \tag{22}$$

where H^*y and $g^*(y)$ are given by (18) and (19) respectively, then the set $Q(G, w_1, w_2)$ is both positively invariant and domain of attraction of the equilibrium $x = 0$ and

$$v^*(x) = \max \left\{ \frac{(Gx)_1}{w_{11}}, \dots, \frac{(Gx)_p}{w_{1p}}, \frac{-(Gx)_1}{w_{21}}, \dots, \frac{-(Gx)_p}{w_{2p}} \right\}$$

is a Lyapunov function.

4. Design techniques

Many different approaches to the unconstrained and constrained stabilisation of bilinear systems can be established using the results stated in Section 3. In this section, we develop systematic design methods that reduce the determination of stabilising control laws in finding a solution to one or a sequence of linear programming problems.

4.1 The unconstrained control problem

A linear control law $u = Kx$ is a solution to the unconstrained control problem if set $Q(G, w_1, w_2)$ is a domain of attraction of the resulting nonlinear closed-loop system. By virtue of Corollary 3.5, such a solution can be obtained by determining matrices $H, D_j,$

$j = 1, 2, \dots, p$ and a positive real number $\varepsilon < 1$ satisfying the linear relations

$$G(A + BK) = HG, \tag{23}$$

$$\sum_{i=1}^n g_{ji} C_i K = G^T D_j G, \quad j = 1, 2, \dots, p, \tag{24}$$

$$\begin{aligned} &\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} D_1^+ \odot W^M + D_1^{\mu^-} \odot W^m \\ \vdots \\ D_p^+ \odot W^M + D_p^{\mu^-} \odot W^m \\ D_1^- \odot W^M + D_1^{\mu^+} \odot W^m \\ \vdots \\ D_p^- \odot W^M + D_p^{\mu^+} \odot W^m \end{bmatrix} \\ &\leq \varepsilon \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \end{aligned} \tag{25}$$

where $W^M = (w_{ij}^M), W^m = (w_{ij}^m)$ are $p \times p$ matrices whose elements are defined by the relations

$$w_{ij}^M \triangleq \max(w_{1i}w_{1j}, w_{2i}w_{2j}),$$

$$w_{ij}^m \triangleq \max(w_{1i}w_{2j}, w_{2i}w_{1j}).$$

A possible approach to the determination of such a solution is to consider these relations as constraints of a linear programming problem with optimisation criterion

$$\min_{K, H, D_1, \dots, D_r, \varepsilon} \{\varepsilon\}. \tag{26}$$

Remark 1: It can be easily shown that from (23)–(25) it follows that

$$v^*(x(t+1)) \leq \varepsilon v^*(x(t)) \quad \forall x \in Q(G, w_1, w_2), \tag{27}$$

where $v^*(x)$ is the Lyapunov function defined by (17). Therefore, minimisation of ε results to a faster transient behaviour for the closed-loop system. Thus, ε can be thought of as a design variable when closed-loop performance requirements are present.

Remark 2: As mentioned in Section 2, the initial unconstrained control problem may not possess any solution. In such a case, the optimisation problem described above does not provide any solution because the optimal value ε_{opt} of ε will be greater than 1. It is clear that for $\varepsilon_{\text{opt}} < 1$ it is necessary that there exist matrices K and H satisfying relations

$$G(A + BK) = HG, \tag{28}$$

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} < \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{29}$$

We first consider the case when matrices K and H resulting from the above linear programming problem satisfy (28) and (29) but $\varepsilon_{\text{opt}} > 1$. Then, there exists $a < 1$ such that $H^*w + ag^*(w) < w$. Thus, by virtue of Theorem 3.4, the linear control law $u = Kx$ makes a subset of $Q(G, w_1, w_2)$, namely $Q(G, aw_1, aw_2)$, domain of attraction of the equilibrium $x = 0$ of the resulting closed-loop system. The largest domain of attraction $Q(G, a_{\text{max}}w_1, a_{\text{max}}w_2)$, where a transient behaviour

