Practical synchronization in networks of diffusively coupled non-identical systems

Supplementary material: Practical synchronization in networks of diffusively coupled non-identical systems

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This document is attached as supplementary material to the manuscript entitled "Training a network of electronic neurons for control of a mobile robot". In this document a general theoretical framework for practical synchronization in networks of diffusively coupled non-identical systems is described.

1 Introduction

We consider a network of $i = 1, 2, \ldots, k$ systems

$$\left\{ \begin{array}{l} \dot{x}_i = f_i(x_i) + B_i u_i \\ y_i = C_i x_i \end{array} \right. \tag{1.1}$$

that interact via diffusive coupling functions

$$u_i = -\sigma \sum_{j=1, j\neq i}^{k} \gamma_{ij}(y_i - y_j), \tag{1.2}$$

with constants $\sigma > 0$ and $\gamma_{ij} \geq 0$ denoting the coupling strength and interaction weights, respectively, $x_i \in \mathbb{R}^n$ is the state of system i, $u_i \in \mathbb{R}^m$ with $1 \leq m \leq n$ is its input, $y_i \in \mathbb{R}^m$ its output, a sufficiently smooth vectorfield $f_i : \mathbb{R}^n \to \mathbb{R}^n$ and constant matrices B_i and C_i are of appropriate dimension. It is assumed that

- $f_i(s) = f(s) + \Delta f_i(s)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\Delta f_i : \mathbb{R}^n \to \mathbb{R}^n$ sufficiently smooth;
- $B_i = B + \Delta B_i$ with $\text{rank}(B_i) = \text{rank}(B) = m$;
- $C_i = C + \Delta C_i$ with $\text{rank}(C_i) = \text{rank}(C) = m$;
- $C_i B_i$ and CB are similar to a positive definite matrix of rank m;
- the matrix

$$\Gamma = \begin{pmatrix} \sum_{j=2}^{k} \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{j=1,j\neq 2}^{k} \gamma_{2j} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\gamma_{(k-1)k} \\ -\gamma_{k1} & \cdots & -\gamma_{k(k-1)} & \sum_{j=1,j\neq 2}^{k-1} \gamma_{kj} \end{pmatrix}$$

  is irreducible.

We remark that Γ is an irreducible matrix if and only if it is the (weighted) Laplacian matrix of a strongly connected digraph [2].

Note that the Hindmarsh-Rose model is written in the form (1.1) and satisfies the assumptions on the system matrices and the coupling functions.

2 Ultimately bounded solutions

Definition 1. The system

$$\begin{cases} \dot{x} = g(x,u) \\ y = h(x) \end{cases} \tag{2.1}$$

with state $x \in \mathbb{R}^n$, inputs and outputs $u, y \in \mathbb{R}^m$, $1 \leq m \leq n$, sufficiently smooth functions $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, h : \mathbb{R}^n \to \mathbb{R}^m$, is called strictly semi-passive with storage function $S : \mathbb{R}^n \to \mathbb{R}_+$ if there exist strictly increasing functions $s_0, s_1, s_2 : \mathbb{R}_+ \to \mathbb{R}_+$, $s_j(0) = 0$ and $\lim_{r\to\infty} s(r) = \infty$, $j = 0, 1, 2$, such that

$$s_0(\|x\|) \leq S(x) \leq s_1(\|x\|)$$

and

$$\dot{S}_{(2.1)}(x) \leq y^\top u - H(x)$$

where $H : \mathbb{R}^n \to \mathbb{R}_+$ satisfies

$$H(x) \geq s_2(\|x\|) - M$$

for some constant $M > 0$.

The Hindmarsh-Rose model of the dynamics of the membrane potential of a neuron is a strictly semi-passive system [7].

Lemma 1. *Suppose that each system* (1.1) *is strictly semi-passive with storage function* $S : \mathbb{R}^n \to \mathbb{R}_+$*, then the solutions of the coupled systems* (1.1), (1.2) *are uniformly ultimately bounded.*

