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Abstract

In this note it is shown how to obtain the Kepler system by geometric reduction of the 4DOF harmonic oscillator. It is shown that this reduction is a reverse KS regularization. It is furthermore shown how the integrals for the Kepler system can be represented as integrals of the harmonic oscillator. Also a representation of the Delaunay elements is computed.

1 Introduction

As is well known the Kepler problem refers to the bounded motion of a particle in $\mathbb{R}^3$ which is influenced by the gravitational field of a second particle fixed at the origin.

We may describe the problem as a Hamiltonian system $(K, T_0\mathbb{R}^3, \tilde{\omega})$, with phase space $T_0\mathbb{R}^3 = (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$, with co-ordinates (x, y), and symplectic form $\tilde{\omega} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3$. The Hamiltonian is given by

$$K(x, y) = \frac{1}{2}\langle y, y\rangle - \frac{\mu}{|x|} \, .$$

Here $\langle \, , \, \rangle$ denotes the Eucledian inner product on $\mathbb{R}^3$ and $|\cdot|$ the induced Euclidian norm.

Consider the action $SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3; (A, x) \to Ax$. This lifts to $SO(3) \times T_0\mathbb{R}^3 \to T_0\mathbb{R}^3; (A, (x, y)) \to (Ax, Ay)$. This group action has momentum mapping $\hat{J} : T_0\mathbb{R}^3 \to \mathfrak{so}(3)^*$. Identifying $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ by taking the transpose, and identifying $\mathfrak{so}(3)$ with $\mathbb{R}^3$ through $\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \to (u_1, u_2, u_3)$, the momentum map becomes $J : (x, y) \to$

$x \times y$, with $\times$ the vector product on $\mathbb{R}^3$. Because the Kepler Hamiltonian is invariant under the $SO(3)$ action $J = (J_1, J_2, J_3) = x \times y$ are integrals of the Kepler system corresponding to the angular momentum vector. Other integrals are given by the Laplace vector or eccentricity vector $A = (A_1, A_2, A_3) = y \times J - \frac{x}{|x|}$.

On (Σ_-, ω) we have the following

$$\{J_i, J_j\} = \sum_k \varepsilon_{ijk} J_k \ , \ \{J_i, A_j\} = \sum_k \varepsilon_{ijk} A_k \ , \ \{A_i, A_j\} = \sum_k \varepsilon_{ijk} J_k \ .$$

Consequently, if we introduce $\xi_i = \frac{1}{2}(J_i + A_i)$, $\eta_i = \frac{1}{2}(J_i - A_i)$, we have

$$\{\xi_i, \xi_j\} = \sum_k \varepsilon_{ijk} \xi_k \ , \ \{\eta_i, \eta_j\} = \sum_k \varepsilon_{ijk} \eta_k \ , \ \{\xi_i, \eta_j\} = 0 \ ,$$

and these relations define a Lie algebra $\mathfrak{so}(3) \times \mathfrak{so}(3) \cong \mathfrak{so}(4)$. Consequently the full symmetry group of the Kepler problem is $SO(4)$. This is a symmetry of the phase space which is not a lift of a symmetry on configuration space.

When using Kustaanheimo-Stiefel regularization regularized perturbed Keplerian systems can be considered as perturbed isotropic harmonic oscillators on $\mathbb{R}^8$, which can be described by a Hamiltonian systems with the standard symplectic form, and co-ordinates (q, Q).

Considerer a perturbed isotropic oscillator oscillator on $\mathbb{R}^8$

$$H(q, Q) = \frac{1}{2}(Q, Q) + \frac{1}{2}(q, q) \ + \text{h.o.t.}$$

with additional symmetry given by the integral

$$\Xi(q, Q) = q_1 Q_2 - Q_1 q_2 + q_3 Q_4 - Q_3 q_4 \ , \tag{1}$$

When such a system is in normal form with respect to its quadratic part H_2 one may reduce with respect the H_2 and Ξ symmetry to obtain a four degree of freedom system with, for $\Xi = 0$, a reduced phase space $S^2 \times S^2$ with Poisson structure equivalent to that of the Lie algebra $\mathfrak{so}(3) \times \mathfrak{so}(3)$. Thus one can consider these systems as generalizations of perturbed Keplerian systems. Systems like this with additional integral $L_1(q, Q) = -q_1 Q_2 + Q_1 q_2 + q_3 Q_4 - Q_3 q_4$ are studied in detail in [6],[7], [5], [8].

In this note it will be shown how the regularized Keplerian system is embedded in the isotropic harmonic oscillator. More precisely the Kepler system is obtained by reparametrizing a reduced harmonic oscillator. It is now clear that after further reduction both systems give rise to identical reduced systems on $S^2 \times S^2$. It will allow us to relate the geometric reduction by invariants to the symplectic reduction of perturbed Kepler systems by using Delaunay coordinates, and to connect the generalized Delaunay coordinates for the perturbed symmetric isotropic harmonic oscillator to the geometric reduction.

The embedding of the Kepler problem in the harmonic oscillator dynamics is performed by combining Kustaanheimo-Stiefel regularization as is discussed in [4] (page 80) with geometric reduction. More precisely it turns out that KS regularization is a reversed reduction. The fact that KS regularization is a reversed reduction is also the topic in [1]. Also already in [10] the regularized Kepler system was considered on an orbit space. However, in these papers the reduction was not extended beyond the KS-map, that is, the embedding in the full reduction of the harmonic oscillator, which allows a full computational construction, was not considered. The computations in this paper make it possible to deal with perturbed Kepler problems trough symmetrically perturbed harmonic oscillators.

In section 2. we review the KS regularization and show how it is related to geometric reduction. In section 3. we further discuss the invariants for the harmonic oscillator and geometric reduction. In section 4. we show how the Kepler system is obtained as a reduced system by formulating the reduced system in a proper chart on the reduced phase space. In section 5. we show how reduction of the Kepler problem fits in further reduction of the harmonic oscillator. Here the integrals of the Kepler problem are represented as integrals on the harmonic oscillator level. In the final section a connection is made to Delaunay variables.

2 KS regularization

We will follow the Kustaanheimo-Stiefel regularization as presented in [4] in complex co-ordinates, which goes back on [10]. We will establish a relation between these complex symplectic spaces and real symplectic spaces by an appropriate choice of real co-ordinates which differs from the one in [10].

