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The Kepler system as a reduced 4D harmonic oscillator

by

J.C. van der Meer

Centre for Analysis, Scientific computing and Applications
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
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The Kepler system as a reduced 4D harmonic oscillator

J.C. van der Meer
Faculteit Wiskunde en Informatica, Technische Universiteit Eindhoven,
P.O.Box 513, 5600 MB Eindhoven, The Netherlands.

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Abstract

In this note it is shown how to obtain the Kepler system by geometric reduction of the 4DOF harmonic oscillator. It is shown that this reduction is a reverse KS regularization. It is furthermore shown how the integrals for the Kepler system can be represented as integrals of the harmonic oscillator. Also a representation of the Delaunay elements is computed.

1 Introduction

As is well known the Kepler problem refers to the bounded motion of a particle in $\mathbb{R}^3$ which is influenced by the gravitational field of a second particle fixed at the origin.

We may describe the problem as a Hamiltonian system $(K, T_0\mathbb{R}^3, \omega)$, with phase space $T_0\mathbb{R}^3 = (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$, with co-ordinates $(x, y)$, and symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3$. The Hamiltonian is given by

$$K(x, y) = \frac{1}{2} \langle y, y \rangle - \frac{\mu}{|x|}.$$ 

Here $\langle , \rangle$ denotes the Euclidean inner product on $\mathbb{R}^3$ and $|\cdot|$ the induced Euclidean norm.

Consider the action $SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3; (A, x) \to Ax$. This lifts to $SO(3) \times T_0\mathbb{R}^3 \to T_0\mathbb{R}^3; (A, (x, y)) \to (Ax, Ay)$. This group action has momentum mapping $J : T_0\mathbb{R}^3 \to so(3)^*$. Identifying $so(3)^*$ with $so(3)$ by taking the transpose, and identifying $so(3)$ with $\mathbb{R}^3$ through

$$\begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \to (u_1, u_2, u_3),$$

the momentum map becomes $J : (x, y) \to$
$x \times y$, with $\times$ the vector product on $\mathbb{R}^3$. Because the Kepler Hamiltonian is invariant under the $SO(3)$ action $J = (J_1, J_2, J_3) = x \times y$ are integrals of the Kepler system corresponding to the angular momentum vector. Other integrals are given by the Laplace vector or eccentricity vector $A = (A_1, A_2, A_3) = y \times J - \frac{y}{|y|}$.

On $(\Sigma, \omega)$ we have the following

$$\{J_i, J_j\} = \sum_k \varepsilon_{ijk} J_k , \quad \{J_i, A_j\} = \sum_k \varepsilon_{ijk} A_k , \quad \{A_i, A_j\} = \sum_k \varepsilon_{ijk} J_k .$$

Consequently, if we introduce $\xi_i = \frac{1}{2}(J_i + A_i)$, $\eta_i = \frac{1}{2}(J_i - A_i)$, we have

$$\{\xi_i, \xi_j\} = \sum_k \varepsilon_{ijk} \xi_k , \quad \{\xi_i, \eta_j\} = \sum_k \varepsilon_{ijk} \eta_k , \quad \{\xi_i, \eta_j\} = 0 ,$$

and these relations define a Lie algebra $\mathfrak{so}(3) \times \mathfrak{so}(3) \cong \mathfrak{so}(4)$. Consequently the full symmetry group of the Kepler problem is $SO(4)$. This is a symmetry of the phase space which is not a lift of a symmetry on configuration space.

When using Kustaanheimo-Stiefel regularization regularized perturbed Keplerian systems can be considered as perturbed isotropic harmonic oscillators on $\mathbb{R}^8$, which can be described by a Hamiltonian systems with the standard symplectic form, and co-ordinates $(q, Q)$.

Considerer a perturbed isotropic oscillator oscillator on $\mathbb{R}^8$

$$H(q, Q) = \frac{1}{2}(Q, Q) + \frac{1}{2}(q, q) + \text{h.o.t.}$$

with additional symmetry given by the integral

$$\Xi(q, Q) = q_1 Q_2 - Q_1 q_2 + q_3 Q_4 - Q_3 q_4 , \quad (1)$$

When such a system is in normal form with respect to its quadratic part $H_2$ one may reduce with respect the $H_2$ and $\Xi$ symmetry to obtain a four degree of freedom system with, for $\Xi = 0$, a reduced phase space $S^2 \times S^2$ with Poisson structure equivalent to that of the Lie algebra $\mathfrak{so}(3) \times \mathfrak{so}(3)$. Thus one can consider these systems as generalizations of perturbed Keplerian systems. Systems like this with additional integral $L_1(q, Q) = -q_1 Q_2 + Q_1 q_2 + q_3 Q_4 - Q_3 q_4$ are studied in detail in [6], [7], [5], [8].

In this note it will be shown how the regularized Keplerian system is embedded in the isotropic harmonic oscillator. More precisely the Kepler system is obtained by reparametrizing a reduced harmonic oscillator. It is now clear that after further reduction both systems give rise to identical reduced systems on $S^2 \times S^2$. It will allow us to relate the geometric reduction by invariants to the symplectic reduction of perturbed Kepler systems by using Delaunay coordinates, and to connect the generalized Delaunay coordinates for the perturbed symmetric isotropic harmonic oscillator to the geometric reduction.
The embedding of the Kepler problem in the harmonic oscillator dynamics is performed by combining Kustaanheimo-Stiefel regularization as is discussed in [4] (page 80) with geometric reduction. More precisely it turns out that KS regularization is a reversed reduction. The fact that KS regularization is a reversed reduction is also the topic in [1]. Also already in [10] the regularized Kepler system was considered on an orbit space. However, in these papers the reduction was not extended beyond the KS-map, that is, the embedding in the full reduction of the harmonic oscillator, which allows a full computational construction, was not considered. The computations in this paper make it possible to deal with perturbed Kepler problems through symmetrically perturbed harmonic oscillators.

In section 2, we review the KS regularization and show how it is related to geometric reduction. In section 3, we further discuss the invariants for the harmonic oscillator and geometric reduction. In section 4, we show how the Kepler system is obtained as a reduced system by formulating the reduced system in a proper chart on the reduced phase space. In section 5, we show how reduction of the Kepler problem fits in further reduction of the harmonic oscillator. Here the integrals of the Kepler problem are represented as integrals on the harmonic oscillator level. In the final section a connection is made to Delaunay variables.

