

## Efficient and robust scale estimation for trended time series

***Citation for published version (APA):***

Caliskan, D., Croux, C., & Gelper, S. E. C. (2009). Efficient and robust scale estimation for trended time series. *Statistics and Probability Letters*, 79(18), 1900-1905. <https://doi.org/10.1016/j.spl.2009.05.019>

***DOI:***

[10.1016/j.spl.2009.05.019](https://doi.org/10.1016/j.spl.2009.05.019)

***Document status and date:***

Published: 01/01/2009

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

***General rights***

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

***Take down policy***

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.



# Efficient and robust scale estimation for trended time series

Derya Caliskan<sup>a,b</sup>, Christophe Croux<sup>a,\*</sup>, Sarah Gelper<sup>c</sup>

<sup>a</sup> LSTAT & Faculty of Business and Economics, K.U.Leuven, Belgium

<sup>b</sup> Department of Statistics, Hacettepe University, Turkey

<sup>c</sup> Erasmus School of Economics, Erasmus University Rotterdam, The Netherlands

## ARTICLE INFO

### Article history:

Received 27 January 2009

Received in revised form 21 May 2009

Accepted 25 May 2009

Available online 31 May 2009

## ABSTRACT

This paper presents a new method for robust online variability extraction in time series. The proposed estimator is simultaneously highly robust and efficient. We derive its breakdown point, influence function, and asymptotic variance and study the finite sample properties in a simulation study.

© 2009 Elsevier B.V. All rights reserved.

## 1. Introduction

In the recent literature, new procedures have been proposed for robust scale estimation in a time series context, see for example Nunkesser et al. (2009) and Gelper et al. (2009). These procedures are designed to monitor the variability of noisy and trended time series. A drawback of the existing methods is that they lose efficiency with respect to the standard non-robust estimator. This paper proposes a new approach which combines the quantile based estimator of Gelper et al. (2009) and the efficient  $\tau$ -scale estimator of Yohai and Zamar (1988). As a result, we obtain a new estimator which is highly robust and at the same time attains high efficiency.

In several application fields data are automatically collected at a high frequency and need to be monitored instantaneously. Assume that we collect observations over time

$$y_t = \mu_t + \sigma_t \epsilon_t \quad \text{for } t = 1, \dots, T \quad (1)$$

where  $\epsilon_t$  is an i.i.d. sequence coming from a symmetric distribution  $F_0$  with mean zero and variance 1. We estimate  $\sigma_t$  using a moving window approach. At each time  $t$ , a scale estimator  $\hat{\sigma}_t$  is computed from a window of width  $n < T$ , containing the observations  $y_{t-n+1}, \dots, y_t$ . The variability is allowed to change slowly over time. More precisely, the variability is only assumed to be constant in a local window which can be fairly short, for example of only 20 observations.

For the ease of notation, we drop the time index  $t$  and the observations within a fixed window are denoted by  $y_1, \dots, y_n$ . We consider vertical heights of the triangles formed by triplets of successive data points. For three successive observations  $y_i, y_{i+1}$  and  $y_{i+2}$ , the height of the triangle formed by these observations is given by

$$h_i = \left| y_{i+1} - \frac{y_i + y_{i+2}}{2} \right|$$

for  $i = 1, \dots, n-2$ . Rousseeuw and Hubert (1996) propose robust scale estimators based on these heights in the context of nonparametric regression, and Gelper et al. (2009) adapted them to the time series context. These estimators are invariant if a trend line  $a + bt$  is added to the data. Moreover, they do not require to model and estimate the signal  $\mu_t$ , and were shown to be applicable for signals containing jumps, trends, trend changes and nonlinearities.

\* Corresponding address: Faculty of Business and Economics, Naamsestraat 69, 3000 Leuven, Belgium.

E-mail address: [christophe.croux@econ.kuleuven.be](mailto:christophe.croux@econ.kuleuven.be) (C. Croux).