Let $z = (z_1, z_2) = (q_1 + iq_2, q_3 + iq_4) \in \mathbb{C}^2_0 = \mathbb{C}^2 - \{0\} = \mathbb{R}^4 - \{0\}$. Let $\langle z, w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2$ be the standard Hermitian inner product on $\mathbb{C}^2$. Furthermore let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli matrices. Set

$$\begin{aligned} x_1 &= \langle z, \sigma_1 z\rangle = z_1\bar{z}_2 + z_2\bar{z}_1 = 2(q_1q_3 + q_2q_4) \ , \\ x_2 &= \langle z, \sigma_2 z\rangle = iz_1\bar{z}_2 - iz_2\bar{z}_1 = 2(q_1q_4 - q_2q_3) \ , \\ x_3 &= \langle z, \sigma_3 z\rangle = z_1\bar{z}_1 - z_2\bar{z}_2 = (q_1^2 + q_2^2) - (q_3^2 + q_4^2) \ . \end{aligned}$$

Then

$$\pi : \mathbb{C}^2_0 \to \mathbb{R}^3; z \to (x_1, x_2, x_3)$$

is the Hopf map. The map π lifts to the Kustaanheimo-Stiefel map $\mathcal{KS}$ on $T^*\mathbb{C}_0^2 = (\mathbb{C}_0^2 - \{0\}) \times \mathbb{C}^2$ with coordinates (z, w)

$$\mathcal{KS} : T^*\mathbb{C}_0^2 \to T^*\mathbb{R}^3 : (z, w) \to (x, y) \; ,$$

where

$$y_1 = \frac{1}{\langle z, z\rangle}(Re\langle w, \sigma_1 z\rangle) \; ,$$
$$y_2 = \frac{1}{\langle z, z\rangle}(Re\langle w, \sigma_2 z\rangle) \; ,$$
$$y_3 = \frac{1}{\langle z, z\rangle}(Re\langle w, \sigma_3 z\rangle) \; .$$

If we set $w_1 = Q_1 + iQ_2$ and $w_2 = Q_3 + iQ_4$ one gets

$$y_1 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)}(Q_2 q_4 + Q_1 q_3 + Q_3 q_1 + Q_4 q_2) \; ,$$
$$y_2 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)}((Q_1 q_4 + Q_4 q_1) - (Q_3 q_2 + Q_2 q_3)) \; ,$$
$$y_3 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)}((Q_1 q_1 + Q_2 q_2) - (Q_3 q_3 + Q_4 q_4)) \; .$$

Define on $T^*\mathbb{C}_0^2$ the 1-form $\Theta = -2\, Re\langle w, dz\rangle$ and symplectic form $\Omega = -d\Theta$. Note that in (q, Q) co-ordinates Ω becomes the two times the standard symplectic form on $\mathbb{R}^4 \times \mathbb{R}^4$.

The Hopf map introduces a nontrivial S^1 bundle over S^2 which is the orbit space of an S^1-action which lifts to the following S^1 action on $T^*\mathbb{C}_0^2$. This action is symplectic with respect to the symplectic form Ω.

$$\varphi : S^1 \times T^*\mathbb{C}_0^2 \to T^*\mathbb{C}_0^2; (e^{it}, z, w) \to (e^{it} z, e^{it} w) \; .$$

On $\mathbb{R}_0^8 = (\mathbb{R}^4 - \{0\}) \times \mathbb{R}^4$ with coordinates (q, Q) this action is given by

$$\varphi : S^1 \times \mathbb{R}_0^8 \to \mathbb{R}_0^*; R_\nu(q, Q) \to (R_\nu q, \mathbb{R}_\nu Q) \; ,$$

with

$$R_\nu = \begin{pmatrix} \cos\nu & -\sin\nu & 0 & 0 \\ \sin\nu & \cos\nu & 0 & 0 \\ 0 & 0 & \cos\nu & -\sin\nu \\ 0 & 0 & \sin\nu & \cos\nu \end{pmatrix} \; .$$

This action has momentum map

$$\mathcal{J} : \mathbb{R}_0^8 \to \mathbb{R}; (q, Q) \to \Xi(q, Q) \; ,$$

with

$$\Xi(q, Q) = (q_1 Q_2 - q_2 Q_1) + (q_3 Q_4 - q_4 Q_3) \; .$$

Let $\tilde{\omega}$ denote the standard symplectic form on $T^*\mathbb{R}^3$.

Proposition 2.1

$$\mathcal{KS}^*\tilde{\omega}|\mathcal{J}_0 = \Omega|\mathcal{J}_0 \ ,$$

where $\mathcal{J}_0 = \mathcal{J}^{-1}(0) - \{0\}$.

Similar statements can be found in [10] and [4], however these are not consistent. Therefore we have reformulated the symplectic form and will give a proof below.

Proof : In stead of the above we will prove the equivalent statement

$$\mathcal{KS}^*\theta|\mathcal{J}_0 = \Theta|\mathcal{J}_0 \ ,$$

with $\theta = \langle y, dx\rangle$ te canonical one-form on $T^*\mathbb{R}^3$. For every u, w, $z \in \mathbb{C}^2$

$$\sum_{j=1}^{3} Re\langle u, \sigma_j(z)\rangle\sigma_j(w) = 2\langle w, z\rangle u - \langle u, z\rangle w \ . \tag{2}$$

Interchanging u with z in eq.(2) gives

$$\sum_{j=1}^{3} Re\langle z, \sigma_j(u)\rangle\sigma_j(w) = 2\langle w, u\rangle z - \langle z, u\rangle w \ . \tag{3}$$

Subtracting eq.(3) from eq.(2) gives

$$\sum_{j=1}^{3} Re\langle u, \sigma_j(z)\rangle\sigma_j(w) = \langle w, z\rangle u - \langle w, u\rangle z - iIm\langle u, z\rangle w \ . \tag{4}$$

Taking the inner product of eq.(4) with z and adding the result to its complex conjugate gives

$$\sum_{j=1}^{3} Re\langle u, \sigma_j(z)\rangle Re\langle z, \sigma_j(w)\rangle = Re\langle z, w\rangle Re\langle u, z\rangle - \langle z, z\rangle Re\langle u, w\rangle \ . \tag{5}$$

Replacing w by dz and u by w in eq.(6) gives

$$\sum_{j=1}^{3} Re\langle w, \sigma_j(z)\rangle Re\langle z, \sigma_j(dz)\rangle = Re\langle z, dz\rangle Re\langle w, z\rangle - \langle z, z\rangle Re\langle w, dz\rangle \ . \tag{6}$$

Now $dx_j = 2Re\langle z, \sigma_j(dz)\rangle$. Consequently

$$\theta = 2\sum_{j=1}^{3} \frac{Re\langle w, \sigma_j(z)\rangle}{\langle z, z\rangle} Re\langle z, \sigma_j(dz)\rangle = 2\frac{Re\langle z, dz\rangle}{\langle z, z\rangle} Re\langle w, z\rangle - 2Re\langle w, dz\rangle \ , \tag{7}$$

which on $\mathcal{J}_0$ equals $-2Re\langle w, dz\rangle$. **q.e.d.**

Consider

$$\tilde{K}(x,y) = \frac{|x|}{k}(K(x,y) + \frac{1}{2}k^2) + \frac{\mu}{k} = \frac{1}{2k}|x|(|y|^2 + k^2) \ . \tag{8}$$

The energy level $K(x,y) = -\frac{1}{2}k^2$ corresponds to the energy level $\tilde{K}(x,y) = \frac{\mu}{k}$. The Hamiltonian vector field corresponding to $\tilde{K}$ is

$$\begin{aligned} \frac{dx}{ds} &= \frac{1}{k}|x|\frac{\partial K}{\partial y} + (K(x,y) + \frac{1}{2}k^2)\frac{\partial}{\partial y}\frac{|x|}{k} \ , \\ \frac{dy}{ds} &= -\frac{1}{k}|x|\frac{\partial K}{\partial x} \ . \end{aligned}$$