Proof. We let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}.$$

Note that $u = -(\Gamma \otimes I_m)y$ with I_m the $m \times m$ identity matrix. For any irreducible Γ there is a vector ν with strictly positive entries ν_i, $i = 1, \ldots, k$, such that $\nu^\top \Gamma = 0$. (This follows immediately from the Perron-Frobenius theorem, cf. [2].) Consider the function

$$V(x) = \nu_1 S(x_1) + \ldots + \nu_k S(x_k).$$

Then the strict semi-passivity property of the systems implies that

$$\dot{S}_{(1.1)}(x_i) = \left(\frac{\partial S}{\partial x_i}\right)^\top (f_i(x_i) + B_i u_i) \leq y_i^\top u_i - H_i(x_i)$$

such that

$$\begin{aligned} \dot{V}(x) &\leq -\nu_1 H_1(x_1) - \ldots - \nu_k H_k(x_k) - y^\top (\mathrm{diag}(\nu) \otimes I_m) u \\ &= -\nu_1 H_1(x_1) - \ldots - \nu_k H_k(x_k) - \sigma y^\top (\mathrm{diag}(\nu)\Gamma \otimes I_m) y, \end{aligned}$$

where $\mathrm{diag}(\nu)$ is a diagonal matrix with the entries of ν on its diagonal. It is not difficult to show that

$$\mathrm{diag}(\nu)\Gamma + \Gamma^\top \mathrm{diag}(\nu)$$

is positive semi-definite, hence

$$\dot{V}(x) \leq -\nu_1 H_1(x_1) - \ldots - \nu_k H_k(x_k).$$

The properties of H_i imply that

$$\dot{V}(x) \leq s_3(\|x\|)$$

for $\|x\| > R$ for some $R > 0$ and $s_3 \in \mathcal{K}_\infty$. An application of Theorem 4.1.16 of [1] proves the lemma. □

For a constant $c > 0$ we let

$$\Omega_c := \{w \in \mathbb{R}^n | S(w) < c\}.$$

Lemma 2. *Suppose that each system* (1.1) *is strictly semi-passive with storage function* $S : \mathbb{R}^n \to \mathbb{R}_+$. *Let constant* c_0 *be such that for all* $i = 1, 2, \ldots, k$

$$H_i(x_i) > 0$$

on $\mathbb{R}^n \backslash \Omega_{c_0}$. *Then the solutions of the coupled systems* (1.1), (1.2) *converge to the compact set*

$$\Omega_{c_0}^k := \underbrace{\Omega_{c_0} \times \cdots \times \Omega_{c_0}}_{k \text{ times}}.$$

Proof. Let

$$c_1 := \sup_{w \in \Omega_{c_0}} S(w)$$

and

$$\tilde{S}(x_i) := \begin{cases} 0 & \text{if } x_i \in \Omega_{c_0}, \\ S(x_i) - c_1 & \text{otherwise.} \end{cases}$$

Consider

$$\tilde{V}(x) = \nu_1 \tilde{S}(x_1) + \ldots + \nu_k \tilde{S}(x_k).$$

It follows from the arguments of the proof of Lemma 1 that there exists a continuous function $W : \mathbb{R}^{kn} \to [0, \infty)$ that is positive definite with respect to the set[1] $\Omega_{c_0}^k$ such that

$$\dot{\tilde{S}}(x) \leq -W(x).$$

An application of Theorem 4.1.18 of [1] proves the lemma. □

[1] We refer to [1] for a definition of a function to be positive definite with respect to a set.

3 Practical network synchronization

Let, for some constant $\epsilon > 0$,

$$\mathcal{M}_\epsilon = \{x = \mathrm{col}(x_1, x_2 \dots, x_k) \in \mathbb{R}^{kn} |\ \|x_i - x_j\| \leq \epsilon \text{ for all } i, j = 1, 2, \dots, k\}.$$

The set $\mathcal{M}_\epsilon$ is the practical synchronization manifold. The set $\mathcal{M}_0$ is the synchronization manifold; For any solution of (1.1), (1.2) on $\mathcal{M}_0$ the corresponding solutions of the individual systems are indistinguishable.

Definition 2. Let $\phi(\cdot; t_0, x_0)$ denote the unique solution of (1.1), (1.2) through $x_0 \in \mathbb{R}^{kn}$ at $t = t_0$ defined on the interval $[t_0, t_1)$, $t_1 > t_0$. The coupled systems (1.1), (1.2) practically synchronize with bound ϵ if for each $\varepsilon > \epsilon$ there is a $T = T(\varepsilon)$, $T < t_1 - t_0$, such that

$$\phi(t; t_0, x_0) \in \mathcal{M}_\varepsilon \quad \forall t \geq t_0 + T$$

Since each C_iB_i is similar to a positive definite matrix, there is a linear, invertible change of variables

$$x_i \mapsto (z_i, y_i)$$

with $z_i \in \mathbb{R}^{n-m}$ and $y_i \in \mathbb{R}^m$. See [6] for the details. In new coordinates the systems' dynamics read

$$\begin{cases} \dot{z}_i = q_i(z_i, y_i) \\ \dot{y}_i = a_i(z_i, y_i) + C_iB_iu_i \end{cases} \tag{3.1}$$