The estimator Gelper et al. (2009) advocate is the  $\alpha$ -quantile of the heights of adjacent triangles

$$Q_{adj}^\alpha(y_1, \dots, y_n) = c_q h_{(\lfloor \alpha(n-2) \rfloor)}, \tag{2}$$

which is the  $\lfloor \alpha(n-2) \rfloor$ th value in the sequence of ordered heights, and where  $c_q$  is a consistency factor. For  $\alpha = 0.25$  the highest value for the breakdown point is attained, and we denote  $Q_{adj} \equiv Q_{adj}^{0.25}$ . While this estimator has good properties, it has a Gaussian efficiency of only 25%. Gelper et al. (2009) proposed to increase the value of  $\alpha$  to get a better efficiency, at the price of a lower breakdown point. The procedure proposed in this paper, the  $\tau$ -adjacent estimator, maintains the high breakdown point of the  $Q_{adj}$ , but can attain an arbitrarily high efficiency.

This robust scale estimator  $\tau_{adj}$  is defined as

$$\tau_{adj}(y_1, \dots, y_n) = c_\tau \left[ S_0^2 \frac{1}{n-2} \sum_{i=1}^{n-2} \rho \left( \frac{h_i}{S_0} \right) \right]^{1/2} \tag{3}$$

with initial scale estimator  $S_0 = Q_{adj}(y_1, \dots, y_n)$ . The loss function  $\rho$  should be bounded, symmetric,  $\rho(0) = 0$ , and non-decreasing on the positive numbers. Its derivative  $\psi = \rho'$  should exist and be positive at zero. In this paper we take Tukey's bisquare  $\rho$  function defined as

$$\rho(x) = \begin{cases} \frac{x^2}{2} \left( 1 - \frac{x^2}{k^2} + \frac{x^4}{3k^4} \right), & \text{if } |x| \leq k \\ \frac{k^2}{6} & \text{if } |x| > k. \end{cases}$$

The value of the tuning constant  $k$  depends on the desired level of efficiency. As will be seen in Section 2,  $k = 5.48$  yields a 95% asymptotic relative efficiency at the Gaussian model.

The performance of the adjacent  $\tau$ -estimator is compared with the square root of the mean of squared adjacent heights,

$$MS_{adj}(y_1, \dots, y_n) = c_s \sqrt{\frac{1}{\lfloor n-2 \rfloor} \sum_{i=1}^{\lfloor n-2 \rfloor} h_i^2} \tag{4}$$

with  $c_s$  again a consistency factor. The  $MS_{adj}$  is not robust, but it is a standard proposal for scale estimation in nonparametric regression (see Gasser et al. (1986)). It is not difficult to check that  $MS_{adj}$  equals the  $\tau_{adj}$  for  $k \rightarrow \infty$ .

Section 2 discusses statistical properties of the  $\tau_{adj}$  estimator. We show that the estimator has a breakdown point of 25%, a bounded influence function, and we compute its asymptotic variance. The good behavior of the estimator is confirmed in Section 3 by a simulation study. Section 4 summarizes the results.

## 2. Statistical properties

In this section, we first derive the expression for the consistency factor  $c_\tau$  for the newly proposed estimator. Expressions for the constants  $c_q$  in (2) and  $c_s$  in (4) are given in Gelper et al. (2009). We then derive the breakdown point and the influence function of the  $\tau_{adj}$  estimator. The breakdown point measures the robustness under larger amounts of outliers, while the influence function measures the sensitivity of the estimator with respect to small amounts of contamination. The last subsection computes the asymptotic variance of the estimator.

### 2.1. Fisher consistency

For all theoretical derivations, we assume local linearity and a constant scale within the considered time window. Hence

$$y_i = a + bi + \sigma \epsilon_i \tag{5}$$

for  $i = 1, \dots, n$  where  $\epsilon_i \sim F_0$ . For an appropriately chosen window width, let  $F$  be the distribution of the data and denote  $H_F$  the distribution of the corresponding triangle heights. The functional form of the  $\tau_{adj}$ -scale estimator (3) corresponds to

$$\tau_{adj}(F) = c_\tau \left[ S_0^2(F) E_{H_F} \rho \left( \frac{h}{S_0(F)} \right) \right]^{1/2} \tag{6}$$

