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by

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# A Sub-Cell Discretization Method for the Convective Terms in the Incompressible Navier-Stokes Equations

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**Abstract** In this contribution we present a sub-cell discretization method for the computation of the interface velocities involved in the convective terms of the incompressible Navier-Stokes equations. We compute the interface velocity by solving a local two-point boundary value problem (BVP) iteratively. To account for the two-dimensionality of the interface velocity we introduce a constant cross-flux term in our computation. The discretization scheme is used to simulate the flow in a lid-driven square cavity.

## 1 Introduction

When solving the incompressible Navier-Stokes equations using a finite-volume method on a staggered grid, it is required to compute the interface velocities involved in the convective terms. Standard methods for computing the interface velocities use linear interpolations (taking the average values of the two neighboring known velocities), or taking the upwind value. For incompressible flows the interface velocities attain the average value in case of diminishing flow ( $\text{Re} \downarrow 0$ ) and the upwind value in the limit  $\text{Re} \rightarrow \infty$ . Using standard methods for computing the interface velocities we tend to ignore the nature of the flow in most cases. In the sub-cell computation, we solve a reduced momentum equation locally over an inter-

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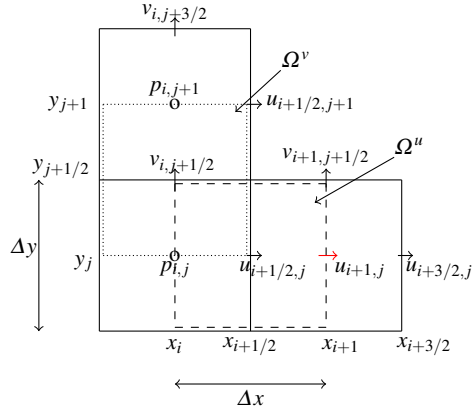
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**Fig. 1** Spatial discretization scheme on a uniform staggered grid, with pressure  $p$  defined in the cell center and the velocity components defined at the center of the cell faces.



val to compute the interface velocities. The interface velocities thus computed are consistent with the equations governing fluid flow.

In [1], we presented the idea of including a piecewise linear pressure gradient in the local BVP for the computation of interface velocities. In the present paper we further extend the method, by including the cross-flux term to the righthand side (RHS) of the local BVP. The inclusion of the cross-flux term provides a two-dimensional character to the computed interface velocities. The sub-cell method is computationally more expensive than the standard methods for computing the interface velocities. However, the accuracy gain allows us to use *coarser* grids as compared to the standard methods.

In the next section we give details of the underlying finite-volume method used for solving the incompressible Navier-Stokes equations. In this article we focus only on the two-dimensional case. The method can also be extended to three dimensions though. In Section 3, we give the details for the integral representation of the interface velocities. In order to account for the nonlinear character of the two-point local BVP, the computation of the interface velocities is done iteratively, which is discussed in Section 4. The results for the proposed discretization scheme are presented in Section 5.

## 2 Convective terms and interface velocities

Consider the incompressible Navier-Stokes equations,

$$\nabla \cdot \mathbf{u} = 0, \quad (1a)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u}, \quad (1b)$$

where  $\mathbf{u} = (u, v)$  is the velocity of the fluid,  $p$  the pressure and  $\text{Re}$  the Reynolds

number. We discretize the above system of equations using a second-order accurate finite volume scheme on a uniform staggered grid, as shown in Figure 1. The discrete system of equations is written as

$$D(\mathbf{u}(t)) = \mathbf{r}_1(t), \quad (2a)$$

$$|\Omega|\mathbf{u}'(t) = -C(\mathbf{u}) + \frac{1}{\text{Re}}L\mathbf{u}(t) - Gp(t) + \mathbf{r}_2(t), \quad (2b)$$

where  $D$ ,  $C$ ,  $L$  and  $G$  represent the discrete divergence, convection, diffusion and gradient operators, respectively, and where  $|\Omega|$  represents the measure of the control volumes. The terms  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(t)$  include the boundary conditions for the system of equations, for details see [3].

