On the expected number of equilibria in a multi-player multi-strategy evolutionary game

On the expected number of equilibria in a multi-player multi-strategy evolutionary game

Manh Hong Duong[α] and The Anh Han[β,γ]

August 19, 2014

[α] Department of Mathematics and Computer Science, Eindhoven University of Technology, Den Dolech 2, PO Box 513, 5600 MB Eindhoven, The Netherlands. Email: m.h.duong@tue.nl

[β] AI lab, Computer Science Department, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussels, Belgium. Email: h.anh@ai.vub.ac.be

[γ] MLG, Département d'Informatique, Université Libre de Bruxelles, Boulevard du Triomphe CP212, 1050 Brussels, Belgium

Abstract

In this paper, we analyze the mean number $E(n,d)$ of internal equilibria in a general d-player n-strategy evolutionary game where the agents' payoffs are normally distributed. First, we give a computationally implementable formula for the general case. Next we characterize the asymptotic behavior of $E(2,d)$, estimating its lower and upper bounds as d increases. Then we provide an exact formula for $E(n,2)$. As a consequence, we show that in both cases the probability to see the maximal possible number of equilibria tends to zero when d or n respectively goes to infinity. Finally, for larger n and d, numerical results are provided and discussed.

Keywords: Multi-player game theory, multiple strategies, random polynomials, number of equilibria, random games.

1 Introduction

Evolutionary game theory is the suitable mathematical framework whenever there is frequency dependent selection – the fitness of an individual does not only depend on its strategy, but also on the composition of the population in relation with (multiple) other strategies [MS82, HS98, Now06]. The payoff from the games is interpreted as individual fitness, naturally leading to a dynamical approach. As in classical game theory with the Nash equilibrium [MB05, McL05], the analysis of properties of equilibrium points in evolutionary game theory has been of special interest [MS82, BCV97, GT10]. Herein, equilibrium points of a dynamical system predict the composition of strategy frequencies where all the strategies have the same average fitness. Biologically, these points can predict a co-existence of different types in a population and the maintenance of polymorphism.

Recently, decent attention has been paid to both numerical and analytical studies of equilibrium points in random evolutionary games [GT10, HTG12]. The focus was on analyzing the probability of observing a certain number of equilibria if the payoff entries are randomly drawn. This probability allows one to predict the complexity of the interactions as the number of strategies and the number of players in the game increase, especially when the environments are unknown or changing rapidly over time [FH92, GRLD09]. Furthermore, these studies have paid substantial attention to the maximal number of equilibrium points, as knowing it is insightful, and historically it has been studied extensively, not only in classical and evolutionary game theory, but also in other fields such as population genetics [MS82, Kar80, VC88a, VC88b, KF70, VC88c, BCV93, BCV97, Alt10, GT10, HTG12]. However, as it deals with the concrete numbers of equilibrium points, the studies have needed to take a direct approach that consists in solving a system of polynomial equations, the degree of which increases with the number of players in a game. As such, the mathematical analysis was mostly restricted to evolutionary games with a small number of players, due to the impossibility of solving general polynomial equations of a high degree [Abe24, HTG12].

In this paper, we ask instead the question: what is the mean or expected number of equilibria that one can observe if the payoff matrix entries of the game are randomly drawn? Knowing the mean number of equilibria not only gives important insights into the overall complexity of the interactions as the number of participating players in the game and the potential strategies the players can adopt are magnified. It also enables us to predict the boundaries of the concrete numbers of equilibrium points such as the maximal one as we show later on in the paper. By connecting to the theory of random polynomials [EK95], we first provide an exact, computationally implementable, formula for the expected number of equilibria in a general multi-player multi-strategy random games when the payoff entries are normally distributed. Secondly, we derive lower and upper bounds of such a formula for the case of two-player games and provide an explicit formula for the case of two-strategy games. Finally, numerical results are provided and discussed when there more players and strategies.

The rest of the paper is structured as follows. In Section 2 we introduce the models and methods: the replicator equation in evolution game theory and the random polynomial theory are summarized in Sections 2.1 and 2.2, respectively. The connection between them, which is the method of this paper, is described in Section 2.3. The main results of this paper are presented in Section 3, starting with two-strategy games in Section 3.1, then with two-player games in Section 3.2, and lastly, with the general case of games with arbitrary numbers of players and strategies in Section 3.3. We compare our results with related ones in the literature and discuss on future perspective in Section 4. Finally, some detailed computations are given in the Appendix.

2 Models and Methods

2.1 Evolutionary game theory and replicator dynamics

The classical approach to evolutionary games is replicator dynamics [TJ78, Zee80, HS98, SS83, Now06], describing that whenever a strategy has a fitness larger the average fitness of the population, it is expected to spread. Formally, let us consider an infinitely large population with n strategies, numerated from 1 to n. They have frequencies x_i, $1 \leq i \leq n$, respectively, satisfying that $0 \leq x_i \leq 1$ and $\sum_{i=1}^{n} x_i = 1$. The interaction of the individuals in the population is in groups of d participants, that is, they play and obtain their fitness from d-player games. We consider here symmetrical games (e.g. the public goods and the common-pool resource game) in which the order of the participants is irrelevant. Let $\alpha^{i_0}_{i_1,\ldots,i_{d-1}}$ be the payoff of the focal player, where i_0 ($1 \leq i_0 \leq n$) is the strategy of the focal player, and let i_k (with $1 \leq i_k \leq n$ and $1 \leq k \leq d-1$) be the strategy of the player in position k. These payoffs form a $(d-1)$-dimensional payoff matrix [GT10], denoted by Π, which satisfies that (because the game symmetry)

$$\alpha^{i_0}_{i_1,\ldots,i_{d-1}} = \alpha^{i_0}_{i'_1,\ldots,i'_{d-1}}, \tag{1}$$

whenever $\{i'_1 \ldots, i'_{d-1}\}$ is a permeation of $\{i_1 \ldots, i_{d-1}\}$. This means only the fraction of each strategy in the game matters.

The average payoff or fitness of the focal player is given by

$$\pi_{i_0} = \sum_{i_1,\ldots,i_{d-1}=1}^{n} \alpha^{i_0}_{i_1,\ldots,i_{d-1}} \prod_{k=1}^{d-1} x_{i_k}. \tag{2}$$

By abuse of notation, let us denote $\alpha^{i_0}_{k_1,\ldots,k_n} := \alpha^{i_0}_{i_1,\ldots,i_{d-1}}$, where k_i, $1 \leq i \leq n$, with $\sum_{i=1}^{n} k_i = d-1$, is the number of players using strategy i in $\{i_1, \ldots, i_{d-1}\}$. Hence, from Eq. (1), the fitness of strategy i_0 can be rewritten as follows

$$\pi_{i_0} = \sum_{\substack{0 \leq k_1,\ldots,k_n \leq d-1, \\ \sum\limits_{i=1}^{n} k_i = d-1}} \alpha^{i_0}_{k_1,\ldots,k_n} \binom{d-1}{k_1,\ldots,k_n} \prod_{i=1}^{n} x_i^{k_i} \quad \text{for } i_0 = 1,\ldots,n, \tag{3}$$

where $\binom{d-1}{k_1,\ldots,k_n} = \frac{(d-1)!}{\prod\limits_{k=1}^{n} k_i!}$ are the multinomial coefficients.

Now the replicator equations for games with n strategies can be written as follows [HS98, Sig10]

$$\dot{x_i} = x_i\,(\pi_i - \langle \pi \rangle) \qquad \text{for } i = 1, \dots, n-1, \tag{4}$$

where $\langle \pi \rangle = \sum_{k=1}^{n} x_k\,\pi_k$ is the average payoff of the population. The equilibrium points of the system are given by the points $(x_1, \dots, x_n)$ satisfying that the fitness of all strategies are the same. That is, they are represented by solutions of the system of equations

$$\pi_i = \pi_n \quad \text{for all } 1 \le i \le n-1. \tag{5}$$

Subtracting from each of the equations the term π_n we obtain a system of $n-1$ polynomials of degree $d-1$

$$\sum_{\substack{0\le k_1,\dots,k_n\le d-1,\\ \sum\limits_{i=1}^{n} k_i = d-1}} \beta^i_{k_1,\dots,k_n} \binom{d-1}{k_1,\dots,k_n} \prod_{i=1}^{n} x_i^{k_i} = 0 \quad \text{for } i = 1,\dots,n-1. \tag{6}$$

where $\beta^i_{k_1,\dots,k_n} := \alpha^i_{k_1,\dots,k_n} - \alpha^n_{k_1,\dots,k_n}$. Assuming that all the payoff entries have the same probability distribution, then all $\beta^i_{k_1,\dots,k_n}$, $i = 1, \dots, n-1$, have symmetric distributions, i.e. with mean 0 (see also proof in [HTG12]). In particular, if $\alpha^i_{k_1,\dots,k_n}$ are normal distributions with mean 0 and variance C then $\beta^i_{k_1,\dots,k_n}$ are normal distributions with mean 0 and variance $2C$.