where  $S_0(F) \equiv Q_{adj}(F) = c_q H_F^{-1}(0.25)$ . The proposed scale estimator is location invariant, invariant when a trend is added to the data, and scale equivariant. Assume that model (5) holds, and take  $a = b = 0$  without loss of generality. Since the constant  $c_q$  is such that  $Q_{adj}$  is Fisher consistent, we have  $S_0(F) = \sigma$ . In order to achieve Fisher consistency, that is  $\tau_{adj}(F) = \sigma$ , we need to take

$$c_\tau = \frac{1}{\sqrt{E_{H_{F_0}} \rho \left( \frac{h}{S_0(F_0)} \right)}} = \frac{1}{\sqrt{E_{H_{F_0}} \rho(h)}}. \tag{7}$$

For  $F_0$  a standard normal  $N(0, 1)$  distribution, it is not difficult to verify that  $E_{H_{F_0}} \rho(h) = E_{F_0} \rho(Z\sqrt{3/2})$ , allowing for immediate calculation of (7). As such, we obtain for  $k = 5.48$  that  $c_\tau = 1.24$ . To make the estimator practically unbiased at finite samples, we propose to replace  $c_\tau$  by a finite sample version  $c_\tau^n$ . By Monte Carlo simulation, following the approach outlined in Gelper et al. (2009), and for  $k = 5.48$ , we obtain

$$c_\tau^n \approx c_\tau \frac{n}{n - 1.34}.$$

## 2.2. Breakdown point

The breakdown point of a scale estimator is the minimal amount of contamination such that the estimated scale becomes either infinite (explosion) or zero (implosion). Let  $\mathbf{y}_n = \{y_1, \dots, y_n\}$  be a sample of size  $n$ . Let  $S$  be a scale estimator. Further, let  $\mathbf{y}_n^m$  be a sample obtained from  $\mathbf{y}_n$  but with a proportion of  $m/n$  observations altered to arbitrary values ( $m \in \{1, \dots, n\}$ ). The *finite sample breakdown point* of  $S$  at the sample  $\mathbf{y}_n$  is defined as

$$\varepsilon^*(S, \mathbf{y}_n) = \min \frac{1}{n} \left\{ m \in \{1, 2, \dots, n\} : \sup_{\mathbf{y}_n^m} \left| \log \left( \frac{S(\mathbf{y}_n^m)}{S(\mathbf{y}_n)} \right) \right| = \infty \right\},$$

Suppose that  $\mathbf{y}_n$  is in general position, meaning that no three observations  $(i, y_i)$  lie on the same line for  $1 \leq i \leq n$ . Gelper et al. (2009) have shown that, for the initial estimator  $S_0 = Q_{adj}$ ,

$$\varepsilon^*(S_0, \mathbf{y}_n) = \frac{1}{n} \min \left\{ \left\lceil \frac{n-1 - \lfloor \alpha(n-2) \rfloor}{3} \right\rceil, \lfloor \alpha(n-2) \rfloor \right\}. \quad (8)$$

The highest possible value for the breakdown point is attained for

$$\alpha = \frac{n+1}{4(n-2)} \approx 0.25. \quad (9)$$

The finite sample breakdown point tends to the *asymptotic breakdown point* of 25%. In the following proposition we state that the breakdown point of the  $\tau_{adj}$  estimator is the same as that of the initial estimator  $S_0$ .

**Proposition 1.** Let  $\mathbf{y}_n$  be a sample in general position. For the  $\tau_{adj}$  estimator with a bounded loss function  $\rho$  we have that

$$\varepsilon^*(\tau_{adj}, \mathbf{y}_n) = \varepsilon^*(S_0, \mathbf{y}_n).$$

**Proof.** From definition (3) of the  $\tau_{adj}$  estimator it is readily seen that the estimator tends to infinity if and only if the initial scale estimator  $S_0$  does, since  $\rho$  is bounded. Implosion of the  $\tau_{adj}$  estimator occurs if either (i)  $S_0$  implodes to zero (ii) all heights  $h_i$  are equal to zero (iii)  $S_0$  is so large that  $\rho(h_i/S_0)$  is arbitrarily small for all  $1 \leq i \leq n-2$ . Since we assumed that the sample is in general position, (ii) cannot occur. Furthermore, by the definition of  $S_0$ , about 75% of the heights is larger than  $S_0$ , and  $\rho(1) > 0$ , hence also (iii) is excluded. We conclude that the  $\tau_{adj}$  implodes if and only if the initial scale estimator implodes.  $\square$