$$v^*(x(t+1)) \leq \varepsilon v^*(x(t)) \tag{30}$$

is guaranteed, can be obtained by solving the optimisation problem

$$\max_{K, H, D_1, \dots, D_p, a} \{a\}, \tag{31}$$

with constraints

$$G(A + BK) = HG, \tag{32}$$

$$\sum_{i=1}^n g_{ji} C_i K = G^T D_j G \quad j = 1, 2, \dots, r, \tag{33}$$

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + a \begin{bmatrix} D_1^+ \odot W^M + D_1^{\mu^-} \odot W^m \\ \vdots \\ D_p^+ \odot W^M + D_p^{\mu^-} \odot W^m \\ D_1^- \odot W^M + D_1^{\mu^+} \odot W^m \\ \vdots \\ D_p^- \odot W^M + D_p^{\mu^+} \odot W^m \end{bmatrix} \leq \varepsilon \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{34}$$

$$a > 0. \tag{35}$$

Remark 3: It should be noticed that the above optimisation problem is convex and can be easily reduced to a sequence of linear programming problems for different values of parameter a .

Next, we consider the case when no matrices K and H satisfying relations (28), (29) exist, that is the case when no linear control law exists making $v^*(x)$ a Lyapunov function for the closed-loop system. In this case we can apply the results stated in Section 3 to make another polyhedral set $Q(G^*, w_1^*, w_2^*)$ domain of attraction of the closed-loop system. To this end, we must first determine matrices K_0 , G_0 and H_0 and a vector $w_0^T = [w_{01}^T \ w_{02}^T]$, $w_0 > 0$ (Blanchini 1999,

Blanchini and Miani 2008) satisfying relations

$$G_0(A + BK_0) = H_0G_0, \tag{36}$$

$$\begin{bmatrix} H_0^+ & H_0^- \\ H_0^- & H_0^+ \end{bmatrix} \begin{bmatrix} w_{01} \\ w_{02} \end{bmatrix} < \begin{bmatrix} w_{01} \\ w_{02} \end{bmatrix}. \tag{37}$$

This can be done, for example, by choosing a feedback gain K_0 placing the eigenvalues inside the unit rhombus. Then, matrix G_0 which transforms matrix $A + BK_0$ to its real Jordan form J_0 , satisfies relation $G_0(A + BK_0) = J_0G_0$. Thus, relation (36) is satisfied for $H_0 = J_0$. Since all eigenvalues $\lambda_i = \sigma_i + j\omega_i$ of matrix H_0 satisfy inequality $|\sigma_i| + |\omega_i| < 1$ the non-negative matrix

$$H_0^* = \begin{bmatrix} H_0^+ & H_0^- \\ H_0^- & H_0^+ \end{bmatrix} \tag{38}$$

has a positive real eigenvalue $\lambda < 1$ associated with a real eigenvector $v_0 > 0$ (Bitsoris 1988). Therefore, inequality (37) is satisfied for $w_0 = v_0$. This, in turn, implies the existence of a positive real number a_0 and matrices D_{0j} , $j = 1, 2, \dots, p$ such that relations (32)–(34) are satisfied for $K = K_0$, $H = H_0$, $D_j = D_{0j}$, $j = 1, 2, \dots, p$ and $a = a_0$. Therefore, by virtue of Theorem 3.4, the polyhedral set $Q(G_0, a_0w_{01}, a_0w_{02})$ is a domain of attraction of the resulting closed-loop system. If the maximal value $a_{0\text{max}}$ of a_0 is such that $Q(G, w_1, w_2) \subseteq Q(G_0, a_{0\text{max}}w_{01}, a_{0\text{max}}w_{02})$ then $u = K_0x$ is a solution to the unconstrained control problem.

In the case when $Q(G, w_1, w_2) \not\subseteq Q(G_0, a_{0\text{max}}w_{01}, a_{0\text{max}}w_{02})$ it is possible to determine another control law $u = Kx$ making a larger set domain of attraction of the corresponding closed-loop system. To this end, we solve the linear programming problem

$$\max_{K, H, D_1, \dots, D_p, a} \{a\}, \tag{39}$$

under constraints

$$G_0(A + BK) = HG_0, \tag{40}$$

$$\sum_{i=1}^n g_{ji} C_i K = G_0^T D_j G_0, \quad j = 1, 2, \dots, p, \tag{41}$$