On $K(x,y) = -\frac{1}{2}k^2$ this corresponds to

$$\frac{dx}{ds} = \frac{1}{k}|x|\frac{\partial K}{\partial y} \ , \ \frac{dy}{ds} = -\frac{1}{k}|x|\frac{\partial K}{\partial x} \ .$$

With $\dfrac{ds}{dt} = \dfrac{k}{|x|}$ this is a time re-scaled version, or re-parametrization, of the Kepler vector field. The integral curves of the pre-regularized Hamiltonian vector field corresponding to $\tilde{K}$ with energy $\frac{\mu}{k}$ agree with the integral curves of the Kepler vector field with energy $-\frac{1}{2}k^2$. The vector fields are not equivalent, as one is a re-parametrization of the other. With $\tilde{K} = \frac{1}{2k}\|x\|(\|y\|^2 + k^2)$ the time re-scaled Kepler Hamiltonian on $(T^*\mathbb{R}^3, \tilde{\omega})$, and

$$\|x\| = q_1^2+q_2^2+q_3^2+q_4^2 \ , \ \|y\|^2 = \frac{Q_1^2+Q_2^2+Q_3^2+Q_4^2}{q_1^2+q_2^2+q_3^2+q_4^2} + \frac{(q_1Q_2 - q_2Q_1) + (q_3Q_4 - q_4Q_3)}{(q_1^2+q_2^2+q_3^2+q_4^2)^2} \ .$$

one finds that the regularized Kepler Hamiltonian is

$$\begin{aligned} \hat{K} = \mathcal{K}\mathcal{S}^*\tilde{K} &= \frac{1}{2k}(\langle w, w\rangle + k^2\langle z, z\rangle) \\ &= \frac{1}{2k}(k^2(q_1^2+q_2^2+q_3^2+q_4^2) + (Q_1^2+Q_2^2+Q_3^2+Q_4^2) + \frac{\Xi}{q_1^2+q_2^2+q_3^2+q_4^2}) \ , \end{aligned}$$

that is, $\hat{K} = H_2$, with

$$H_2 = \frac{1}{2k}(k^2(q_1^2+q_2^2+q_3^2+q_4^2) + (Q_1^2+Q_2^2+Q_3^2+Q_4^2)) \ ,$$

when restricted to $\mathcal{J}_0$. The regularization is now obtained by extending to all of $\Xi^{-1}(0)$ and applying Ξ reduction, that is, dividing out the S^1-action generated by Ξ.

In order to simplify the the computations, in the following we take $\mu = k = 1$. Actually this can be obtained by re-scaling.

3 H_2 and Ξ invariants, Ξ-reduction

Consider the isotropic harmonic oscillator $H_2(q, Q)$ on all of $\mathbb{R}^8$ with standard symplectic form ω. Furthermore consider the integral $\Xi(q, Q)$. Then any Ξ invariant higher order perturbation of H_2 restricts on $\mathcal{J}_0$ to a perturbation of the regularized Kepler Hamiltonian. In order to relate the Kepler system to the H_2 and Ξ invariants we will discuss these in this section. We will start with presenting the invariants for the H_2 action as they will be used in the sequel. There are 16 invariants for the action corresponding to H_2 (see [4]):

$$\begin{aligned}
\pi_1 &= Q_1^2 + q_1^2 & \pi_2 &= Q_2^2 + q_2^2 \\
\pi_3 &= Q_3^2 + q_3^2 & \pi_4 &= Q_4^2 + q_4^2 \\
\pi_5 &= Q_1 Q_2 + q_1 q_2 & \pi_6 &= Q_1 Q_3 + q_1 q_3 \\
\pi_7 &= Q_1 Q_4 + q_1 q_4 & \pi_8 &= Q_2 Q_3 + q_2 q_3 \\
\pi_9 &= Q_2 Q_4 + q_2 q_4 & \pi_{10} &= Q_3 Q_4 + q_3 q_4 \\
\pi_{11} &= q_1 Q_2 - q_2 Q_1 & \pi_{12} &= q_1 Q_3 - q_3 Q_1 \\
\pi_{13} &= q_1 Q_4 - q_4 Q_1 & \pi_{14} &= q_2 Q_3 - q_3 Q_2 \\
\pi_{15} &= q_2 Q_4 - q_4 Q_2 & \pi_{16} &= q_3 Q_4 - q_4 Q_3 \, .
\end{aligned}$$

These invariants can be easily derived using complex conjugate co-ordinates and are subjected a set of relations which can be found in [6, 7] , defining the first reduced phase space. This first reduced phase space is diffeomorphic to $\mathbb{CP}^3$. However, it is more convenient to use the invariants

$$\begin{aligned}
H_2 &= \tfrac{1}{2}\left(\pi_1 + \pi_2 + \pi_3 + \pi_4\right) & K_2 &= \pi_8 - \pi_7 \\
L_2 &= \pi_{12} + \pi_{15} & K_3 &= -\pi_6 - \pi_9 \\
K_1 &= \tfrac{1}{2}\left(-\pi_1 - \pi_2 + \pi_3 + \pi_4\right) & J_3 &= \pi_8 + \pi_7 \\
J_7 &= \pi_{12} - \pi_{15} & J_6 &= \pi_6 - \pi_9 \\
J_1 &= \tfrac{1}{2}\left(\pi_1 - \pi_2 - \pi_3 + \pi_4\right) & J_4 &= \pi_5 + \pi_{10} \\
L_3 &= \pi_{14} - \pi_{13} & \Xi &= \pi_{16} + \pi_{11} \\
J_2 &= \tfrac{1}{2}\left(\pi_1 - \pi_2 + \pi_3 - \pi_4\right) & J_5 &= \pi_5 - \pi_{10} \\
J_8 &= \pi_{14} + \pi_{13} & L_1 &= \pi_{16} - \pi_{11}
\end{aligned} \tag{9}$$