## 2.3. Influence function

The influence function of the functional  $S$  at the distribution  $F$  measures the effect on  $S$  of adding a small mass at the point  $w$ , standardized by the mass of the contamination. If we denote the point mass at  $w$  by  $\Delta_w$  and consider the contaminated distribution  $F_\varepsilon = (1 - \varepsilon)F + \varepsilon\Delta_w$  then the influence function is given by

$$IF(w; S, F) = \lim_{\varepsilon \rightarrow 0} \left[ \frac{S(F_\varepsilon) - S(F)}{\varepsilon} \right]. \quad (10)$$

In the Appendix we derive an explicit expression for the influence function of  $\tau_{adj}$ , assuming model (5) holds with  $F_0 = N(0, 1)$ . Fig. 1 pictures this influence function at this normal model with  $\sigma = 1$ . We see that the  $\tau_{adj}$  has the desirable property of a bounded IF, hence it is B-robust. Its influence function is smooth and has a quadratic shape close to the center of the distribution.

## 2.4. Asymptotic variance

The estimators are based on heights of triangles. While the observations themselves are assumed to be independent, the heights will be autocorrelated up to order two. As in Genton (1998), the asymptotic variance of an estimator  $S$  based on the heights  $h_i$  is given by

$$ASV(S, F) = E(IF^2(h_i; S, H_F)) + 2E(IF(h_i; S, H_F)IF(h_{i+1}; S, H_F)) + 2E(IF(h_i; S, H_F)IF(h_{i+2}; S, H_F)).$$

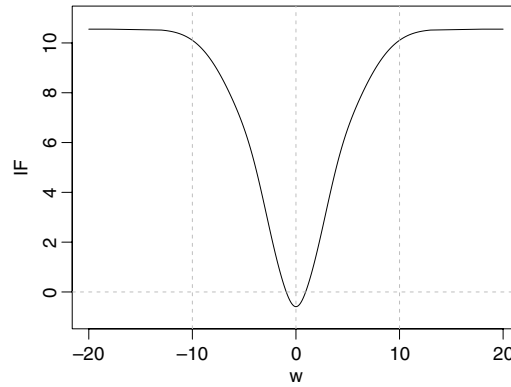


Fig. 1. Influence function of the  $\tau_{adj}$  estimator at the normal model with  $\sigma = 1$ .

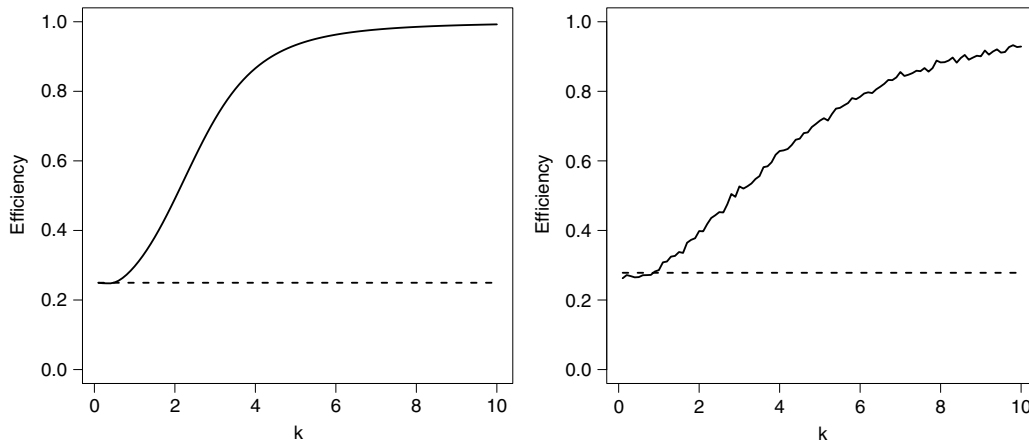


Fig. 2. Asymptotic efficiencies (left) and finite sample efficiencies for  $n = 20$  (right) of the  $\tau_{adj}$  estimator (solid line) and the  $Q_{adj}$  estimator (dashed line) as a function of the tuning constant  $k$  at the normal model.