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} w_{01} \\ w_{02} \end{bmatrix} + a \begin{bmatrix} D_1^+ \odot W_0^M + D_1^{\mu^-} \odot W_0^m \\ \vdots \\ D_r^+ \odot W_0^M + D_r^{\mu^-} \odot W_0^m \\ D_1^- \odot W_0^M + D_1^{\mu^+} \odot W_0^m \\ \vdots \\ D_r^- \odot W_0^M + D_r^{\mu^+} \odot W_0^m \end{bmatrix} < \begin{bmatrix} w_{01} \\ w_{02} \end{bmatrix}, \tag{42}$$

$$a > 0. \tag{43}$$

It is clear that the optimal solution a_{\max} of this problem satisfies inequality $a_{\max} \geq a_{0\max}$. Thus, the new domain of attraction $Q(G_0, a_{\max}w_{01}, a_{\max}w_{02})$ satisfies relation $Q(G_0, a_{0\max}w_{01}, a_{0\max}w_{02}) \subseteq Q(G_0, a_{\max}w_{01}, a_{\max}w_{02})$.

Remark 4: We can determine a better polyhedral estimate of the domain of attraction of the closed-loop system with the control law resulting from the optimisation problem (39)–(43): if G^* is the matrix that transforms matrix $A + BK$ to its real Jordan form J , then

$$G^*(A + BK) = HG^*, \tag{44}$$

for $H=J$ and there exists a vector $w^T = [w_1^T \ w_2^T]$, $w > 0$ satisfying the inequality

$$\begin{bmatrix} H^+ & H^- \\ H^- & H^+ \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} < \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{45}$$

Therefore, there exist positive real numbers a and matrices D_j such that relations (30)–(34) are satisfied. Thus, with the control law $u = Kx$, besides $Q(G_0, a_{\max}w_{01}, a_{\max}w_{02})$, the polyhedral set $Q(G^*, a_{\max}w_1, a_{\max}w_2)$ is also positive invariant and domain of attraction of the closed-loop system. Consequently, with the control law $u = Kx$, set

$$Q(G_0, a_{0\max}w_{01}, a_{0\max}w_{02}) \cup Q(G^*, a_{\max}w_1, a_{\max}w_2)$$

is also a domain of attraction of the corresponding closed-loop system.

4.2 The constrained control problem

Let us now consider the constrained control problem, that is the case where control constraints of the form (4) are also imposed. The problem consists in the determination of a linear state-feedback control law $u(t) = Kx(t)$ such that all initial states belonging to the set defined by inequalities (3) are transferred asymptotically to the origin while the control constraints (4) are satisfied. As has been noticed in Section 2, this problem may not possess any solution. According to a general result concerning the control of nonlinear systems (Bitsoris and Gravalou 1995), a necessary and sufficient condition for a linear control law $u(t) = Kx(t)$ to be a solution to the constrained control problem is the existence of a subset Ω of the state space which is both a positively invariant set and domain of attraction of the resulting closed-loop system and satisfies the set relation

$$Q(G, w_1, w_2) \subseteq \Omega \subseteq Q(K, \rho_1, \rho_2). \tag{46}$$

By combining these conditions with the results of Section 3 relative to the stability and the positive invariance of polyhedral sets for systems with second-order polynomial nonlinearities, many different approaches for the determination of such a control law can be developed. An interesting special case is when $Q(G, w_1, w_2) = \Omega$, that is when the stabilising linear control law $u(t) = Kx(t)$ renders the desired domain of attraction positively invariant w.r.t. the closed-loop system. Then, set relation (46) becomes

$$Q(G, w_1, w_2) \subseteq Q(K, \rho_1, \rho_2). \tag{47}$$

By virtue of Farkas Lemma, set relation (47) is equivalent to the existence of a nonnegative matrix $L \in \mathbb{R}^{2m \times 2r}$ such that

$$L \begin{bmatrix} G \\ -G \end{bmatrix} = \begin{bmatrix} K \\ -K \end{bmatrix}, \tag{48}$$

$$L \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \leq \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}. \tag{49}$$

Combining these relations with the conditions of positive invariance and attractivity of the set $Q(G, w_1, w_2)$ stated in Theorem 3.2, we establish the following result:

Theorem 4.1: *The control law $u(t) = Kx(t)$ is a solution to the constrained stabilisation problem if there exist matrices $H \in \mathbb{R}^{p \times p}$, $D_j \in \mathbb{R}^{r \times r}$, $j = 1, 2, \dots, r$, $L \in \mathbb{R}^{2m \times 2r}$ and $L \geq 0$ such that (23)–(25), (48) and (49) are satisfied.*

Thus, for the constrained control problem we can use similar linear programming design techniques by considering (48) and (49) as additional linear constraints.