2 KS regularization

We will follow the Kustaanheimo-Stiefel regularization as presented in [4] in complex co-ordinates, which goes back on [10]. We will establish a relation between these complex symplectic spaces and real symplectic spaces by an appropriate choice of real co-ordinates which differs from the one in [10].

Let $z = (z_1, z_2) = (q_1 + iq_2, q_3 + iq_4) \in \mathbb{C}_0^2 = \mathbb{C}^2 - \{0\} = \mathbb{R}^4 - \{0\}$. Let $\langle z, w \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2$ be the standard Hermitian inner product on $\mathbb{C}^2$. Furthermore let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the Pauli matrices. Set

$$x_1 = \langle z, \sigma_1 z \rangle = z_1 \bar{z}_2 + z_2 \bar{z}_1 = 2(q_1 q_3 + q_2 q_4),$$
$$x_2 = \langle z, \sigma_2 z \rangle = iz_1 \bar{z}_2 - iz_2 \bar{z}_1 = 2(q_1 q_4 - q_2 q_3),$$
$$x_3 = \langle z, \sigma_3 z \rangle = z_1 \bar{z}_1 - z_2 \bar{z}_2 = (q_1^2 + q_2^2) - (q_3^2 + q_4^2).$$

Then

$$\pi : \mathbb{C}_0^2 \to \mathbb{R}^3; z \to (x_1, x_2, x_3)$$
is the Hopf map. The map $\pi$ lifts to the Kustaanheimo-Stiefel map $KS$ on $T^*\mathbb{C}^2_0 = (\mathbb{C}^2_0 - \{0\}) \times \mathbb{C}^2$ with coordinates $(z, w)$

$$KS : T^*\mathbb{C}^2_0 \to T^*\mathbb{R}^3 : (z, w) \to (x, y),$$

where

$$y_1 = \frac{1}{\langle z, z \rangle} (Re \langle w, \sigma_1 z \rangle),$$

$$y_2 = \frac{1}{\langle z, z \rangle} (Re \langle w, \sigma_2 z \rangle),$$

$$y_3 = \frac{1}{\langle z, z \rangle} (Re \langle w, \sigma_3 z \rangle).$$

If we set $w_1 = Q_1 + iQ_2$ and $w_2 = Q_3 + iQ_4$ one gets

$$y_1 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)} (Q_2q_4 + Q_1q_3 + Q_3q_1 + Q_4q_2),$$

$$y_2 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)} ((Q_1q_4 + Q_4q_1) - (Q_3q_2 + Q_2q_3)), $$

$$y_3 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)} ((Q_1q_1 + Q_2q_2) - (Q_3q_3 + Q_4q_4)).$$

Define on $T^*\mathbb{C}^2_0$ the 1-form $\Theta = -2Re\langle w, dz \rangle$ and symplectic form $\Omega = -d\Theta$. Note that in $(q, Q)$ co-ordinates $\Omega$ becomes the two times the standard symplectic form on $\mathbb{R}^4 \times \mathbb{R}^4$.

The Hopf map introduces a nontrivial $S^1$ bundle over $S^2$ which is the orbit space of an $S^1$-action which lifts to the following $S^1$ action on $T^*\mathbb{C}^2_0$. This action is symplectic with respect to the symplectic form $\Omega$.

$$\varphi : S^1 \times T^*\mathbb{C}^2_0 \to T^*\mathbb{C}^2_0; (e^{it}, z, w) \to (e^{it}z, e^{it}w).$$

On $\mathbb{R}^8_0 = (\mathbb{R}^4 - \{0\}) \times \mathbb{R}^4$ with coordinates $(q, Q)$ this action is given by

$$\varphi : S^1 \times \mathbb{R}^8_0 \to \mathbb{R}^8; R_\nu(q, Q) \to (R_\nu q, R_\nu Q),$$

with

$$R_\nu = \begin{pmatrix} \cos \nu & -\sin \nu & 0 & 0 \\ \sin \nu & \cos \nu & 0 & 0 \\ 0 & 0 & \cos \nu & -\sin \nu \\ 0 & 0 & \sin \nu & \cos \nu \end{pmatrix}.$$

This action has momentum map

$$J : \mathbb{R}^8_0 \to \mathbb{R}; (q, Q) \to \Xi(q, Q),$$

with

$$\Xi(q, Q) = (q_1Q_2 - q_2Q_1) + (q_3Q_4 - q_4Q_3).$$

Let $\tilde{\omega}$ denote the standard symplectic form on $T^*\mathbb{R}^3$. 

4
Proposition 2.1

\[ KS^* \tilde{\omega}|_{J_0} = \Omega|_{J_0} , \]

where \( J_0 = J^{-1}(0) - \{0\} \).

Similar statements can be found in [10] and [4], however these are not consistent. Therefore we have reformulated the symplectic form and will give a proof below.

**Proof:** In stead of the above we will prove the equivalent statement

\[ KS^* \theta|_{J_0} = \Theta|_{J_0} , \]

with \( \theta = \langle y, dx \rangle \) te canonical one-form on \( T^*\mathbb{R}^3 \). For every \( u, w, z \in \mathbb{C}^2 \)

\[ \sum_{j=1}^{3} Re\langle u, \sigma_j(z)\rangle \sigma_j(w) = 2\langle w, z\rangle u - \langle u, z\rangle w . \] (2)

Interchanging \( u \) with \( z \) in eq.(2) gives

\[ \sum_{j=1}^{3} Re\langle z, \sigma_j(u)\rangle \sigma_j(w) = 2\langle w, u\rangle z - \langle u, z\rangle w . \] (3)

Subtracting eq.(3) from eq.(2) gives

\[ \sum_{j=1}^{3} Re\langle u, \sigma_j(z)\rangle \sigma_j(w) = \langle w, z\rangle u - \langle w, u\rangle z - iIm\langle u, z\rangle w . \] (4)

Taking the inner product of eq.(4) with \( z \) and adding the result to its complex conjugate gives

\[ \sum_{j=1}^{3} Re\langle u, \sigma_j(z)\rangle Re\langle z, \sigma_j(w)\rangle = Re\langle z, w\rangle Re\langle u, z\rangle - \langle z, z\rangle Re\langle u, w\rangle . \] (5)

Replacing \( w \) by \( dz \) and \( u \) by \( w \) in eq.(6) gives

\[ \sum_{j=1}^{3} Re\langle w, \sigma_j(z)\rangle Re\langle z, \sigma_j(dz)\rangle = Re\langle z, dz\rangle Re\langle w, z\rangle - \langle z, z\rangle Re\langle w, dz\rangle . \] (6)

Now \( dx_j = 2Re\langle z, \sigma_j(dz)\rangle \). Consequently

\[ \theta = 2 \sum_{j=1}^{3} \frac{Re\langle w, \sigma_j(z)\rangle}{\langle z, z\rangle} Re\langle z, \sigma_j(dz)\rangle = 2 \frac{Re\langle z, dz\rangle}{\langle z, z\rangle} Re\langle w, z\rangle - 2Re\langle w, dz\rangle , \] (7)
which on $J_0$ equals $-2Re \langle w, dz \rangle$.

q.e.d.