where  $IF(h; S, H_F)$  is the influence function of the estimator  $S$  at the distribution  $H_F$ . At model (5) with  $F_0 = N(0, 1)$ , the influence function of the heights can be obtained by straightforward calculus. Without loss of generality we may further assume that  $\sigma = 1$ . Then,

$$IF(h; S, H_N) = IF(h; S_0, H_N) \left( c_\tau \sqrt{d} - \frac{c_\tau}{2\sqrt{d}} E[\psi(Z)Z] \right) + \frac{\sigma_0}{2} \left( \frac{c_\tau}{\sqrt{d}} \rho \left( \frac{h}{\sigma_0} \right) - \sqrt{d} c_\tau \right)$$

where  $\sigma_0 = \sqrt{\frac{3}{2}}$ ,  $d = E[\rho(Z)]$ ,  $Z \sim N(0, 1)$ ,  $\psi = \rho'$ , and  $N$  is an index referring to the assumption of normality. The influence function of the initial scale estimator  $Q_{adj}$  is given by

$$IF(h; S_0, H_N) = c_q \left[ \frac{0.25 - I(h < Q_N^\alpha)}{2\sqrt{2/3}\varphi(\sqrt{2/3}Q_N^{0.25})} \right]$$

where  $Q_N^{0.25} = H_N^{-1}(0.25)$  is the first quartile of the distribution of the heights at the standard normal distribution, and  $I$  is the indicator function, see Gelper et al. (2009).

The exact value of the ASV for the non-trimmed mean-squared-heights estimator  $MS_{adj}$  is  $35/36$ , at the normal model with  $\sigma = 1$ . For the  $\tau$ -scale estimator, the ASV is obtained by numerical integration. The asymptotic relative efficiency of the  $\tau_{adj}$  estimator w.r.t.  $MS_{adj}$  is then defined as

$$Eff(S, F) = \frac{ASV(MS_{adj}, F)}{ASV(S, F)}.$$

Fig. 2 present the asymptotic efficiency of the  $\tau_{adj}$  as a function of  $k$ , at the normal model. An efficiency of 0.95 is attained at  $k = 5.48$ . We also simulated, over  $M = 10\,000$  simulation runs, finite sample efficiencies for a window width of  $n = 20$ . One sees that the asymptotic results provide a good approximation for the finite sample setting. Furthermore, for  $k$  converging to zero, the efficiency of the  $Q_{adj}$  and  $\tau_{adj}$  estimators coincide.

**Table 1**

Simulated RMSE for clean data, 1%, 5% and 10% outliers, for window width  $n = 20$ ,  $T = 1000$ , and averaged over  $M = 10\,000$  simulation runs. Three types of outlier configurations are considered.

$\varepsilon$	Additive outliers			Patches of outliers			Innovation outliers		
	$Q_{adj}$	$\tau_{adj}$	$MS_{adj}$	$Q_{adj}$	$\tau_{adj}$	$MS_{adj}$	$Q_{adj}$	$\tau_{adj}$	$MS_{adj}$
0.00	0.44	0.24	0.22	0.44	0.24	0.22	0.41	0.30	0.29
0.01	0.45	0.29	0.36	0.45	0.28	0.29	0.40	0.30	0.32
0.05	0.51	0.46	0.70	0.49	0.38	0.50	0.40	0.30	0.44
0.10	0.61	0.67	1.01	0.55	0.48	0.70	0.40	0.37	0.62