In the following analysis, we focus only on internal equilibrium points [GT10, HTG12], i.e. $0 < x_i < 1$ for all $1 \le i \le n-1$. Hence, by using the transformation $y_i = \frac{x_i}{x_n}$, with $0 < y_i < +\infty$ $(1 \le i \le n-1)$, dividing the left hand side of the above equation by x_n^{d-1} we obtain the following equation in terms of $(y_1, \dots, y_{n-1})$ that is equivalent to (6)

$$\sum_{\substack{0\le k_1,\dots,k_{n-1}\le d-1,\\ \sum\limits_{i=1}^{n-1} k_i \le d-1}} \beta^i_{k_1,\dots,k_{n-1}} \binom{d-1}{k_1,\dots,k_n} \prod_{i=1}^{n-1} y_i^{k_i} = 0 \qquad \text{for } i = 1,\dots,n-1. \tag{7}$$

As stated, the goal of this article is to compute the expected number of (internal) equilibria in a general n-strategy d-player random evolutionary game. That consists in computing the expected number of solutions $(y_1, \dots, y_{n-1}) \in {\mathbf{R}_+}^{n-1}$ of the system of $(n-1)$ polynomials of degree $(d-1)$ in (7). Furthermore, herein our analysis focuses on the case where the payoff matrix entries have identically normal distributions. It is known that, even for $n = 2$ it is impossible to analytically solve the system whenever $d > 5$ [Abe24], as seen in [HTG12]. Hence, it is not feasible to use this direct approach of analytically solving the system if one wants to deal with the games with a large number of players and with multiple strategies. In this work, we address this issue by connecting to the theory of random polynomials described in the following section.

2.2 Random polynomial theory

Keeping the form of Eq. (7) in mind, we consider a system of $n-1$ random polynomials of degree $d-1$,

$$\sum_{\substack{0\le k_1,\dots,k_{n-1}\le d-1,\\ \sum_{i=1}^{n-1}k_i\le d-1}} a^i_{k_1,\dots,k_{n-1}}\prod_{i=1}^{n-1}y_i^{k_i}=0 \qquad \text{for } i=1,\dots,n-1, \tag{8}$$

where $a^i_{k_1,\dots,k_{n-1}}$ are independent and identically distributed (i.i.d.) multivariate normal random vectors with mean zero and covariance matrix C. Denote by $v(y)$ the vector whose components are all the monomials $\{\prod_{i=1}^{n-1}y_i^{k_i}\}$ where $0\le k_i\le d-1$ and $\sum_{i=1}^{n-1}k_i\le d-1$. Let A denote the random matrix whose $i-$th row contains all coefficients $a^i_{k_1,\dots,k_{n-1}}$. Then (8) can be re-written as

$$Av(y)=0. \tag{9}$$

The following theorem is the starting point of the analysis of this paper.

Theorem 2.1. *[EK95, Theorem 7.1] Let U be any measurable subset of $\mathbb{R}^{n-1}$. Assume that the rows of A are iid multivariate normal random vectors with mean zero and covariance matrix C. The expected number of real roots of the system of equations (9) that lie in the set U is given by*

$$\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\int_U\left(\det\left[\frac{\partial^2}{\partial x_i\partial y_j}(\log v(x)^TCv(y))\Big|_{y=x=t}\right]_{ij}\right)^{\frac{1}{2}}dt.$$

2.3 From random polynomial theory to evolutionary game theory

Let $E(n,d)$ be the number of internal equilibria in a $d-$player random game with n strategies. As has been shown in Section 2.1, $E(n,d)$ is the same as the number of positive solutions of Equation (15). Suppose for now that all $\beta^i_{k_1,\dots,k_n}$ are independent Gaussian distributions with mean 0 and variance 1, then for each i, $\beta^i_{k_1,\dots,k_{n-1}}\begin{pmatrix}d-1\\k_1,\dots,k_n\end{pmatrix}$ are i.i.d. multivariate normal random vectors with mean zero and covariance matrix C given by

$$C=\operatorname{diag}\left(\begin{pmatrix}d-1\\k_1,\dots,k_n\end{pmatrix}^2\right)_{0\le k_i\le d-1,\ \sum\limits_{i=1}^{n-1}k_i\le d-1}. \tag{10}$$

Now we can apply Theorem 2.1 with $a^i_{k_1,\dots,k_{n-1}}=\beta^i_{k_1,\dots,k_{n-1}}\begin{pmatrix}d-1\\k_1,\dots,k_n\end{pmatrix}$ and $U=[0,\infty)^{n-1}\subset\mathbf{R}^{n-1}$. We obtain the following lemma.

Lemma 2.2. *The expected number of internal equilibria in a* d*-player* n*-strategy random game is* *given by*

$$E(n,d) = \pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\underbrace{\int_0^\infty\dots\int_0^\infty}_{n-1\ times}\left(\det\left[\frac{\partial^2}{\partial x_i y_j}(\log v(x)^T C v(y))\big|_{y=x=t}\right]_{ij}\right)^{\frac{1}{2}} dt, \tag{11}$$

where

$$v(x)^T C v(y) = \sum_{\substack{0\leq k_1,\dots,k_{n-1}\leq d-1,\\ \sum_{i=1}^{n-1} k_i\leq d-1}} \begin{pmatrix} d-1 \\ k_1,\dots,k_n\end{pmatrix}^2 \prod_{i=1}^n x_i^{k_i} y_i^{k_i}. \tag{12}$$

Denote by L the matrix with entries

$$L_{ij} = \frac{\partial^2}{\partial x_i y_j}(\log v(x)^T C v(y))\big|_{y=x=t},$$

then $E(n,d)$ can be written as

$$E(n,d) = \pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\underbrace{\int_0^\infty\dots\int_0^\infty}_{n-1\text{ times}}(\det L)^{\frac{1}{2}}\, dt. \tag{13}$$

It has been shown that a d-player n-strategy game has at most $(d-1)^{n-1}$ isolated internal equilibria (and this bound is sharp) [HTG12]. We denote by p_i, $1\leq i\leq (d-1)^{n-1}$, the probability that the game has exactly i such equilibria. Then $E(n,d)$ can also be defined through p_i as follows

$$E(n,d) = \sum_{i=1}^{(d-1)^{n-1}} i\cdot p_i. \tag{14}$$

3 Results

We start with the case where there are two strategies ($n = 2$), analytically deriving the upper and lower bounds for $E(2,d)$. Next we derive exact results for games with two players ($d = 2$). Finally, we provide numerical results and discussion for the general case with arbitrary number of players and strategies. We start by considering that the coefficients $\beta^i_{k_1,\dots,k_n}$ are standard normal distributions, and at the end show that the results do not change if these are arbitrary normal distributions [1].

3.1 Multi-player two-strategy games

We first consider games with arbitrary number of players, but having only two strategies, i.e. $n = 2$. In this case, Eq. (7) is simplified to the following univariate polynomial equation of degree

[1]For simplicity, we only consider normal distributions with mean zero since normal distributions with mean non-zero can also be shifted to the former case.

$d-1$ with $y \in \mathbf{R}_+$

$$\sum_{k=0}^{d-1} \beta_k \binom{d-1}{k} y^k = 0. \tag{15}$$

The following lemma provides a formula for $E(2,d)$.