5. Numerical examples

Example 1: We consider a second-order bilinear system with system matrices

$$A = \begin{bmatrix} 1 & 0.01 \\ 0.01 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.09 \\ 0.09 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.001 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ -0.004 \end{bmatrix}.$$

The state vector is constrained to satisfy linear inequalities

$$-4 \leq x_i \leq 4 \quad i = 1, 2. \tag{50}$$

Bounds are also imposed on the control input:

$$-2 \leq u \leq 2. \tag{51}$$

The problem to be solved is the determination of a linear state-feedback stabilising control law $u = Kx$ and of a domain of attraction $D \subset \mathbb{R}^2$ of the resulting closed-loop system so that all initial states $x_0 \in D$ are transferred asymptotically to the origin while both state and control constraints (50) and (51) are respected.

The largest admissible domain of attraction is the state constraints set $Q(G_x, w_x, w_x) = \{x \in \mathbb{R}^2 : -w_x \leq G_x x \leq w_x\}$, where

$$G_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad w_x = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

Solving the linear programming problem (23)–(25), (48) and (49) with $G = G_x$, $w_1 = -w_2 = w_x$ and $\rho_1 = \rho_2 = 2$ we obtain

$$K^* = \begin{bmatrix} -0.1111 & -0.1270 \end{bmatrix},$$

and $\varepsilon_{\min} = 0.99$. Thus, with the linear control law $u = K^*x$ the whole state constraints set $Q(G_x, w_x, w_x)$ becomes positively invariant and a domain of attraction where both state and control constraints are respected.

In Figure 1, the trajectories of the closed-loop system starting from the vertices of the positively invariant polyhedral set $Q(G_x, w_x, w_x)$ are shown. This problem has also been investigated by Cannon et al. (2003). For comparison purposes the invariant set obtained by that approach is also shown in Figure 1.

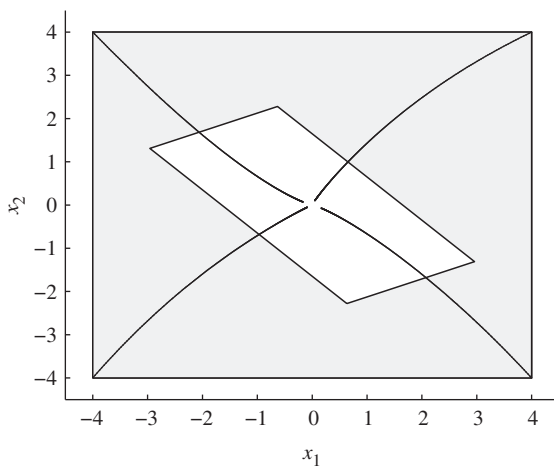


Figure 1. Invariant set with polyhedral Lyapunov function (grey); invariant set using the method described in Cannon et al. (2003) (white) and trajectories emanating from vertices of invariant set S_x .

Example 2: We consider a third-order bilinear system with two inputs and system matrices

$$A = \begin{bmatrix} 1.10 & -0.20 & -0.34 \\ -0.06 & 0.70 & -0.41 \\ 0.41 & 0.41 & 0.90 \end{bmatrix}, \quad B = \begin{bmatrix} 3.75 & 0 \\ 1.05 & -1.33 \\ -0.85 & -0.49 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.12 & -0.18 \\ -0.22 & 0.30 \\ 0.36 & 0.07 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -0.32 & -0.03 \\ 0.48 & -0.18 \\ 0.36 & -0.38 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} -0.35 & 0.55 \\ 0.36 & -0.74 \\ -0.18 & -0.77 \end{bmatrix}.$$

The control inputs have to respect the linear constraints

$$-\rho_2 \leq u \leq \rho_1,$$

where $\rho_1 = \rho_2 = [1 \ 1]^T$. In this example, no initial condition set is given. The problem to be investigated is the determination of a subset $Q(G, w_1, w_2)$ of the state space, as well as a corresponding linear feedback gain K , such that $Q(G, w_1, w_2)$ is a positively invariant set of the closed-loop system and a domain of attraction while the input constraints are satisfied.