with sufficiently smooth functions $q_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$ and $a_i : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$. Since $f_i(x_i) = f(x_i) + \Delta f_i(x_i)$ with f and Δf_i sufficiently smooth, $q_i(z_i, y_i) = q(z_i, y_i) + \Delta q_i(z_i, y_i)$ and $a_i(z_i, y_i) = a(z_i, y_i) + \Delta a_i(z_i, y_i)$ with $q, \Delta q_i, a, \Delta a_i$ sufficiently smooth. Without loss of generality we assume that C_iB_i is a diagonal matrix, hence a diagonal matrix with strictly positive entries. In addition we assume that:

Assumption 1. There are sets $\mathcal{Z} \subset \mathbb{R}^{n-m}$ and $\mathcal{Y} \subset \mathbb{R}^m$ such that

$$z_i(t) \in \mathcal{Z}, \quad y_i(t) \in \mathcal{Y}, \quad \forall t \geq t_0 \quad \forall i.$$

It follows from Lemmas 1 and 2 that this assumption is satisfied, possibly after re-defining t_0, if the systems are strictly semi-passive with a common storage function. In addition we assume that:

Assumption 2. There exist a positive definite matrix $P \in \mathbb{R}^{(n-m)\times(n-m)}$ such that the symmetric matrix

$$Q(z_i, y_i) := \left(\frac{\partial q}{\partial z_i}(z_i, y_i)\right)^\top P + P\left(\frac{\partial q}{\partial z_i}(z_i, y_i)\right)$$

is uniformly negative definite on $\mathbb{R}^{n-m} \times \mathcal{Y}^m$, i.e. for any $(z_i, y_i) \in \mathbb{R}^{n-m} \times \mathcal{Y}^m$ the eigenvalues of $Q(z_i, y_i)$ are negative and separated away from zero.

It can easily be shown, cf. [5], that this assumption implies the existence of a positive constant α such that

$$(z_i - z_j)^\top P[q(z_i, y_i) - q(z_j, y_i)] \leq -\alpha \|z_i - z_j\|^2 \quad \forall z_i, z_j \in \mathbb{R}^{n-1} \quad \forall y_i \in \mathcal{Y}.$$

Assumption 2 holds for the Hindmarsh-Rose neuron with P the identity matrix [7]. Thus both assumptions hold for Hindmarsh-Rose neurons.

Theorem 1. *Consider the network of diffusively coupled systems* (3.1), (1.2) *and suppose that assumptions 1 and 2 hold. There exists a constant* $\bar{\sigma} > 0$ *such that for any* $\sigma \geq \bar{\sigma}$ *the network of diffusively coupled systems* (3.1), (1.2) *practically synchronizes with bound* ϵ.

Proof. Let

$$\bar{\Gamma} = \mathrm{diag}(C_1 B_1, \ldots, C_k B_k)(\Gamma \otimes I_m)$$

and note that $\bar{\Gamma}$

- is irreducible since Γ is irreducible and each $C_i B_i$ diagonal;
- has m zero eigenvalues since Γ has a simple zero eigenvalue, hence $\Gamma \otimes I_m$ has m zero eigenvalues;
- has all non-zero eigenvalues in $\mathbb{C}_+$, the open right half-plane of the complex plane.

The second property of $\bar{\Gamma}$ follows from the fact that Γ is the Laplacian matrix of a strongly connected weighted digraph and the multiplicity of the zero eigenvalue equals the number of strongly connected components. The third property of $\bar{\Gamma}$ is a direct consequence of Gerschgorin's Disc theorem, cf. [2].

Let

$$U = \begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{1} & -I_{k-1} \end{pmatrix} \in \mathbb{R}^{k \times k}$$

where $\mathbf{0}$ ($\mathbf{1}$) is a vector of appropriate dimension with all entries equal 0 (1). Then

$$(U \otimes I_m)\bar{\Gamma}(U \otimes I_m)^{-1} = \begin{pmatrix} \mathbf{0} & \star \\ \mathbf{0} & \tilde{\Gamma} \end{pmatrix}$$

with $\tilde{\Gamma} \in \mathbb{R}^{(k-1)m \times (k-1)m}$ and $\star$ denotes some $m \times (k-1)m$ matrix. It is straightforward that the eigenvalues of $\tilde{\Gamma}$ are the non-zero eigenvalues of $\bar{\Gamma}$. Let

$$\tilde{z}_j = z_1 - z_j, \quad \tilde{y}_j = y_1 - y_j, \quad j = 2, \ldots, k.$$

