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An improved 1D model for liquid slugs travelling in pipelines

by

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AN IMPROVED 1D MODEL FOR LIQUID SLUGS TRAVELLING IN PIPELINES

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ABSTRACT

An improved one-dimensional (1D) model – compared to previous work by the authors – is proposed which is able to predict the acceleration and shortening of a single liquid slug propagating in a straight pipe with a downstream bend. The model includes holdup at the slug’s tail and flow separation at the bend. The obtained analytical and numerical results are validated against experimental data. The effects of the improvement and of holdup are examined in a parameter variation study.

INTRODUCTION

Isolated liquid slugs travelling in pipelines form a potential danger that needs to be assessed. The slugs may accelerate to high velocities and damage pipe anchors and hydraulic machinery when hitting obstacles like bends and valves. The impact force is proportional to the square of the slug’s speed, and the impact duration is proportional to the slug’s length and inversely proportional to its speed. Therefore a good assessment of both speed and length is essential in risk and post-accident analyses.

A literature review of the subject is given in Ref. [1]. An alternative derivation (using Leibniz’s rule) and refinement of the 1D model used in [1, 2] is presented in the Appendix herein. The refinement is that in the equation of motion the slug velocity is not taken uniform, but linearly increasing from front to tail. Symbolic solutions are provided where possible. The solutions are used to examine the influence of the refinement and compared with laboratory measurements. The obtained results may also be of value in the modelling of pipe draining [3] and filling [4, 5].

SLUG MOTION

Consider the schematised liquid slug travelling at speed \( v \) in a straight pipe with circular cross section \( A \) as sketched in Fig. 1. The planar front is at position \( x_1 \) and the planar tail is at position \( x_2 \). The pressure is \( P_1 \) at the front of the slug and \( P_2 \) at its tail. The travelling slug leaves liquid behind which is referred to as holdup. This holdup is proportional to a coefficient \( \beta \) to be defined later. The slug has mass \( m \), length \( L \) and constant density \( \rho \) related by
\[ m = \rho AL \quad (1) \]

For the sake of clarity, three propulsion and five resistance mechanisms are identified. These are:

**Propulsion**
1) When there is a positive pressure difference between slug tail and front, and there is no holdup, the slug accelerates according to \( a = (P_2 - P_1) A / m \).
2) When there is no pressure difference between slug tail and front, but there is stagnant holdup, the slug acceleration is \( a = - (v / m) \mathrm{dm} / \mathrm{dt} \); that is, \( a \) increases because \( m \) decreases.
3) Gravity in downward sloping pipes (with angle \( \theta \)) gives an acceleration \( a = g \sin \theta \).

**Resistance**
1) Gravity in upward sloping pipe.
2) Skin friction, turbulence.
3) Pressure build-up in front of the slug, trapped air.
4) The slug may break up because of obstacles and/or air entrainment.
5) The slug may pick up additional liquid on its way.

To explain the key mechanisms of slug motion, two idealised cases are considered, before introducing the 1D model developed in this paper.

**Frictional acceleration without holdup**

The governing equation for a slug driven by a constant pressure difference and opposed by quasi-steady turbulent friction is:

\[ \frac{dv}{dt} = \frac{P_2 - P_1}{\rho L} - \frac{f}{2D} v^2. \quad (2) \]

The analytical solution is [6, 7]:

\[ v(t) = v_\infty \tanh \left( \text{artanh}(v_0 / v_\infty) + \sqrt{C_1 C_2} t \right). \quad (3) \]

where \( C_1 \equiv \frac{P_2 - P_1}{\rho L} > 0, \quad C_2 \equiv \frac{f}{2D}, \quad v_\infty = \sqrt{C_1 / C_2}, \quad v_0 \) is the initial velocity at \( t = 0 \), \( D \) = pipe diameter and \( f \) = Darcy-Weisbach friction coefficient.

The distance travelled from \( x(0) = 0 \) is:

\[ x(t) = \int_0^t v(t) \, dt = \frac{1}{C_2} \ln \left( \cosh \left( \text{artanh}(v_0 / v_\infty) + \sqrt{C_1 C_2} \, t \right) \right). \quad (4a) \]

Slug arrival times follow from the inverse of relation (4a):

\[ t(x) = \frac{1}{\sqrt{C_1 C_2}} \left( \text{arcosh} \left( e^{C_2 x} \right) - \text{artanh} \left( v_0 / v_\infty \right) \right). \quad (4b) \]

The velocity of the slug when it has travelled a distance \( L_{\text{pipe}} \) is

\[ v(L_{\text{pipe}}) = v_\infty \sqrt{1 - e^{-2C_2 L_{\text{pipe}}}}. \quad (5) \]

The formulas (3), (4) and (5) provide insight in the time and length scales involved, give order-of-magnitude estimates, and they can be used to validate numerical solutions.

**Frictionless acceleration with holdup**

Even when there is no pressure difference across the slug, a moving slug will accelerate when it loses mass. This self-propulsion is similar to that of a space rocket and the same governing equation applies [8]:

\[ \frac{dv}{dt} = - \frac{v \, dm}{m \, dt}. \quad (6) \]

It is assumed that the holdup is stagnant, meaning that it has zero velocity, so that its velocity relative to the slug is \(-v\). Substituting Eq. (1) in Eq. (6) gives

\[ L \frac{dv}{dt} = -v \frac{dL}{dt} = -v(v_1 - v_2), \quad (7) \]

where \( v_1 \) and \( v_2 \) are the velocities of the slug front and tail, respectively (Fig. 1) and \( L = x_1 - x_2 \). Replacing the velocity \( v \) in \( dv/dt \) on the left by the centroid velocity \((v_1 + v_2)/2\) and the velocity \( v \) on the right by \( v_1 \), one obtains

\[ L \left( \frac{dv_1}{dt} + \frac{dv_2}{dt} \right) = -v_1 (v_1 - v_2) = \frac{\beta}{1 - \beta} v_1^2 = \beta (1 - \beta) v_1^2. \quad (8) \]

The expressions with \( \beta \) are for later use.