with relations

$$
\begin{aligned}
K_1\,L_1 + K_2\,L_2 + K_3\,L_3 - \Xi\,n = 0, &\quad J_6\,L_1 + J_4\,L_2 - J_1\,L_3 - J_8\,n = 0,\\
J_3\,L_1 - J_2\,L_2 - J_5\,L_3 + J_7\,n = 0, &\quad J_8\,K_3 + J_2\,L_1 + J_3\,L_2 + J_1\,\Xi = 0,\\
J_7\,K_3 - J_4\,L_1 + J_6\,L_2 - J_5\,\Xi = 0, &\quad J_8\,K_2 - J_5\,L_1 - J_3\,L_3 - J_4\,\Xi = 0,\\
J_7\,K_2 - J_1\,L_1 - J_6\,L_3 - J_2\,\Xi = 0, &\quad J_5\,K_2 - J_2\,K_3 + J_8\,L_1 - J_6\,n = 0,\\
J_1\,K_2 + J_4\,K_3 + J_7\,L_1 + J_3\,n = 0, &\quad J_8\,K_1 + J_5\,L_2 - J_2\,L_3 - J_6\,\Xi = 0,\\
J_7\,K_1 + J_1\,L_2 + J_4\,L_3 + J_3\,\Xi = 0, &\quad J_6\,K_1 + J_4\,K_2 - J_1\,K_3 - J_8\,\Xi = 0,\\
J_5\,K_1 + J_3\,K_3 - J_8\,L_2 + J_4\,n = 0, &\quad J_4\,K_1 - J_6\,K_2 - J_7\,L_3 + J_5\,n = 0,\\
J_3\,K_1 - J_2\,K_2 - J_5\,K_3 + J_7\,\Xi = 0, &\quad J_2\,K_1 + J_3\,K_2 + J_8\,L_3 + J_1\,n = 0,\\
J_1\,K_1 + J_6\,K_3 - J_7\,L_2 + J_2\,n = 0, &\quad J_6\,J_7 + J_3\,J_8 + K_3\,L_2 - K_2\,L_3 = 0,\\
J_5\,J_7 - J_1\,J_8 + K_3\,\Xi - L_3\,n = 0, &\quad J_4\,J_7 - J_2\,J_8 - K_3\,L_1 + K_1\,L_3 = 0,\\
J_3\,J_7 - J_6\,J_8 - K_1\,\Xi + L_1\,n = 0, &\quad J_2\,J_7 + J_4\,J_8 + K_2\,\Xi - L_2\,n = 0,\\
J_1\,J_7 + J_5\,J_8 - K_2\,L_1 + K_1\,L_2 = 0, &\quad J_3\,J_5 + J_1\,J_6 - K_1\,K_3 + L_1\,L_3 = 0,\\
J_3\,J_4 + J_2\,J_6 - L_3\,\Xi + K_3\,n = 0, &\quad J_2\,J_4 - J_1\,J_5 - J_3\,J_6 - J_7\,J_8 = 0,\\
J_1\,J_4 - J_2\,J_5 - K_2\,K_3 + L_2\,L_3 = 0, &\quad J_2\,J_3 - J_4\,J_6 - K_1\,K_2 + L_1\,L_2 = 0,\\
J_1\,J_3 - J_5\,J_6 - L_2\,\Xi + K_2\,n = 0, &\quad J_1\,J_2 + J_4\,J_5 - L_1\,\Xi + K_1\,n = 0,
\end{aligned}
\tag{10}
$$

joint with

$$
\begin{aligned}
{K_1}^2 + {K_2}^2 + {K_3}^2 + {L_1}^2 + {L_2}^2 + {L_3}^2 - \Xi^2 - n^2 = 0\ ,\\
{J_7}^2 + {J_8}^2 - {L_1}^2 - {L_2}^2 - {L_3}^2 + \Xi^2 = 0\ ,\\
{J_4}^2 - {J_5}^2 + {J_6}^2 - {J_8}^2 - {K_3}^2 + {L_3}^2 = 0\ ,\\
{J_3}^2 + {J_6}^2 - {K_2}^2 - {K_3}^2 - {L_1}^2 + \Xi^2 = 0\ ,\\
{J_2}^2 + {J_5}^2 - {J_6}^2 + {J_8}^2 + {K_2}^2 + {K_3}^2 + {L_1}^2 - n^2 = 0\ ,\\
{J_1}^2 + {J_5}^2 + {K_3}^2 + {L_1}^2 + {L_2}^2 - n^2 = 0\ ,
\end{aligned}
\tag{11}
$$

and

$$
H_2 = n. \tag{12}
$$

Let us furthermore consider the Ξ-action. In [6] we find that there are 16 invariants for the action corresponding to Ξ:

$$\begin{aligned}
s_1 &= q_1^2 + q_2^2 & s_2 &= q_3^2 + q_4^2 \\
s_3 &= Q_1^2 + Q_2^2 & s_4 &= Q_3^2 + Q_4^2 \\
s_5 &= q_1 Q_1 + q_2 Q_2 & s_6 &= q_3 Q_3 + q_4 Q_4 \\
s_7 &= q_1 Q_2 - q_2 Q_1 & s_8 &= q_3 Q_4 - q_4 Q_3 \\
s_9 &= q_1 q_4 - q_2 q_3 & s_{10} &= q_1 q_3 + q_2 q_4 \\
s_{11} &= Q_1 Q_4 - Q_2 Q_3 & s_{12} &= Q_1 Q_3 + Q_2 Q_4 \\
s_{13} &= q_1 Q_4 - q_2 Q_3 & s_{14} &= q_1 Q_3 + q_2 Q_4 \\
s_{15} &= Q_1 q_4 - Q_2 q_3 & s_{16} &= Q_1 q_3 + Q_2 q_4 \; .
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
K_1 &= \tfrac{1}{2}\left(-s_1 - s_3 + s_2 + s_4\right) \, , \\
K_2 &= -s_9 - s_{11} \, , \\
K_3 &= -s_{10} - s_{12} \, , \\
L_1 &= -s_7 + s_8 \, , \\
L_2 &= s_{14} - s_{16} \, , \\
L_3 &= s_{15} - s_{13} \, , \\
\Xi &= s_7 + s_8 \, , \\
H_2 &= \tfrac{1}{2}\left(s_1 + s_2 + s_3 + s_4\right) \, . \qquad (13)
\end{aligned}$$

That is, K_i, L_i, Ξ, and H_2 are the generating invariants for the combined Ξ action and H_2 action. Consider the orbit map

$$\tau_{\Xi} : \mathbb{R}^8 \to \mathbb{R}^{16}; (q, Q) \to (s_1, \cdots, s_{16}) \, .$$

One can check that, after an appropriate permutation, the s_i fulfill the same set of relations as the π_i (see table 1.). These relations define the orbit space, which therefore is geometrically the same as the orbit space defined by the π_i. The Poisson structure, however, will be different, which means that as Poisson spaces these orbit spaces will differ. When we set $\Xi = s_7 + s_8 = \xi$, $\xi \in \mathbb{R}$ we obtain the reduced phase spaces $V_\xi = \tau_{\Xi}(\Xi^{-1}(\xi))$. Consequently $\tau_\Xi(\mathcal{J}_0) \subset V_0$ and the regularized system is the reduced system on all of V_0. By construction the Kepler coordinates (x, y) are functions in the invariants s_i and provide a chart for $\tau_\Xi(\mathcal{J}_0) \subset V_0$.

4 Constrained dynamics and reduction

Note that, in order to obtain the Kepler dynamics, on $\mathbb{R}^8$ the dynamics of X_{H_2} has to be constrained to $\mathcal{J}_0$.(see [11]) Because $\{H_2, \Xi\} = 0$ the dynamics actually restricts to $\mathcal{J}_0$ in a natural way.

