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A GENERALIZED RAMANUJAN-NAGELL EQUATION RELATED TO CERTAIN STRONGLY REGULAR GRAPHS

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Abstract
A quadratic-exponential Diophantine equation in 4 variables, describing certain strongly regular graphs, is completely solved. Along the way we encounter different types of generalized Ramanujan-Nagell equations whose complete solution can be found in the literature, and we come across a problem on the order of the prime ideal above 2 in the class groups of certain imaginary quadratic number fields, which is related to the size of the squarefree part of $2^n - 1$ and to Wieferich primes, and the solution of which can be based on the abc-conjecture.

1. Introduction
The question to determine the strongly regular graphs with parameters\(^1\) $(v, k, \lambda, \mu)$ with $v = 2^n$ and $\lambda = \mu$, was recently posed by Natalia Tokareva\(^2\). Somewhat later Tokareva noted\(^3\) that the problem had already been solved by Bernasconi, Codenotti and Vanderkam [2], but nevertheless we found it, from a Diophantine point of view, of some interest to study a ramification of this problem.

We note the following facts about strongly regular graphs, see [5]. They satisfy $(v-k-1)\mu = k(k-\lambda-1)$. With $v = 2^n$ and $\lambda = \mu$ this becomes $2^n = 1 + k(k-1)/\mu$. In this case their eigenvalues are $k$ and $\pm t$ with $t^2 = k - \mu$, with $t$ an integer. From these data Bernasconi and Codenotti [1] derived the diophantine equation $k^2 - 2^n k + t^2(2^n - 1) = 0$, which was subsequently solved in [2]. The only solutions turned out to be $(k, t) = (0, 0), (1, 1), (2^n - 1, 1), (2^n, 0)$ for all $n$, and additionally $(k, t) = (2^{n-1} - 2\frac{1}{2}n-1, 2\frac{1}{2}n-1), (2^{n-1} + 2\frac{1}{2}n-1, 2\frac{1}{2}n-1)$ for even $n$. As a result, the only nontrivial strongly regular graphs of the desired type $(2^n, k, \mu, \mu)$ are those

\(^1\)See [5] for the definition of strongly regular graphs with these parameters.
\(^2\)Personal communication to Andries Brouwer, March 2013.
\(^3\)Personal communication to BdW, April 2013.
with even \( n \) and \((k, \mu) = (2^{n-1} \pm 2^{\frac{n-1}{2}}, 2^{n-2} \pm 2^{\frac{n-2}{2}})\). These are precisely the graphs associated to so-called bent functions, see [1].

In studying this diophantine problem we take a somewhat deviating path\(^4\). Without loss of generality we may assume that there are three distinct eigenvalues, i.e., \( t \geq 1 \) and \( k > 1 \). The multiplicity of \( t \) then is \((2^n - 1 - k/t)/2\), so \( t \mid k \). It follows that also \( t \mid \mu \). We write \( k = at \) and \( \mu = bt \). Then we find \( t = a - b \) and \( 2^n = (a^2 - 1)t/b \).

Let \( g = \gcd(a, b) = \gcd(b, t) \), and write \( a = cg, b = dg \). It then follows that \( 2^n \) is the product of the integers \((a^2 - 1)/d\) and \( t/g \), which therefore are both powers of 2. Let \((a^2 - 1)/d = 2^m\). Then we have \( m \leq n \).

Since \( 2^n - 1 = a(at - 1)/b = a(a^2 - ab - 1)/b \), the question now has become to determine the solutions in positive integers \( n, m, c, g \) of the diophantine equation

\[
2^n - 1 = c (2^m - cg^2).
\]

(1)

For the application at hand only \( n \geq m \) is relevant, but we will study \( n < m \) as well. With \( n \geq m \) there obviously are the four families of Table 1. Our first, completely elementary, result is that there are no others.

\[
\begin{array}{|c|c|c|c|}
\hline
\text{I} & \text{II} & \text{III} & \text{IV} \\
\hline
n & m & c & g \\
\hline
\text{any} & n & 1 & n \\
\text{I} & n & 2^n - 1 & 1 \\
\text{II} & \frac{1}{2}n + 1 & 2^{\frac{n}{2}} - 1 & 1 \\
\text{III} & \frac{1}{2}n + 1 & 2^{\frac{n}{2}} + 1 & 1 \\
\text{IV} & \frac{1}{2}n + 1 & 2^{\frac{n}{2}} - 1 & 1 \\
\hline
\end{array}
\]

Table 1: Four families of solutions of (1) with \( n \geq m \).