The procedure described in Section 4 provides a solution to this problem. By applying a standard eigenvalue assignment approach we determine a gain matrix

$$K_0 = \begin{bmatrix} -0.1745 & 0.0073 & 0.1040 \\ 0.1628 & 0.3219 & -0.1291 \end{bmatrix}$$

placing the eigenvalues of matrix $A + BK_0$, at 0.1, 0.6 and 0.9. Matrix G_0 that transforms $A + BK_0$ to its Jordan form and vectors w_{01}, w_{02} given by the relations

$$G_0 = \begin{bmatrix} 0.763 & 0.643 & 0.066 \\ 0.600 & -0.665 & -0.445 \\ 0.6679 & 0.106 & 0.737 \end{bmatrix},$$

$$w_{01} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad w_{02} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

satisfy conditions (36) and (37). Solving the optimisation problem (39)–(43), (48), (49) we obtain the optimal values $a_{\text{opt}} = 0.5715$ and

$$K = \begin{bmatrix} -0.3324 & 0.0304 & 0.0728 \\ -0.0576 & 0.4751 & 0.1901 \end{bmatrix}.$$

Thus, with the control input $u = Kx$, the polyhedral set $Q(G, w_1, w_2) = Q(G_0, a_{\text{opt}}w_{01}, a_{\text{opt}}w_{02})$ is both positively

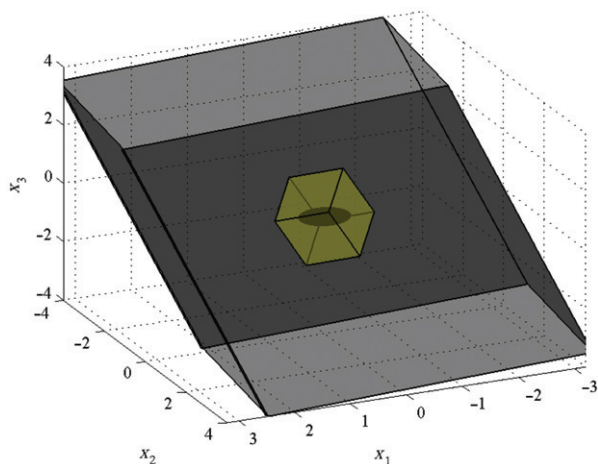


Figure 2. Input constraint set $R(Kx, u_m, u_M)$ (black), polytopic invariant set (yellow) and ellipsoidal invariant set (black). Available in colour online.

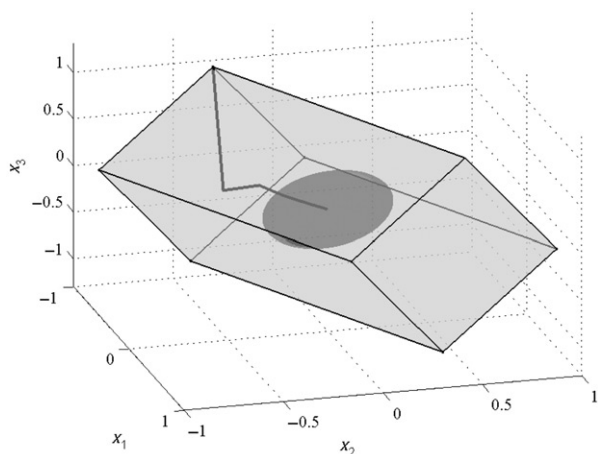


Figure 3. Computed polytopic invariant set $Q(G, w_1, w_2)$ (white), ellipsoidal invariant set from Bloemen, van den Boom, and Verbruggen (2002) (black) and trajectory for initial state $x_0 = [-0.4891 \ -0.4406 \ 1.2831]^T$.

invariant and domain of attraction of the resulting closed-loop system.