$s_1 s_3 = s_5^2 + s_7^2$	$s_1 s_2 = s_{10}^2 + s_9^2$
$s_1 s_4 = s_{14}^2 + s_{13}^2$	$s_2 s_3 = s_{16}^2 + s_{15}^2$
$s_3 s_4 = s_{12}^2 + s_{11}^2$	$s_2 s_4 = s_6^2 + s_8^2$
$s_1 s_{16} = s_5 s_{10} + s_7 s_9$	$s_1 s_{12} = s_5 s_{14} + s_7 s_{13}$
$s_1 s_6 = s_{10} s_{14} + s_9 s_{13}$	$s_3 s_{10} = s_5 s_{16} - s_7 s_{15}$
$s_3 s_{14} = s_5 s_{12} - s_7 s_{11}$	$s_3 s_6 = s_{16} s_{12} + s_{15} s_{11}$
$s_2 s_5 = s_{10} s_{16} + s_9 s_{15}$	$s_2 s_{14} = s_{10} s_6 - s_9 s_8$
$s_2 s_{12} = s_{16} s_6 - s_{15} s_8$	$s_4 s_5 = s_{14} s_{12} + s_{13} s_{11}$
$s_4 s_{10} = s_{14} s_6 + s_{13} s_8$	$s_4 s_{16} = s_{12} s_6 + s_{11} s_8$
$s_7 s_8 = s_9 s_{11} - s_{13} s_{15}$	$s_2 s_7 = s_{16} s_9 - s_{10} s_{15}$
$s_2 s_{14} = s_{10} s_6 - s_9 s_8$	$s_2 s_{13} = s_6 s_9 + s_{10} s_8$
$s_2 s_{12} = s_{16} s_6 - s_{15} s_8$	$s_2 s_{11} = s_6 s_{15} + s_{16} s_8$
$s_4 s_5 = s_{14} s_{12} + s_{13} s_{11}$	$s_4 s_7 = s_{12} s_{13} - s_{14} s_{11}$
$s_4 s_{10} = s_{14} s_6 + s_{13} s_8$	$s_4 s_9 = s_6 s_{13} - s_{14} s_8$
$s_4 s_{16} = s_{12} s_6 + s_{11} s_8$	$s_4 s_{15} = s_6 s_{11} - s_{12} s_8$
$s_5 s_6 - s_{14} s_{16} = s_{13} s_{15} + s_7 s_8$	$s_6 s_7 + s_5 s_8 = s_{16} s_{13} - s_{14} s_{15}$
$s_{14} s_{16} - s_{10} s_{12} = s_{13} s_{15} - s_9 s_{11}$	$s_{12} s_9 + s_{10} s_{11} = s_{14} s_{15} + s_{16} s_{13}$
$s_5 s_6 + s_7 s_8 = s_{10} s_{12} + s_9 s_{11}$	$s_6 s_7 - s_5 s_8 = s_{12} s_9 - s_{10} s_{11}$
$s_7 + s_8 = \xi$	

Table 1: Relations for the s_i invariants

Let $\mathcal{A}_\Xi$ denote the function algebra $C^\infty(\mathbb{R}^8, \mathbb{R})_\Xi$ of smooth φ_Ξ invariants functions. Let $\mathcal{I}_\Xi$ denote the ideal in $\mathcal{A}_\Xi$ generated by Ξ. Then for $f, g \in \mathcal{A}_\Xi$ we have $f|_{\mathcal{J}_0} = g|_{\mathcal{J}_0}$ if and only if $f - g \in \mathcal{I}_\Xi$. Suppose $f - g \in \mathcal{I}_\Xi$, that is, $f - g = \Xi k$ for $k \in \mathcal{A}_\Xi$. Then $\{f - g, h\} = \{\Xi k, h\} = k\{\Xi, h\} + \Xi\{k, h\}$. Consequently $\{f, h\} - \{g, h\} \in \mathcal{I}_\Xi$. Thus also the Poisson bracket has a natural restriction.

By construction x_i, and y_i must be Ξ invariant. We have

$$
\begin{aligned}
x_1 &= 2 s_{10} \ , \\
x_2 &= 2 s_9 \ , \\
x_3 &= s_1 - s_2 \ , \\
y_1 &= \frac{1}{s_1 + s_2}(s_{14} + s_{16}) \ , \\
y_2 &= \frac{1}{s_1 + s_2}(s_{13} + s_{15}) \ , \\
y_3 &= \frac{1}{s_1 + s_2}(s_5 - s_6) \ .
\end{aligned}
$$

Thus

$$
< x, x >= (s_1 + s_2)^2 \ , \ < y, y >= \frac{s_3 + s_4}{s_1 + s_2} - \frac{\Xi^2}{(s_1 + s_2)^2} \ .
$$

$\{,\}$	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}
s_1	0	0	$4s_5$	0	$2s_1$	0	0	0	0	0	$2s_{13}$	$2s_{14}$	0	0	$2s_9$	$2s_{10}$
s_2	0	0	0	$4s_6$	0	$2s_2$	0	0	0	0	$2s_{15}$	$2s_{16}$	$2s_9$	$2s_{10}$	0	0
s_3	$-4s_5$	0	0	0	$-2s_3$	0	0	0	$-2s_{15}$	$-2s_{16}$	0	0	$-2s_{11}$	$-2s_{12}$	0	0
s_4	0	$-4s_6$	0	0	0	$-2s_4$	0	0	$-2s_{13}$	$-2s_{14}$	0	0	0	0	$-2s_{11}$	$-2s_{12}$
s_5	$-2s_1$	0	$2s_3$	0	0	0	0	0	$-s_9$	$-s_{10}$	s_{11}	s_{12}	$-s_{13}$	$-s_{14}$	s_{15}	s_{16}
s_6	0	$-2s_2$	0	$2s_4$	0	0	0	0	$-s_9$	$-s_{10}$	s_{11}	s_{12}	s_{13}	s_{14}	$-s_{15}$	$-s_{16}$
s_7	0	0	0	0	0	0	0	0	s_{10}	$-s_9$	s_{12}	$-s_{11}$	s_{14}	$-s_{13}$	s_{16}	$-s_{15}$
s_8	0	0	0	0	0	0	0	0	$-s_{10}$	s_9	$-s_{12}$	s_{11}	$-s_{14}$	s_{13}	$-s_{16}$	s_{15}
s_9	0	0	$2s_{15}$	$2s_{13}$	s_9	s_9	$-s_{10}$	s_{10}	0	0	s_5+s_6	s_7-s_8	s_1	0	s_2	0
s_{10}	0	0	$2s_{16}$	$2s_{14}$	s_{10}	s_{10}	s_9	$-s_9$	0	0	s_8-s_7	s_5+s_6	0	s_1	0	s_2
s_{11}	$-2s_{13}$	$-2s_{15}$	0	0	$-s_{11}$	$-s_{11}$	$-s_{12}$	s_{12}	$-s_5-s_6$	s_7-s_8	0	0	$-s_4$	0	$-s_3$	0
s_{12}	$-2s_{14}$	$-2s_{16}$	0	0	$-s_{12}$	$-s_{12}$	s_{11}	$-s_{11}$	s_8-s_7	$-s_5-s_6$	0	0	0	$-s_4$	0	$-s_3$
s_{13}	0	$-2s_9$	$2s_{11}$	0	s_{13}	$-s_{13}$	$-s_{14}$	s_{14}	$-s_1$	0	s_4	0	0	0	$-s_5+s_6$	s_8-s_7
s_{14}	0	$-2s_{10}$	$2s_{12}$	0	s_{14}	$-s_{14}$	s_{13}	$-s_{13}$	0	$-s_1$	0	s_4	0	0	s_7-s_8	$-s_5+s_6$
s_{15}	$-2s_9$	0	0	$2s_{11}$	$-2s_{15}$	s_{15}	$-s_{16}$	s_{16}	$-s_2$	0	s_3	0	s_5-s_6	s_8-s_7	0	0
s_{16}	$-2s_{10}$	0	0	$2s_{12}$	$-2s_{16}$	s_{16}	s_{15}	$-s_{15}$	0	$-s_2$	0	s_3	s_7-s_8	s_5-s_6	0	0