**Theorem 1.** All the solutions of (1) with \( n \geq m \) are given in Table 1.

**Proof.** Note that \( c \) and \( g \) are odd, and that \( cg^2 < 2^m \).

For \( m \leq 2 \) the only possibilities for \( cg^2 < 2^m \) are \( c = g = 1 \), leading to \( m = n \), fitting in [I], and for \( m = 2 \) also \( c = 3, g = 1 \), leading to \( n = 2 \), fitting in [II].

For \( m \geq 3 \) we look at (1) modulo \( 2^m \). Using \( n \geq m \) we get \((cg)^2 \equiv 1 \pmod{2^m}\), and by \( m \geq 3 \) this implies \( cg \equiv \pm 1 \pmod{2^{m-1}} \). So either \( c = g = 1 \), immediately leading to \( m = n \) and thus to [I], or \( cg \geq 2^{m-1} - 1 \). Since also \( cg^2 \leq 2^m - 1 \) we get \( g \leq \frac{2^m - 1}{2^{m-1} - 1} < 3 \), hence \( g = 1 \). We now have \( c \equiv \pm 1 \pmod{2^{m-1}} \) and \( 1 < c < 2^m \), implying \( c = 2^{m-1} - 1 \) or \( c = 2^{m-1} + 1 \) or \( c = 2^m - 1 \), leading to exactly [III], [IV], [II] respectively. \( \square \)

Note that this result implies the result of [2].

---

\(^4\text{I owe this idea to Andries Brouwer.}\)
When \( m > n \), a fifth family and seven isolated solutions are easily found, see Table 2. For \( n = 3 \) and \( c = 1 \) equation (1) is precisely the well known Ramanujan-Nagell equation [6].

<table>
<thead>
<tr>
<th></th>
<th>( n )</th>
<th>( m )</th>
<th>( c )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>any ( \geq 3 )</td>
<td>( 2n - 2 )</td>
<td>1</td>
<td>( 2^{n-1} - 1 )</td>
</tr>
<tr>
<td>VI</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7</td>
<td>3</td>
<td></td>
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<tr>
<td></td>
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<td>1</td>
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<td></td>
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<td></td>
<td>15</td>
<td>1</td>
<td>181</td>
<td></td>
</tr>
<tr>
<td>VII</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
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<td>7</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>3</td>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: One family and seven isolated solutions of (1) with \( m > n \).

In Sections 2, 3 and 4 we will prove the following result, which is not elementary anymore, and works for both cases \( n \geq m \) and \( m > n \) at once.

**Theorem 2.** All the solutions of (1) with \( m > n \) are given in Table 2.

2. Small \( n \)

The cases \( n \leq 2 \) are elementary.

*Proof of Theorems 1 and 2 when \( n \leq 2 \).* Clearly \( n = 1 \) leads to \( c = 1 \) and \( 2^m - g^2 = 1 \), which for \( m \geq 2 \) is impossible modulo 4. So there is only the trivial solution \( m = g = 1 \). And for \( n = 2 \) we find \( 3 = c (2^m - cg^2) \), so \( c = 1 \) or \( c = 3 \). With \( c = 1 \) we have \( 2^m - g^2 = 3 \), which for \( m \geq 3 \) is impossible modulo 8. So we are left with the trivial \( m = 2, g = 1 \) only. And with \( c = 3 \) we have \( 2^m - 3g^2 = 1 \), which also for \( m \geq 3 \) is impossible modulo 8. So we are left with the trivial \( m = 2, g = 1 \) only. \( \square \)

3. Recurrence Sequences

From now on we assume \( n \geq 3 \). Let us write \( D = 2^n - 1 \).

**Lemma 3.** For any solution \((n, m, c, g)\) of (1) there exists an integer \( h \) such that

\[
h^2 + Dg^2 = 2^\ell \quad \text{with} \quad \ell = 2m - 2,
\]

\[
c = \frac{2^{m-1} \pm h}{g^2}.
\]
Proof. We view equation (1) as a quadratic equation in $c$. Its discriminant is $2^{2m} - 4Dg^2$, which must be an even square, say $4h^2$. This immediately gives the result. □