Consider

$$\tilde{K}(x, y) = \left| x \right| (K(x, y) + \frac{1}{2}k^2) + \frac{\mu}{k} = \frac{1}{2k} \left| x \right| (|y|^2 + k^2).$$  \hspace{1cm} (8)

The energy level $K(x, y) = -\frac{1}{2}k^2$ corresponds to the energy level $\tilde{K}(x, y) = \frac{\mu}{k}$. The Hamiltonian vector field corresponding to $\tilde{K}$ is

$$\begin{align*}
\frac{dx}{ds} &= \frac{1}{k} \left| x \right| \frac{\partial K}{\partial y} + (K(x, y) + \frac{1}{2}k^2) \frac{\partial}{\partial y} \left| x \right|, \\
\frac{dy}{ds} &= -\frac{1}{k} \left| x \right| \frac{\partial K}{\partial x}.
\end{align*}$$

On $K(x, y) = -\frac{1}{2}k^2$ this corresponds to

$$\begin{align*}
\frac{dx}{ds} &= \frac{1}{k} \left| x \right| \frac{\partial K}{\partial y}, \\
\frac{dy}{ds} &= -\frac{1}{k} \left| x \right| \frac{\partial K}{\partial x}.
\end{align*}$$

With $\frac{ds}{dt} = \frac{k}{\left| x \right|}$ this is a time re-scaled version, or re-parametrization, of the Kepler vector field. The integral curves of the pre-regularized Hamiltonian vector field corresponding to $\tilde{K}$ with energy $\frac{\mu}{k}$ agree with the integral curves of the Kepler vector field with energy $-\frac{1}{2}k^2$. The vector fields are not equivalent, as one is a re-parametrization of the other.

With $\tilde{K} = \frac{1}{2k} \left| x \right| (||y||^2 + k^2)$ the time re-scaled Kepler Hamiltonian on $(T^*\mathbb{R}^3, \tilde{\omega})$, and

$$||x|| = q_1^2 + q_2^2 + q_3^2 + q_4^2, \quad ||y||^2 = \frac{Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2}{q_1^2 + q_2^2 + q_3^2 + q_4^2} + \frac{(q_1Q_2 - q_2Q_1) + (q_3Q_4 - q_4Q_3)}{q_1^2 + q_2^2 + q_3^2 + q_4^2},$$

one finds that the regularized Kepler Hamiltonian is

$$\hat{K} = KS^*\tilde{K} = \frac{1}{2k} \left( \langle w, w \rangle + k^2 \langle z, z \rangle \right)$$

$$= \frac{1}{2k} \left( k^2 (q_1^2 + q_2^2 + q_3^2 + q_4^2) + (Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) + \frac{\Xi}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \right),$$

that is, $\hat{K} = H_2$, with

$$H_2 = \frac{1}{2k} \left( k^2 (q_1^2 + q_2^2 + q_3^2 + q_4^2) + (Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) \right),$$

when restricted to $J_0$. The regularization is now obtained by extending to all of $\Xi^{-1}(0)$ and applying $\Xi$ reduction, that is, dividing out the $S^1$-action generated by $\Xi$.

In order to simplify the computations, in the following we take $\mu = k = 1$. Actually this can be obtained by re-scaling.
Consider the isotropic harmonic oscillator $H_2(q, Q)$ on all of $\mathbb{R}^8$ with standard symplectic form $\omega$. Furthermore consider the integral $\Xi(q, Q)$. Then any $\Xi$ invariant higher order perturbation of $H_2$ restricts on $J_0$ to a perturbation of the regularized Kepler Hamiltonian. In order to relate the Kepler system to the $H_2$ and $\Xi$ invariants we will discuss these in this section. We will start with presenting the invariants for the $H_2$ action as they will be used in the sequel. There are 16 invariants for the action corresponding to $H_2$ (see [4]):

\[
\begin{align*}
\pi_1 &= Q_1^2 + q_1^2 \\
\pi_3 &= Q_3^2 + q_3^2 \\
\pi_5 &= Q_1Q_2 + q_1q_2 \\
\pi_7 &= Q_1Q_4 + q_1q_4 \\
\pi_9 &= Q_2Q_4 + q_2q_4 \\
\pi_{11} &= q_1Q_2 - q_3Q_1 \\
\pi_{13} &= q_1Q_4 - q_3Q_1 \\
\pi_{15} &= q_2Q_4 - q_4Q_2 \\
\pi_2 &= Q_2^2 + q_2^2 \\
\pi_4 &= Q_4^2 + q_4^2 \\
\pi_6 &= Q_1Q_3 + q_1q_3 \\
\pi_8 &= Q_2Q_3 + q_2q_3 \\
\pi_{10} &= Q_3Q_4 + q_3q_4 \\
\pi_{12} &= q_1Q_3 - q_3Q_1 \\
\pi_{14} &= q_2Q_3 - q_3Q_2 \\
\pi_{16} &= q_3Q_4 - q_4Q_3.
\end{align*}
\]