Table 2: Poisson structure for the s_i invariants

Furthermore, considered as functions on $\mathbb{R}^8$, with co-ordinates (q, Q), we have

$$
\begin{aligned}
&\{x_1, x_1\} = \{x_1, x_2\} = \{x_1, x_3\} = \{x_2, x_3\} = \{x_2, x_4\} = \{x_3, x_4\} = 0 \ , \\
&\{x_1, y_2\} = \{x_1, y_3\} = \{x_2, y_1\} = \{x_2, y_3\} = \{x_3, y_1\} = \{x_3, y_2\} = 0 \ , \\
&\{x_1, y_1\} = \{x_2, y_2\} = \{x_3, y_3\} = 2 \ , \\
&\{y_1, y_1\} = \{y_2, y_2\} = \{y_3, y_3\} = 0 \ , \\
&\{y_1, y_2\} = \frac{2(s_1 - s_2)\Xi}{(s_1 + s_2)^2} \ , \\
&\{y_1, y_3\} = \frac{4s_9\Xi}{(s_1 + s_2)^2} \ , \\
&\{y_2, y_3\} = \frac{4s_{10}\Xi}{(s_1 + s_2)^2} \ .
\end{aligned} \tag{14}
$$

This shows that if $\Xi = 0$ the standard Poisson bracket on $\mathbb{R}^8$ induces 2 times the standard Poisson bracket on $\mathbb{R}^6$ with co-ordinates (x, y). Which provides another proof of Proposition 2. As the expressions x_i, y_i are functionally independent as functions of the s_i, the coordinates (x, y) define a chart for $\tau_{\Xi}(\mathcal{J}_0) \subset \tau_{\Xi}(\Xi^{-1}(0))$.

We may now compute the reduced H_2 vector field on $\tau_{\Xi}(\mathcal{J}_0)$ in the chart with coordinates (x, y) by computing the Poisson brackets of the co-ordinates (x, y) with H_2. Because the reduction mapping is a Poisson map it is sufficient to compute the brackets in (q, Q) co-ordinates. Because we will constrain the result to $\Xi = 0$ we will consider the result modulo Ξ.

$$
\begin{aligned}
\{x_1, H_2\} &= 2(Q_1q_3 + q_1Q_3 - Q_2q_4 - q_2Q_4) = 2|x|y_1 \ , \\
\{x_2, H_2\} &= 2(Q_2q_3 + q_2Q_3 - Q_1q_4 - q_1Q_4) = 2|x|y_2 \ , \\
\{x_3, H_2\} &= 2(q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4) = 2|x|y_3 \ , \\
\{y_1, H_2\} &= -2\frac{(q_1^2+q_2^2+q_3^2+q_4^2+Q_1^2+Q_2^2+Q_3^2+Q_4^2)(q_1q_3+q_2q_4)}{(q_1^2+q_2^2+q_3^2+q_4^2)^2} \\
&\quad - 2\frac{(Q_2q_3+q_2Q_3-Q_1q_4-q_1Q_4)(q_1Q_2-Q_1q_2+q_3Q_4-Q_3q_4)}{(q_1^2+q_2^2+q_3^2+q_4^2)^2} \\
&= -2\frac{H_2}{|x|^2}x_1 - 2\frac{\Xi}{|x|^2}(s_{15}-s_{13}) \ , \\
\{y_2, H_2\} &= -2\frac{(q_1^2+q_2^2+q_3^2+q_4^2+Q_1^2+Q_2^2+Q_3^2+Q_4^2)(q_1q_4-q_2q_3)}{(q_1^2+q_2^2+q_3^2+q_4^2)^2} \\
&\quad - 2\frac{(-Q_1q_3+q_1Q_3-Q_2q_4+q_2Q_4)(q_1Q_2-Q_1q_2+q_3Q_4-Q_3q_4)}{(q_1^2+q_2^2+q_3^2+q_4^2)^2} \\
&= -2\frac{H_2}{|x|^2}x_2 - 2\frac{\Xi}{|x|^2}(s_{14}-s_{16}) \ , \\
\{y_3, H_2\} &= -2\frac{(q_1^2+q_2^2+q_3^2+q_4^2+Q_1^2+Q_2^2+Q_3^2+Q_4^2)(q_1^2+q_2^2-q_3^2-q_4^2-q_2q_3)}{(q_1^2+q_2^2+q_3^2+q_4^2)^2} \\
&\quad - 2\frac{(Q_1q_2-q_1Q_2-Q_3q_4+q_3Q_4)(q_1Q_2-Q_1q_2+q_3Q_4-Q_3q_4)}{(q_1^2+q_2^2+q_3^2+q_4^2)^2} \\
&= -2\frac{H_2}{|x|^2}x_3 - 2\frac{\Xi}{|x|^2}(s_8-s_7) \ .
\end{aligned} \tag{15}
$$

Consequently on $\Xi = 0$ and $H_2 = 1$ we get two times the re-scaled Kepler vector field corresponding to the Hamiltonian $\tilde{\mathcal{K}}$ as given by formula (8).

The full regularized Kepler system is obtained by considering the reduced H_2 vector field on V_0, where $V_0 \backslash \tau_\Xi(\mathcal{J}_0)$ corresponds to the collision set. let $C = \{(q,Q) \in \mathbb{R}^8 | \|q\| = 0\}$, and $C_0 = C \cap \Xi^{-1}(0)$. Then $\tau_\Xi(C_0) = V_0 \backslash \tau_\Xi(\mathcal{J}_0)$. Consider on the image of the Ξ orbit map a new set of coordinates $H_2, \Xi, K_i, L_j, P_\ell$ similar to the (K, L, J) coordinates for the H_2 reduction. Here K_i and L_j are as in formula (13), and

$$
\begin{aligned}
P_1 &= \tfrac{1}{2}\left(-s_1 - s_2 + s_3 + s_4\right) \ , \\
P_2 &= \tfrac{1}{2}\left(-s_1 + s_2 + s_3 - s_4\right) \ , \\
P_3 &= s_9 - s_{11} \ , \\
P_4 &= s_5 + s_6 \ , \\
P_5 &= s_5 - s_6 \ , \\
P_6 &= s_{10} - s_{12} \ , \\
P_7 &= s_{14} + s_{16} \ , \\
P_8 &= s_{13} + s_{15} \ .
\end{aligned} \tag{16}
$$

The reduced H_2 flow on $\mathbb{R}^{16}$ now becomes a simultaneous rotation in the (P_1, P_4)-plane, the (P_2, P_5)-plane, the (P_3, P_8)-plane, and the (P_6, P_7)-plane. It is easy to check that the collision set $\tau_\Xi(C_0)$ is not invariant under this action. The collision orbits of the Kepler set are regularized to reduced H_2 orbits that intersect the set $\tau_\Xi(C_0)$.