So $\ell$ is even, but when studying (2) we will also allow odd $\ell$ for the moment. Note the ‘basic’ solution $(h,g,\ell) = (1, 1, n)$ of (2). In the quadratic field $K = \mathbb{Q} \left( \sqrt{-D} \right)$ we therefore look at

$$\alpha = \frac{1}{2} \left( 1 + \sqrt{-D} \right),$$

which is an integer of norm $2^{n-2}$. Note that $D$ is not necessarily squarefree (e.g. $n = 6$ has $D = 63 = 3^2 \cdot 7$), so the order $\mathcal{O}$ generated by the basis $\{1, \alpha\}$, being a subring of the ring of integers (the maximal order of $K$), may be a proper subring. The discriminant of $K$ is the squarefree part of $-D$, which, just like $-D$ itself, is congruent to 1 (mod 8). So in the ring of integers the prime 2 splits, say (2) = $\mathfrak{p} \mathfrak{q}$, and without loss of generality we can say $(\alpha) = \mathfrak{p}^{n-2}$. Note that it may happen that a smaller power of $\mathfrak{p}$ already is principal. Indeed, for $n = 6$ we have $\mathfrak{p} = (\frac{1}{2} (1 - \sqrt{-7}))$ itself already being principal, where $(\alpha) = (\frac{1}{2} (1 + \sqrt{-63})) = \mathfrak{p}^4$. But note that $\mathfrak{p}, \mathfrak{p}^2, \mathfrak{p}^3$ are not in the order $\mathcal{O}$, and it is the order which interests us. We have the following result.

**Lemma 4.** The smallest positive $s$ such that $\mathfrak{p}^s$ is a principal ideal in $\mathcal{O}$ is $s = n - 2$.

In a later section we further comment on the order of $\mathfrak{p}$ in the full class group for general $n$. In particular we gather some evidence for the following conjecture, showing (among other things) that it follows from (an effective version of) the abc-conjecture (at least for large enough $n$).

**Conjecture 5.** For $n \neq 6$ the smallest positive $s$ such that $\mathfrak{p}^s$ is a principal ideal in the maximal order of $K$ is $s = n - 2$.

**Proof of Lemma 4.** There exists a minimal $s > 0$ such that $\mathfrak{p}^s$ is principal and is in the order $\mathcal{O}$. Let $\frac{1}{2} (a + b\sqrt{-D})$ be a generator of $\mathfrak{p}^s$, then $a, b$ are coprime and both odd, and

$$a^2 + D b^2 = 2^{s+2}.$$

(4)

Since $\mathfrak{p}^{n-2} = (\alpha)$ is principal and in $\mathcal{O}$, we now find that $s|n - 2$, and

$$\left( a + b\sqrt{-D} \right)^k = \pm 2^{k-1} \left( 1 + \sqrt{-D} \right), \quad \text{with} \ k = \frac{n-2}{s}.$$

(5)

Comparing imaginary parts in (5) gives that $b \mid 2^{k-1}$, and from the fact that $b$ is odd it follows that $b = \pm 1$. Equation (4) then becomes $a^2 + D = 2^{s+2}$, which is $a^2 = 2^{s+2} - n + 1$. This equation, which is a generalization of the Ramanujan-Nagell equation that occurs for $n = 3$, has, according to Szalay [8], only the solutions given in Table 3. Only in case [ii] we have $k = \frac{n-2}{s}$ integral, and this proves $k = 1$, $s = n - 2$. □
We next show that the solutions $h, g$ of (2) are elements of certain binary recurrence sequences. We define for $k \geq 0$

\[
h_k = \alpha^k + \bar{\alpha}^k, \quad \text{with } h_0 = 2, h_1 = 1, \text{ and } h_{k+1} = h_k - 2^{n-2}h_{k-1} \text{ for } k \geq 1,
\]

\[
g_k = \frac{\alpha^k - \bar{\alpha}^k}{\sqrt{-D}}, \quad \text{with } g_0 = 0, g_1 = 1, \text{ and } g_{k+1} = g_k - 2^{n-2}g_{k-1} \text{ for } k \geq 1.
\]

For even $n$, say $n = 2r$, we can factor $D$ as $(2^r - 1)(2^r + 1)$. Now we define

\[
\lambda = \frac{1}{2} \left(2^r + 1 + \sqrt{-D}\right), \quad \mu = \frac{1}{2} \left(2^r - 1 + \sqrt{-D}\right),
\]

satisfying $N(\lambda) = 2^{2^r-1} + 2^{r-1}$ and $N(\mu) = 2^{2^r-1} - 2^{r-1}$, $\lambda \bar{\mu} = -\alpha \sqrt{-D}$, $\lambda \mu = 2^{r-1} \sqrt{-D}$, $\lambda^2 = (2^r + 1)\alpha$, and $\mu^2 = -(2^r - 1)\bar{\alpha}$. For $n = 2r$ and $\kappa \geq 0$ we define

\[
u_{\kappa} = \frac{1}{2^r + 1} \left(\lambda \alpha^\kappa + \bar{\lambda} \bar{\alpha}^\kappa\right), \quad \text{with } u_0 = 1, u_1 = -(2^{r-1} - 1),
\]

\[
u_{\kappa+1} = \nu_\kappa - 2^{n-2}u_{\kappa-1} \text{ for } \kappa \geq 1,
\]

\[
u_0 = 1, \nu_1 = 2^{r-2} - 1,
\]

\[
u_{\kappa+1} = \nu_\kappa - 2^{n-2}v_{\kappa-1} \text{ for } \kappa \geq 1.
\]

We present a few useful properties of these recurrence sequences.