Then

$$\begin{pmatrix} \dot{\tilde{z}}_1 \\ \vdots \\ \dot{\tilde{z}}_{k-1} \end{pmatrix} = \begin{pmatrix} q(z_1, y_1) - q(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ q(z_1, y_1) - q(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix} + \tilde{q}(z_1, y_1 \tilde{z}, \tilde{y})$$

and

$$\begin{pmatrix} \dot{\tilde{y}}_1 \\ \vdots \\ \dot{\tilde{y}}_{k-1} \end{pmatrix} = \begin{pmatrix} a(z_1, y_1) - a(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ a(z_1, y_1) - a(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix} - \tilde{\Gamma} \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_{k-1} \end{pmatrix} + \tilde{a}(z_1, y_1 \tilde{z}, \tilde{y})$$

with $\tilde{z} = \mathrm{col}(\tilde{z}_1, \ldots, \tilde{z}_{k-1})$, $\tilde{y} = \mathrm{col}(\tilde{y}_1, \ldots, \tilde{y}_{k-1})$,

$$\tilde{q}(z_1, y_1 \tilde{z}, \tilde{y}) = \begin{pmatrix} \Delta q_1(z_1, y_1) - \Delta q_2(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ \Delta q_1(z_1, y_1) - \Delta q_k(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix}$$

and

$$\tilde{a}(z_1, y_1 \tilde{z}, \tilde{y}) = \begin{pmatrix} \Delta a_1(z_1, y_1) - \Delta a_2(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ \Delta a_1(z_1, y_1) - \Delta a_k(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix}.$$

Consider the positive definite function

$$V(\tilde{z}, \tilde{y}) = \tilde{z}^\top (I_k \otimes P)\tilde{z} + \tilde{y}^\top P_1 \tilde{y}$$

with P as in Assumption 1 and positive definite matrix P_1 is such that $\|P_1\| = 1$[2] and

$$-\tilde{\Gamma} P_1 - P_1 \tilde{\Gamma}^\top \leq -\eta I$$

for some positive constant η. The existence of such matrix P_1 is guaranteed by the fact the $-\tilde{\Gamma}$ is a stable (Hurwitz) matrix.

Assumption 1 and smoothness of the functions q_i, a_i implies the existence of positive constants B_q and B_a such that

$$\|(I_k \otimes P)\tilde{q}(z_1, y_1 \tilde{z}, \tilde{y})\| \leq B_q, \quad \|P\tilde{a}(z_1, y_1 \tilde{z}, \tilde{y})\| \leq B_a.$$

Assumptions 1, 2 and smoothness of the function q implies the existence of positive constants κ_0, κ_1 such that

$$\tilde{z}^\top (I_k \otimes P) \begin{pmatrix} q(z_1, y_1) - q(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ q(z_1, y_1) - q(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix} \leq -\kappa_0 \|\tilde{z}\|^2 + \kappa_1 \|\tilde{z}\| \|\tilde{y}\|.$$

Assumption 1 and smoothness of the function a implies the existence of positive constants κ_2, κ_3 such that

$$\tilde{y}^\top P_1 \begin{pmatrix} a(z_1, y_1) - a(z_1 - \tilde{z}_1, y_1 - \tilde{y}_1) \\ \vdots \\ a(z_1, y_1) - a(z_1 - \tilde{z}_{k-1}, y_1 - \tilde{y}_{k-1}) \end{pmatrix} \leq \kappa_2 \|\tilde{z}\| \|\tilde{y}\| + \kappa_3 \|\tilde{y}\|^2.$$

Thus

$$\begin{aligned} \dot{V}(\tilde{z}, \tilde{y}) \leq -\kappa_0 \|\tilde{z}\|^2 + \kappa_1 \|\tilde{z}\| \|\tilde{y}\| + \kappa_2 \|\tilde{z}\| \|\tilde{y}\| \\ + (\kappa_3 - \sigma\eta) \|\tilde{y}\|^2 + B_q \|\tilde{z}\| + B_a \|\tilde{y}\| \\ = - \begin{pmatrix} \tilde{z} \\ \tilde{y} \end{pmatrix}^\top W \begin{pmatrix} \tilde{z} \\ \tilde{y} \end{pmatrix} + B_q \|\tilde{z}\| + B_a \|\tilde{y}\| \end{aligned}$$

with

$$W = \begin{pmatrix} \kappa_0 & -\frac{\kappa_1+\kappa_2}{2} \\ -\frac{\kappa_1+\kappa_2}{2} & \sigma\eta - \kappa_3 \end{pmatrix}.$$

Note that W is positive definite for

$$\sigma > \bar{\sigma} = \frac{1}{\eta}\left(\kappa_3 + \frac{(\kappa_1+\kappa_2)^2}{4\kappa_0}\right).$$

[2]Given a matrix A, $\|A\| = \max_{\|x\|=1} \|Ax\|$.