Lemma 3.1. *Assume that β_k are independent Gaussian distributions with variance 1 and mean 0. Then the number of internal equilibria, $E(2,d)$, in a $d-$player random game with two strategies is given by*

$$E(2,d) = \int_0^\infty f(t)\,dt, \tag{16}$$

where

$$f(t) = \frac{1}{\pi}\left[\frac{\sum\limits_{k=1}^{d-1} k^2 \binom{d-1}{k}^2 t^{2(k-1)}}{\sum\limits_{k=0}^{d-1}\binom{d-1}{k}^2 t^{2k}} - \left(\frac{\sum\limits_{k=1}^{d-1} k \binom{d-1}{k}^2 t^{2k-1}}{\sum\limits_{k=0}^{d-1}\binom{d-1}{k}^2 t^{2k}}\right)^2\right]^{\frac{1}{2}}. \tag{17}$$

Proof. Since β_k has Gaussian distribution with variance 1 and mean 0, $\beta_k \binom{d-1}{k}$ has Gaussian distribution with variance $\binom{d-1}{k}^2$ and mean 0. According to Lemma 2.2, the equality (16) holds with

$$f(t) = \frac{1}{\pi}\left[\frac{\partial^2}{\partial x \partial y}\left(\log v(x)^T C v(y)\right)\Big|_{y=x=t}\right]^{\frac{1}{2}}, \tag{18}$$

where the vector v and the matrix C (covariance matrix) are given by

$$v(x) = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{d-1} \end{pmatrix}, \quad C = (C_{ij})_{i,j=1,\ldots,d-1} \quad \text{with} \quad C_{ij} = \delta_{ij}\binom{d-1}{i}\binom{d-1}{j}. \tag{19}$$

A straightforward calculation gives

$$v(x)^T C v(y) = \sum_{k=0}^{d-1}\binom{d-1}{k}^2 x^k y^k,$$

and

$$\frac{\partial^2}{\partial x \partial y}\left(\log v(x)^T C v(y)\right) = \frac{\sum\limits_{k=1}^{d-1} k^2\binom{d-1}{k}^2 x^{k-1}y^{k-1}}{\sum\limits_{k=0}^{d-1}\binom{d-1}{k}^2 x^k y^k} - \frac{\left(\sum\limits_{k=1}^{d-1} k\binom{d-1}{k}^2 x^k y^{k-1}\right)\left(\sum\limits_{k=1}^{d-1} k\binom{d-1}{k}^2 x^{k-1} y^k\right)}{\left(\sum\limits_{k=0}^{d-1}\binom{d-1}{k}^2 x^k y^k\right)^2}.$$

Substituting this expression into (18), we obtain (17). ◻

Example 3.1. For the cases $d = 2$ and $d = 3$, we have

$$E(2,2) = \frac{1}{\pi}\int_0^\infty \frac{1}{1+t^2}\,dt = \frac{1}{\pi}\lim_{t\to+\infty}\tan^{-1}(t) = 0.5,$$
$$E(2,3) = \frac{2}{\pi}\int_0^\infty \frac{\sqrt{t^4+t^2+1}}{t^4+4t^2+1}\,dt \approx 0.77.$$

The following proposition presents some properties of the density function $f(t)$.

Proposition 3.2. *The following properties hold*

1)

$$f(t) = \frac{d-1}{\pi}\frac{\sqrt{\sum\limits_{k=0}^{2d-4} a_k\, t^{2k}}}{\sum\limits_{k=0}^{d-1}\binom{d-1}{k}^2 t^{2k}}, \tag{20}$$

where

$$a_k = \sum_{\substack{1\le i\le d-1\\ 1\le j\le d-2\\ i+j=k}} \binom{d-1}{i}^2\binom{d-2}{j}^2 - \sum_{\substack{1\le i'\le d-2\\ 1\le j'\le d-2\\ i'+j'=k}} \binom{d-2}{i'}\binom{d-1}{i'+1}\binom{d-2}{j'}\binom{d-1}{j'+1}, \tag{21}$$

$$a_k = a_{2d-4-k}, \quad \textit{for all}\ \ 0\le k\le 2d-4, \quad a_0 = a_{2d-4} = 1, \quad a_k \ge 1. \tag{22}$$

2) $f(0) = \frac{d-1}{\pi}, \quad f(1) = \frac{d-1}{2\pi}\frac{1}{\sqrt{2d-3}}.$

3) $f(t) = \frac{1}{2\pi}\left[\frac{1}{t}G'(t)\right]^{\frac{1}{2}}$, *where*

$$G(t) = t\frac{\frac{d}{dt}M_d(t)}{M_d(t)} = t\frac{d}{dt}\log M_d(t).$$

4) $t \mapsto f(t)$ *is a decreasing function.*

5) $f\left(\frac{1}{t}\right) = t^2 f(t)$.

6)

$$E(2,d) = 2\int_0^1 f(t)dt = 2\int_1^\infty f(t)\,dt. \tag{23}$$

Proof. 1) Set

$$M_d(t) = \sum_{k=0}^{d-1}\binom{d-1}{k}^2 t^{2k}, \quad A_d(t) = \sum_{k=1}^{d-1}k^2\binom{d-1}{k}^2 t^{2(k-1)}, \quad B_d(t) = \sum_{k=1}^{d-1}k\binom{d-1}{k}^2 t^{2k-1}. \tag{24}$$

Then

$$f(t) = \frac{1}{\pi}\frac{\sqrt{A_d(t)M_d(t) - B_d(t)^2}}{M_d(t)}.$$

Using

$$k\binom{d-1}{k}=(d-1)\binom{d-2}{k-1},$$

we can transform

$$A_d(t)=(d-1)^2\sum_{k=1}^{d-1}\binom{d-2}{k-1}^2 t^{2(k-1)}=(d-1)^2\sum_{k=0}^{d-2}\binom{d-2}{k}^2 t^{2k}=(d-1)^2M_{d-1}(t),$$
$$B_d(t)=(d-1)\sum_{k=1}^{d-1}\binom{d-2}{k-1}\binom{d-1}{k}t^{2k-1}=(d-1)t\sum_{k=0}^{d-2}\binom{d-2}{k}\binom{d-1}{k+1}t^{2k}.$$

Therefore,

$$A_d(t)M_d(t)-B_d(t)^2=(d-1)^2\left[\left(\sum_{k=0}^{d-2}\binom{d-2}{k}^2t^{2k}\right)\left(\sum_{k=0}^{d-1}\binom{d-1}{k}^2t^{2k}\right)-t^2\left(\sum_{k=0}^{d-2}\binom{d-2}{k}\binom{d-1}{k+1}t^{2k}\right)^2\right]$$
$$=(d-1)^2\sum_{k=0}^{2d-4}a_k\,t^{2k},$$

where

$$a_k=\sum_{\substack{0\le i\le d-1\\0\le j\le d-2\\ i+j=k}}\binom{d-1}{i}^2\binom{d-2}{j}^2-\sum_{\substack{0\le i'\le d-2\\0\le j'\le d-2\\ i'+j'=k-1}}\binom{d-2}{i'}\binom{d-1}{i'+1}\binom{d-2}{j'}\binom{d-1}{j'+1}.$$

For the detailed computations of a_k and the proof of (22), see Appendix 5.1.

2) The value of $f(0)$ is found directly from (20). For the detailed computations of $f(1)$, see Appendix 5.2.

3) It follows from (24) that

$$B_d(t)=\frac{1}{2}M_d'(t),\quad A_d(t)=\frac{1}{4t}(tM_d'(t))', \tag{25}$$

where $'$ is derivative w.r.t t. Hence

$$f(t)=\frac{1}{2\pi}\left(\frac{\frac{1}{t}(tM_d'(t))'M_d(t)-M_d'(t)^2}{M_d(t)^2}\right)^{\frac{1}{2}}$$
$$=\frac{1}{2\pi}\left(\frac{(tM_d'(t))'M_d(t)-tM_d'(t)^2}{tM_d(t)^2}\right)^{\frac{1}{2}}$$
$$=\frac{1}{2\pi}\left(\frac{1}{t}G'(t)\right)^{\frac{1}{2}},$$

where $G(t)=t\frac{M_d'(t)}{M_d(t)}$.