**Lemma 6.**

(a) For any $n \geq 3$ we have $g_{2\kappa} = g_\kappa h_\kappa$ for all $\kappa \geq 0$.

(b) For even $n = 2r$ we have $g_{2\kappa+1} = u_\kappa v_\kappa$ for all $\kappa \geq 0$.

(c) For any $n$ and even $k = 2\kappa$, we have

\[
2^{(n-2)\kappa+1} + h_{2\kappa} = h_\kappa^2, \quad 2^{(n-2)\kappa+1} - h_{2\kappa} = (2^n - 1)g_\kappa^2.
\]

(d) For any even $n = 2r$ and odd $k = 2\kappa + 1$, we have

\[
2^{(r-1)(2\kappa+1)+1} + h_{2\kappa+1} = (2^r + 1)u_\kappa^2, \quad 2^{(r-1)(2\kappa+1)+1} - h_{2\kappa+1} = (2^r - 1)v_\kappa^2.
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>$s$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[i]</td>
<td>any $\geq 2$</td>
<td>$2n - 4$</td>
</tr>
<tr>
<td>[ii]</td>
<td>$n - 2$</td>
<td>$2^{n-1} - 1$</td>
</tr>
<tr>
<td>[iii]</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>181</td>
</tr>
</tbody>
</table>

Table 3: The solutions of $a^2 = 2^{n+2} - 2^n + 1$ with $a > 0$. 

[Lemma 6 is provided as a statement of properties related to binary recurrence sequences.]


Proof. Trivial by writing out all equations and using the mentioned properties of $\lambda, \mu$.

For curiosity only, note that $(2^r + 1)u_\kappa^2 + (2^r - 1)v_\kappa^2 = 2^{(r+1)(2\kappa+1)+2}$.

Now that we have introduced the necessary binary recurrence sequences, we can state the relation to the solutions of (2).

**Lemma 7.** Let $(h, g, \ell)$ be a solution of (2).

(a) There exists a $k \geq 0$ such that $h = \pm h_k$, $g = \pm g_k$ and $(n - 2)k = \ell - 2$.

(b) If $\ell$ is even and equation (3) holds with $m = \frac{1}{2}(n - 2)k + 2$ and integral $c$, then one of the four cases [A], [B], [C], [D] as shown in Table 4 applies, according to $k$ being even or odd, and the $\pm$ in (3) being $+$ or $-$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k$</th>
<th>$\pm$</th>
<th>condition</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[A]</td>
<td>any $2\kappa$</td>
<td>+</td>
<td>$g_\kappa = \pm 1$</td>
<td>$\frac{1}{2n-1} h_\kappa^2$</td>
</tr>
<tr>
<td>[B]</td>
<td>$2\kappa + 1$</td>
<td>$\pm h_\kappa^2</td>
<td>2^n - 1$</td>
<td>$\frac{1}{2^n - 1} v_\kappa^2$</td>
</tr>
<tr>
<td>[C]</td>
<td>$2\kappa - 1$</td>
<td>$\pm v_\kappa^2</td>
<td>2^r + 1$</td>
<td>$\frac{1}{2^n - 1} u_\kappa^2$</td>
</tr>
<tr>
<td>[D]</td>
<td>$2\kappa + 1$</td>
<td>$\pm u_\kappa^2</td>
<td>2^r - 1$</td>
<td>$\frac{1}{2^n - 1} v_\kappa^2$</td>
</tr>
</tbody>
</table>

Table 4: The four cases.

Proof.

(a) Equation (2) implies that $g, h$ are coprime, so that $\left(\frac{1}{2} (h \pm g\sqrt{-D})\right) = \varphi^{\ell-2}$.

Lemma 4 then implies that $n - 2 | \ell - 2$. We take $k = \frac{\ell - 2}{n - 2}$ and thus have $\frac{1}{2} (h \pm g\sqrt{-D}) = \alpha^k$ or $\tilde{\alpha}^k$, and the result follows.

(b) Note that $\ell$ being even implies that at least one of $n, k$ is even.

For even $k = 2\kappa$, (a) and Lemma 6(a) say that $g = \pm g_k = \pm g_\kappa h_\kappa$.

If $\pm = +$ then equation (3) and Lemma 6(a,c) say that $c = \frac{2^{n-2} + h_\kappa}{g_\kappa^2}$.

If $\pm = -$ then equation (3) and Lemma 6(a,c) say that $c = \frac{2^{n-2} - h_\kappa}{g_\kappa^2}$.