**Frictional acceleration with holdup**

The combined model (roughly Eqs (2) and (7)) is derived in Appendix A. In terms of the slug front velocity \( v_1 \), slug length \( L \) and slug front position \( x_1 \), the governing equations read:
\[ \frac{1 - \frac{1}{2} \beta}{1 - \beta} L(t) \frac{dv_1}{dr}(t) = \frac{\beta}{1 - \beta} v_1^2(t) + \frac{P_2(t) - P_1(t)}{\rho} \]
\[ - \frac{f}{2D} \frac{1 - \frac{1}{2} \beta^2}{(1 - \beta)^2} L(t) v_1^2(t), \]
(9a)

\[ \frac{dL}{dr}(t) = -\frac{\beta}{1 - \beta} v_1(t), \]
(9b)

\[ \frac{dx_1}{dr}(t) = v_1(t). \]
(9c)

An analytical solution is derived as follows. Take a constant pressure difference \( P_2 - P_1 > 0, \beta > 0 \) and \( f = 0 \). The case \( f > 0 \) is dealt with in Appendix A. Use Eq. (9b) to eliminate \( v_1 \) in Eq. (9a). The result is

\[ L \frac{d^2L}{dr^2} = -\frac{1 - \frac{1}{2} \beta}{1 - \beta} \left( \frac{dL}{dr} \right)^2 - \frac{\beta}{1 - \frac{1}{2} \beta} \frac{P_2 - P_1}{\rho}. \]
(11)

Consider \( \frac{dL}{dr} \) as a function of \( L \). That is, \( \frac{dL}{dr} = w(L) < 0 \), so that

\[ \frac{d^2L}{dr^2} = \frac{dw}{dL} \frac{dL}{dr} = \frac{dw}{dL} = 1 \frac{dw^2}{dL}. \]
(12)

Equation (11) then becomes a linear first-order ODE in \( w^2 \),

\[ \frac{dw^2}{dL} + 2 \frac{1 - \frac{1}{2} \beta}{L} w^2 = -\frac{2}{L} \frac{\beta}{1 - \frac{1}{2} \beta} \frac{P_2 - P_1}{\rho}. \]
(13)

The solution for the initial condition \( v_1(t_0) = 0 \) or \( w(L_0) = 0 \) is

\[ w^2 = \left( \frac{L_0}{L} \right)^{\alpha} - 1 \frac{\beta}{1 - \frac{1}{2} \beta} \frac{P_2 - P_1}{\rho} \]
with \( \alpha := 2 \frac{1 - \beta}{1 - \frac{1}{2} \beta}. \)
(14)

Using Eqs (9b) and (10), the solution for \( v_1 \) as a function of \( L_{pipe} \) is

\[ v_1 = \sqrt{\frac{1}{1 - \frac{1}{2} \beta} \left( \frac{L_{pipe}}{L_0} \right)^{\alpha} - 1} \frac{1 - \frac{1}{2} \beta}{\beta} \frac{P_2 - P_1}{\rho}. \]
(15)

The corresponding formula for \( f > 0 \) is in terms of incomplete gamma functions:

\[ v_1 = e^{f \cdot L} \sqrt{\frac{\Gamma(\alpha, 2L f^* \beta)}{(2L f^* \beta)^{\alpha}}} \frac{\Gamma(\alpha, 2L_0 f^* \beta)}{(2L_0 f^* \beta)^{\alpha}} \left( \frac{L_0}{L} \right)^{\alpha}, \]
(16)

where \( f^* := \frac{f}{2D} \frac{1 - \frac{1}{2} \beta^2}{\beta(1 - \frac{1}{2} \beta)} \) and \( L = L_0 - \frac{\beta}{1 - \beta} L_{pipe} \) (Eq. (10)).

See Appendix A for the derivation. If \( 2L_0 f^* > 37 \) (for \( 1.7 \leq \beta \leq 2 \)) the incomplete gamma functions are of the order of
where \( C_c \) is the flow contraction coefficient, \( K_e \) is the elbow's minor loss coefficient and \( H \) is the Heaviside step function. The build-up of the resistance is assumed to take place linear in time, where the time interval \( t_2-t_1 \) is approximated by \( (L_e+L_{\text{front}})/v_1 \), where \( L_{\text{front}} \) is the length of a non-orthogonal slug front (Fig. 4b). The factor containing the Heaviside functions is replaced by unity for instantaneous impacts \( (t_2=t_1) \) [1]; in that case analytical solutions can be derived.

**NUMERICAL INTEGRATION**

Numerical integration is required when the slug hits the elbow non-instantaneously and when the pressure difference \( P_2-P_1 \) is not constant but given by measured or calculated (gas dynamics) values. The governing equations (9) can be casted in the standard form

\[
\frac{dy}{dt} = f(t, y), \quad \text{with } y := \begin{pmatrix} v_1 \\ x_t \end{pmatrix}.
\]

This ODE is autonomous \([ f(t, y) = f(y) \)] when the pressure difference \( P_2-P_1 \) is constant. The local stability of the solutions is investigated through the Jacobian matrix \( J \) of \( f \) with respect to \( y \). Any suitable numerical integration scheme can be used to solve Eq. (19). The forward Euler method will do the job if the time step \( \Delta t \) is taken sufficiently small; the modulus of the (complex valued) eigenvalues of \( I + \Delta t J \) must then be smaller than 1 [9].

**LABORATORY EXPERIMENT**

The experimental setup described in Ref. [10] is used as test problem, see Fig. 4a. The key part is a 12 m long inclined pipe of 0.1 m diameter leading upwards to an open elbow. Slugs of water are at rest in the lower elbow, see Fig 4b, before they start to accelerate due to a suddenly applied pressure difference. Initial slug lengths were \( L_0 = 3, 4, 5 \) or 6 m undergoing driving pressure differences of \( \Delta P = 2, 3, 4 \) or 5 bar. It is noted that the driving pressure in the laboratory experiments was not constant.

**PARAMETER VARIATION STUDY**

Here we ignore the pipe inclination, the non-orthogonal slug front and the upper elbow. The slugs are allowed to travel as far as they can, until they vanish (say at \( L < D \)). The pipe diameter is taken ten times smaller (\( D = 0.01 \) m) to enhance the effect of friction.

Case 1 concerns a slug of initial length 3 m and mass 24 kg driven by a pressure difference of 5 bar. In Case 2, these values are 5 m, 40 kg and 2 bar. The input values for the simulations are: pipe diameter \( D = 0.01 \) m, friction factor \( f = 0.016 \), mass...
density \( \rho = 1000 \text{ kg/m}^3 \), initial slug length \( L_0 = 3 \text{ (5) m} \), driving pressure difference \( \Delta P = 5 \text{ (2) bar} \), and holdup coefficient \( \beta = 0, 0.05, 0.1, 0.15 \text{ or } 0.2 \). The slug is at rest before it starts to accelerate at \( t = 0 \) due the sudden pressure difference \( \Delta P \). The calculated velocity and acceleration histories are presented below.

The differences between the old model [1, 2] and the new improved model herein are shown in Fig. 5 for \( \beta = 0.2 \). The differences are evident; the old model gives more conservative results, that is, higher velocities and hence larger dynamic pressures.

\[
\text{(a)}
\]

\[
\begin{align*}
\text{(Not To Scale)}
\end{align*}
\]

\[
\text{(b)}
\]

\[
\begin{align*}
\text{Low inertia, low wall friction portion of the slug}
\end{align*}
\]

\[
\begin{align*}
\text{High inertia, high wall friction portion of the slug}
\end{align*}
\]

\[
\begin{align*}
\text{Assumed uniformly distributed forces}
\end{align*}
\]

**Figure 4** Experimental setup (a) and initial slug (b) [10].

The influence of the holdup coefficient in the new model is examined through the slug velocities in Fig. 6. Holdup slightly decreases the initial acceleration and largely increases the final acceleration. The five lines meet nearly at the same (inflection) point. Because the governing equations are strongly nonlinear, not much can be said about this most interesting point. The inflection point distinguishes two regimes: large acceleration before it and acceleration only due to holdup after it. The value of the velocity at the inflection point is more or less independent of the hold up and only depends on friction and inertia. This velocity is very close to \( v_\infty \) defined in Eq. (3). After the inflection holdup becomes important, because the mass lost per unit time increases with velocity and becomes large compared to the slug mass, Eq. (6).