4) Since $M_d(t)$ contains only even powers of t with positive coefficients, all of its roots are purely

imaginary. Suppose that

$$M_d(t) = \prod_{i=1}^{d-1}(t^2 + r_i),$$

where $r_i > 0$ for all $1 \le i \le d-1$. It follows that

$$G(t) = t\frac{M_d'(t)}{M_d(t)} = \sum_{i=1}^{d-1} \frac{2t^2}{t^2 + r_i},$$

and hence

$$(2\pi f(t))^2 = \frac{1}{t}G'(t) = \sum_{i=1}^{d-1} \frac{4r_i}{(t^2 + r_i)^2}.$$

Since $r_i > 0$ for all $i = 1, \ldots, d-1$, the above equality implies that $f(t)$ is decreasing in $t \in [0, \infty)$.

5) Set

$$g(t) := \sqrt{A_d(t)M_d(t) - B_d(t)^2},$$

Then

$$f(t) = \frac{1}{\pi}\frac{g(t)}{M_d(t)}. \tag{26}$$

It follows from the symmetric properties of the binomial coefficients that

$$M_d\left(\frac{1}{t}\right) = \frac{1}{t^{2(d-1)}} M_d(t).$$

Similarly, from (22) we have

$$g\left(\frac{1}{t}\right) = \frac{1}{t^{2(d-2)}} g(t).$$

Therefore

$$f\left(\frac{1}{t}\right) = \frac{1}{\pi}\frac{g(1/t)}{M_d(1/t)} = \frac{1}{\pi}t^2\frac{g(t)}{M_d(t)} = t^2 f(t).$$

6) By change of variable, $s = \frac{1}{t}$, and from 3), we have

$$\int_1^\infty f(t)\,dt = \int_1^0 f\left(\frac{1}{s}\right)\frac{-1}{s^2}\,ds = \int_0^1 f(s)\,ds.$$

Therefore

$$E(2,d) = \int_0^\infty f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^\infty f(t)\,dt = 2\int_0^1 f(t)\,dt.$$

□

Remark 3.3. We provide an alternative proof of the fifth property in the above lemma in Appendix 5.3. □

Remark 3.4. Besides enabling a significantly less complex numerical computation of $E(2,d)$ (see already our numerical results using this formula in Table 1), the equality (23) reveals an interesting property: the expected number of zeros of the polynomial $P(y)$ in two intervals $(0,1)$ and $(1,\infty]$

are the same. Equivalently, the expected numbers of internal equilibria in two interval $(0, \frac{1}{2}]$ and $(\frac{1}{2}, 1)$ are equal since $y = \frac{x}{1-x}$. □

Based on the analytical formula of $E(2,d)$, we now provide upper and lower bounds for the mean number of equilibria as the payoff entries of the game are randomly drawn.

Theorem 3.5. *$E(2,d)$ satisfies the following estimate*

$$\frac{d-1}{2\pi\sqrt{2d-3}} \leq E(2,d) \leq \frac{1}{\pi}\sqrt{d-1}\sqrt{1+\frac{\pi}{2}\sqrt{d-1}}. \tag{27}$$

Proof. Since $f(t)$ is decreasing, we have $f(0) \geq f(t) \geq f(1)$ for $t \in [0,1]$. As a consequence,

$$E(2,d) \geq 2\int_0^1 f(1)\,dt = 2f(1) = \frac{d-1}{\pi\sqrt{2d-3}}.$$

To obtain the upper bound, we proceed as follows.

$$\begin{aligned}
E(2,d)^2 &= 4\left(\int_0^1 f(t)\,dt\right)^2\\
&\leq 4\int_0^1 f(t)^2\,dt \quad \text{(by Jensen's inequality)}\\
&= \frac{1}{\pi^2}\int_0^1 \frac{1}{t}G'(t)\,dt\\
&= \frac{1}{\pi^2}\int_0^1 \sum_{i=1}^{d-1}\frac{4r_i}{(t^2+r_i)^2}\,dt\\
&= \frac{1}{\pi^2}\sum_{i=1}^{d-1}\int_0^1 \frac{4r_i}{(t^2+r_i)^2}\,dt\\
&= \frac{1}{\pi^2}\sum_{i=1}^{d-1}\left(\frac{2}{r_i+1}+\frac{\cot^{-1}(\sqrt{r_i})}{\sqrt{r_i}}\right) \qquad (28)
\end{aligned}$$

Note that if $ab = 1$, then

$$\frac{1}{a+1}+\frac{1}{b+1} = 1. \tag{29}$$

We observe that if z is a zero of $M_d(t)$, then $\frac{1}{z}$ is also a zero because $M_d(t) = t^{2(d-1)}M_d(1/t)$. This implies that the sequence $\{r_i, i = 1, \ldots, d-1\}$ can be grouped into $\frac{d-1}{2}$ pairs of the form $\left(a, \frac{1}{a}\right)$. Using (29), we obtain

$$\frac{1}{\pi^2}\sum_{i=1}^{d-1}\frac{2}{r_i+1} = \frac{1}{\pi^2}(d-1). \tag{30}$$

For the second term, since $\cot^{-1}(z) \leq \frac{\pi}{2}$ for all $z \geq 0$, we have

$$\begin{aligned}
\sum_{i=1}^{d-1} \frac{\cot^{-1}(\sqrt{r_i})}{\sqrt{r_i}} &\leq \frac{\pi}{2} \sum_{i=1}^{d-1} \frac{1}{\sqrt{r_i}} \\
&= \frac{\pi}{2} \frac{\sum\limits_{i=1}^{d-1} \prod\limits_{j\neq i} \sqrt{r_j}}{\prod\limits_{i=1}^{d-1} \sqrt{r_i}} \\
&= \frac{\pi}{2} \sum_{i=1}^{d-1} \prod_{j\neq i} \sqrt{r_j} \\
&\leq \frac{\pi}{2} \sqrt{(d-1) \sum_{i=1}^{d-1} \prod_{j\neq i} r_j} \\
&= \frac{\pi}{2} (d-1)^{\frac{3}{2}}, \qquad (31)
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} b_i \right)^2 \leq n \sum_{i=1}^{n} b_i^2,$$

and the fact that $\prod\limits_{i=1}^{d-1} r_i = 1$ and $\sum\limits_{i=1}^{d-1} \prod\limits_{j\neq i} r_j = (d-1)^2$ according to the Vieta's theorem for the roots $\{r_i\}$ of M_d.

From (28), (30) and (31), we have

$$E(2,d)^2 \leq \frac{1}{\pi^2} \left((d-1) + \frac{\pi}{2} (d-1)^{\frac{3}{2}} \right) = \frac{1}{\pi^2} (d-1)(1 + \frac{\pi}{2}\sqrt{d-1}),$$

or equivalently

$$E(2,d) \leq \frac{1}{\pi} \sqrt{d-1} \sqrt{1 + \frac{\pi}{2}\sqrt{d-1}}.$$

□

In Figure 1a, we show the numerical results for $E(d)$ in comparison with the obtained upper and lower bounds.

Corollary 3.6. *1) The expected number of equilibria increases unboundedly when d tends to infinity*

$$\lim_{d\to\infty} E(2,d) = +\infty. \qquad (32)$$

2) The probability p_m of observing m equilibria, $1 \leq m \leq d-1$, is bounded by

$$p_m \leq \frac{E(2,d)}{m} \leq \frac{1}{\pi m} \sqrt{d-1} \sqrt{1 + \frac{\pi}{2}\sqrt{d-1}}. \qquad (33)$$

In particular,

$$p_{d-1} \leq \frac{1}{\pi}\frac{\sqrt{1+\frac{\pi}{2}\sqrt{d-1}}}{\sqrt{d-1}}, \quad \textit{and} \quad \lim_{d\to\infty} p_{d-1} = 0.$$

Proof. 1) This is a direct consequence of (27), as the lower bound of $E(2,d)$ tends to infinity when d tends to infinity.

2) This is again a direct consequence of (27) and definition of $E(2,d)$. For any $1 \leq m \leq d-1$, we have

$$E(2,d) = \sum_{i=1}^{d-1} p_i i \geq p_m m.$$

In particular,

$$p_{d-1} \leq \frac{E(2,d)}{d-1} \leq \frac{1}{\pi}\frac{\sqrt{1+\frac{\pi}{2}\sqrt{d-1}}}{\sqrt{d-1}}.$$

As a consequence, $\lim_{d\to\infty} p_{d-1} = 0$. Similarly, we can show that this limit is actually true for p_k for any $k = O(d)$ as $d \to \infty$.