Then $c$ being integral implies $g_\kappa = \pm 1$ and $c = 1$.

Then $c$ being integral implies $h_\kappa^2 | 2^n - 1$. 


For even $n = 2r$ and odd $k = 2\kappa + 1$, (a) and Lemma 6(b) say that $g = \pm g_k = \pm u_k v_k$.

If $\pm = +$ then equation (3) and Lemma 6(b,d) say that $c = \frac{2^{r-1}(2\kappa+1)+h_{2\kappa+1}}{g_{2\kappa+1}^2} = \frac{2^r+1}{v_k^2}$. Then $c$ being integral implies $v_k^2 \mid 2^r + 1$.

If $\pm = -$ then equation (3) and Lemma 6(b,d) say that $c = \frac{2^{r-1}(2\kappa+1)-h_{2\kappa+1}}{g_{2\kappa+1}^2} = \frac{2^r-1}{u_k^2}$. Then $c$ being integral implies $u_k^2 \mid 2^r - 1$.

\[
\square
\]

Let’s trace the known solutions.

Families [I] and [II] have $k = 2$, so $\kappa = 1$, and $c = 1$ or $c = 2^n - 1$, so they are in cases [A] and [B] with $g_1 = 1$ and $h_1 = 1$, respectively.

Cases [III] and [IV] have $k = 1$, so $\kappa = 0$, and $c = 2^r - 1$ or $c = 2^r + 1$, so they are in cases [D] and [C] with $u_0 = 1$ and $v_0 = 1$, respectively.

Family [V] has $k = 4$, so $\kappa = 2$, and $c = \frac{2^{(n-2)2^2+1} + h_4}{g_4^2} = \frac{h_2^2}{g_2^2 h_2^2} = \frac{1}{g_2^2} = 1$, so it is in case [A].

The known solutions with $n = 3$ and even $k = 2\kappa$ are presented Table 5, and the known solutions with $n = 4$ and even $k = 2\kappa$ resp. odd $k = 2\kappa + 1$ are presented in Table 6.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>…</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_\kappa$</td>
<td>2</td>
<td>(1)</td>
<td>-3</td>
<td>-5</td>
<td>(1)</td>
<td>11</td>
<td>9</td>
<td>-13</td>
<td>…</td>
<td>67</td>
<td>-47</td>
<td>-181</td>
</tr>
<tr>
<td>$g_\kappa$</td>
<td>0</td>
<td>(1)</td>
<td>(1)</td>
<td>(1)</td>
<td>(1)</td>
<td>11</td>
<td>9</td>
<td>-13</td>
<td>…</td>
<td>67</td>
<td>-47</td>
<td>-181</td>
</tr>
<tr>
<td>$m$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>…</td>
<td>13</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

| [A] | c | (1) | (1) | (1) | (1) | (1) |
| [B] | c | (7) | (1) | (1) | (1) | (1) |

Table 5: Tracing the solutions with $n = 3$ and even $k = 2\kappa$ to elements in recurrence sequences.

4. Solving the Four Cases

All four cases [A], [B], [C] and [D] can be reduced to diophantine equations known from the literature.

**Lemma 8.** Case [A] leads to only the solutions from families [I] and [V], and the three isolated solutions from [VI] with odd $m$. 
Table 6: Tracing the solutions with \( n = 4 \) and even \( k = 2\kappa \), resp. odd \( k = 2\kappa + 1 \), to elements in recurrence sequences.

**Proof.** Table 4 gives \( g_k = \pm 1 \) and \( \epsilon = 1 \). Then Equation (1) becomes the generalized Ramanujan-Nagell equation \( g^2 = 2^m - 2^n + 1 \), which was completely solved by Szalay [8].

**Lemma 9.** Case [B] leads to only the solutions from family [II], and the isolated solution from [VI] with \( m \) even.

**Proof.** Note that we have \( \kappa \geq 1 \), and then \( h_\kappa \equiv 1 \pmod{2^{n-2}} \), so we have either \( h_\kappa = 1 \) or \( |h_\kappa| \geq 2^{n-2} - 1 \). In the latter case the condition in Table 4 implies \( (2^{n-2} - 1)^2 \leq h_\kappa^2 \leq 2^n - 1 \), leading to \( n \leq 4 \). If \( n = 3 \) we must have \( h_\kappa = \pm 1 \). But \( h_\kappa \) is never congruent to \(-1 \pmod{8} \), so \( h_\kappa = 1 \). If \( n = 4 \) then we must have \( |h_\kappa| = 1 \). Note that (when \( \kappa \geq 1 \)) we always have \( h_\kappa \equiv 1 \pmod{4} \). So we find that \( h_\kappa = 1 \) always, and it follows from Table 4 that \( c = 2^n - 1 \), and Equation (1) now becomes \( g^2 = \frac{2^m - 1}{2^n - 1} \). Hence \( n \mid m \). The equation \( g^2 = \frac{x^t - 1}{x - 1} \) has been treated by Ljunggren [4], proving (among other results) that for even \( x \), always \( t \leq 2 \). Hence either \( m = n \), \( g = 1 \) leading to family [II], or \( m = 2n \), in which case \( 2^n + 1 \) must be a square. This happens only for \( n = 3 \), leading to \( m = 6 \), thus to the only solution from [VI] with even \( m \).