Thus there is a positive constant $\mu = \mu(\sigma) = \lambda_{\min}(W)$ such that for $\sigma > \bar{\sigma}$

$$\dot{V}(\tilde{z}, \tilde{y}) \le -\mu(\|\tilde{z}\|^2 + \|\tilde{y}\|^2) + B_q \|\tilde{z}\| + B_a \|\tilde{y}\|$$

such that

$$\dot{V}(\tilde{z}, \tilde{y}) \le -\mu(1-\rho)(\|\tilde{z}\|^2 + \|\tilde{y}\|^2), \quad \rho \in (0,1),$$

if

$$\|\tilde{z}\| = \frac{B_q}{\rho\mu} \text{ and } \|\tilde{y}\| = \frac{B_a}{\rho\mu}.$$

Standard techniques, cf. Lemma 9.2 of [3], imply that for any $\sigma > \bar{\sigma}$ the systems practically synchronize with bound

$$\epsilon = \sqrt{\frac{\lambda_{\max}(\mathrm{diag}(I_k \otimes P), P_1)}{\lambda_{\min}(\mathrm{diag}(I_k \otimes P), P_1)} \frac{B_q^2 + B_a^2}{\rho^2\mu^2}},$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the smallest, respectively, largest eigenvalue of a symmetric matrix. □

4 Practical output-synchronization of two coupled systems

Consider two systems (1.1) that interact via coupling

$$\begin{aligned} u_1 &= \sigma(1-\eta)[y_2 - y_1] \\ u_2 &= \sigma\eta[y_1 - y_2] \end{aligned} \tag{4.1}$$

with constant $\eta \in (0,1)$.

Definition 3. Let $y_i(\cdot; t_0, x_0)$, $i = 1, 2$, denote the unique output solution of (1.1), (4.1) through $x_0 \in \mathbb{R}^{2n}$ at $t = t_0$ defined on the interval $[t_0, t_1)$, $t_1 > t_0$. The coupled systems (1.1), (4.1) practically output-synchronize with bound ϵ_y if for each $\varepsilon_y > \epsilon_y$ there is a $T = T(\varepsilon_y)$, $T < t_1 - t_0$, such that

$$|y_1(t; t_0, x_0) - y_2(t; t_0, x_0)| < \varepsilon_y \quad \forall t \ge t_0 + T$$

Of course, whenever the conditions of Lemma 1 and Theorem 1 are satisfied the two coupled systems (1.1), (4.1) practically output-synchronize with some bound ϵ_y, provided that $\sigma > \bar{\sigma}$.

We assume for the systems (1.1)

$$C_1 B_1 = C_2 B_2$$

and let

$$\begin{aligned} \tilde{z} &= z_1 - z_2, \\ \tilde{y} &= y_1 - y_2. \end{aligned}$$

Then we get

$$\begin{aligned} \dot{\tilde{z}} &= q_1(z_1, y_i) - q_2(z_2, y_2), \\ \dot{\tilde{y}} &= a_1(z_1, y_i) - a_2(z_2, y_2) - \sigma\tilde{y}. \end{aligned}$$

We observe that these dynamics, which describe the synchronization error, do not depend on the weight parameter η. Consider

$$V(\tilde{y}) = \tfrac{1}{2}\tilde{y}^2.$$

Using similar reasoning as in the proof of Theorem 1, we deduct that

$$\dot{V}(\tilde{y}) \leq c_1 + c_2\tilde{y} - \sigma\tilde{y}^2$$

where constants c_1 and c_2 depend on the functions $q_1(z_1, y_i) - q_2(z_2, y_2)$ and $a_1(z_1, y_i) - a_2(z_2, y_2)$ given the bounds of the solutions $y_i(\cdot)$ and $z_i(\cdot)$. A direct application of LaSalle's invariance principle, cf. [4], shows that we can find a $\bar{\sigma} > 0$ such that if $\sigma > \bar{\sigma}$ then $\dot{V}(\tilde{y}) < 0$ for all $\|\tilde{y}\| > \epsilon_y = \epsilon_y(\sigma)$ and, moreover, the larger σ the smaller ϵ_y with

$$\lim_{\sigma\to\infty} \epsilon_y(\sigma) = 0.$$

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