□

From this corollary we can see that, interestingly, although the mean number of equilibria tends to infinity when the number of players d increases, the probability to see the maximal number of equilibria in a d-player system converges to 0. There has been extensive research studying the maximal number of equilibrium points of a system [MS82, Kar80, VC88a, VC88b, KF70, VC88c, BCV93, BCV97, Alt10]. Our results suggest that the possibility to reach such a maximal number is very small when d is sufficiently large.

3.2 Two-player multi-strategy games

In this section, we consider games with two players, i.e. $d = 2$, and arbitrary strategies. In this case (7) is simplified to a linear system

$$\begin{cases} \sum_j \beta_j^i\, y_j = 0, \quad \text{for} \quad i = 1, \ldots, n-1 \\ \sum_j y_j = 1 \end{cases} \tag{34}$$

where β_j^i have Gaussian distributions with mean 0 and variance 1. The main result of this section is the following explicit formula for $E(n,2)$.

Theorem 3.7. *We have*

$$E(n,2) = \frac{1}{2^{n-1}}. \tag{35}$$

Proof. According to Lemma 2.2, we have

$$E(n,2) = \pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\underbrace{\int_0^\infty \ldots \int_0^\infty}_{n-1\text{ times}} (\det L)^{-n/2}\, dt_1 \ldots dt_{n-1}, \tag{36}$$

where L is the matrix with entries

$$L_{ii} = \frac{1}{1+\sum\limits_{k=1}^{n-1} t_k^2} - \frac{t_i^2}{(1+\sum\limits_{k=1}^{n-1} t_k^2)^2},$$
$$L_{ij} = \frac{-t_i t_j}{(1+\sum\limits_{k=1}^{n-1} t_k^2)^2} \quad \forall i \neq j$$

The determinant of L is computed in the following auxiliary lemma whose proof is given in the Appendix 5.4.

Lemma 3.8. *It holds that*

$$\det L = \frac{1}{\left(1+\sum\limits_{k=1}^{n-1} t_k^2\right)^n}. \tag{37}$$

We continue with the computation of $E(n,2)$.

$$\begin{aligned}
E(n,2) &= \pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\underbrace{\int_0^\infty \dots \int_0^\infty}_{n-1\,\text{times}} \left(1+\sum_{k=1}^{n-1} t_k^2\right)^{-n/2} dt_1 \dots dt_{n-1} \\
&= \frac{1}{2}\pi^{-\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\underbrace{\int_0^\infty \dots \int_0^\infty}_{n-2\,\text{times}} \left(1+\sum_{k=1}^{n-2} t_k^2\right)^{-(n-1)/2} dt_1 \dots dt_{n-2} \\
&\dots \\
&= \frac{1}{2^{n-1}}\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{1}{2^{n-1}},
\end{aligned}$$

where we have repeatedly used the equality (with $a > 0$ and $p > 1$)

$$\int_0^\infty (a+t^2)^{-p} dt = \frac{\sqrt{\pi}\,\Gamma(p-\frac{1}{2})}{2\,\Gamma(p)} a^{\frac{1}{2}-p}$$

□

As a corollary we recover the following result obtained in [HTG12, Theorem 1].

Corollary 3.9. *In a random two-player game with n strategies, the probability that there exists a (unique) isolated internal equilibrium is 2^{1-n}.*

3.3 Multi-player multi-strategy games

We now move to the general case of multi-player games with multiple strategies. We provide numerical results for this case.

For simplicity of notation in this section we write $\sum\limits_{k_1,\ldots,k_{n-1}}$ instead of $\sum\limits_{\substack{0\leq k_1,\ldots,k_{n-1}\leq d-1,\\ \sum_{i=1}^{n-1} k_i\leq d-1}}$ and $\sum\limits_{k_1,\ldots,k_{n-1}|k_{i_1},..,k_{i_m}}$ instead of $\sum\limits_{\substack{0\leq k_i\leq d-1\ \forall i\in S\setminus R,\\ 1\leq k_j\leq d-1\ \forall j\in R,\\ \sum\limits_{i=1}^{n-1} k_i\leq d-1}}$ where $S=\{1,\ldots,n-1\}$ and $R=\{i_1,\ldots,i_m\}$.

According to Lemma 2.2, the expected number of internal equilibria in a $d-$player random game with n strategies is given by

$$E(n,d)=\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\int_U\left(\det\left[\frac{\partial^2}{\partial x_i y_j}(\log v(x)^T C v(y))\big|_{y=x=t}\right]_{ij}\right)^{\frac{1}{2}}dt=\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\int_U(\det L)^{\frac{1}{2}}\,dt,$$

where Γ is the Gamma function, and L denotes the matrix with entries

$$L_{ij}=\frac{\partial^2}{\partial x_i y_j}(\log v(x)^T C v(y))\big|_{y=x=t}.$$

We have

$$v(x)^T C v(y)=\sum_{k_1,\ldots,k_{n-1}}\begin{pmatrix}d-1\\k_1,\ldots,k_n\end{pmatrix}^2\prod_{i=1}^n x_i^{k_i}y_i^{k_i},$$

Set $\Pi(x,y):=\prod\limits_{l=1}^{n-1}x_l^{k_l}y_l^{k_l}$. Then

$$\frac{\partial^2}{\partial x_i y_j}(\log v(x)^T C v(y))=\frac{\frac{1}{x_iy_j}\sum\limits_{k_1,\ldots,k_{n-1}|k_i,k_j}k_ik_j\begin{pmatrix}d-1\\k_1,\ldots,k_{n-1}\end{pmatrix}^2\Pi(x,y)}{\sum\limits_{k_1,\ldots,k_{n-1}}\begin{pmatrix}d-1\\k_1,\ldots,k_{n-1}\end{pmatrix}^2\Pi(x,y)}$$
$$-\frac{\left(\frac{1}{x_i}\sum\limits_{k_1,\ldots,k_{n-1}|k_i}k_i\begin{pmatrix}d-1\\k_1,\ldots,k_n\end{pmatrix}^2\Pi(x,y)\right)\left(\frac{1}{y_j}\sum\limits_{k_1,\ldots,k_n|k_j}k_j\begin{pmatrix}d-1\\k_1,\ldots,k_n\end{pmatrix}^2\Pi(x,y)\right)}{\left(\sum\limits_{k_1,\ldots,k_{n-1}}\begin{pmatrix}d-1\\k_1,\ldots,k_n\end{pmatrix}^2\Pi(x,y)\right)^2}.$$

Therefore

$$L_{ij} = \frac{\frac{1}{t_i t_j} \sum_{k_1,\ldots,k_{n-1}|k_i,k_j} k_i k_j \binom{d-1}{k_1,\ldots,k_n}^2 \prod_l t_l^{2k_l}}{\sum_{k_1,\ldots,k_{n-1}} \binom{d-1}{k_1,\ldots,k_n}^2 \prod_l t_l^{2k_l}} \tag{38}$$

$$- \frac{\left(\frac{1}{t_i} \sum_{k_1,\ldots,k_{n-1}|k_i} k_i \binom{d-1}{k_1,\ldots,k_n}^2 \prod_l t_l^{2k_l}\right)\left(\frac{1}{t_j} \sum_{k_1,\ldots,k_{n-1}|k_j} k_j \binom{d-1}{k_1,\ldots,k_n}^2 \prod_l t_l^{2k_l}\right)}{\left(\sum_{k_1,\ldots,k_{n-1}} \binom{d-1}{k_1,\ldots,k_n}^2 \prod_l t_l^{2k_l}\right)^2}. \tag{39}$$

So far in the paper we assume that all the $\beta^i_{k_1,\ldots,k_{n-1}}$ in Eq.(7) are standard normal distributions. The following lemma shows, as a consequence of the above described formula, that all the results obtained so far remain valid if they have a normal distribution with mean zero and arbitrary variance (i.e. the entries of the game payoff matrix have a same, arbitrary normal distribution).