These invariants can be easily derived using complex conjugate co-ordinates and are subjected a set of relations which can be found in [6, 7], defining the first reduced phase space. This first reduced phase space is diffeomorphic to $\mathbb{CP}^3$. However, it is more convenient to use the invariants

\[
\begin{align*}
H_2 &= \frac{1}{2} (\pi_1 + \pi_2 + \pi_3 + \pi_4) \\
L_2 &= \pi_{12} + \pi_{15} \\
K_1 &= \frac{1}{2} (-\pi_1 - \pi_2 + \pi_3 + \pi_4) \\
J_1 &= \pi_{12} - \pi_{15} \\
J_3 &= \pi_8 + \pi_7 \\
J_4 &= \pi_6 - \pi_9 \\
J_6 &= \pi_6 + \pi_9 \\
J_7 &= \pi_8 - \pi_7 \\
J_8 &= \frac{1}{2} (\pi_1 - \pi_2 + \pi_3 - \pi_4) \\
J_9 &= \pi_{14} + \pi_{13} \\
J_{10} &= \pi_{15} + \pi_{14} \\
J_{11} &= \pi_{16} + \pi_{11} \\
J_{12} &= \pi_{16} - \pi_{11}.
\end{align*}
\]
with relations

\[ K_1 L_1 + K_2 L_2 + K_3 L_3 - \Xi n = 0, \quad J_6 L_1 + J_4 L_2 - J_1 L_3 - J_8 n = 0, \]
\[ J_3 L_1 - J_2 L_2 - J_5 L_3 + J_6 n = 0, \quad J_8 K_3 + J_2 L_1 + J_5 L_2 + J_1 \Xi = 0, \]
\[ J_7 K_3 - J_1 L_1 + J_6 L_2 - J_5 \Xi = 0, \quad J_8 K_2 - J_5 L_1 - J_3 L_3 - J_4 \Xi = 0, \]
\[ J_7 K_2 - J_1 L_1 - J_6 L_3 - J_2 \Xi = 0, \quad J_5 K_2 - J_2 K_3 + J_8 L_1 - J_6 n = 0, \]
\[ J_1 K_2 + J_3 K_3 + J_7 L_1 + J_3 n = 0, \quad J_8 K_1 + J_3 L_2 - J_2 L_3 - J_6 \Xi = 0, \]
\[ J_7 K_1 + J_1 L_2 + J_4 L_3 + J_5 \Xi = 0, \quad J_6 K_1 + J_4 K_2 - J_1 K_3 - J_8 \Xi = 0, \]
\[ J_5 K_1 + J_3 K_3 - J_8 L_2 + J_4 n = 0, \quad J_4 K_1 - J_6 K_2 - J_7 L_3 + J_5 n = 0, \]
\[ J_3 K_1 - J_2 K_2 - J_5 K_3 + J_7 \Xi = 0, \quad J_2 K_1 + J_3 K_2 + J_8 L_3 + J_1 n = 0, \]
\[ J_1 K_1 + J_6 K_3 - J_7 L_2 + J_2 n = 0, \quad J_6 J_7 + J_3 J_8 + K_3 L_2 - K_2 L_3 = 0, \]
\[ J_5 J_7 - J_1 J_8 + K_3 \Xi - L_3 n = 0, \quad J_4 J_7 - J_2 J_8 - K_3 L_1 + K_1 L_3 = 0, \]
\[ J_3 J_7 - J_6 J_8 - K_1 \Xi + L_1 n = 0, \quad J_2 J_7 + J_4 J_8 + K_2 \Xi - L_2 n = 0, \]
\[ J_1 J_7 + J_5 J_8 - K_2 L_1 + K_1 L_2 = 0, \quad J_3 J_5 + J_1 J_6 - K_1 K_3 + L_1 L_3 = 0, \]
\[ J_3 J_4 + J_2 J_6 - L_3 \Xi + K_3 n = 0, \quad J_2 J_4 - J_1 J_5 - J_3 J_6 - J_7 J_8 = 0, \]
\[ J_1 J_4 - J_2 J_5 - K_2 K_3 + L_2 L_3 = 0, \quad J_2 J_3 - J_4 J_6 - K_1 K_2 + L_1 L_2 = 0, \]
\[ J_1 J_5 - J_5 J_6 - L_2 \Xi + K_2 n = 0, \quad J_1 J_2 + J_4 J_5 - L_1 \Xi + K_1 n = 0, \]  \hspace{1cm} (10)

joint with

\[ K_1^2 + K_2^2 + K_3^2 + L_1^2 + L_2^2 + L_3^2 - \Xi^2 - n^2 = 0, \]
\[ J_1^2 + J_3^2 - L_1^2 - L_2^2 - L_3^2 + \Xi^2 = 0, \]
\[ J_1^2 - J_5^2 + J_6^2 - J_8^2 - K_3^2 + L_4^2 = 0, \]
\[ J_3^2 + J_6^2 - K_2^2 - K_3^2 - L_1^2 + \Xi^2 = 0, \]
\[ J_2^2 + J_5^2 - J_6^2 + J_8^2 + K_2^2 + K_3^2 + L_4^2 - n^2 = 0, \]
\[ J_1^2 + J_5^2 + K_3^2 + L_1^2 + L_2^2 - n^2 = 0, \] \hspace{1cm} (11)

and

\[ H_2 = n. \] \hspace{1cm} (12)

Let us furthermore consider the $\Xi$-action. In [6] we find that there are 16 invariants for
the action corresponding to $\Xi$:

\[
\begin{align*}
    s_1 &= q_1^2 + q_2^2 \\
    s_2 &= q_3^2 + q_4^2 \\
    s_3 &= Q_1^2 + Q_2^2 \\
    s_4 &= Q_3^2 + Q_4^2 \\
    s_5 &= q_1 Q_1 + q_2 Q_2 \\
    s_6 &= q_3 Q_3 + q_4 Q_4 \\
    s_7 &= q_1 Q_2 - q_2 Q_1 \\
    s_8 &= q_3 Q_4 - q_4 Q_3 \\
    s_9 &= q_1 q_4 - q_2 q_3 \\
    s_{10} &= q_1 q_3 + q_2 q_4 \\
    s_{11} &= Q_1 Q_4 - Q_2 Q_3 \\
    s_{12} &= Q_1 Q_3 + Q_2 Q_4 \\
    s_{13} &= q_1 Q_4 - q_2 Q_3 \\
    s_{14} &= q_1 Q_3 + q_2 Q_4 \\
    s_{15} &= Q_1 q_4 - Q_2 q_3 \\
    s_{16} &= Q_1 q_3 + Q_2 q_4 .
\end{align*}
\]

Furthermore we have

\[
\begin{align*}
    K_1 &= \frac{1}{2} (-s_1 - s_3 + s_2 + s_4) , \\
    K_2 &= -s_9 - s_{11} , \\
    K_3 &= -s_{10} - s_{12} , \\
    L_1 &= -s_7 + s_8 , \\
    L_2 &= s_{14} - s_{16} , \\
    L_3 &= s_{15} - s_{13} , \\
    \Xi &= s_7 + s_8 , \\
    H_2 &= \frac{1}{2} (s_1 + s_2 + s_3 + s_4) .
\end{align*}
\]