Let us consider the  $u$ -component of the convective term  $C(\mathbf{u})$  at  $(x_{i+1/2}, y_j)$  i.e.,

$$(C^u(\mathbf{u}))_{i+1/2,j} = \Delta y (u_{i+1,j}^2 - u_{i,j}^2) + \Delta x (v_{i+1/2,j+1/2} u_{i+1/2,j+1/2} - v_{i+1/2,j-1/2} u_{i+1/2,j-1/2}). \quad (3)$$

In order to compute the above term we need the interface velocities  $u_{i+1,j}$ ,  $u_{i+1/2,j+1/2}$  and  $v_{i+1/2,j+1/2}$ , the rest can be computed in similar way. We will focus primarily on the computation of the interface velocity  $u_{i+1,j}$  using the local momentum equation,

$$(u^2)_x - \varepsilon u_{xx} = -p_x - ((uv)_y - \varepsilon u_{yy}) \quad (x_{i+1/2} < x < x_{i+3/2}; y = y_j), \quad (4)$$

where the flow is assumed to be locally steady ( $u_t = 0$ ) and  $\varepsilon = 1/\text{Re}$ . Let  $F^{u,y} = (uv)_y - \varepsilon u_{yy}$ , then equation (4) becomes

$$(u^2)_x - \varepsilon u_{xx} = -p_x - F^{u,y}. \quad (5)$$

The above equation resembles a steady viscous Burgers equation. We assume that the pressure  $p$  is piecewise linear over  $(x_{i+1/2}, x_{i+3/2})$ , thus the pressure gradient  $p_x$  is piecewise constant with a jump at  $x_{i+1}$ . On the other hand, the cross-flux term  $F^{u,y}$  is constant over the domain. We now suppress the  $y$ -dependence of equation (5) and denote  $u(x_i, y_j)$  by  $u_i$ . Thus we have to solve the equation for  $x \in (x_{i+1/2}, x_{i+3/2})$  subject to the boundary conditions

$$u(x_{i+1/2}) = u_{i+1/2}, \quad u(x_{i+3/2}) = u_{i+3/2}, \quad (6)$$

in order to compute  $u_{i+1} = u(x_{i+1})$  (indicated in red in Figure 1).

Further, we linearize equation (5) by replacing the nonlinear term  $(u^2)_x$  by  $Uu_x$ , where  $U$  is an estimate for the interface velocity  $u_{i+1,j}$ . The linearized equation is then solved iteratively, in order to account for the nonlinearity of the problem. The details for solving the linearized local BVP under the assumption that  $F^{u,y} = 0$  can be found in [1]. In this paper, we briefly outline the method used in [1] and then extend it by including a constant cross flux term  $F^{u,y}$ .

### 3 Integral representation of the interface velocities

In the local BVP (5)–(6), we get the  $y$ -dependence of the velocity component  $u$  as a result of the inclusion of the constant cross-flux term  $F^{u,y}$ . We introduce the following notation:  $u' = u_x$ ,  $p' = p_x$ ,  $a = U/\varepsilon$  and  $P = a\Delta x$ , where  $\Delta x$  is the width of the control volume shown in Figure 1 and  $P$  the local *Péclet number* for the control volume. Then equation (5) can be linearized and rewritten as

$$\varepsilon(u' - au)' = p' + F^{u,y}, \quad (\varepsilon > 0). \quad (7)$$

Using the integrating factor formulation  $u' - au = e^{ax}(e^{-ax}u)'$  and integrating equation (7), we find

$$\varepsilon(e^{-ax}u)' = e^{-ax}(I(x) + F^{u,y}(x - x_{i+1}) + K), \quad I(x) = \int_{x_{i+1}}^x p'(\xi) d\xi.$$