Lemma 3.10. *Suppose $\beta^i_{k_1,\ldots,k_{n-1}}$ have normal distributions with mean 0 and arbitrary variance σ^2. Then, L_{ij} as defined in (38) does not depend on σ^2.*

Proof. In this case, $\beta^i_{k_1,\ldots,k_{n-1}} \binom{d-1}{k}$ has Gaussian distribution with variance $\sigma^2 \binom{d-1}{k}^2$ and mean 0. Hence,

$$v(x)^T C v(y) = \sum_{k_1,\ldots,k_{n-1}} \sigma^2 \binom{d-1}{k_1,\ldots,k_n}^2 \prod_{i=1}^n x_i^{k_i} y_i^{k_i},$$

Repeating the same calculation we obtain the same L_{ij} as in (38), which is independent of σ. □

This result suggests that when dealing with random games as in this article, it is sufficient to consider that payoff entries are from interval $[0, 1]$ instead off from an arbitrary one, as done numerically in [GT10]. A similar behaviour has been observed in [HTG12] for the analysis and computation with small d or n, showing that results are not dependent on the interval where the payoff entries are drawn.

Example 3.2 (d-players with n=3 strategies).

$$L_{11} = \frac{\frac{1}{t_1^2}\sum\limits_{k_1=1,k_2=0} k_1^2 \binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}}{\sum\limits_{k_1,k_2}\binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}} - \frac{\left(\frac{1}{t_1}\sum\limits_{k_1=1,k_2=0} k_1 \binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}\right)^2}{\left(\sum\limits_{k_1,k_2}\binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}\right)^2}$$

$$L_{12} = \frac{\frac{1}{t_1t_2}\sum\limits_{k_1=1,k_2=1} k_1k_2 \binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}}{\sum\limits_{k_1,k_2}\binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}} - \frac{\left(\frac{1}{t_1}\sum\limits_{k_1=1,k_2=0} k_1 \binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}\right)\left(\frac{1}{t_2}\sum\limits_{k_1=0,k_2=1} k_2 \binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}\right)}{\left(\sum\limits_{k_1,k_2}\binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}\right)^2}$$

$$L_{21} = L_{12}$$

$$L_{22} = \frac{\frac{1}{t_2^2}\sum\limits_{k_1=0,k_2=1} k_2^2 \binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}}{\sum\limits_{k_1,k_2}\binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}} - \frac{\left(\frac{1}{t_2}\sum\limits_{k_1=0,k_2=1} k_2 \binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}\right)^2}{\left(\sum\limits_{k_1,k_2}\binom{d-1}{k_1,k_2}^2 t_1^{2k_1}t_2^{2k_2}\right)^2}$$

Therefore,

$$E(3,d) = \pi^{-\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)\int_0^\infty\int_0^\infty \sqrt{L_{11}L_{22} - L_{12}L_{21}}\, dt_1dt_2.$$

Next we provide some numerical results. We numerically compute $E(n,d)$ for $n \in \{2,3,4\}, d \leq 20$, and show them in Table 1 and Figure 1b. We also plot the lower and upper bound for $E(2,d)$ obtained in Theorem 3.5 and compare them with its numerical computation, see Figure 1a . We note that for small n and d (namely, $n \leq 5$ and $d \leq 4$), $E(n,d)$ can also be computed numerically via the probabilities p_i of observing exactly i equilibria using (14). This direct approach have been used in [HTG12] and [GT10], and our results are compatible.

Table 1: Expected number of internal equilibria for some values of d and n.

d	2	3	4	5	6	7	8	9	10
$n=2$	0.5	0.77	0.98	1.16	1.31	1.47	1.60	1.73	1.84
$n=3$	0.25	0.57	0.92	1.29	1.67	2.06	2.46	2.86	3.27
$n=4$	0.125	0.41	0.84	1.39	2.05	2.81	3.67	4.62	5.66

4 Discussion

In evolutionary game literature, mathematical study has been mostly carried out for the pairwise game scenarios [HS98, Now06], despite the abundance of multi-player game examples in nature [BCV97, GT10]. The mathematical study for multi-player games has only attracted more attention recently, apparently because the resulting complexity and dynamics from the games are

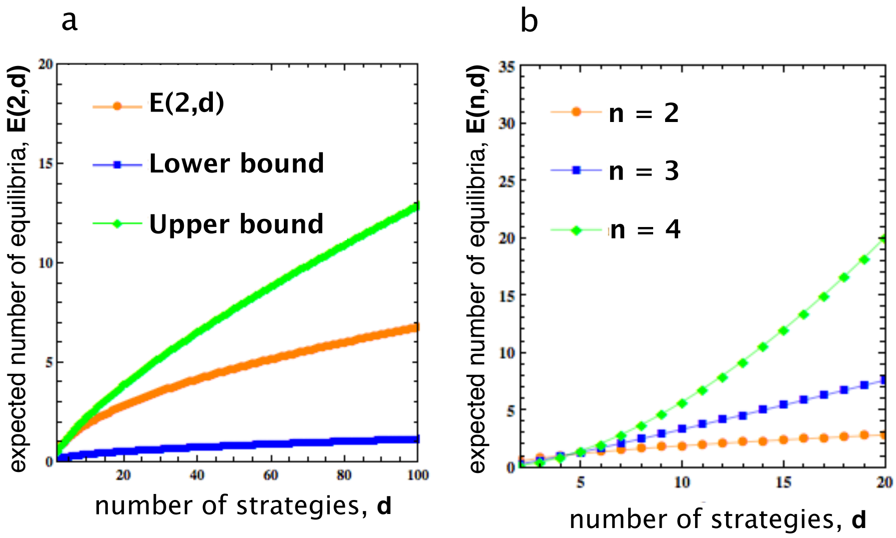

Figure 1: **(a) Expected number of equilibria, $E(2, d)$, for varying the number of players in the game, and the upper and lower bounds obtained in Theorem 3.5**. **(b) Expected number of equilibria, $E(n, d)$, for varying the number of players and strategies in the game**. We plot for different number of strategies: $n = 2$, 3 and 4. The payoff entries of the d-player n-strategy game are randomly drawn from the normal distribution. In general, the larger the number of players in the game, the higher the number of equilibria one can expect if the payoff entries of the game are randomly chosen. We observe that for small d, $E(n, d)$ increases with n while it decreases for large d (namely, $d \geq 5$, see also Table 1). The results were obtained numerically using Mathematica.

significantly magnified with the number of the game participants [BCV97, HTG12, WTG13]. As seen from our analysis, multi-player games introduce non-linearity into the inducing equations, because the fitness functions are polynomial instead of being linear as in the pairwise interactions [HS98, GT10, WTG13]. In addition, as the number of strategies increases, one needs to deal with systems of multivariate polynomials. This seemingly complexity of the general multi-player multi-strategy games make the analysis of the equilibrium points extremely difficult. For instance, in [HTG12], as the analysis was based on an explicit calculation of the zeros of systems of polynomials, it cannot go beyond games with a small number of players (namely, $d \leq 5$), even for the simplest case of $n = 2$. Here we have taken a different approach based on the well-established theory of random polynomials. We have derived a computationally implementable formula, $E(n, d)$, of the mean number of equilibria in a general random d-player game with n strategies. For $n = 2$ and with an arbitrary d, we have derived asymptotic upper and lower bounds of $E(2, d)$. An interesting implication of these results is that although the expected number of equilibria tends to infinity when d increases, the probability to see the maximal possible number of equilibria tends to 0. This is an notable observation since knowing the maximal number of equilibrium points in an evolutionary process is insightful and has been of special focus in biological contexts [Lev00, GT10]. Furthermore, for $d = 2$ with an arbitrary n, we have derived the exact value $E(n, 2) = 2^{1-n}$, recovering results obtained in [HTG12]. In the general case, based on the formula provided we have been able to numerically calculate $E(n, d)$, thereby lifting the analytical and numerical computations from previous work that made use of the direct approach.