That is, $K_i$, $L_i$, $\Xi$, and $H_2$ are the generating invariants for the combined $\Xi$ action and $H_2$ action. Consider the orbit map

\[
\tau_\Xi : \mathbb{R}^8 \rightarrow \mathbb{R}^{16}; (q, Q) \rightarrow (s_1, \cdots, s_{16}) .
\]

One can check that, after an appropriate permutation, the $s_i$ fulfill the same set of relations as the $\pi_i$ (see table 1.). These relations define the orbit space, which therefore is geometrically the same as the orbit space defined by the $\pi_i$. The Poisson structure, however, will be different, which means that as Poisson spaces these orbit spaces will differ. When we set $\Xi = s_7 + s_8 = \xi$, $\xi \in \mathbb{R}$ we obtain the reduced phase spaces $V_\xi = \tau_\Xi(\Xi^{-1}(\xi))$. Consequently $\tau_\Xi(J_0) \subset V_0$ and the regularized system is the reduced system on all of $V_0$. By construction the Kepler coordinates $(x, y)$ are functions in the invariants $s_i$ and provide a chart for $\tau_\Xi(J_0) \subset V_0$.

4 Constrained dynamics and reduction

Note that, in order to obtain the Kepler dynamics, on $\mathbb{R}^8$ the dynamics of $X_{H_2}$ has to be constrained to $J_0$ (see [11]). Because $\{H_2, \Xi\} = 0$ the dynamics actually restricts to $J_0$ in a natural way.
The Poisson bracket has a natural restriction. 

Thus, by construction, we have

\[ f, g \in A_{\Xi} \]

and only if \( f, g \in I_{\Xi} \). Thus also \( f - g \in I_{\Xi} \), that is, \( f - g = \Xi k \) for \( k \in A_{\Xi} \). Then \( \{ f - g, h \} = \{ \Xi k, h \} = k\{ \Xi, h \} + \Xi\{ k, h \} \). Consequently \( \{ f, h \} - \{ g, h \} \in I_{\Xi} \). Thus also the Poisson bracket has a natural restriction.

By construction \( x_i \), and \( y_i \) must be \( \Xi \) invariant. We have

\[
\begin{align*}
x_1 &= 2s_{10}, \\
x_2 &= 2s_9, \\
x_3 &= s_1 - s_2, \\
y_1 &= \frac{1}{s_1 + s_2}(s_{14} + s_{16}), \\
y_2 &= \frac{1}{s_1 + s_2}(s_{13} + s_{15}), \\
y_3 &= \frac{1}{s_1 + s_2}(s_5 - s_6).
\end{align*}
\]

Thus

\[
< x, x > = (s_1 + s_2)^2, \quad < y, y > = \frac{s_3 + s_4}{s_1 + s_2} - \frac{\Xi^2}{(s_1 + s_2)^2}.
\]

Table 1: Relations for the \( s_i \) invariants

| \( s_1 s_1 = s_3^2 + s_4^2 \) | \( s_1 s_2 = s_3 s_4 + s_5 s_6 \) |
| \( s_2 s_2 = s_7^2 + s_8^2 \) | \( s_2 s_3 = s_7 s_8 + s_9 s_10 \) |
| \( s_3 s_3 = s_7^2 + s_8^2 \) | \( s_3 s_4 = s_7 s_8 + s_9 s_10 \) |
| \( s_4 s_4 = s_7^2 + s_8^2 \) | \( s_4 s_5 = s_7 s_8 + s_9 s_10 \) |
| \( s_5 s_5 = s_7^2 + s_8^2 \) | \( s_5 s_6 = s_7 s_8 + s_9 s_10 \) |
| \( s_6 s_6 = s_7^2 + s_8^2 \) | \( s_6 s_7 = s_7 s_8 + s_9 s_10 \) |
| \( s_7 s_7 = s_7^2 + s_8^2 \) | \( s_7 s_8 = s_7 s_8 + s_9 s_10 \) |
| \( s_8 s_8 = s_7^2 + s_8^2 \) | \( s_8 s_9 = s_7 s_8 + s_9 s_10 \) |

Let \( A_{\Xi} \) denote the function algebra \( C^\infty(\mathbb{R}^8, \mathbb{R})_{\Xi} \) of smooth \( \varphi_{\Xi} \) invariants functions. Let \( I_{\Xi} \) denote the ideal in \( A_{\Xi} \) generated by \( \Xi \). Then for \( f, g \in A_{\Xi} \) we have \( f|_{\Xi_0} = g|_{\Xi_0} \) if and only if \( f - g \in I_{\Xi} \). Suppose \( f - g \in I_{\Xi} \), then \( f - g = \Xi k \) for \( k \in A_{\Xi} \). Then \( \{ f - g, h \} = \{ \Xi k, h \} = k\{ \Xi, h \} + \Xi\{ k, h \} \). Consequently \( \{ f, h \} - \{ g, h \} \in I_{\Xi} \). Thus also the Poisson bracket has a natural restriction.
Table 2: Poisson structure for the $s_i$ invariants

Furthermore, considered as functions on $\mathbb{R}^8$, with co-ordinates $(q, Q)$, we have

\[
\begin{align*}
\{x_1, x_1\} &= \{x_1, x_2\} = \{x_1, x_3\} = \{x_2, x_3\} = \{x_2, x_4\} = \{x_3, x_4\} = 0 , \\
\{x_1, y_2\} &= \{x_1, y_3\} = \{x_2, y_1\} = \{x_2, y_3\} = \{x_3, y_1\} = \{x_3, y_2\} = 0 , \\
\{x_1, y_1\} &= \{x_2, y_2\} = \{x_3, y_3\} = 2 , \\
\{y_1, y_1\} &= \{y_2, y_2\} = \{y_3, y_3\} = 0 , \\
\{y_1, y_2\} &= \frac{2(s_1 - s_2)\Xi}{(s_1 + s_2)^2} , \\
\{y_1, y_3\} &= \frac{4s_0\Xi}{(s_1 + s_2)^2} , \\
\{y_2, y_3\} &= \frac{4s_{10}\Xi}{(s_1 + s_2)^2} .
\end{align*}
\]

This shows that if $\Xi = 0$ the standard Poisson bracket on $\mathbb{R}^8$ induces 2 times the standard Poisson bracket on $\mathbb{R}^6$ with co-ordinates $(x, y)$. Which provides another proof of Proposition 2. As the expressions $x_i$, $y_i$ are functionally independent as functions of the $s_i$, the coordinates $(x, y)$ define a chart for $\tau_{\Xi}(J_0) \subset \tau_{\Xi}(\Xi^{-1}(0))$.