### 3. Simulation

In this section, we simulate a time series generated from model (1) of length  $T = 1000$ , with constant location and scale  $\sigma_t = 1$ . We consider three types of outliers: (a) isolated additive outliers, (b) patches of outliers, where a patch is a group of 3 consecutive outliers having the same value, and (c) innovation outliers. For schemes (a) and (b) we consider independent error terms  $\varepsilon_t \sim N(0, 1)$ , and induce outliers by replacing a proportion  $\varepsilon$  of the observations, for  $\varepsilon = 0, 0.01, 0.05$ , and  $0.10$ , by values coming from an  $N(0, 5)$ . In simulation scheme (c) we consider a first order autoregressive model  $\varepsilon_t = \theta\varepsilon_{t-1} + v_t$ , with  $v_t \sim N(0, 1)$ , and  $\theta = 0.5$ . Outliers are then induced by replacing a proportion  $\varepsilon$  of the  $v_t$  by values coming from an  $N(0, 5)$ . For every simulated series, we compute the root mean squared error (RMSE)

$$RMSE = \left( \frac{1}{T - n + 1} \sum_{t=n}^T \frac{(\hat{\sigma}_t - \sigma_t)^2}{\sigma_t^2} \right)^{1/2}.$$

Here  $n = 20$  is the window width,  $T$  is the length of the time series, and  $\hat{\sigma}_t$  is the estimated scale.

In Table 1 the average RMSE over the  $M = 10\,000$  simulated time series is reported. In the absence of outliers, the robust  $\tau_{adj}$  is almost as efficient as the  $MS_{adj}$  estimator. This is as expected, since the tuning constant  $k$  was selected to achieve a 95% relative efficiency. This is in contrast with the  $Q_{adj}$  estimator, having an RMSE which is almost twice as large. In the presence of even only 1% of contamination, the  $MS_{adj}$  is no longer the most precise, and the  $\tau$ -estimator behaves best. For larger amounts of contamination, i.e. 10%, the  $Q_{adj}$  estimator is slightly better than the  $\tau_{adj}$ , at least for additive outliers. For patches of outliers and innovation outliers, the  $\tau_{adj}$  remains to have the smallest RMSE. We conclude from the simulation study that in the presence of amounts of contamination up to 10%, the  $\tau_{adj}$  estimator is to be preferred. If we have a larger (but less than 25%) proportion of outliers, then the bias of the  $\tau_{adj}$  remains bounded, given its high breakdown point, but the  $Q_{adj}$  estimator tends to perform better. Since we expect that most univariate time series contain few outliers and that it is in practice quite rare to have large amounts of outliers, we recommend the  $\tau_{adj}$  estimator.

### 4. Conclusion

Robust scale estimators based on the heights of triangles formed by triplets of successive observations are used for monitoring the scale of nonlinear noisy time series, as is documented in Gelper et al. (2009). In this paper we propose to use  $\tau$ -estimators. These estimators keep the high breakdown point of the initial estimator, while they may have an arbitrarily high efficiency. The efficiency of the  $\tau_{adj}$  estimator depends on a tuning constant  $k$ . We computed the value of  $k$  yielding a 95% relative efficiency with respect to the standard estimator. Monte Carlo simulations illustrate the good performance of the proposed procedure.

A major question we did not address is the choice of the window width  $n$ . It needs to be small enough for (5) to hold, but large enough to still provide accurate estimates. Another topic for future research is to investigate the properties of the  $\tau_{adj}$  estimator for dependent data. The scale estimator can then still be applied, and will maintain the high breakdown point. The asymptotic variance, however, will depend on the dependency structure in the data.

### Appendix

*Derivation of the influence function of the  $\tau_{adj}$  estimator at the normal model:*