5 Invariants and reduction of the Kepler problem

The reduction of the Kepler problem, that is, the reduction with respect to the S^1-action given by the periodic flow of the Kepler system (see [3]), goes through the orbit map defined by the invariants given by $J + LA$, and $J - LA$, where $J = x \times y$ is the momentum vector and $A = y \times J - \frac{x}{|x|}$ is the Laplace vector, also called Rung-Lenz vector or eccentricity vector. In the following we will write $R = q_1^2 + q_2^2 + q_3^2 + q_4^2$. We have:

$$
\begin{aligned}
J_1 &= \frac{(-q_1^2 - q_2^2 + q_3^2 + q_4^2)(q_1 Q_4 + Q_1 q_4 - q_2 Q_3 - Q_2 q_3)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} + \\
&\quad \frac{2(q_1 q_4 - q_2 q_3)(q_1 Q_1 + q_2 Q_2 - q_3 Q_3 - q_4 Q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \\
&= (s_{15} - s_{13}) + 2\frac{s_{10}}{s_1 + s_2}(s_7 + s_8) = L_3 + \frac{x_1}{R}\Xi \, , \\
J_2 &= \frac{(q_1^2 + q_2^2 - q_3^2 - q_4^2)(q_1 Q_3 + Q_1 q_3 + q_2 Q_4 + Q_2 q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} - \\
&\quad \frac{2(q_1 q_3 + q_2 q_4)(q_1 Q_1 + q_2 Q_2 - q_3 Q_3 - q_4 Q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \\
&= (s_{14} - s_{16}) + 2\frac{s_9}{s_1 + s_2}(s_7 + s_8) = L_2 + \frac{x_2}{R}\Xi \, , \\
J_3 &= \frac{2((q_1^2 + q_2^2)(q_3 Q_4 - Q_3 q_4) + (q_3^2 + q_4^2)(Q_1 q_2 - q_1 Q_2))}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \\
&= \frac{2(s_1 s_8 - s_2 s_7)}{s_1 + s_2} = L_1 + \frac{x_3}{R}\Xi \, .
\end{aligned} \tag{17}
$$

Consequently the "regularized" momentum vector is on $\mathcal{J}_0$ equivalent to (L_3, L_2, L_1).

Furthermore

$$
\begin{aligned}
A_1 =& \frac{1}{(q_1^2+q_2^2+q_3^2+q_4^2)^2}\big(-2(q_1q_3+q_2q_4)(q_1^2+q_2^2+q_3^2+q_4^2) \\
&+2(-Q_2q_3-q_2Q_3+Q_1q_4+q_1Q_4)((q_3^2+q_4^2)(Q_1q_2-q_1Q_2)+(q_1^2+q_2^2)(-Q_3q_4+q_3Q_4)) \\
&-(q_1Q_1+q_2Q_2-q_3Q_3-q_4Q_4)\big((q_1^2+q_2^2-q_3^2-q_4^2)(Q_1q_3+q_1Q_3+Q_2q_4+q_2Q_4) \\
&-2(q_1q_3+q_2q_4)(q_1Q_1+q_2Q_2-q_3Q_3-q_4Q_4)))\\
=& \frac{1}{(s_1+s_2)^2}(-2s_{10}(s_1+s_2)+2(s_{13}+s_{15})(s_1s_8-s_2s_7) \\
&-(s_5-s_6)((s_1-s_2)(s_{14}+s_{16})-2s_{10}(s_5-s_6))) \\
=& K_3+\frac{x_1H_2-x_1}{R}-\frac{J_1}{R}\Xi\,,
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
A_2 =& \frac{1}{(q_1^2+q_2^2+q_3^2+q_4^2)^2}\big(2(q_2q_3-q_1q_4)(q_1^2+q_2^2+q_3^2+q_4^2) \\
&-2(Q_1q_3+q_1Q_3+Q_2q_4+q_2Q_4)((q_3^2+q_4^2)(Q_1q_2-q_1Q_2)+(q_1^2+q_2^2)(-Q_3q_4+q_3Q_4)) \\
&+(q_1Q_1+q_2Q_2-q_3Q_3-q_4Q_4)\big(-(q_1^2+q_2^2-q_3^2-q_4^2)(-Q_2q_3-q_2Q_3+Q_1q_4+q_1Q_4)\big) \\
&+2(-q_2q_3+q_1q_4)(q_1Q_1+q_2Q_2-q_3Q_3-q_4Q_4)))\\
=& \frac{1}{(s_1+s_2)^2}(-2s_9(s_1+s_2)+2(s_{14}+s_{16})(s_1s_8-s_2s_7) \\
&+(s_5-s_6)(-(s_1-s_2)(s_{13}+s_{15})+2s_9(s_5-s_6))) \\
=& K_2+\frac{x_2H_2-x_2}{R}-\frac{J_2}{R}\Xi\,,
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
A_3 =& \frac{1}{(q_1^2+q_2^2+q_3^2+q_4^2)^2}\big((-q_1^2-q_2^2+q_3^2+q_4^2)(q_1^2+q_2^2+q_3^2+q_4^2) \\
&-(-Q_2q_3-q_2Q_3+Q_1q_4+q_1Q_4)\big(-(q_1^2+q_2^2-q_3^2-q_4^2)(-Q_2q_3-q_2Q_3+Q_1q_4+q_1Q_4) \\
&+2(-q_2q_3+q_1q_4)(q_1Q_1+q_2Q_2-q_3Q_3-q_4Q_4)) \\
&+(Q_1q_3+q_1Q_3+Q_2q_4+q_2Q_4)\big((q_1^2+q_2^2-q_3^2-q_4^2)(Q_1q_3+q_1Q_3+Q_2q_4+q_2Q_4) \\
&-2(q_1q_3+q_2q_4)(q_1Q_1+q_2Q_2-q_3Q_3-q_4Q_4)))\\
=& \frac{1}{(s_1+s_2)^2}((-s_1+s_2)(s_1+s_2)-(s_{13}+s_{15})((-s_1+s_2)(s_{13}+s_{15}) \\
&+2s_9(s_5-s_6)) \\
&+(s_{14}+s_{16})((s_1-s_2)(s_{14}+s_{16})-2s_{10}(s_5-s_6))) \\
=& K_1+\frac{x_3H_2-x_3}{R}-\frac{J_3}{R}\Xi\,,
\end{aligned} \tag{20}
$$

Consequently the "regularized" Laplace vector, i.e. $\mathcal{KS}^*(A)$, is on $H_2^{-1}(1)$ and $\mathcal{J}_0$ equivalent to (K_3, K_2, K_1).