We may now compute the reduced $H_2$ vector field on $\tau_{\Xi}(J_0)$ in the chart with coordinates $(x, y)$ by computing the Poisson brackets of the co-ordinates $(x, y)$ with $H_2$. Because the reduction mapping is a Poisson map it is sufficient to compute the brackets in $(q, Q)$ co-ordinates. Because we will constrain the result to $\Xi = 0$ we will consider the result
corresponding to the Hamiltonian \( \tilde{H} \) on \( V \)

The full regularized Kepler system is obtained by considering the reduced 

Consequently on \( \Xi = 0 \) and 

modulo \( \Xi \).

\[
\begin{align*}
\{ x_1, H_2 \} &= 2(q_1q_3 + q_1q_4 - q_2q_4 - q_2q_3) = 2|x_1|, \\
\{ x_2, H_2 \} &= 2(q_2q_3 + q_2q_4 - q_1q_4 - q_1q_3) = 2|x_2|, \\
\{ x_3, H_2 \} &= 2(q_1q_1 + q_2q_2 - q_3q_3 - q_4q_4) = 2|x_3|, \\
\{ y_1, H_2 \} &= -2(q_1^2 + q_2^2 + q_3^2 + q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2)(q_1q_3 + q_2q_4) \\
&\quad - 2(q_2q_4 - q_1q_3 - Q_2q_4 + q_2q_4)(q_1q_2 - q_1q_2 + q_3q_4 - Q_3q_4) \\
&\quad (q_1^2 + q_2^2 + q_3^2 + q_4^2)^2 \\
&= -2 \frac{H_2}{x^2} x_1 - 2 \Xi (s_{15} - s_{13}), \\
\{ y_2, H_2 \} &= -2(q_1^2 + q_2^2 + q_3^2 + q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2)(q_1q_4 - q_2q_3) \\
&\quad - 2(-Q_1q_3 + q_1q_3 - Q_2q_4 + q_2q_4)(q_1q_2 - q_1q_2 + q_3q_4 - Q_3q_4) \\
&\quad (q_1^2 + q_2^2 + q_3^2 + q_4^2)^2 \\
&= -2 \frac{H_2}{x^2} x_2 - 2 \Xi (s_{14} - s_{16}), \\
\{ y_3, H_2 \} &= -2(q_1^2 + q_2^2 + q_3^2 + q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2)(q_1^2 + q_2^2 - q_3^2 - q_4^2 - q_2q_3) \\
&\quad - 2(q_1q_2 - q_1q_2 - Q_3q_4 + q_3q_4)(q_1q_2 - q_1q_2 + q_3q_4 - Q_3q_4) \\
&\quad (q_1^2 + q_2^2 + q_3^2 + q_4^2)^2 \\
&= -2 \frac{H_2}{x^2} x_3 - 2 \Xi (s_8 - s_7). \quad (15)
\end{align*}
\]

Consequently on \( \Xi = 0 \) and \( H_2 = 1 \) we get two times the re-scaled Kepler vector field corresponding to the Hamiltonian \( \tilde{K} \) as given by formula (8).

The full regularized Kepler system is obtained by considering the reduced \( H_2 \) vector field on \( V_0 \), where \( V_0 \setminus \tau_\Xi(J_0) \) corresponds to the collision set. Let \( C = \{(q, Q) \in \mathbb{R}^3 \mid \|q\| = 0\} \), and \( C_0 = C \cap \Xi^{-1}(0) \). Then \( \tau_\Xi(C_0) = V_0 \setminus \tau_\Xi(J_0) \). Consider on the image of the \( \Xi \) orbit map a new set of coordinates \( H_2, \Xi, K_i, L_j, P_t \) similar to the \( (K, L, J) \) coordinates for the \( H_2 \) reduction. Here \( K_i \) and \( L_j \) are as in formula (13), and

\[
\begin{align*}
P_1 &= \frac{1}{2} (-s_1 - s_2 + s_3 + s_4), \\
P_2 &= \frac{1}{2} (-s_1 + s_2 + s_3 - s_4), \\
P_3 &= s_9 - s_{11}, \\
P_4 &= s_5 + s_6, \\
P_5 &= s_5 - s_6, \\
P_6 &= s_{10} - s_{12}, \\
P_7 &= s_{14} + s_{16}, \\
P_8 &= s_{13} + s_{15}. \quad (16)
\end{align*}
\]
The reduced $H_2$ flow on $\mathbb{R}^{16}$ now becomes a simultaneous rotation in the $(P_1, P_4)$-plane, the $(P_2, P_5)$-plane, the $(P_3, P_8)$-plane, and the $(P_6, P_7)$-plane. It is easy to check that the collision set $\tau_\Xi(C_0)$ is not invariant under this action. The collision orbits of the Kepler set are regularized to reduced $H_2$ orbits that intersect the set $\tau_\Xi(C_0)$.

5 Invariants and reduction of the Kepler problem

The reduction of the Kepler problem, that is, the reduction with respect to the $S^1$-action given by the periodic flow of the Kepler system (see [3]), goes through the orbit map defined by the invariants given by $J + LA$, and $J - LA$, where $J = x \times y$ is the momentum vector and $A = y \times J - \frac{x}{|x|}$ is the Laplace vector, also called Rung-Lenz vector or eccentricity vector. In the following we will write $R = q_1^2 + q_2^2 + q_3^2 + q_4^2$. We have:

$$J_1 = \frac{(-q_1^2 - q_2^2 + q_3^2 + q_4^2)(q_1 q_4 + Q_1 q_4 - q_2 Q_3 - Q_2 q_3)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} + \frac{2(q_1 q_4 - q_2 q_3)(q_1 Q_1 + q_2 Q_2 - q_3 Q_3 - q_4 Q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

$$= (s_{15} - s_{13}) + 2\frac{s_{10}}{s_1 + s_2} (s_7 + s_8) = L_3 + \frac{x_1}{R} \Xi ,$$

$$J_2 = \frac{(q_1^2 + q_2^2 - q_3^2 - q_4^2)(q_1 Q_3 + Q_1 q_3 + q_2 Q_4 + Q_2 q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2} - \frac{2(q_1 q_3 + q_2 q_4)(q_1 Q_1 + q_2 Q_2 - q_3 Q_3 - q_4 Q_4)}{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

$$= (s_{14} - s_{16}) + 2\frac{s_9}{s_1 + s_2} (s_7 + s_8) = L_2 + \frac{x_2}{R} \Xi ,$$

$$J_3 = \frac{2((q_1^2 + q_3^2)(q_3 Q_4 - Q_3 q_4) + (q_2^2 + q_4^2)(Q_1 q_2 - q_1 Q_2))}{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

$$= \frac{2(s_1 s_8 - s_2 s_7)}{s_1 + s_2} = L_1 + \frac{x_3}{R} \Xi . \quad (17)$$