This problem has also been studied by Bloemen et al. (2002) in the context of MPC. The authors computed an ellipsoidal invariant set that served as the target set in the MPC algorithm. In Figure 2, the unbounded input constraint set, translated in the state space for the computed feedback gain together with the invariant set $Q(G, w_1, w_2)$ and the one computed in Bloemen et al. (2002) are shown. In Figure 3, the last two sets are shown, together with the trajectory of the closed-loop with initial state $x_0 = [-0.4891 \ -0.4406 \ 1.2831]^T$. It is clearly shown that the polytopic set is much larger than the ellipsoidal one. In Figure 4, the control inputs for the same initial state $x_0 = [-0.4891 \ -0.4406 \ 1.2831]^T$ are shown.

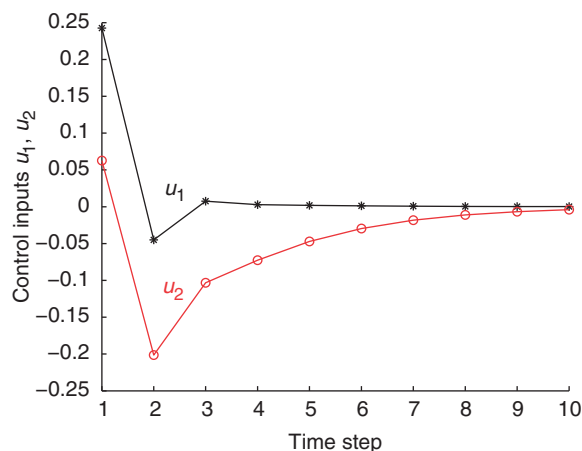


Figure 4. Control inputs for $x_0 = [-0.4891 \ -0.4406 \ 1.2831]^T$.

6. Conclusions

A new approach to the constrained and unconstrained stabilisation of discrete-time bilinear systems by linear state-feedback has been presented. In contrast to all known Lyapunov oriented methods which are based on quadratic functions, in this article polyhedral Lyapunov functions have been used. Since the use of polyhedral functions allows the construction of polyhedral positively invariant sets and domains of attraction, this type of function seem to be the natural Lyapunov functions for studying control problems under linear constraints. This approach leads to an analytic way of computing linear state-feedback gains and fixed-complexity polytopic positively invariant sets. The first step in this direction has been the development of the necessary theoretical background, namely the establishment of conditions guaranteeing the positive invariance of polyhedral sets w.r.t. to nonlinear systems with second-order polynomial nonlinearities. Using known results on the connection between comparison systems and positively invariant sets (Bitsoris and Gravalou 1995; Bitsoris and Truffet 2006), it has been shown that a polyhedral set is positively invariant w.r.t. this class of nonlinear systems if an associated linear algebraic problem is feasible. Then, systematic methods for the determination of stabilising linear state-feedback control laws for both constrained and unconstrained bilinear systems have been developed. The case where no feasible solution rendering the initial condition set a domain of attraction exists has also been investigated. For this case, an approach for the determination of a domain of attraction, possibly smaller than the pre-specified set, has been established, thus extending considerably previous work (Bitsoris and Athanasopoulos 2008). By all these methods the stabilisation problem of bilinear systems is reduced to linear programming

problems where performance issues reflecting the transient behaviour rate of the system are introduced as design parameters. It should be noticed that all these approaches which can be applied to multiple-input multiple-output, open-loop stable or unstable systems with any linear input and/or state constraints are just some of many different design approaches that can be developed using the general results stated in Section 3.

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Appendix A: Proof of Theorem 3.2