The corresponding acceleration histories are shown in Fig. 7. At \( t = 0, v_1 = 0 \), so that the initial acceleration determined by Eq. (9a) is

\[
\frac{dv_1}{dt}(0) = \frac{1 - \beta}{1 - \frac{1}{2} \beta} \frac{\Delta P}{\rho L_0},
\]

where the factor \( (1 - \beta)/(1 - \frac{1}{2} \beta) \) is 1 for \( \beta = 0 \) and 8/9 for \( \beta = 0.2 \).

\[
\text{(a)}
\]

\[
\begin{align*}
\text{New model}
\end{align*}
\]

\[
\text{Old model}
\]

\[
\begin{align*}
\text{New model versus old model. Velocity history for } \beta = 0.2 \text{ for: (a) Case 1; (b) Case 2.}
\end{align*}
\]

\[
\text{(b)}
\]

\[
\begin{align*}
\text{New model}
\end{align*}
\]

\[
\text{Old model}
\]

\[
\begin{align*}
\text{Velocity (m/s)}
\end{align*}
\]

\[
\begin{align*}
\text{Time (s)}
\end{align*}
\]

\[
\begin{align*}
\text{Velocity (m/s)}
\end{align*}
\]

\[
\begin{align*}
\text{Time (s)}
\end{align*}
\]
Figure 6 New model. Velocity ($v_1$) as function of distance ($x_1$) for different $\beta$ for: (a) Case 1; (b) Case 2. Solid lines: analytical solutions (Eqs (5), (16) and (17)); broken line (coinciding with red solid line): numerical solution.

Figure 7 New model. Acceleration history for different $\beta$ for: (a) Case 1; (b) Case 2.

VALIDATION

In the simulation of the laboratory experiment [10], the upper elbow is added and has a length of say $L_\epsilon = D$. When the wedge-shaped slug front (Fig. 4b) reaches the elbow at $t = t_1$, the slab starts to experience an additional "resistance" according to Eq. (18). The estimated distance travelled by the slug front before it arrives at the elbow is 9.5 m. The input values for the simulations are: pipe diameter $D = 0.1$ m, travelled distance $L_{\text{pipe}} = 9.5$ m, elbow length $L_\epsilon = D$, elbow loss coefficient $K_e = 0.9$, flow contraction coefficient $C_c = 0.51$, slug front length $L_{\text{front}} = 3D$, friction factor $f = 0.016$, mass density $\rho = 1000$ kg/m$^3$, initial slug length $L_0 = 3$ m, driving pressure difference $\Delta P = 2, 3, 4$ or 5 bar, and an estimated holdup coefficient $\beta = 0.05$. The initial slug has a mass of 24 kg. For the highest driving pressure of 5 bar, it hits the elbow after $t_1 = 0.37$ s with a mass of 20 kg ($L = 2.5$ m) and a velocity $v_1$ of 45 m/s. The pressure in the elbow itself is strongly non-uniform, both along and perpendicular to streamlines. Here we take the conservative estimate $P = \rho v_1^2$. Calculated (shifted in time) and measured pressures are shown in Fig. 8. The observed magnitudes and shapes agree well in view of the many uncertainties in mathematical model and physical experiment. Once again, different from the simulation, in the experiment the driving pressure was not constant but decreasing, the slug-front wedge-shaped and the pipe inclined slightly upwards.

Figure 8 Pressure history at the elbow for a 24 kg slug ($L_0 = 3$ m) (100 psi $\approx$ 7 bar): Prediction by new model (top); Measurement [10] (bottom).
CONCLUSION

A refined 1D model for slug propagation has been derived. The differences with the old model are significant and less conservative results are obtained when the slug travels a sufficiently long distance. As a by-product, improved governing equations for pipe emptying and filling have been derived in the Appendix. Analytical solutions for limit cases ($\beta = 0$ and $f = 0$) provide useful guidance. A full analytical solution ($\beta > 0$ and $f > 0$) has been found for the case that the driving pressure difference is constant. The model was able to correctly predict orders of magnitudes and trends in measured data.

NOMENCLATURE

\begin{align*}
a & = \text{acceleration (m/s}^2) \\
A & = \text{cross-sectional pipe area (m}^2) \\
A_f & = \text{cross-sectional flow area (m}^2) \\
c & = \text{propagation speed (m/s)} \\
C & = \text{constant} \\
C_c & = \text{flow contraction coefficient} \\
D & = \text{inner pipe diameter (m)} \\
e & = \text{exponential function} \\
Ei & = \text{exponential integral} \\
f & = \text{Darcy-Weisbach friction coefficient} \\
f' & = \text{constant (1/m)} \\
f & = \text{vector function} \\
g & = \text{gravitational acceleration (m/s}^2) \\
h & = \text{flow depth (m)} \\
H_R & = \text{reservoir head (m)} \\
H & = \text{Heaviside step function} \\
K & = \text{entrance or exit loss coefficient} \\
K_e & = \text{elbow loss coefficient} \\
L & = \text{length of slug (m)} \\
L_{\text{front}} & = \text{length of slug front (m)} \\
L_{\text{pipe}} & = \text{distance travelled by slug front (m)} \\
m & = \text{mass of slug (kg)} \\
P & = \text{pressure (Pa)} \\
t & = \text{time (s)} \\
T & = \text{top width of flow section (m)} \\
v & = \text{velocity (m/s)} \\
w & = \text{time derivative of } L \text{ (m/s)} \\
x & = \text{axial position (m)} \\
y & = \text{vector of unknowns} \\
\alpha & = \text{constant} \\
\beta & = \text{holdup coefficient} \\
\gamma & = \text{constant} \\
\Delta P & = P_1 - P_2 \text{ (Pa)} \\
\Gamma & = \text{incomplete gamma function} \\
\Gamma^* & = \text{self-defined function} \\
\lambda & = \text{dummy variable} \\
\theta & = \text{angle of inclination of pipe (rad)} \\
\rho & = \text{mass density of liquid (kg/m}^3) \\
\end{align*}

Subscripts

\begin{align*}
c & = \text{contraction} \\
e & = \text{elbow} \\
hu & = \text{holdup} \\
l & = \text{liquid} \\
R & = \text{reservoir} \\
0 & = \text{initial value; constant value} \\
1 & = \text{slug front} \\
2 & = \text{slug tail} \\
3 & = \text{air intrusion front} \\
\infty & = \text{final value} \\
\end{align*}

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APPENDIX A

DERIVATION OF GOVERNING EQUATIONS AND ANALYTICAL SOLUTIONS

The moving liquid slug loses mass at its tail at a rate proportional to the distance travelled and leaves behind a liquid layer – the holdup – occupying a constant fraction $\beta$ of the pipe cross-sectional area $A$.