Note that we begin the integration from  $x_{i+1}$ , as  $p'$  has a jump at  $x = x_{i+1}$ . Further integrating the above from  $x_{i+1/2}$  to  $x$  and using the boundary condition  $u(x_{i+1/2}) = u_{i+1/2}$  yields

$$\begin{aligned} u(x) &= e^{a(x-x_{i+1/2})}u_{i+1/2} + \frac{1}{\varepsilon} \int_{x_{i+1/2}}^x e^{a(x-\xi)}I(\xi)d\xi + \\ &\quad \frac{1}{\varepsilon}F^{u,y} \int_{x_{i+1/2}}^x e^{a(x-\xi)}(\xi - x_{i+1})d\xi + \frac{1}{\varepsilon}K \int_{x_{i+1/2}}^x e^{a(x-\xi)}d\xi. \end{aligned}$$

We now introduce the scaled  $x$ -coordinate  $\sigma$  defined as

$$\sigma := \sigma(x) = \frac{x - x_{i+1/2}}{\Delta x}, \quad (0 \leq \sigma \leq 1).$$

Using the scaled coordinate we get

$$\begin{aligned} u(\sigma) &= e^{P\sigma}u_{i+1/2} + \frac{1}{\varepsilon}\Delta x J(\sigma) + \frac{1}{\varepsilon}F^{u,y}\Delta x^2 \int_0^\sigma e^{P(\sigma-\eta)}\left(\eta - \frac{1}{2}\right)d\eta + \\ &\quad \frac{K}{U}(e^{P\sigma} - 1), \end{aligned}$$

where  $J(\sigma)$  is defined as

$$J(\sigma) = \int_0^\sigma e^{P(\sigma-\eta)}I(x_{i+1/2} + \eta\Delta x)d\eta.$$

The integral in the RHS which gives the contribution of the cross flux term in the interface velocity, is given by

$$\int_0^\sigma e^{P(\sigma-\eta)}\left(\eta - \frac{1}{2}\right)d\eta = G(\sigma; P),$$

where  $G(\sigma; P)$  is defined as

$$G(\sigma; P) := \frac{1}{P^2} \left( \left( 1 - \frac{1}{2}P \right) (e^{P\sigma} - 1) - \sigma P \right).$$

Thus we get

$$u(\sigma) = e^{P\sigma} u_{i+1/2} + \frac{1}{\varepsilon} \Delta x J(\sigma) + \frac{1}{\varepsilon} F^{u,y} \Delta x^2 G(\sigma; P) + \frac{K}{U} (e^{P\sigma} - 1).$$

Applying the boundary condition  $u(x_{i+3/2}) = u_{i+3/2}$  we obtain

$$u(\sigma) = W(1 - \sigma; -P) u_{i+1/2} + W(\sigma; P) u_{i+3/2} + \frac{1}{\varepsilon} \Delta x \left( J(\sigma) - W(\sigma; P) J(1) \right) + \frac{1}{\varepsilon} F^{u,y} \Delta x^2 \left( G(\sigma; P) - W(\sigma; P) G(1; P) \right),$$

where

$$W(\sigma; P) = \frac{e^{P\sigma} - 1}{e^P - 1}, \quad \left( 0 \leq W(\sigma; P) \leq 1; \quad W(1 - \sigma; -P) + W(\sigma; P) = 1 \right).$$

The details for the computation of  $J(\sigma)$  and  $J(1)$  can be found in [1]. At this point we rewrite  $u(\sigma)$  as a sum of components arising from terms in the RHS of equation (7),

$$u(\sigma) = u^h(\sigma) + u^p(\sigma) + u^f(\sigma), \quad (10)$$

where

$$\begin{aligned} u^h(\sigma) &:= W(1 - \sigma; -P) u_{i+1/2} + W(\sigma; P) u_{i+3/2}, \\ u^p(\sigma) &:= \frac{1}{\varepsilon} \Delta x \left( J(\sigma) - W(\sigma; P) J(1) \right), \\ u^f(\sigma) &:= \frac{1}{\varepsilon} F^{u,y} \Delta x^2 \left( \frac{1}{P} (W(\sigma; P) - \sigma) \right). \end{aligned}$$

In case of no pressure gradient and no cross flux, we get  $u(\sigma) = u^h(\sigma)$  on solving the homogeneous local BVP. Including the pressure gradient  $p'$  in the RHS of the homogeneous local BVP gives us  $u(\sigma) = u^h(\sigma) + u^p(\sigma)$ . Similarly, including the constant cross flux term  $F^{u,y}$  gives us the additional component  $u^f(\sigma)$ .