On the other side, the study of distribution of zeros of system of random polynomials as described in (8), especially in the univariate case, has been carried out by many authors, see for instance [EK95] for a nice exposition and [TV14] for the most recent results. The most well-known classes of polynomials are: flat polynomials or Weyl polynomials for $a_i := \frac{1}{i!}$, elliptic polynomials or binomial polynomials for $a_i := \sqrt{\begin{pmatrix} d-1 \\ i \end{pmatrix}}$ and Kac polynomials for $a_i := 1$. We emphasize the difference between the polynomial studied in this paper with the elliptic case: $a_i = \begin{pmatrix} d-1 \\ i \end{pmatrix}$ are binomial coefficients, not their square root. In the elliptic case, $v(x)^T C v(y) = \sum_{i=1}^{d-1} \begin{pmatrix} d-1 \\ i \end{pmatrix} x^i y^i = (1+xy)^{d-1}$, and as a result $E = E(2,d) = \sqrt{d-1}$. While in our case, because of the square in the coefficients, $v(x)^T C v(y) = \sum_{i=1}^{d-1} \begin{pmatrix} d-1 \\ i \end{pmatrix}^2 x^i y^i$ is no longer a generating function. Whether one can find a compact or asymptotic formula for $E(2,d)$ is unclear. For the multivariate situation, the exact formula for $E(n,2)$ is interesting by itself and we could not be able to find it in the literature. Due to the complexity in the general case $d, n \geq 3$, further research is required.

In short, we have described a novel approach to calculating and analyzing the (expected) number of equilibrium points in a general random evolutionary game, giving insights into the overall complexity of such dynamical system as the players and the strategies in the game increase. Since the theory of random polynomials is rich, we envisage that our method could be extended to obtain results for other more complex scenarios such as games having a payoff matrix with dependent entries and/or with general distributions.

5 Appendix

5.1 Properties of a_k

In the following we prove that $a_{2d-4-k} = a_k$ for all $0 \le k \le 2d-4$. Indeed,

$$\begin{aligned}
a_k &= \sum_{\substack{0\le i\le d-1\\0\le j\le d-2\\i+j=k}} \binom{d-1}{i}^2\binom{d-2}{j}^2 - \sum_{\substack{0\le i'\le d-2\\0\le j'\le d-2\\i'+j'=k-1}} \binom{d-2}{i'}\binom{d-1}{i'+1}\binom{d-2}{j'}\binom{d-1}{j'+1}.\\
&= \binom{d-2}{k}^2 + \sum_{\substack{0\le i\le d-2\\0\le j\le d-2\\i+j=k-1}} \binom{d-1}{i+1}^2\binom{d-2}{j}^2 - \sum_{\substack{0\le i'\le d-2\\0\le j'\le d-2\\i'+j'=k-1}} \binom{d-2}{i'}\binom{d-1}{i'+1}\binom{d-2}{j'}\binom{d-1}{j'+1}.\\
&= \binom{d-2}{k}^2 + \sum_{\substack{0\le i\le d-2\\0\le j\le d-2\\i+j=k-1}} \binom{d-1}{i+1}\binom{d-2}{j}\left(\binom{d-1}{i+1}\binom{d-2}{j} - \binom{d-2}{i}\binom{d-1}{j+1}\right)\\
&= \binom{d-2}{k}^2 + \sum_{\substack{0\le i\le j\le d-2\\i+j=k-1}} \left(\binom{d-1}{i+1}\binom{d-2}{j} - \binom{d-2}{i}\binom{d-1}{j+1}\right)^2
\end{aligned}$$

We prove that $a_{2d-4-k} = a_k$. Indeed, we have

$$\begin{aligned}
a_{2d-4-k} &= \binom{d-2}{2d-4-k}^2 + \sum_{\substack{0\le i\le j\le d-2\\i+j=2d-k-5}} \left(\binom{d-2}{i+1}\binom{d-2}{j} - \binom{d-2}{i}\binom{d-1}{j+1}\right)^2
\end{aligned}$$

(we use here the transformations $i = d-3-i$ and $j = d-3-j$)

$$\begin{aligned}
&= \binom{d-2}{2d-4-k}^2 + \sum_{\substack{-1\le j\le i\le d-3\\i+j=k-1}} \left(\binom{d-1}{i+1}\binom{d-2}{j+1} - \binom{d-2}{i+1}\binom{d-1}{j+1}\right)^2\\
&= \binom{d-2}{2d-4-k}^2 + \sum_{\substack{-1\le j\le i\le d-3\\i+j=k-1}} \left(\binom{d-1}{i+1}\binom{d-2}{j} - \binom{d-2}{i}\binom{d-1}{j+1}\right)^2
\end{aligned}$$

Since for $j=-1$ and $i=k$ we have

$$\left(\binom{d-1}{i+1}\binom{d-2}{j} - \binom{d-2}{i}\binom{d-1}{j+1}\right)^2 = \binom{d-1}{k}^2,$$

and for $j = d-2$ and $i = k-1-(d-2)$, we have

$$\left(\binom{d-1}{i+1}\binom{d-2}{j} - \binom{d-2}{i}\binom{d-1}{j+1}\right)^2 = \left(\binom{d-1}{k-(d-2)} - \binom{d-2}{k-1-(d-2)}\right)^2$$
$$= \binom{d-2}{k-(d-2)}^2$$
$$= \binom{d-2}{2d-4-k}^2,$$

it follows that $a_{2d-4-k} = a_k$ for all $0 \leq k \leq 2d-4$.

5.2 Detailed computation of $f(1)$

We use the following identities involving the square of binomial coefficients.

$$M_d(1) = \sum_{k=0}^{d-1} \binom{d-1}{k}^2 = \binom{2(d-1)}{d-1},$$
$$A_d(1) = (d-1)^2 M_{d-1}(1) = (d-1)^2 \binom{2(d-2)}{d-2},$$
$$B_d(1) = \sum_{k=1}^{d-1} k \binom{d-1}{k}^2 = \frac{d-1}{2}\binom{2(d-1)}{d-1}.$$

Therefore, we have

$$f(1) = \frac{1}{\pi}\frac{\sqrt{A_d(1)M_d(1) - B_d(1)^2}}{M_d(1)}$$
$$= \frac{1}{\pi}\frac{\sqrt{(d-1)^2\binom{2(d-1)}{d-1}\left[\binom{2(d-2)}{d-2} - \frac{1}{4}\binom{2(d-1)}{d-1}\right]}}{\binom{2(d-1)}{d-1}}$$
$$= \frac{d-1}{\pi} \times \sqrt{\frac{\binom{2(d-2)}{d-2}}{\binom{2(d-1)}{d-1}} - \frac{1}{4}}$$
$$= \frac{d-1}{\pi} \times \sqrt{\frac{d-1}{2(2d-3)} - \frac{1}{4}}$$
$$= \frac{d-1}{2\pi\sqrt{2d-3}},$$

where we have used the identity

$$\binom{2(n+1)}{n+1} = \frac{n+1}{2(2n+1)}\binom{2n}{n}.$$

5.3 Alternative proof of the fifth property in Lemma 3.1

We show here an alternative proof of the fifth property without using the first one in Lemma 3.1. Since $M_d\left(\frac{1}{t}\right) = t^{2(1-d)M_d(t)}$, we have

$$\begin{aligned}
M_d'\left(\frac{1}{t}\right) &= -t^{3-2d}\left[2(1-d)M_d(t) + tM_d'(t)\right],\\
M_d''\left(\frac{1}{t}\right) &= t^{4-2d}\left[2(3-2d)(1-d)M_d(t) + 2(3-2d)tM_d'(t) + t^2M_d''(t)\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
B_d\left(\frac{1}{t}\right) &= \frac{1}{2}M_d'\left(\frac{1}{t}\right)\\
&= -t^{3-2d}\left[(1-d)M_d(t) + \frac{1}{2}tM_d'(t)\right],\\
A_d\left(\frac{1}{t}\right) &= \frac{t}{4}\left[M_d'\left(\frac{1}{t}\right) + \frac{1}{t}M_d''\left(\frac{1}{t}\right)\right]\\
&= \frac{1}{4}t^{4-2d}\left[4(1-d)^2 + (5-4d)tM_d'(t) + t^2M_d''(t)\right],\\
A_d\left(\frac{1}{t}\right)M_d\left(\frac{1}{t}\right) - B_d\left(\frac{1}{t}\right)^2 &= \frac{1}{4}t^{6-2d}\left[4(1-d)^2M_d(t)^2 + (5-4d)tM_d'(t)M_d(t) + t^2M_d''(t)M_d(t)\right]\\
&\quad - t^{6-4d}\left[(1-d)M_d(t) + \frac{1}{2}tM_d'(t)\right]^2\\
&= \frac{1}{4}t^{6-2d}\left[tM_d(t)M_d'(t) + t^2M_d(t)M_d''(t) - t^2M_d'(t)^2\right]\\
&= \frac{1}{4}t^{8-2d}\left[\frac{1}{t}M_d(t)M_d'(t) + M_d(t)M_d''(t) - M_d'(t)^2\right],\\
f\left(\frac{1}{t}\right) &= \frac{1}{\pi}\frac{\sqrt{A_d\left(\frac{1}{t}\right)M_d\left(\frac{1}{t}\right) - B_d\left(\frac{1}{t}\right)^2}}{M_d\left(\frac{1}{t}\right)}\\
&= \frac{1}{\pi}\frac{t^{4-2d}\sqrt{\frac{1}{4t}\left[M_d(t)M_d'(t) + tM_d(t)M_d''(t) - tM_d'(t)^2\right]}}{t^{2-2d}M_d(t)}\\
&= t^2\frac{1}{\pi}\frac{\sqrt{\frac{1}{4t}\left[M_d(t)M_d'(t) + tM_d(t)M_d''(t) - tM_d'(t)^2\right]}}{M_d(t)}\\
&= t^2f(t).
\end{aligned}$$