Conservation of mass

The moving mass balance is (Fig. 1)

$$\frac{d}{dt} \int_{x(t)}^{x(t)} \rho A \, dx = - \rho A \beta \frac{dx_2}{dt}(t), \quad (A1)$$

which directly leads to

$$\frac{dx_1}{dt}(t) - \frac{dx_2}{dt}(t) = - \beta \frac{dx_2}{dt}(t) \quad \text{or} \quad \frac{dx_1}{dt}(t) = (1 - \beta) \frac{dx_2}{dt}(t). \quad (A2)$$

Integration gives

$$L(t) - L(t_0) = - \beta \left( x_2(t) - x_2(t_0) \right) \quad \text{with} \quad L(t) = x_1(t) - x_2(t). \quad (A3)$$

In terms of $v_1$ and $L$, the governing equation is:

$$\frac{dL}{dt}(t) = - \beta \frac{v_1(t)}{1 - \beta}. \quad (A4)$$

This is exactly the same as in previous work [1, 2].

Conservation of momentum

The momentum balance (that is consistent with the moving mass balance) is

$$\frac{d}{dt} \int_{x(t)}^{x(t)} \rho A v(x,t) \, dx = - \rho A \beta \left( v_2(t) - v_{hu}(t) \right) \frac{dx_2}{dt}(t)$$

$$+ \left( P_2(t) - P_1(t) \right) A$$

$$- \frac{f}{2D} \int_{x_2(t)}^{x_1(t)} \rho A v^2(x,t) \, dx \quad (A5)$$

where $v_{hu}(t)$ is the velocity of the holdup directly behind the slug tail. For the sake of simplicity, and because it is an unknown factor, $v_{hu}(t)$ is taken zero herein, i.e., the holdup sticks to the pipe wall. Applying Leibniz’s rule results in ($v_{hu} = 0$)

$$v(x_1(t),t) \frac{dx_1}{dt}(t) - v(x_2(t),t) \frac{dx_2}{dt}(t) + \int_{x_1(t)}^{x_2(t)} \frac{\partial v}{\partial t}(x,t) \, dx =$$

$$-\beta v_2^2(t) + \frac{P_2(t) - P_1(t)}{\rho} - \frac{f}{2D} \int_{x_2(t)}^{x_1(t)} v^2(x,t) \, dx \quad (A6)$$

Rearranging gives

$$\int_{x_2(t)}^{x_1(t)} \frac{\partial v}{\partial t}(x,t) \, dx =$$

$$= -v_1^2(t) + (1 - \beta) v_2^2(t) + \frac{P_2(t) - P_1(t)}{\rho} - \frac{f}{2D} \int_{x_2(t)}^{x_1(t)} v^2(x,t) \, dx$$

$$= \beta v_1(t) v_2(t) + \frac{P_2(t) - P_1(t)}{\rho} - \frac{f}{2D} \int_{x_2(t)}^{x_1(t)} v^2(x,t) \, dx$$

$$= \frac{\beta}{1 - \beta} v_1^2(t) + \frac{P_2(t) - P_1(t)}{\rho} - \frac{f}{2D} \int_{x_2(t)}^{x_1(t)} v^2(x,t) \, dx$$

$$= \beta (1 - \beta) v_2^2(t) + \frac{P_2(t) - P_1(t)}{\rho} - \frac{f}{2D} \int_{x_2(t)}^{x_1(t)} v^2(x,t) \, dx \quad (A7)$$

8
The two integrals are approximated. Assuming that the velocity increases linearly from $v_1$ at $x_1$ to $v_2$ at $x_2$ yields
\[
\int_{x_1(t)}^{x_2(t)} \frac{\partial v}{\partial t} (x,t) \, dx \approx \frac{1}{2} \left( \frac{\partial v}{\partial t} (x_1(t),t) + \frac{\partial v}{\partial t} (x_2(t),t) \right) (x_1(t) - x_2(t)) \]
\[
= \frac{1}{2} \left( \frac{dv_1}{dt} (t) + \frac{dv_2}{dt} (t) \right) (x_1(t) - x_2(t)) \]
\[
= \frac{1}{2} \left( \frac{dv_1}{dt} (t) - \frac{1}{1 - \beta} L(t) \frac{dv_1}{dt} (t) \right) \]
\[
= (1 - \frac{1}{2} \beta) L(t) \frac{dv_1}{dt} (t)
\]
(A8)

\[
\int_{x_1(t)}^{x_2(t)} v^2 (x,t) \, dx \approx \frac{1}{3} \left( v^2 (x_1(t),t) + v(x_1(t),t)v(x_2(t),t) + v^2 (x_2(t),t) \right) \]
\[
= \frac{1}{3} \left( v_1^2 (t) + v_1(t)v_2 (t) + v_2^2 (t) \right) (x_1(t) - x_2(t)) \]
\[
= \frac{1}{3} L(t) v_1^2 (t) \]
\[
= (1 - \frac{1}{2} \beta + \frac{1}{2} \beta^2) L(t) v_1^2 (t)
\]

In terms of $v_1$ and $L$, the governing equation is:
\[
\frac{1 - \frac{1}{2} \beta}{1 - \beta} L(t) \frac{dv_1}{dt} (t) = \frac{\beta}{1 - \beta} v_1^2 (t) + \frac{P_2(t) - P_1(t)}{\rho} \]
\[
- \frac{f}{2D} \frac{1 - \frac{1}{2} \beta + \frac{1}{3} \beta^2}{(1 - \beta)^2} L(t) v_1^2 (t)
\]
(A10)

This is different from previous work [1, 2]; there are first-order (in $\beta$) corrections to both the inertia and friction term, and a factor 2 in the holdup term is absent here (conform Eq. (8)).

A minor improvement would be to make the friction factor $f$ dependent on $V$. However, for accelerating flows an unsteady friction model is strongly advised [11-14]. Starting from rest, the flow is initially laminar.