Finally the interface velocity  $u_{i+1,j}$  can be computed as

$$u_{i+1,j} = u_{i+1,j}^h + u_{i+1,j}^p + u_{i+1,j}^f.$$

For  $x = x_{i+1}$ , we have  $\sigma = 0.5$ , for which  $W := W(0.5; P) = (1 + e^{P/2})^{-1}$ . Now the velocity components are given by

$$u_{i+1,j}^h = (1 - W) u_{i+1/2,j} + W u_{i+3/2,j}, \quad (11a)$$

$$u_{i+1,j}^p = -\frac{1}{4\varepsilon} \Delta x^2 \left( A(-P/2) (\delta_x p)_{i+1/2,j} + A(P/2) (\delta_x p)_{i+3/2,j} \right), \quad (11b)$$

$$u_{i+1,j}^f = \frac{1}{\varepsilon} F_{i+1,j}^{u,y} \Delta x^2 \left( W - \frac{1}{2} \right) \frac{1}{P}, \quad (11c)$$

where



$$(\delta_x p)_{i+1/2,j} = \frac{1}{\Delta x} (p_{i+1,j} - p_{i,j}) \quad \text{and} \quad A(z) = \frac{e^z - 1 - z}{z^2(e^z + 1)}.$$

#### 4 Computation of the interface velocity

For solving the discretized momentum equation (2b), we need to compute  $F^{u,y} = (uv)_y - \varepsilon u_{yy}$  at  $(x_{i+1/2}, y_j)$  and  $(x_{i+3/2}, y_j)$ . Thus the terms  $F_{i+1/2,j}^{u,y}$  and  $F_{i+3/2,j}^{u,y}$  are computed at each time step. In order to compute  $F_{i+1,j}^{u,y}$  we take the weighted average of  $F_{i+1/2,j}^{u,y}$  and  $F_{i+3/2,j}^{u,y}$  analogous to equation (11a), i.e.,

$$F_{i+1,j}^{u,y} = (1 - W)F_{i+1/2,j}^{u,y} + WF_{i+3/2,j}^{u,y}. \quad (12)$$

The cross-flux  $F_{i+1,j}^{u,y}$  can also be computed in other ways. The method proposed here is favored because of its computational efficiency.

In order to account for the nonlinearity of equation (5), we iteratively compute the integral representation (equation (11)) of the interface velocities. Algorithm 1 describes the iterative computation of the interface velocity  $u_{i+1,j}$ .

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#### Algorithm 1 Iterative computation of the interface velocity $u_{i+1,j}$

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Input:  $u_{i+1/2,j}$ ,  $u_{i+3/2,j}$ ,  $(\delta_x p)_{i+1/2,j}$ ,  $(\delta_x p)_{i+3/2,j}$ ,  $F_{i+1/2,j}^{u,y}$ ,  $F_{i+3/2,j}^{u,y}$ ,  $\Delta x$  and  $\varepsilon$

Initialization: Set  $u^h = \frac{1}{2}(u_{i+1/2,j} + u_{i+3/2,j})$ ,  $u^p = 0$  and  $u^f = 0$

$$u_{i+1,j} = u^h + u^p + u^f$$

Set,  $u_{i+1,j}^{(k-1)} = 0$  (the interface velocity from previous iteration)

Define  $TOL$ , as a control parameter for the convergence of the iterative procedure