**Lemma 10.** Cases [C] and [D] lead to only the solutions from families [III] and [IV], and the isolated solutions [VII].

**Proof.** It is easy to see that \( u_\kappa \equiv 1 - 2^{r-1} \pmod{2^{2r-2}} \), \( v_\kappa \equiv 1 + 2^{r-1} \pmod{2^{2r-2}} \) for all \( \kappa \geq 1 \). If \( r \geq 3 \) then it follows that \( |v_\kappa| \geq 2^{r-1} + 1 \) and \( |u_\kappa| \geq 2^{r-1} - 1 \), so the condition in Table 4 shows that in case [C] \( (2^{r-1} + 1)^2 \leq 2^r + 1 \) and in case [D] \( (2^{r-1} - 1)^2 \leq 2^r - 1 \), which both are impossible. Thus \( r = 2 \) or \( \kappa = 0 \).

The case \( \kappa = 0 \) gives \( k = 1 \), so \( g = 1 \), and \( m = \frac{1}{2}n + 1 \), and this gives exactly families [III] and [IV]. So we are left with \( r = 2 \) and \( \kappa \geq 1 \), so \( n = 4 \).

In case [C] the condition in Table 4 shows that \( v^2_\kappa \leq 5 \), but also we always have \( v_\kappa \equiv 3 \pmod{4} \), leaving only room for \( v_\kappa = -1, c = 5 \). This leaves us with solving \( 3 = 2^n - 5g^2 \). This equation is a special case of the generalized Ramanujan-Nagell
equation treated in [9, Chapter 7], from which it can easily be deduced that the only solutions are \( (m, g) = (3, 1), (7, 5) \) (solutions nrs. 72 and 223 in [9, Chapter 7, Table I]). It might occur elsewhere in the literature as well.

In case [D] the condition in Table 4 shows that \( u_k^2 \leq 3 \), but also always \( u_k \equiv 3 \pmod{4} \), leaving only room for \( u_k = -1 \), \( c = 3 \). This leaves us with solving \( 5 = 2^n - 3g^2 \). Again this equation is a special case of the generalized Ramanujan-Nagell equation treated in [9, Chapter 7], and it can easily be deduced that the only solutions are \( (m, g) = (3, 1), (5, 3), (9, 13) \) (solutions nrs. 43, 123 and 257 in [9, Chapter 7, Table I]). It might also occur elsewhere in the literature as well.

\[
\text{Proof of Theorems 1 and 2 when } n \geq 3. \text{ This is done in Lemmas 3, 7, 8, 9 and 10.}
\]

\[
\square
\]

5. The Order of the Prime Ideal Above 2 in the Ideal Class Group of \( \mathbb{Q} \left( \sqrt{-\left(2^n - 1\right)} \right) \), and Wieferich Primes

We cannot fully prove Conjecture 5, but we will indicate why we think it is true. We will deduce it from the abc-conjecture, and we have a partial result.

Recall that a Wieferich prime is a prime \( p \) for which \( 2^{p-1} \equiv 1 \pmod{p^2} \). For any odd prime \( p \) we introduce \( w_{p,k} \) as the order of 2 in the multiplicative group \( \mathbb{Z}_{p^k}^* \), and \( \ell_p \) as the number of factors \( p \) in \( 2^{p-1} - 1 \). Fermat’s theorem shows that \( \ell_p \geq 1 \), and Wieferich primes are those with \( \ell_p \geq 2 \).

**Theorem 11.** Let \( n \geq 3 \), \( 2^n - 1 = D = e^2D' \) with \( D' \) squarefree and \( e \geq 1 \). Let \( \varphi \) be a prime ideal above 2 in \( \mathbb{K} = \mathbb{Q} \left( \sqrt{-D} \right) \).

\[(a) \text{ If } e < 2^{n/4-3/5} \text{ then the smallest positive } s \text{ such that } \varphi^s \text{ is a principal ideal}
\]

\text{in the maximal order of } \mathbb{K} \text{ is } s = n - 2.

\[(b) \text{ The condition } e < 2^{n/4-3/5} \text{ holds at least in the following cases:}
\]

\[ (1) \text{ } n \neq 6 \text{ and } n \leq 200,
\]

\[ (2) \text{ } n \text{ is not a multiple of } w_{p,2} \text{ for some Wieferich prime } p.
\]

In particular Conjecture 5 is true for all \( n \neq 6 \) with \( 3 \leq n \leq 200 \).