**Analytical solution**

An analytical solution can be derived for $v_1$ when the pressure difference $P_2 - P_1 > 0$ is constant. Equation (A4) is used to eliminate $v_1$ from Eq. (A10), so that
\[
L \frac{d^2 L}{dt^2} = - \frac{1 - \beta}{1 - \frac{1}{2} \beta} \left( \frac{dL}{dt} \right)^2 - \frac{\beta}{1 - \frac{1}{2} \beta} \frac{P_2(t) - P_1(t)}{\rho} \]
\[
+ \frac{f}{2D} \frac{1 - \frac{1}{2} \beta + \frac{1}{3} \beta^2}{(1 - \frac{1}{2} \beta)^2} \left( \frac{dL}{dt} \right)^2
\]
(A11)

Define $\frac{dL}{dt} = w(L)$, $\alpha = 2 \frac{1 - \beta}{1 - \frac{1}{2} \beta}$ and $f^* = \frac{f}{2D} \frac{1 - \frac{1}{2} \beta + \frac{1}{3} \beta^2}{(1 - \frac{1}{2} \beta)^2}$, use Eq. (12) and find the following linear first-order ODE in $w^*$,
\[
\frac{d w^*}{dL} + (\alpha - 2 f^*) w^* = - \frac{2 \beta}{L} \frac{P_2(t) - P_1(t)}{\rho}
\]
(A12)

The solution for the initial condition $w(L_0) = 0$,
\[
w^* = \frac{2 \beta}{L} \frac{P_2(t) - P_1(t)}{\rho} \left( \Gamma(\alpha, 2L f^*) \right) \left( \frac{L_0}{L} \right)^{\alpha} e^{2 f^* L},
\]
(A13)

is in terms of upper incomplete gamma functions $\Gamma$. The slug length $L$ is replaced by $L_0 - \frac{\beta}{1 - \beta} L_{pipe}$ (Eq. 10), which is the solution for $L$ as a function of $L_{pipe}$ (the distance travelled by the slug front). The symbolic formula (16) for $v_1$ (instead of $w^*$) follows then directly from Eq. (A4).

**Pipe emptying**

In pipe emptying [3]: $x_1(t) = x_L = \text{constant pipe length and} \beta$ represents liquid holdup, see Fig. 9. The slug length $L$ decreases from $x_L$ to zero. Note that $\frac{dx}{dt} = 0 \Rightarrow v_1 = (1 - \beta) v_2$.

In terms of $v_2$ and $L$, the governing equations are
\[
\frac{dL}{dt} = -v_2(t),
\]
\[
(1 - \frac{1}{2} \beta) L(t) \frac{dv_2}{dt} (t) = \beta (1 - \beta) v_2^2 (t) + \frac{P_2(t) - P_1(t)}{\rho}
\]
\[
- \frac{f}{2D} \frac{1 - \frac{1}{2} \beta + \frac{1}{3} \beta^2}{(1 - \beta)^2} L(t) v_2^2 (t) - \frac{K_2}{(1 - \beta)^2} v_2^2 (t)
\]
(A15)

**Figure 9** Sketch of pipe emptying.
where the last term represents resistance at the exit (at $x_1$).

The analytical solution for constant pressure difference $P_2 - P_1 > 0$ and initial condition $w(L_0) = 0$ is,

$$w^2 = \frac{2}{1 - \frac{1}{2} \beta} \frac{P_2 - P_1}{\rho} \left( \frac{\Gamma(\gamma, 2L \tilde{f})}{(2L \tilde{f})^\gamma} - \frac{\Gamma(\gamma, 2L_0 \tilde{f})}{(2L_0 \tilde{f})^\gamma} \right) e^{2\gamma L},$$  
(A15a)

where $\gamma := \alpha - \frac{K (1 - \beta)^2}{1 - \frac{1}{2} \beta}$ and $\tilde{f} := \beta f^*$.

The first argument of the incomplete gamma function $\Gamma$ must be positive. To allow for $-1 < \gamma < 0$, integrating by parts leads to

$$w^2 = \frac{2}{1 - \frac{1}{2} \beta} \frac{P_2 - P_1}{\rho} \left( \Gamma^*(L) - \Gamma^*(L_0) \right) e^{2\gamma L},$$  
(A15b)

with

$$\Gamma^*(L) := \frac{1}{\gamma} \left( \frac{\Gamma(\gamma + 1, 2L \tilde{f})}{(2L \tilde{f})^\gamma} - e^{-2\gamma L} \right).$$

The slug length $L$ is to be replaced by $L_0 - L_{pipe}$, where $L_{pipe}$ is the distance travelled by the slug tail. The symbolic formula for $v_2$ (instead of $w^2$) follows then directly from Eq. (A14).

The special – or limiting – case $f = 0$ gives the solution

$$v_2 = \frac{1}{\sqrt{(1 - \frac{1}{2} \beta)\gamma}} \frac{P_2 - P_1}{\rho} \left( \frac{L_0}{L} \right)^\gamma - 1 .$$  
(A15c)

For the case $\beta = 0$ and $K = 0$ the solution is

$$v_2 = e^{C_1 L \sqrt{2 \frac{P_2 - P_1}{\rho} \left( \text{Ei}(-2C_2 L_0) - \text{Ei}(-2C_2 L) \right)}},$$  
(A15d)

where $C_2 := \frac{f}{2D}$ and the difference of exponential integrals $\text{Ei}$ is herein calculated numerically according to

$$v_2 = e^{C_1 L \sqrt{2 \frac{P_2 - P_1}{\rho} \int_L^{L_0} e^{-2C_2 \xi} \, d\xi}}.$$  
(A15e)

### Pipe filling

In pipe filling [4, 5, 15, 16]: $x_2(t) = x_0 = 0$ and $\beta$ represents (gravity driven) air intrusion at the slug front, see Fig. 10. The slug length $L$ increases from $L_0 > 0$ to pipe length $x_1$. Note that $\frac{dx_2}{dt} = 0 \neq v_2$ and $\frac{dx_3}{dt} \neq v_3$, because $v$ is discontinuous at $x_3$ and therefore $v_3$ is not defined.

![Figure 10](https://via.placeholder.com/150)

**Figure 10** Sketch of pipe filling.

The mass balance is

$$\frac{d}{dt} \int_{x_2}^{x_1} \rho A_v(x,t) \, dx = \rho A v_2(t) ,$$  
(A16)

where $A_v$ is the cross-sectional flow area of the liquid. Separating (at $x_3$) the two regions sketched in Fig. 10 gives

$$\frac{d}{dt} \int_{x_2}^{x_1} \rho A \, dx + \frac{d}{dt} \int_{x_3}^{x_1} \rho (1 - \beta) A \, dx = \rho A v_2(t).$$  
(A17)

Downstream of $x_3$, either the flow area (in A16) or the mass density (not necessarily stratified flow) is reduced by a factor $1 - \beta$ (in A17); this depends on the chosen control volume. Take the derivatives of the integrals to find

$$\frac{d}{dt} \frac{dx_3}{dt} + (1 - \beta) \frac{d}{dt} (\frac{dx_1}{dt} - \frac{dx_3}{dt}) = v_2(t) \quad \text{or}$$

$$(1 - \beta) v_1(t) + \beta \frac{d}{dt} (\frac{dx_3}{dt}) = v_2(t) \quad \text{with}$$  
(A18)

$$\frac{d}{dt} (\frac{dx_3}{dt}) = v_1(t) - c \quad \text{so that}$$

$$v_2(t) = v_1(t) - \beta c,$$  
(A19)

where $c$ is the speed with which the air front propagates relative to the slug front (Fig. 10). The value of $c$ is estimated by [17]:

$$c = \sqrt{g \cos \theta (\frac{1 - \beta}{T})},$$  
(A20)
where $T$ is the top width of the prismatic flow section, $g$ is the gravitational acceleration and $\theta$ is the angle of pipe inclination. The speed $c$ is $\sqrt{gh}$, when $\theta = 0$ and the dimensionless flow depth $h/D$ (a function of $\beta$) equals 0.769 [5]. The speed $c$ grows to infinity when $h/D$ approaches unity.