Define  $err := |u_{i+1,j} - u_{i+1,j}^{(k-1)}|$

**do**

$$u_{i+1,j}^{(k-1)} = u_{i+1,j}$$

$$P := \frac{1}{\varepsilon} u_{i+1,j} \Delta x \quad \text{and} \quad W = (1 + \exp(P/2))^{-1}$$

$$u^h = (1 - W)u_{i+1/2,j} + Wu_{i+3/2,j}$$

$$u^p = -\frac{\Delta x^2}{4\varepsilon} (A(-P/2)(\delta_x p)_{i+1/2,j} + A(P/2)(\delta_x p)_{i+3/2,j})$$

$$F_{i+1,j}^{u,y} = (1 - W)F_{i+1/2,j}^{u,y} + WF_{i+3/2,j}^{u,y}$$

$$u^f = \frac{1}{\varepsilon} F_{i+1,j}^{u,y} \Delta x^2 (W - \frac{1}{2}) \frac{1}{P}$$

$$u_{i+1,j} = u^h + u^p + u^f$$

$$err = |u_{i+1,j} - u_{i+1,j}^{(k-1)}|$$

**while**  $err \leq TOL$

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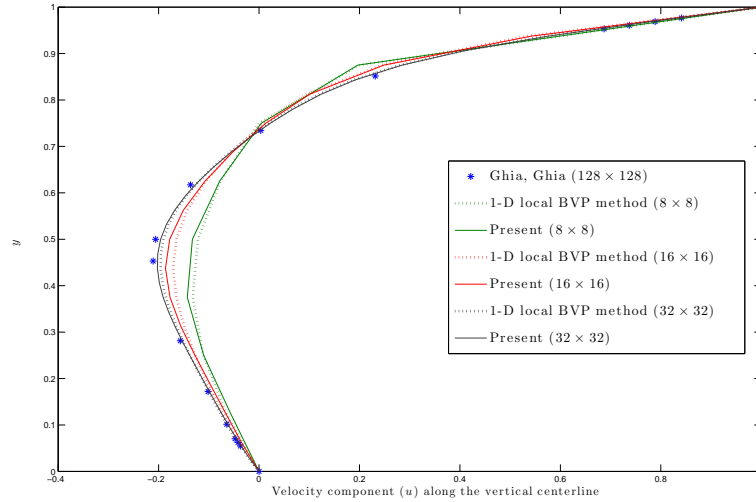
For computing the convective term as given by equation (3), besides  $u_{i+1,j}$  we also need  $u_{i+1/2,j+1/2}$  and  $v_{i+1/2,j+1/2}$ . These velocities can also be computed using the iterative local BVP method. Further details for the iterative computation of these interface velocities can be found in [1].

It should be noted that the above discussion should be analogously used for the computation of the interface velocity  $v_{i,j+1}$  involved in the convective term  $C^v(\mathbf{u})$ .

## 5 Numerical Results

We now use the proposed discretization scheme for computing the interface velocities involved in the convective terms of the incompressible Navier-Stokes equations applied to lid-driven cavity flow.

In this contribution we present the results for the flow in a lid-driven square cavity with  $Re = 100$ , on a hierarchy of rather coarse grids ( $8 \times 8$ ), ( $16 \times 16$ ) and ( $32 \times 32$ ). The results obtained using the present method are compared with those obtained using the 1-D local BVP method (absence of the cross flux term) described in [1]. We take the results from Ghia, Ghia and Shin [2] on a ( $128 \times 128$ ) grid as a reference. Figure 2 shows the velocity profiles for  $u$  along the vertical line passing through the geometric center of the cavity. It can be seen that the present method exhibits higher accuracy than the 1-D local BVP method for all grid sizes. Thus we have improved the 1-D local BVP method by including the cross-flux term, which provides a two-dimensional character to the interface velocities.



**Fig. 2** Comparison of the velocity component  $u$  along the vertical centerline of the cavity for  $Re = 100$ .

## 6 Conclusions

In the preceding sections we proposed a method for the sub-cell computation of interface velocities using local two-point BVPs. We presented an integral representation for the interface velocities as a sum of the components  $u^h$ ,  $u^p$  and  $u^f$  given by equation (11). The integral representation is then solved iteratively to give us the interface velocities. As observed from Figure 2, the present method exhibits higher accuracy than the 1-D local BVP method. Thus inclusion of the cross-flux term in the local BVP gives us higher accuracy by taking into account the two-dimensionality of the interface velocities.

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