**Proof of Theorem 11.**

\[(a) \text{ We start as in the proof of Lemma 4. There exists a minimal } s > 0 \text{ such that}
\]

\[ \varphi^s \text{ is principal in the ring of integers of } \mathbb{K} = \mathbb{Q} \left( \sqrt{-D'} \right). \]

\text{Let } \frac{1}{2} \left( a + b \sqrt{-D'} \right) \text{ be a generator of this principal ideal, then } a, b \text{ are both odd and coprime, and}

\[ a^2 + D'b^2 = 2e^2. \]

\text{Since } \varphi^{n-2} = (\alpha) \text{ (with } \alpha = \frac{1}{2} \left( 1 + e \sqrt{-D'} \right) \text{) is principal}

\text{with norm } 2^{n-2}, \text{ we now find that } s | n - 2. \text{ Let us write } k = \frac{n - 2}{s}.
\]
The condition \( e < 2^{n/4-3/5} \) implies \( D' > \frac{2^n - 1}{2^{n/2-6/5}} \). As \( ks = n - 2 \) and we don’t know much about \( s \) we estimate \( k \leq n - 2 \). We may however assume \( k \geq 2 \), as \( k = 1 \) is what we want to prove. This means that we get \( s \leq \frac{1}{2} n - 1 \), and from \( a^2 + D' b^2 = 2^{s+2} \) we get \( 1 \leq |b| \leq \frac{2^{n/4+1/2}}{\sqrt{D'}} < \frac{2^{n/2-1/10}}{\sqrt{2^n - 1}} \). And this contradicts \( n \geq 3 \).

(b) We would like to get more information on how big \( e \) can become. To get an idea of what happens we computed \( e \) for all \( n \leq 200 \). Table 7 shows the cases with \( e > 1 \). Note that in all these cases \( e \mid n \), and that in all of these cases except \( n = 6 \) we have \( e < 2^{n/4-3/5} \), with for larger \( n \) an ample margin. This proves that condition (1) is sufficient.

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Table 7: The values of \( e > 1 \) for all \( n \leq 200 \).

Next let condition (2) hold, i.e., \( n \) is not a multiple of \( w_{p,2} \) for some Wieferich prime \( p \). We will prove that in this case \( e \mid n \), as was already observed in Table 7. This then is sufficient, as \( e \mid n \) implies \( e \leq n \), and \( n < 2^{n/4-3/5} \) is true for \( n \geq 20 \), and for \( 3 \leq n \leq 19 \) with the exception of \( n = 6 \) we already saw that \( e < 2^{n/4-3/5} \).

The following result is easy to prove: if \( p \) is an odd prime and \( a \equiv 1 \pmod{p^t} \) for some \( t \geq 1 \) but \( a \not\equiv 1 \pmod{p^{t+1}} \), then \( a^p \equiv 1 \pmod{p^{t+1}} \) but \( a^p \not\equiv 1 \pmod{p^{t+2}} \). By the obvious \( w_{p,\ell_p} \mid p-1 \) it now follows that \( p \nmid w_{p,\ell_p} \), and the above result used with induction now gives \( w_{p,k} = w_{p,\ell_p} p^{k-\ell_p} \) for \( k \geq \ell_p \).

Now assume that \( p \) is a prime factor of \( e \), and \( p^k \mid e \) but \( p^{k+1} \nmid e \). Then \( 2^n \equiv 1 \pmod{p^{2k}} \), \( 2^{p-1} \not\equiv 1 \pmod{p^{\ell_p+1}} \), and \( w_{p,2k} = w_{p,\ell_p} p^{2k-\ell_p} \) has \( w_{p,2k} \mid n \). Hence \( p^{2k-\ell_p} \mid n \). When \( k \geq \ell_p \) for all \( p \) we find that \( e \mid n \). But condition (2) implies that \( \ell_p = 1 \) for all \( p \mid e \), and we’re done.

Extending Table 7 soon becomes computationally challenging, as \( 2^n - 1 \) has to be factored. However, we can easily compute a divisor of \( e \), and thus a lower bound, for many more values of \( n \), by simply trying only small prime factors. We computed for all primes up to \( 10^5 \) to which power they appear in \( 2^n - 1 \) for all \( n \) up to 12000.
We conjecture that the resulting lower bounds for $e$ are the actual values. In most cases we found them to be divisors of $n$ indeed. But interestingly we found a few exceptions.