The momentum balance is (assuming that $P_2 = P_1$)

$$\frac{d}{dt} \int_{x_2}^{x_1} \rho A_v(x,v) \, dx = \rho A_v^2 \nu (t) + (P_2(t) - P_1(t)) A$$

so that

$$\frac{d}{dt} \int_{x_2}^{x_1} \rho A_v(x,v) \, dx + \frac{d}{dt} \int_{x_1}^{x_2} \rho (1-\beta) A_v(x,v) \, dx =$$

$$\rho A_v^2 \nu (t) + (P_2(t) - P_1(t)) A$$

(A21)

Apply Leibniz’s rule to arrive at

$$v(x(t),t) \frac{dx}{dt} + \int_{x_2}^{x_1} \frac{\partial v}{\partial t} (x,v) \, dx +$$

$$(1-\beta) \left[ v(x_1(t),t) \frac{dx_1}{dt} - v(x_1^*(t),t) \frac{dx_1^*}{dt} \right] +$$

$$\int_{x_1}^{x_1^*} \frac{\partial v}{\partial t} (x,v) \, dx = v_2^2(t) + \frac{P_2(t) - P_1(t)}{\rho}$$

(A22)

$$- \frac{f}{2D} (x_3(t) - x_2^*) v_2^2(t) - \frac{f}{2D} (x_1(t) - x_3(t)) (1-\beta) v_1^2(t)$$

or (using A19 and $x_1(t) = L(t)$)

$$\int_{x_2}^{x_1} \frac{\partial v}{\partial t} (x,v) \, dx - \beta \int_{x_1}^{x_1^*} \frac{\partial v}{\partial t} (x,v) \, dx =$$

$$(v_1(t) - \beta c)^2 - (1-\beta) v_1^2(t) - \beta (v_1(t) - c)^2$$

$$+ \frac{P_2(t) - P_1(t)}{\rho}$$

$$- \frac{f}{2D} \left[ L(t) (v_1(t) - \beta c)^2 - \beta c t (v_1(t) - c)^2 + \beta (1-\beta) c^3 t \right]$$

(A23)

Note that $\frac{dv_1}{dt} = \frac{dv_2}{dt} = \frac{d^2 x_3}{dt^2}$, because $\frac{dc}{dt} = 0$.

The governing equations in terms of $L$ and $v_1$ are

$$\frac{dL}{dt} = v_1(t)$$

(A24)

$$\left( L(t) - \beta c t \right) \frac{dv_1}{dt} = -\beta (1-\beta) c^2 + \frac{P_2(t) - P_1(t)}{\rho}$$

$$- \frac{f}{2D} \left[ L(t) (v_1(t) - \beta c)^2 - \beta c t (v_1(t) - c)^2 + \beta (1-\beta) c^3 t \right]$$

(A25)

where $x_1(t) - x_3(t) = ct$ has been used.

These formulas are an improvement with respect to the 1D model used in [5], where $\beta$ was 0. One open question remains: how to calculate $\beta$?

Analytical solutions are derived for the case $\beta = 0$ (no air intake, hence $c = 0$ and $v_1 = v_2$). For filling from a reservoir with constant head $H_R$ into an empty pipe, the governing equations are (A24) and:

$$L(t) \frac{dv_1}{dt} = gH_R \left( \frac{K+1}{2} \right) v_1^2(t) - \frac{f}{2D} L(t) v_1^2(t)$$

(A25a)

where $P_2(t) - P_1(t) = \rho g H_R$ and the $K+1$ term accounts for entrance loss and velocity head [4, 5]. The semi-analytical solution for initial conditions $v_1(t_0) = 0$ and $L(t_0) = L_0 > 0$ is

$$v_1 = L^\frac{K+1}{2} e^{-\frac{f}{2D} \int_{L_0}^{L} L^K e^{2C_2L'} dL'}$$

(A26)
with \( C_2 = \frac{f}{2D} \). The slug length \( L \) is to be replaced by \( L_0 + L_{\text{pipe}} \), where \( L_{\text{pipe}} \) is the distance travelled by the slug front.

The special case \( f = 0 \) (and \( \beta = 0 \)) gives the solution

\[
v_1 = \sqrt{\frac{2}{K+1} g \frac{H_R}{K+1} \left(1 - \left(\frac{L_0}{L}\right)^{K+1}\right)},
\]

\( \text{(A27)} \)

with limit value \( v_{1,\infty} = \sqrt{\frac{2g H_R}{K+1}} \)

**APPENDIX B**

**MATHCAD COMPUTATIONS**

B1 Slug – Parameter variation

B2 Slug – Validation

B3 Pipe emptying

B4 Pipe filling
Parameter variation study

Input data (Bozkus, Bara, Ger, 2004, JPTV 126, 241-249) Case 1

\( p = 1000 \)
\( \beta = 0.20 \)
\( f = 0.006 \)
\( D = 0.61 \)
\( t_{b_{pp}} = 8.5 \)
\( L_0 = 1.0 \)
\( P_1 = 500000 \)
\( L_g = 3 \)
\( v_g = 0 \)
\( C_f = 0.51 \)
\( K_w = 0.9 \)
\( P_2 = 0 \)
\( \eta_g = 0.2506 \)
\( C_f = \frac{P_2}{4D} \)
\( C_2 = \frac{4}{D} \)
\( t_{s_{b_{pp}}} = 3.0 \)

Exact solution for \( \beta = 0 \)

\[ v_d = \frac{3}{2} \]  
\[ t_{b_{pp}} = 14.434 \]

Exact solution for \( f = 0 \)

\[ \alpha(0) = \frac{1}{1 - \beta} \]
\[ \alpha(L) = \frac{1}{1 - \beta} \]
\[ v_{dy}(L) = \frac{1}{1 - \beta} \]
\[ v_{dy}(0) = 0 \]
\[ v_{dy}(t_{b_{pp}}) = 154.684 \]

Ode RHS

\[ E(1,y) = \begin{bmatrix} 0 \\ \frac{1 - \beta}{\beta} \frac{dy}{dt} \\ \frac{1}{4} \frac{D}{\beta} \frac{d^2y}{dt^2} \end{bmatrix} \]