Setting

$$v(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix} = \begin{bmatrix} Gx \\ -Gx \end{bmatrix}$$

inequalities

$$-w_2 \leq Gx \leq w_1 \tag{A1}$$

can be equivalently written in the form

$$v(x) \leq w.$$

Consequently, adopting the notations (5) and (10), the polyhedral set defined by inequalities (A1) can be written as

$$Q(G, w_1, w_2) = P(v, w).$$

On the other hand,

$$v_i(x(t+1)) = (-1)^{i+1}G(A+BK)x(t) + (-1)^{i+1}G \times \begin{bmatrix} x^T(t)C_1Kx(t) \\ x^T(t)C_2Kx(t) \\ \vdots \\ x^T(t)C_nKx(t) \end{bmatrix}, \quad i = 1, 2$$

and taking into account (11) and (12) we establish the relations

$$v_i(x(t+1)) = (-1)^{i+1}HGx(t) + (-1)^{i+1} \times \begin{bmatrix} x^T(t)G^TD_1Gx(t) \\ x^T(t)G^TD_2Gx(t) \\ \vdots \\ x^T(t)G^TD_rGx(t) \end{bmatrix}, \quad i = 1, 2. \tag{A2}$$

Since $H = H^+ - H^-$,

$$HGx = H^+(Gx) + H^-(-Gx), \tag{A3}$$

$$H(-Gx) = H^-(Gx) + H^+(-Gx). \tag{A4}$$

Using notations (7), relations (A3) and (A4) can be equivalently written as

$$HGx = H^+y_1 + H^-y_2, \tag{A5}$$

$$-HGx = H^-y_1 + H^+y_2. \tag{A6}$$

Also,

$$\begin{aligned} x^TG^TD_jGx &= x^TG^TD_j^\delta Gx + x^TG^TD_j^\mu Gx \\ &= x^TG^TD_j^\delta Gx + x^TG^TD_j^{\mu^+} Gx - x^TG^TD_j^{\mu^-} Gx, \end{aligned} \tag{A7}$$

and

$$-x^TG^TD_jGx = -x^TG^TD_j^\delta Gx - x^TG^TD_j^{\mu^+} Gx + x^TG^TD_j^{\mu^-} Gx \tag{A8}$$

because

$$D_j = D_j^\delta + D_j^\mu, \tag{A9}$$

$$D_j^\mu = D_j^{\mu^+} - D_j^{\mu^-}.$$

Using notations (7)–(9), from (A7) it follows that

$$x^TG^TD_jGx \leq D_j^{\delta^+} \odot Y^M + D_j^{\mu^+} \odot Y^M + D_j^{\mu^-} \odot Y^m,$$

or, by virtue of (A9),

$$x^TG^TD_jGx \leq D_j^+ \odot Y^M + D_j^{\mu^-} \odot Y^m \tag{A10}$$

because matrices $D_j^{\delta^+}$, $D_j^{\delta^-}$, $D_j^{\mu^+}$ and $D_j^{\mu^-}$ have nonnegative elements and for a nonnegative matrix D

$$\begin{aligned} x^TG^TDGx &= \sum_{i=1}^p \sum_{j=1}^p d_{ij}(Gx)_i(Gx)_j \\ &= \sum_{i=1}^p \sum_{j=1}^p d_{ij}(-Gx)_i(-Gx)_j \\ &\leq \sum_{i=1}^p \sum_{j=1}^p d_{ij} \max\{(Gx)_i(Gx)_j, (-Gx)_i(-Gx)_j\} \\ &= D \odot Y^M, \end{aligned}$$

and

$$\begin{aligned} -x^TG^TDGx &= \sum_{i=1}^p \sum_{j=1}^p d_{ij}(Gx)_i(-Gx)_j \\ &= \sum_{i=1}^p \sum_{j=1}^p d_{ij}(-Gx)_i(Gx)_j \\ &\leq \sum_{i=1}^p \sum_{j=1}^p d_{ij} \max\{(Gx)_i(-Gx)_j, (-Gx)_i(Gx)_j\} \\ &= D \odot Y^m. \end{aligned}$$

Using similar arguments, it can be shown that

$$-x^TG^TD_jGx \leq D_j^- \odot Y^M + D_j^{\mu^+} \odot Y^m. \tag{A11}$$

Thus, taking into account (A5), (A6), (A10) and (A11)–(A8), from (A2) it follows that

$$y(t+1) \leq h(y(t))$$

or, equivalently,

$$v(x(t+1)) \leq h[v(x(t))]$$

with function $h(y)$ defined by (14). By construction, this function is nondecreasing. Therefore, by virtue of Lemma 1, from (13) it follows that set $Q(G, w_1, w_2) = P(v, w)$ is positively invariant with respect to the closed-loop nonlinear system (2). \square