The only cases for $n$ where we are not yet sure that the conditions of Theorem 11(b) are fulfilled are related to Wieferich primes. Only two such primes are known: $1093$ and $3511$, with $w_{1093,2} = 364$, $w_{3511,2} = 1755$. So the multiples of $364$ and $1755$ are interesting cases for $n$. Indeed, we found that the value for $e$ in those cases definitely does not divide $n$. See Table 8 for those values for $n \leq 12000$.

Most probably $364$ is the smallest $n$ for which the conditions of Theorem 11(b) do not hold, but we are not entirely sure, as there might exist a Wieferich prime $p$ with exceptionally small $w_{p,1}$.  

If $n$ is divisible by $w_{p,2}$ for a Wieferich prime $p$, then the above proof actually shows that when $n$ is multiplied by at most $p^{e-1}$ (for each such $p$) it will become a multiple of $e$. It seems quite safe to conjecture the following.

**Conjecture 12.** For all $n \geq 7$ we have $e < 2^{n/4} - 3/5$.  

Most probably a much sharper bound is true, probably a polynomial bound, maybe even $e < n^2$.

According to the Wieferich prime search\(^5\), there are no other Wieferich primes up to $10^{17}$. A heuristic estimate for the number of Wieferich primes up to $x$ is

log log x, see [3]. This heuristic is based on the simple expectation estimate \( \sum_{p \leq x} p^{-1} \) for the number of \( p \) such that the second \( p \)-ary digit from the right in \( 2^{p-1} - 1 \) is zero. A similar argument for higher powers of \( p \) indicates that the number of primes \( p \) such that \( 2^{p-1} \equiv 1 \pmod{p^3} \) (i.e., \( \ell_p \geq 3 \)) is finite, probably at most 1, because \( \sum_{p} p^{-2} \approx 0.4522 \). This gives some indication that \( e \) probably always divides \( n \) times \( p \), a not too large factor. However, \( w_{p,\ell_p} \) might be much smaller than \( p \), and thus a multiplication factor of \( p \) might already be large compared to \( n \). We do not know how to find a better lower bound for \( w_{p,2} \) than the trivial \( w_{p,2} > 2 \log_2 p \).

6. Connection to the abc-Conjecture

Miller\(^6\) gives an argument that an upper bound for \( e \) in terms of \( n \) follows from the \( abc \)-conjecture. The \( abc \)-conjecture states that if \( a + b = c \) for coprime positive integers, and \( N \) is the product of the prime numbers dividing \( a, b \) or \( c \), then for every \( \epsilon > 0 \) there are only finitely many exceptions to \( c < N^{1+\epsilon} \). Indeed, assuming \( e \geq 2n/4-3/5 \) for infinitely many \( n \) contradicts the \( abc \)-conjecture, namely \( 2^{n/2} = 1 + e^{2D} \) has \( c = 2^n \) and \( N \leq 2e^{D/2} < (2^{n/2})^{(4/3)} \), which contradicts the conjecture. Indeed, assuming that the \( abc \)-conjecture is true, there is for every \( \epsilon > 0 \) a constant \( K = K(\epsilon) \) such that \( c < KN^{1+\epsilon} \), and we get \( e < K^{1/(1+\epsilon)}2^{1+\epsilon}/2^{1+\epsilon} \). This shows that any \( \epsilon < 1/3 \) will for sufficiently large \( n \) give the truth of Conjecture 5 via Theorem 11(a).

Robert, Stewart and Tenenbaum [7] formulate a strong form of the \( abc \)-conjecture, implying that \( \log e < \log N + C \sqrt{\log N \over \log \log N} \) for a constant \( C \) (asymptotically \( 4\sqrt{3} \)). Using \( c = 2^n \) and \( N \leq 2^{n+1} \) we then obtain \( n \log 2 < (n + 1) \log 2 - \log e + C \sqrt{\log (n+1) \log 2} \), hence \( e < \exp \left(C' \sqrt{n \log n} \right) \) for a constant \( C' \) slightly larger than \( C \), probably \( C' < 7.5 \). Not exactly polynomial, but this is a general form of the \( abc \)-conjecture, not using the special form of our \( abc \)-example, and it does of course imply Conjecture 5.

Even though Conjecture 5 follows from an effective version of the \( abc \)-conjecture, it might be possible to prove it in some other way.

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\(^6\)“Re: Order of an ideal in a class group”, message to the NMBRTHRY mailing list, April 7, 2013, https://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind1304&L=NMBRTHRY&F=RSS&P=5692
References