Consequently the ”regularized” momentum vector is on $J_0$ equivalent to $(L_3, L_2, L_1)$.
Furthermore

\[ A_1 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)^2} \left( -2(q_1q_3 + q_2q_4)(q_1^2 + q_2^2 + q_3^2 + q_4^2) \right. \\
+ 2(-Q_2q_3 - q_2Q_3 + Q_1q_4 + q_1Q_4)(\xi_1q_2 - q_1Q_2) + (q_1^2 + q_2^2)(-Q_3q_4 + q_3Q_4) \\
- (q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4) \left. (q_1^2 + q_2^2 - q_3^2 - q_4^2)(q_1q_3 + q_1Q_3 + q_2Q_4 + q_2Q_4) \right) \\
- 2(q_1q_3 + q_2q_4)(q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4) \right) \\
= \frac{1}{(s_1 + s_2)^2} \left( -2s_1(s_1 + s_2) + 2(s_1 + s_2)(s_1s_8 - s_2s_7) \\
- (s_5 - s_6)((s_1 - s_2)(s_1s_8 + s_2s_7) - 2s_1(s_5 - s_6)) \right) \\
= K_3 + \frac{x_1H_2 - x_1}{R} - \frac{J_1}{R} \Xi , \quad (18) \]

\[ A_2 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)^2} \left( 2(q_2q_3 - q_1q_4)(q_1^2 + q_2^2 + q_3^2 + q_4^2) \right. \\
- 2(Q_1q_3 + q_1Q_3 + Q_2q_4 + q_2Q_4)(\xi_2q_1 - q_2Q_1) + (q_1^2 + q_2^2)(-Q_3q_4 + q_3Q_4) \\
+ (q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4) \left. (q_1^2 + q_2^2 - q_3^2 - q_4^2)(-Q_2q_3 - q_2Q_3 + Q_1q_4 + q_1Q_4) \right) \\
+ 2(-q_2q_3 + q_1q_4)(q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4) \right) \\
= \frac{1}{(s_1 + s_2)^2} \left( -2s_9(s_1 + s_2) + 2(s_1 + s_2)(s_1s_8 - s_2s_7) \\
+ (s_5 - s_6)((s_1 - s_2)(s_1s_8 + s_2s_7) + 2s_9(s_5 - s_6)) \right) \\
= K_2 + \frac{x_2H_2 - x_2}{R} - \frac{J_2}{R} \Xi , \quad (19) \]

\[ A_3 = \frac{1}{(q_1^2 + q_2^2 + q_3^2 + q_4^2)^2} \left( (-q_1^2 - q_2^2 + q_3^2 + q_4^2)(q_1^2 + q_2^2 + q_3^2 + q_4^2) \right. \\
- (Q_2q_3 - q_2Q_3 + Q_1q_4 + q_1Q_4) \left. (-q_1^2 - q_2^2 + q_3^2 - q_4^2)(Q_2q_3 - q_2Q_3 + Q_1q_4 + q_1Q_4) \right) \\
+ 2(-q_2q_3 + q_1q_4)(q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4) \\
+ (Q_1q_3 + q_1Q_3 + Q_2q_4 + q_2Q_4) \left. (q_1^2 + q_2^2 - q_3^2 - q_4^2)(Q_1q_3 + q_1Q_3 + Q_2q_4 + q_2Q_4) \right) \\
- 2(q_1q_3 + q_2q_4)(q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4) \right) \\
= \frac{1}{(s_1 + s_2)^2} \left( (-s_1 + s_2)(s_1 + s_2) - (s_1 + s_2)((-s_1 + s_2)(s_1s_8 + s_2s_7) \\
+ 2s_9(s_5 - s_6)) \\
+ (s_1 + s_2)((s_1 - s_2)(s_1s_8 + s_2s_7) - 2s_1(s_5 - s_6)) \right) \\
= K_1 + \frac{x_3H_2 - x_3}{R} - \frac{J_3}{R} \Xi , \quad (20) \]

Consequently the ”regularized” Laplace vector, i.e. \( KS^*(A) \), is on \( H_2^{-1}(1) \) and \( J_0 \) equivalent to \( (K_3, K_2, K_1) \).

When we consider the combined \( H_2 \) and \( \Xi \) orbit map

\[ \tau_{H_2, \Xi} : \mathbb{R}^8 \rightarrow \mathbb{R}^8, (q, Q) \rightarrow (K_1, K_2, K_3, L_1, L_2, L_3, H_2, \Xi) . \]
Then the reduced phase spaces are given by setting \( H_2 \) and \( \Xi \) equal to a constant. For \( H_2 = 1 \) and \( \Xi = 0 \) the reduced phase space is \( S^2 \times S^2 \) (see [7]) given by

\[
\|L + K\|^2 = \|L - K\|^2 = 1 .
\]

(21)

This corresponds to the reduced phase space for the Kepler problem which is \( S^2 \times S^2 \) given by

\[
\|J + A\|^2 = \|J - A\|^2 = 1 .
\]

(22)

For \( \Xi = 0, H_2 = 1 \) the sets defined by equations (21) and (22) are equal. Furthermore one can easily check that also the Poisson structures induced by the orbit mappings on these spaces are the same. Note that points in the image of the set \( C_0 = \{(q, Q) \in \mathbb{R}^8||q|| = 0\} \cap \Xi^{-1}(0) \) have to be excluded when considering the Kepler problem. On \( C_0 \) we have that \( L_1 = L_2 = L_3 = 0 \). Consequently \( \tau_{H_2, \Xi}(C_0) \) is the set \( S^2 = \{(K, -K) \in S^2 \times S^2|K_i^2 + K_j^2 + K_3^2 = 1\} \) (see also [3]).