Assume that model (5) holds, with  $F_0 = N(0, 1)$ . Without loss of generality, assume  $a = b = 0$ , and  $\sigma = 1$ , such that  $F = N(0, 1)$ . From (6) it follows that

$$\begin{aligned} IF(w; \tau_{adj}, F) &= \frac{\partial}{\partial \varepsilon} \tau_{adj}(F_\varepsilon) \Big|_{\varepsilon=0} \\ &= c_\tau \frac{\partial}{\partial \varepsilon} \left[ S_0^2(F_\varepsilon) E_{H_{F_\varepsilon}} \rho \left( \frac{h}{S_0(F_\varepsilon)} \right) \right]^{1/2} \Big|_{\varepsilon=0} \\ &= \frac{c_\tau}{2} E_{H_F} \rho(h)^{-1/2} \frac{\partial}{\partial \varepsilon} \left[ S_0^2(F_\varepsilon) E_{H_{F_\varepsilon}} \rho \left( \frac{h}{S_0(F_\varepsilon)} \right) \right] \Big|_{\varepsilon=0} \end{aligned}$$

$$= \frac{c_\tau}{2} E_{H_F} \rho(h)^{-1/2} \left( 2IF(w; S_0, F) E_{H_F} \rho(h) + \underbrace{\frac{\partial}{\partial \varepsilon} E_{H_{F_\varepsilon}} \rho \left( \frac{h}{S_0(F_\varepsilon)} \right) \Big|_{\varepsilon=0}}_A \right)$$

where we used that  $S(F_0) = 1$ . In the above expression, with  $\psi = \rho'$ ,

$$\begin{aligned} A &= \frac{\partial}{\partial \varepsilon} \int_0^\infty \rho \left( \frac{h}{S_0(F_\varepsilon)} \right) dH_{F_\varepsilon}(h) \Big|_{\varepsilon=0} \\ &= -IF(w; S_0, F) \int_0^\infty \psi(h) h dH_F(h) + \int_0^\infty \rho(h) d \underbrace{\frac{\partial}{\partial \varepsilon} H_{F_\varepsilon}(h)}_B \Big|_{\varepsilon=0}. \end{aligned}$$

One has  $H_F(u) = \Phi(\sqrt{2/3}u) - \Phi(-\sqrt{2/3}u)$ , giving us for  $B$ :

$$\begin{aligned} \frac{\partial H_{F_\varepsilon}(u)}{\partial \varepsilon} \Big|_{\varepsilon=0} &= -3(2\Phi(\sqrt{2/3}h) - 1) + \Phi(\sqrt{2}(h+w)) - \Phi(\sqrt{2}(w-h)) \\ &\quad + 2(\Phi(\sqrt{4/5}(w/2+h)) - \Phi(\sqrt{4/5}(w/2-h))) \\ &:= G(h, w). \end{aligned}$$

So we can write  $A$

$$A = -IF(w; S_0, F) \int_0^\infty \psi(h) h dH_F(h) + \int_0^\infty \rho(h) dG(h, w).$$

We conclude that, with  $c_\tau^2 = 1/E_{H_{F_0}} \rho(h)$ ,

$$IF(w; \tau_{adj}, F) = IF(w; S_0, F) \left[ 1 - c_\tau^2 \int_0^\infty \psi(h) h dH_F(h) \right] + \frac{c_\tau^2}{2} \int_0^\infty \rho(h) dG(h, w). \quad (11)$$

The above integrals can be computed either analytically or numerically. An expression for  $IF(w; Q_{adj}, F)$  is given in Gelper et al. (2009).

## References

- Gasser, T., Sroka, L., Jennen-Steinmetz, C., 1986. Residual variance and residual pattern in nonlinear regression. *Biometrika* 73 (3), 625–633.
- Gelper, S., Schettlinger, K., Croux, C., Gather, U., 2009. Robust online scale estimation in time series: A regression-free approach. *Journal of Statistical Planning and Inference* 139, 335–339.
- Genton, M.G., 1998. Asymptotic variance of M-estimators for dependent Gaussian random variables. *Statistics & Probability Letters* 38 (3), 255–261.
- Nunkesser, R., Fried, R., Schettlinger, K., Gather, U., 2009. Online analysis of time series by the  $q_n$  estimator. *Computational Statistics and Data Analysis* 53, 2354–2362.
- Rousseeuw, P., Hubert, M., 1996. Regression-free and robust estimation of scale for bivariate data. *Computational Statistics and Data Analysis* 21, 67–85.
- Yohai, J., Zamar, H., 1988. High breakdown-point estimates of regression by means of the minimization of an efficient scale. *Journal of the American Statistical Association* 83–402, 406–413.