When we consider the combined H_2 and Ξ orbit map

$$\tau_{H_2,\Xi} : \mathbb{R}^8 \to \mathbb{R}^8; (q, Q) \to (K_1, K_2, K_3, L_1, L_2, L_3, H_2, \Xi)\,.$$

Then the reduced phase spaces are given by setting H_2 and Ξ equal to a constant. For $H_2 = 1$ and $\Xi = 0$ the reduced phase space is $S^2 \times S^2$ (see [7]) given by

$$\|L + K\|^2 = \|L - K\|^2 = 1 \; . \tag{21}$$

This corresponds to the reduced phase space for the Kepler problem which is $S^2 \times S^2$ given by

$$\|J + A\|^2 = \|J - A\|^2 = 1 \; . \tag{22}$$

For $\Xi = 0$, $H_2 = 1$ the sets defined by equations (21) and (22) are equal. Furthermore one can easily check that also the Poisson structures induced by the orbit mappings on these spaces are the same. Note that points in the image of the set $C_0 = \{(q, Q) \in \mathbb{R}^8 | \|q\| = 0\} \cap \Xi^{-1}(0)$ have to be excluded when considering the Kepler problem. On C_0 we have that $L_1 = L_2 = L_3 = 0$. Consequently $\tau_{H_2,\Xi}(C_0)$ is the set $\mathcal{S}^2 = \{(K, -K) \in S^2 \times S^2 | K_1^2 + K_2^2 + K_3^2 = 1\}$ (see also [3]).

6 Delaunay elements

Now consider the Delaunay elements (L, G, H, ℓ, g, h). These can be considered as a set of action-angle coordinates designed especially for the Kepler problem ([9]). We know that $L = \frac{1}{\sqrt{-2K_0}}$, where K_0 is the Kepler energy. For the re-scaled Kepler Hamiltonian we therefore get $L = \tilde{K}$. That is we may take $L = H_2$ because on $\mathcal{J}_0$ this is equivalent to $L = \tilde{K}$. Furthermore we know that $G^2 = |J|^2$ with $J = x \times y$ the angular momentum vector. In (q, Q) co-ordinates we have

$$\begin{aligned} J_1 =& \frac{(-q_1^2 - q_2^2 + q_3^2 + q_4^2)(q_1Q_4 + Q_1q_4 - q_2Q_3 - Q_2q_3)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} + \\ & \frac{2(q_1q_4 - q_2q_3)(q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \; , \\ J_2 =& \frac{(q_1^2 + q_2^2 - q_3^2 - q_4^2)(q_1Q_3 + Q_1q_3 + q_2Q_4 + Q_2q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} - \\ & \frac{2(q_1q_3 + q_2q_4)(q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \; , \\ J_3 =& \frac{2((q_1^2 + q_2^2)(q_3Q_4 - Q_3q_4) + (q_3^2 + q_4^2)(Q_1q_2 - q_1Q_2))}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \; . \end{aligned}$$

and

$$G^2 = \sum_{1 \leqslant i, < j \leqslant 4} S_{ij}^2 + \Xi^2 \; ,$$

with

$$S_{ij}(q,Q) = q_iQ_j - q_jQ_i \ , \ i<j \ , \ i=1,2,3,4 \ , \ j=1,2,3,4 \ ,$$

Because

$$\sum_{1\leqslant i,<j\leqslant 4} S_{ij}^2 = \Xi^2 + L_1^2 + L_2^2 + L_3^2 + J_7^2 + J_8^2 \ ,$$

and

$$S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = \Xi^2 - L_1^2 - L_2^2 - L_3^2 + J_7^2 + J_8^2 = 0 \ .$$

We find

$$G^2 = 2(L_1^2 + L_2^2 + L_3^2) \ .$$

Finally we have

$$H = J_3 = \frac{2(s_1S_{34} - s_2S_{12})}{s_1+s_2} = L_1 + \frac{s_1-s_2}{s_1+s_2}\Xi \ .$$

Consequently

$$\mathcal{KS}^*(L) = H_2 \ , \ \mathcal{KS}^*(G) = \sqrt{L_1^2+L_2^2+L_3^2} \ , \ \mathcal{KS}^*(H) = L_1 \ .$$

Note that the expression for G corresponds with the one in [2](formula (28)), while Ξ and H correspond to U_1 and U_3 introduced in [2].

We will study the flow corresponding to these "action coordinates" on the twice reduced orbit space. Recall that the orbit map

$$\tau_{H_2,\Xi} : \mathbb{R}^8 \to \mathbb{R}^8; (q,Q) \to (K_1,K_2,K_3,L_1,L_2,L_3,H_2,\Xi)$$

defines an orbit space through the relations

$$|K|^2 + |L|^2 = H_2^2 - \Xi^2 \ , \ \langle K, L\rangle = H_2\Xi \ ,$$

which, when $\Xi = 0$ and $H_2 = 1$, defines a 4-dimensional reduced phase space in $\mathbb{R}^6$ with coordinates (K,L) given by

$$|K|^2 + |L|^2 = 1 \ , \ \langle K, L\rangle = 0 \ .$$

Note that this is also a reduced phase space for the Kepler problem because of the equivalence of K and L with A and J. The Poisson bracket on K,L space is given by table 3. Write $\tilde{G} = \mathcal{KS}^*(G) = |L|$, and let

$\{\cdot,\cdot\}$	K_1	K_2	K_3	L_1	L_2	L_3
K_1	0	$-2L_3$	$2L_2$	0	$-2K_3$	$2K_2$
K_2	$2L_3$	0	$-2L_1$	$2K_3$	0	$-2K_1$
K_3	$-2L_2$	$2L_1$	0	$-2K_2$	$2K_1$	0
L_1	0	$-2K_3$	$2K_2$	0	$-2L_3$	$2L_2$
L_2	$2K_3$	0	$-2K_1$	$2L_3$	0	$-2L_1$
L_3	$-2K_2$	$2K_1$	0	$-2L_2$	$2L_1$	0

Table 3: Bracket relations for the (K, L) variables.

$$\sigma(K) = \begin{pmatrix} 0 & -2K_3 & 2K_2 \\ 2K_3 & 0 & -2K_1 \\ -2K_2 & 2K_1 & 0 \end{pmatrix}$$

then the reduced vector field corresponding to $\tilde{G}$ is

$$\begin{cases} \dot{K} = \sigma(K)\frac{L}{|L|} \ , \\ \dot{L} = 0 \ . \end{cases}$$

That is, with L fixed the remaining equation is

$$\dot{K} = -\frac{2}{|L|} L \times K = -\frac{2}{|L|}\sigma(L)K \ .$$

Now $\exp(-t\frac{2}{|L|}\sigma(L))$ is a rotation about L by the angle $-2t$ ([12]). Consequently the flow of $\tilde{G}$ is a rotation in $\mathbb{R}^3$ with axis L.

For the flow corresponding to L_1 we find the vector field

$$\dot{K} = \begin{pmatrix} 0 \\ 2K_3 \\ -2K_2 \end{pmatrix} \ , \ \dot{L} = \begin{pmatrix} 0 \\ 2L_3 \\ -2L_2 \end{pmatrix} \ .$$

In this case the flow is a simultaneous rotation about the K_1-axis in K-space and the L_1-axis in L-space.

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