Euler forward Numerical solution

\[ y_{n+1} = y_n + \Delta t \frac{dy}{dt} \]

\[ k = 0 \]
\[ x = k \Delta t \]
\[ t_{b_{pp}} = \frac{28.526}{148.148} \]
\[ \frac{28.526}{148.148} \times 3 \]

Acceleration

\[ a_y = \frac{dy}{dt} \]

\[ a_y = \frac{\Delta y}{\Delta t} \]

Slag velocity as a function of \( x \)

Slag velocity as a function of \( t \)
Mathcad 15 CASA-14-16_App-B1.xmcd 5

Results (t, x, L, v)

\[ s_2 = [s_2]_2 \]
\[ s_1 = [s_1]_1 \]
\[ s_0 = [s_0]_0 \]

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Mathcad 15 CASA-14-16_App-B1.xmcd 6

Slug length as function of t

Slug position as function of t

Slug velocity as function of x and \( \beta \)

Acceleration as function of t

Same figure
Validation

Input data  (Bozkus, Bara, Ger, 2004, JPVT 126, 241-249)

\[ \beta = 0.98 \]
\[ \ell = 0.005 \]
\[ D = 0.1 \]
\[ T_p = 500000 \]
\[ L_p = 3 \]
\[ y = 0 \]
\[ C_p = 0.5 \]
\[ \mu = 0.8 \]
\[ \rho = 0 \]
\[ C_1 = \frac{P_2}{L_p} \]
\[ C_2 = \frac{L_0}{2D} \]
\[ \tan \theta = 3 \]
\[ \mu_0 = 8.5 \]
\[ \mu_1 = 1 \]

Elbow "resistance"

\[ \nu_0 = 0.371 \]
\[ \nu_1 = 44.8 \]
\[ \nu_2 = 3.08 \]
\[ \nu_3 = 2.36 \]

\[ \nu_{\text{sr}} = \text{slug arrival time} \]
\[ \nu_{\text{v}} = \text{slug velocity} \]

\[ \nu_{\text{v}} = \nu_{\text{v}} \text{ from Table of Results} \]

\[ R_{\text{v}}(1) = \left( \frac{1}{C_1^2} + C_2^2 L_0 \right) \]
\[ = 7.34 \]

\[ R_{\text{v}}(2) = 1.07 \]

Exact solution for \( \beta = 0 \)

\[ \nu_{\text{v}}(x) = \left( \frac{1}{L_0} - \beta L_0 \right) \]
\[ \nu_{\text{v}}(L_0) = 3.08 \]
\[ \nu_{\text{v}}(0) = 2.36 \]

Exact solution for \( \beta = 0 \)

\[ \nu_{\text{v}}(x) = \left( \frac{1}{L_0} - \beta L_0 \right) \]
\[ \nu_{\text{v}}(L_0) = 3.08 \]
\[ \nu_{\text{v}}(0) = 2.36 \]

Numerical solution

\[ y = \left[ \begin{array}{c} \frac{y_1}{y_2} \\ \frac{y_0}{y_1} \end{array} \right] \]
\[ R_{\text{v}}(1) = \left( \frac{1}{C_1^2} + C_2^2 L_0 \right) \]
\[ = 7.34 \]

\[ R_{\text{v}}(2) = 1.07 \]

ODE RHS  (18)

\[ \nu_{\text{v}}(x) = \left( \frac{1}{L_0} - \beta L_0 \right) \]

Euler forward: Numerical solution

\[ y_k = \left( \begin{array}{c} y_0 \\ y_1 \end{array} \right) \]
\[ \Delta x = 0.001 \]
\[ N = 1000 \]

\[ \alpha = 0.989 \]
\[ \beta = 0.98 \]
\[ \ell = 0.005 \]
\[ D = 0.1 \]

Approximate solution for \( \beta = 0 \) and \( 2 \leq L_0 \leq 37 \)

\[ y_{\text{v}}(x) = \left[ \begin{array}{c} \frac{y_1}{y_2} \\ \frac{y_0}{y_1} \end{array} \right] \]
\[ \nu_{\text{v}}(x) = \left( \frac{1}{L_0} - \beta L_0 \right) \]
\[ \nu_{\text{v}}(L_0) = 3.08 \]
\[ \nu_{\text{v}}(0) = 2.36 \]

Approximate solution for \( \beta = 0 \) and \( 2 \leq L_0 \leq 37 \)

\[ y_{\text{v}}(x) = \left[ \begin{array}{c} \frac{y_1}{y_2} \\ \frac{y_0}{y_1} \end{array} \right] \]
\[ \nu_{\text{v}}(x) = \left( \frac{1}{L_0} - \beta L_0 \right) \]
\[ \nu_{\text{v}}(L_0) = 3.08 \]
\[ \nu_{\text{v}}(0) = 2.36 \]
Slug velocity as function of x:

\[ \frac{\nu_s}{\nu_{s0}} = \frac{n_s}{n_{s0}} \]

Slug velocity as function of t:

\[ \nu_t = \frac{\nu_{t0}}{\sqrt{n_t(n_{t0})}} \]

Slug length as function of t:

\[ L_t = \frac{L_{t0}}{n_t(n_{t0})} \]

Slug position as function of t:

\[ y_t = \frac{y_{t0}}{n_t(n_{t0})} \]

Results (t, x, L, v):

\[ a_t = \frac{a_{t0}}{n_t(n_{t0})}, \quad b_t = \frac{b_{t0}}{n_t(n_{t0})}, \quad c_t = \frac{c_{t0}}{n_t(n_{t0})} \]

Table:

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 Slug impact pressure as function of t:

\[ P_t = \frac{n_t(n_{t0})}{\sqrt{\Phi_t}} \]

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Measurements
Pipe emptying

Input data (Test case)

\[ \beta = 0.2 \quad f = 0.016 \quad D = 0.01 \quad L_{\text{pipe}} = 12 \quad L_0 = 1 \times 10^6 \]

\[ P_2 = 300000 \quad L_0 = L_{\text{pipe}} \quad v_0 = 0 \quad C_1 = 0.7 \quad K_p = 1.0 \]

\[ P_1 = 0 \quad m_0 = \rho \cdot \pi \cdot D^2 \cdot L_0 \quad m_0 = 0.942 \quad C_1 = P_2 - P_0 \quad C_2 = 0.9 \]

\[ C_1 = 16.447 \quad C_2 = 0.8 \]

\[ \text{inlet} = 3 \times 10^6 \]

"Exact" solution for \( \beta = 0 \) and \( K_p = 0 \)

\[ m_{\text{in}}(t) := \frac{\beta}{2} \cdot \frac{L_0}{C_1} \cdot \left( \frac{P_2 - P_0}{\rho} \right) \cdot \left( \frac{L_0}{L_0} \right)^2 \]

\[ \gamma_{\text{in}}(t) := \gamma_{\text{in}}(t) = \frac{\beta}{2} \cdot \frac{L_0}{C_1} \cdot \left( \frac{P_2 - P_0}{\rho} \right) \cdot \left( \frac{L_0}{L_0} \right)^2 \]