6 Delaunay elements

Now consider the Delaunay elements \((L, G, H, \ell, g, h)\). These can be considered as a set of action-angle co-ordinates designed especially for the Kepler problem ([9]). We know that \( L = \frac{1}{\sqrt{-K_0}} \), where \( K_0 \) is the Kepler energy. For the re-scaled Kepler Hamiltonian we therefore get \( L = \hat{K} \). That is we may take \( L = H_2 \) because on \( J_0 \) this is equivalent to \( L = \hat{K} \). Furthermore we know that \( G^2 = |J|^2 \) with \( J = x \times y \) the angular momentum vector. In \((q, Q)\) co-ordinates we have

\[
J_1 = \frac{(-q^2_1 - q^2_2 + q^2_3 + q^3_4)(q_1Q_4 + Q_1q_4 - q_2Q_3 - Q_2q_3)}{q^2_1 + q^2_2 + q^2_3 + q^2_4},
\]

\[
J_2 = \frac{(q^2_1 + q^2_2 - q^2_3 - q^3_4)(q_1Q_3 + Q_1q_3 + q_2Q_4 + Q_2q_4)}{q^2_1 + q^2_2 + q^2_3 + q^2_4},
\]

\[
J_3 = \frac{2(q^2_1)(q_1Q_1 + q_2Q_2 - q_3Q_3 - q_4Q_4)}{q^2_1 + q^2_2 + q^2_3 + q^2_4},
\]

\[
G^2 = \sum_{1 \leq i, j \leq 4} S^2_{ij} + \Xi^2 ,
\]
with

\[ S_{ij}(q, Q) = q_i Q_j - q_j Q_i , \quad i < j \quad i = 1, 2, 3, 4 \quad j = 1, 2, 3, 4 \]

Because

\[ \sum_{1 \leq i < j \leq 4} S_{ij}^2 = \Xi^2 + L_1^2 + L_2^2 + L_3^2 + J_7^2 + J_8^2 , \]

and

\[ S_{12}S_{34} - S_{13}S_{24} + S_{14}S_{23} = \Xi^2 - L_1^2 - L_2^2 - L_3^2 + J_7^2 + J_8^2 = 0 . \]

We find

\[ G^2 = 2(L_1^2 + L_2^2 + L_3^2) . \]

Finally we have

\[ H = J_3 = \frac{2(s_1 S_{34} - s_2 S_{12})}{s_1 + s_2} = L_1 + \frac{s_1 - s_2}{s_1 + s_2} \Xi . \]

Consequently

\[ \mathcal{KS}^*(L) = H_2 , \quad \mathcal{KS}^*(G) = \sqrt{L_1^2 + L_2^2 + L_3^2} , \quad \mathcal{KS}^*(H) = L_1 . \]

Note that the expression for \( G \) corresponds with the one in [2](formula (28)), while \( \Xi \) and \( H \) correspond to \( U_1 \) and \( U_3 \) introduced in [2].

We will study the flow corresponding to these "action coordinates" on the twice reduced orbit space. Recall that the orbit map

\[ \tau_{H_2, \Xi} : \mathbb{R}^8 \to \mathbb{R}^8, (q, Q) \to (K_1, K_2, K_3, L_1, L_2, L_3, H_2, \Xi) \]

defines an orbit space through the relations

\[ |K|^2 + |L|^2 = H_2^2 - \Xi^2 , \quad \langle K, L \rangle = H_2 \Xi , \]

which, when \( \Xi = 0 \) and \( H_2 = 1 \), defines a 4-dimensional reduced phase space in \( \mathbb{R}^6 \) with coordinates \( (K, L) \) given by

\[ |K|^2 + |L|^2 = 1 , \quad \langle K, L \rangle = 0 . \]

Note that this is also a reduced phase space for the Kepler problem because of the equivalence of \( K \) and \( L \) with \( A \) and \( J \). The Poisson bracket on \( K, L \) space is given by table 3. Write \( \tilde{G} = \mathcal{KS}^*(G) = |L| \), and let
\[ \{\cdot, \cdot\} \begin{array}{c|cccccc} \hline K_1 & K_2 & K_3 & L_1 & L_2 & L_3 \\ \hline K_1 & 0 & -2L_3 & 2L_2 & 0 & -2K_3 & 2K_2 \\ K_2 & 2L_3 & 0 & -2L_1 & 2K_3 & 0 & -2K_1 \\ K_3 & -2L_2 & 2L_1 & 0 & -2K_2 & 2K_1 & 0 \\ L_1 & 0 & -2K_3 & 2K_2 & 0 & -2L_3 & 2L_2 \\ L_2 & 2K_3 & 0 & -2K_1 & 2L_3 & 0 & -2L_1 \\ L_3 & -2K_2 & 2K_1 & 0 & -2L_2 & 2L_1 & 0 \\ \hline \end{array} \]

Table 3: Bracket relations for the \((K, L)\) variables.

\[ \sigma(K) = \begin{pmatrix} 0 & -2K_3 & 2K_2 \\ 2K_3 & 0 & -2K_1 \\ -2K_2 & 2K_1 & 0 \end{pmatrix} \]

then the reduced vector field corresponding to \(\tilde{G}\) is

\[ \begin{cases} \dot{K} = \sigma(K) \frac{L}{|L|} \\ \dot{L} = 0 \end{cases} \]

That is, with \(L\) fixed the remaining equation is

\[ \dot{K} = -\frac{2}{|L|} L \times K = -\frac{2}{|L|} \sigma(L) K. \]

Now \(\exp(-t \frac{2}{|L|} \sigma(L))\) is a rotation about \(L\) by the angle \(-2t\) ([12]). Consequently the flow of \(\tilde{G}\) is a rotation in \(\mathbb{R}^3\) with axis \(L\).

For the flow corresponding to \(L_1\) we find the vector field

\[ \begin{pmatrix} 0 \\ 2K_3 \\ -2K_2 \end{pmatrix}, \dot{L} = \begin{pmatrix} 0 \\ 2L_3 \\ -2L_2 \end{pmatrix}. \]

In this case the flow is a simultaneous rotation about the \(K_1\)-axis in \(K\)-space and the \(L_1\)-axis in \(L\)-space.

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<th>Month</th>
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<td>13-19</td>
<td>A. Muntean</td>
<td>Pedestrians moving in dark: Balancing measures and playing games on lattices</td>
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<td>E.N.M. Cirillo</td>
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<td>Q. Hou</td>
<td>SPH simulation of free overfall in open channels with even and uneven bottom</td>
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<td>P.J.P. van Meurs</td>
<td>Upscaling of dislocation walls in finite domains</td>
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<td>13-22</td>
<td>C.R. Prins</td>
<td>A numerical method for the design of free-form reflectors for lighting applications</td>
<td>Sept.’13</td>
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<td>J.C. van der Meer</td>
<td>The Kepler system as a reduced 4D harmonic oscillator</td>
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