5.4 Proof of Lemma 3.8

Denoting $\Sigma = 1 + \sum_{k=1}^{n-1} t_k^2$, we have

$$\begin{aligned}
\det L &= \frac{1}{\Sigma^{2(n-1)}} \begin{pmatrix} \Sigma - t_1^2 & -t_1 t_2 & \dots & -t_1 t_{n-1} \\ -t_2 t_1 & \Sigma - t_2^2 & \dots & -t_2 t_{n-1} \\ \dots & \dots & \dots & \dots \\ -t_{n-1} t_1 & -t_{n-1} t_2 & \dots & \Sigma - t_{n-1}^2 \end{pmatrix} \\
&= \frac{1}{t_1 \dots t_{n-1}} \frac{1}{\Sigma^{2(n-1)}} \begin{pmatrix} t_1(\Sigma - t_1^2) & -t_1 t_2^2 & \dots & -t_1 t_{n-1}^2 \\ -t_2 t_1^2 & t_2(\Sigma - t_2^2) & \dots & -t_2 t_{n-1}^2 \\ \dots & \dots & \dots & \dots \\ -t_{n-1} t_1^2 & -t_{n-1} t_2^2 & \dots & t_{n-1}(\Sigma - t_{n-1}^2) \end{pmatrix} \\
&= \frac{1}{t_1 \dots t_{n-1}} \frac{1}{\Sigma^{2(n-1)}} \begin{pmatrix} t_1 & -t_1 t_2^2 & \dots & -t_1 t_{n-1}^2 \\ t_2 & t_2(\Sigma - t_2^2) & \dots & -t_2 t_{n-1}^2 \\ \dots & \dots & \dots & \dots \\ t_{n-1} & -t_{n-1} t_2^2 & \dots & t_{n-1}(\Sigma - t_{n-1}^2) \end{pmatrix} \\
&= \frac{1}{\Sigma^{2(n-1)}} \begin{pmatrix} 1 & -t_2^2 & \dots & -t_{n-1}^2 \\ 1 & \Sigma - t_2^2 & \dots & -t_{n-1}^2 \\ \dots & \dots & \dots & \dots \\ 1 & -t_2^2 & \dots & \Sigma - t_{n-1}^2 \end{pmatrix} \\
&= \frac{1}{\Sigma^{2(n-1)}} \begin{pmatrix} 1 & -t_2^2 & \dots & -t_{n-1}^2 \\ 0 & \Sigma & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Sigma \end{pmatrix} \\
&= \frac{1}{\Sigma^n}.
\end{aligned}$$

Acknowledgment We would like to thank members of the WMS seminar at the CASA at Eindhoven University of Technology, especially Dr. R. van Hassel and Prof. A. Blokhuis for useful discussion. TAH acknowledges the generous support provided by the F.W.O. Belgium (postdoctoral fellowship 05_05 TR 7142).

References

[Abe24] Niels Henrik Abel. Mémoire sur les équations algébriques, où l'on démontre l'impossibilité de la résolution de l'équation générale du cinquiéme degré. *Abel's Ouvres*, (1):28–33, 1824.

[Alt10] Lee Altenberg. Proof of the Feldman-Karlin conjecture on the maximum number of equilibria in an evolutionary system. *Theoretical Population Biology*, 77(4):263 – 269, 2010.

[BCV93] M. Broom, C. Cannings, and G. T. Vickers. On the number oflocal maxima of a constrained quadratic form. *Proc. R. Soc. Lond. A*, 443:573–584, 1993.

[BCV97] M. Broom, C. Cannings, and G.T. Vickers. Multi-player matrix games. *Bull. Math. Biol.*, 59(5):931–952, 1997.

[EK95] Alan Edelman and Eric Kostlan. How many zeros of a random polynomial are real? *Bull. Amer. Math. Soc. (N.S.)*, 32(1):1–37, 1995.

[FH92] D. Fudenberg and C. Harris. Evolutionary dynamics with aggregate shocks. *J. Econ. Theory*, 57:420–441, 1992.

[GRLD09] Thilo Gross, Lars Rudolf, Simon A. Levin, and Ulf Dieckmann. Generalized models reveal stabilizing factors in food webs. *Science*, 325, 2009.

[GT10] Chaitanya S. Gokhale and Arne Traulsen. Evolutionary games in the multiverse. *Proc. Natl. Acad. Sci. U.S.A.*, 107(12):5500–5504, 2010.

[HS98] J. Hofbauer and K. Sigmund. *Evolutionary Games and Population Dynamics*. Cambridge University Press, Cambridge, 1998.

[HTG12] The Anh Han, Arne Traulsen, and Chaitanya S. Gokhale. On equilibrium properties of evolutionary multi-player games with random payoff matrices. *Theoretical Population Biology*, 81(4):264 – 272, 2012.

[Kar80] Samuel Karlin. The number of stable equilibria for the classical one-locus multiallele slection model. *J. Math. Biology*, 9:189–192, 1980.

[KF70] Samuel Karlin and Marcus W. Feldman. Linkage and selection: Two locus symmetric viability model. *Theoretical Population Biology*, 1:39–71, 1970.

[Lev00] Simon A. Levin. Multiple scales and the maintenance of biodiversity. *Ecosystems*, 3:498–506, 2000.

[MB05] Andrew McLennan and Johannes Berg. Asymptotic expected number of nash equilibria of two-player normal form games. *Games and Economic Behavior*, 51(2):264–295, 2005.

[McL05] Andrew McLennan. The expected number of nash equilibria of a normal form game. *Econometrica*, 73(1):141–174, 2005.

[MS82] J. Maynard-Smith. *Evolution and the Theory of Games*. Cambridge University Press, Cambridge, 1982.

[Now06] M. A. Nowak. *Evolutionary Dynamics*. Harvard University Press, Cambridge, MA, 2006.

[Sig10] K. Sigmund. *The calculus of selfishness*. Princeton Univ. Press, 2010.

[SS83] P. Schuster and K. Sigmund. Replicator dynamics. *J. Theo. Biol.*, 100:533–538, 1983.

[TJ78] P. D. Taylor and L. Jonker. Evolutionary stable strategies and game dynamics. *Math. Biosci.*, 40:145–156, 1978.

[TV14] T. Tao and V. Vu. Local universality of zeroes of random polynomials. *Int Math Res Notices*, 2014.

[VC88a] G. T. Vickers and C. Cannings. On the number of equilibria in a one-locus, multi allelic system. *J. theor. Biol.*, 131:273–277, 1988.

[VC88b] G. T. Vickers and C. Cannings. Patters of ESS's I. *J. theor. Biol.*, 132:387–408, 1988.

[VC88c] G. T. Vickers and C. Cannings. Patters of ESS's II. *J. theor. Biol.*, 132:409–420, 1988.

[WTG13] Bin Wu, Arne Traulsen, and Chaitanya S Gokhale. Dynamic properties of evolutionary multi-player games in finite populations. *Games*, 4(2):182–199, 2013.

[Zee80] E. C. Zeeman. Population dynamics from game theory. *in Lecture Notes in Mathematics*, 819:471–497, 1980.