Exact solution for \( f = 0 \)

\[ m_{\text{in}}(t) := \frac{\beta}{2} \cdot \frac{L_0}{C_1} \cdot \left( \frac{P_2 - P_0}{\rho} \right) \cdot \left( \frac{L_0}{L_0} \right)^2 \]

\[ \gamma_{\text{in}}(t) := \gamma_{\text{in}}(t) = \frac{\beta}{2} \cdot \frac{L_0}{C_1} \cdot \left( \frac{P_2 - P_0}{\rho} \right) \cdot \left( \frac{L_0}{L_0} \right)^2 \]

ODE RHS

\[ f(t, y, \beta) := \begin{bmatrix} \frac{d}{dt} y_1 \\ \frac{d}{dt} y_2 \\ \frac{d}{dt} y_3 \\ \frac{d}{dt} y_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{1 - \frac{f}{\beta}} y_2 \\ \frac{1}{1 - \frac{f}{\beta}} y_3 \\ \frac{1}{1 - \frac{f}{\beta}} y_4 \\ \frac{1}{1 - \frac{f}{\beta}} \left( \frac{1 - \frac{f}{\beta}}{y_4} \right) \end{bmatrix} \]

Euler forward

Numerical solution

\[ y_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \Delta t = 0.0002 \quad N = 9665 \quad \rho \left( \frac{P_2}{L_0} \right) = 18.518519 \]

\[ k = 0 \quad N = 0 \quad \Delta t = 1.883 \quad y_0 = \begin{bmatrix} 18.518519 \\ 0 \end{bmatrix} \]

\[ y_{k+1} = y_k + \Delta t \cdot f(y_k, \beta) \]

\[ y_0 = \begin{bmatrix} 3.704 \times 10^{-5} \\ 1.284 \end{bmatrix} \quad y_1 = \begin{bmatrix} 7.004 \times 10^{-3} \\ 0 \end{bmatrix} \]

\[ y_2 = \begin{bmatrix} 7.004 \times 10^{-3} \\ 0 \end{bmatrix} \]

\[ y_3 = 34.719565 \times 10^{-3} \times \beta \]

\[ y_4 = 12.00710786 \]

\[ \frac{P_2}{L_0} = \beta \]

Acceleration

\[ a = \frac{f(y_2)}{\Delta t} \]

\[ a_1 = 18.518519 \quad a_2 = 18.518519 \]

\[ a_3 = 18.518519 \quad a_4 = 18.518519 \]

Exact solution for \( f > 0 \) and \( 2 \times f < 26 \)

\[ \Gamma(t) = \left( \frac{m_0}{\rho} \right) \left( \frac{P_2 - P_0}{\rho} \right) \left( \frac{L_0}{L_0} \right)^2 \]

\[ v_{\text{ex}}(t) := \gamma_{\text{ex}}(t) = \gamma_{\text{ex}}(t) = \frac{\beta}{2} \cdot \frac{L_0}{C_1} \cdot \left( \frac{P_2 - P_0}{\rho} \right) \cdot \left( \frac{L_0}{L_0} \right)^2 \]

\[ \gamma_{\text{ex}}(t) := \gamma_{\text{ex}}(t) \]

\[ v_{\text{ex}}(t) := \gamma_{\text{ex}}(t) = \gamma_{\text{ex}}(t) = \frac{\beta}{2} \cdot \frac{L_0}{C_1} \cdot \left( \frac{P_2 - P_0}{\rho} \right) \cdot \left( \frac{L_0}{L_0} \right)^2 \]

Slug velocity as function of \( x \)

\[ \beta = 0.2 \quad f = 0.016 \quad K_p = 1 \]

Slag velocity as function of \( x \) and \( \beta \)
Slug length as function of $t$}

![Slug length graph]

Acceleration as function of $t$}

![Acceleration graph]

Results \((b, x, L, v)\)

\[
\begin{align*}
a_k &= \left[ y_k \right]_2 \\
b_k &= \left[ y_k \right]_1 \\
c_k &= \left[ y_k \right]_0
\end{align*}
\]

<table>
<thead>
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<th>3</th>
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<tbody>
<tr>
<td>a</td>
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<td>c</td>
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Table \(a, b, c\)
**Pipe filling**

**Input data (Test case)**

- \( p = 1000 \)
- \( f = 0.046 \)
- \( D = 0.01 \)
- \( L_{\text{pipe}} = 12 \)
- \( C_1 = 1.5 \)

**Input data**

- \( v_{bf0} = 0 \) m/s
- \( v_{bf0} = 2 \) m/s
- \( P_2 = 300000 \)
- \( \rho = 0 \)
- \( C_1 = 0.5 \)
- \( C_2 = 0.8 \)

**Euler forward**

\[
\begin{align*}
\Delta t &= \frac{L}{v_{bf0}}, \\
C_1 &= 200, \\
C_2 &= 0.8, \\
L_{\text{pipe}} &= 10.
\end{align*}
\]

**Results**

<table>
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<tr>
<th>( x )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( y_3 )</th>
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<tr>
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<td>0.15</td>
<td>0.15</td>
</tr>
</tbody>
</table>

**Slug velocity as function of x**

\[
\begin{align*}
v_0 &= \frac{\rho_0}{\rho} \left( \frac{P_2}{P_1} \right)^{\frac{1}{2}}, \\
\gamma &= \frac{\left( \frac{P_2}{P_1} \right)^{\frac{1}{2}}}{L}, \\
\beta &= \frac{\rho_0}{\rho} \left( \frac{P_2}{P_1} \right)^{\frac{1}{2}} \cdot \sqrt{\frac{L}{v_{bf0}}},
\end{align*}
\]

**Table:**

<table>
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<tr>
<th>Table 1:</th>
<th>( x )</th>
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<th>( y_2 )</th>
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**Table 3:**

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**Results**

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# Previous Publications in This Series:

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<th>Number</th>
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<th>Title</th>
<th>Month</th>
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<tbody>
<tr>
<td>14-12</td>
<td>O. Krehel, A. Muntean, P. Knabner</td>
<td>Multiscale modeling of colloidal dynamics in porous media: Capturing aggregation and deposition effects</td>
<td>Apr. ’14</td>
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<tr>
<td>14-13</td>
<td>J.C. van der Meer, F. Crespo, S. Ferrer</td>
<td>Generalized Hopf fibration and geometric SO(3) reduction of the 4DOF harmonic oscillator</td>
<td>Apr. ’14</td>
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<tr>
<td>14-14</td>
<td>J. de Graaf</td>
<td>Matrix Gauge fields and Noether’s theorem</td>
<td>May ’14</td>
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<tr>
<td>14-16</td>
<td>A.S. Tijsseling, Q. Hou, Z. Bozkus</td>
<td>An improved 1D model for liquid slugs travelling in pipelines</td>
<td>May ’14</td>
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