Plasticity of strain-hardening materials

Citation for published version (APA):

Document status and date:
Published: 01/01/1967

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

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Download date: 30. Jan. 2020
**technische hogeschool eindhoven**
laboratorium voor mechanische technologie en werkplaatstechniek

**rapport van de sectie:** Workshop Engineering

**titel:**
Plasticity of Strain-hardening Materials

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**samenvatting**
This report is a summary of theory and applications concerning the mechanics of plasticity of strain-hardening materials subjected to finite strains. It aims at giving a review of methods of calculation such as may be used for research in workshop engineering. It may also serve as a basis for lectures on mechanics of plasticity.

*(translation of report No. 0168)*

**prognose**
Several of the results obtained may in the future be verified by experiments and by means of numerical calculations.
Design entailing plastic flow is

sometimes dangerous
often prohibited
always unavoidable

(From a lecture of Prof. Dr. J.B. Alblas)
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Mechanics of plasticity for finite deformations, applied on strain-hardening materials is mathematically very difficult to deal with. In technical respect, however, the subject is rather important, especially in the case of metal processing. This Thesis tries to develop a relatively simple method of calculating these problems.

In doing so, it wishes to serve two purposes: First, it is intended to be a summary of the applied theory of plasticity as it may be used for research purposes. Secondly, it might be the basis for a series of lectures for third-year students in the Technological University of Eindhoven.

The material has been collected from the literature and from original research.

As for the manner of treating the material, our basic thought was that a - sometimes rough - mathematical approximation of the actual situation (that is: strain hardening and finite strains) often makes more sense than an exact treatment starting from incorrect assumptions (that is: ideal plastic material and infinitesimal strains).

Finally, it should be emphasised that part of this Thesis is considered to be a starting point for experiments. Some of the theories have already been verified, while others are being verified at the moment. Hence, it may appear in due course that some of the material offered will have to be adapted afterwards.

LITERATURE INDEX


6. Rheology, vol.I, chapter 4 D.C. Drucker - Stress Strain relations in the plastic range of metals - Experiments and basic concepts.

8. Ir.W. Grijm - Plasticiteit.
   (Technological University Delft, Netherlands).

1. Stresses \[1\] *)

1.1. Stress vector and stress tensor.

The state of stress in a point \(P\) of a medium is mathematically described by the stress tensor. We consider a plane-element \(dS\), through \(P\), and parallel to the \(YOZ\) plane of a Cartesian coordinate system \(XYZ\).

The material on one side of the element generally transmits a force to the material on the other side. We will refer to this force as \(\overrightarrow{dK}\).

We define the stress vector in \(P\) as:

\[
\overrightarrow{P} = \frac{\overrightarrow{dK}}{dS}
\]  

(1-1)

We shall decompose this stress vector into a normal stress, perpendicular to the plane, and a shear stress, parallel to it. We shall call the normal stress \(\sigma_x\), the components of the shear stress in the directions \(Y\) and \(Z\) \(\tau_{xy}\) and \(\tau_{xz}\) respectively.

Similarly, we have for a plane perpendicular to the \(Y\)-axis \(\sigma_y\), \(\tau_{yx}\) and \(\tau_{yz}\), and for a plane perpendicular to the \(Z\)-axis \(\sigma_z\), \(\tau_{zy}\) and \(\tau_{zx}\) (Fig. 1-1).

These 9 quantities are the components of the stress tensor:

\[
\begin{pmatrix}
\sigma_x & \tau_{yx} & \tau_{zx} \\
\tau_{xy} & \sigma_y & \tau_{zy} \\
\tau_{xz} & \tau_{yz} & \sigma_z
\end{pmatrix}
\]

Fig. 1-1

Components of the stress tensor near \(P\).

Thus, the first index of \(\tau\) refers to the plane along which it works, the second determines its direction.

*) Numbers in square brackets refer to the literature index.
1.2. The conditions of equilibrium

Fig. 1-2. Equilibrium of stresses near P.
Cartesian coordinates.

In order to derive the conditions of equilibrium we consider an infinitesimal element (Fig. 1-2), loaded with stresses. We assume that all stresses can be differentiated with respect to their place. If, e.g., the normal stress on the YOZ plane is \( \sigma_x \), then the normal stress on a parallel plane at a distance \( dx \) will be \( \sigma_x + \frac{\partial \sigma_x}{\partial x} dx \), etc.

For the equilibrium of moments with respect to the Z-axis, we derive

\[
2\tau_{xy} \frac{\partial x}{\partial x} \frac{dx}{dy} \frac{dz}{dy} - 2\tau_{yx} \frac{\partial x}{\partial y} \frac{dx}{dy} \frac{dz}{dy} + \frac{\partial \tau_{xy}}{\partial x} \frac{dy}{dy} \frac{dz}{dy} - \frac{\partial \tau_{yx}}{\partial y} \frac{dx}{dx} \frac{(dy)^2}{dz} = 0
\]

(1-2)

The last two terms are small of higher order than the first two and may be neglected. Then we find

\[
\tau_{xy} = \tau_{yx}
\]

(1-3)

and, similarly, \( \tau_{yz} = \tau_{zy} \), and \( \tau_{zx} = \tau_{xz} \). The stress tensor is symmetrical.

From the equilibrium of forces in direction \( x \) we find

\[
(\sigma_x + \frac{\partial \sigma_x}{\partial x} \frac{dx}{dy} \frac{dz}{dy})dydz + (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \frac{dz}{dy} \frac{dy}{dy} \frac{dz}{dy})dx dy + (\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \frac{dy}{dy} \frac{dz}{dy} \frac{dy}{dy})dx dz +
\]

\(- \sigma_x \frac{dy}{dy} \frac{dz}{dz} - \tau_{zx} \frac{dx}{dx} \frac{dy}{dy} - \tau_{yx} \frac{dx}{dx} \frac{dz}{dz} = 0
\]

(1-4)

From (1-4) we can derive

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0
\]

(1-5a)
and, by cyclic changing of indices,

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (1-5b)
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0. \quad (1-5c)
\]

When we use a cylindrical coordinate system, we find for a wedge-shaped element (Fig. 1-3)

\[
\frac{\partial \sigma}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta \phi}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau \theta - \tau_{\theta}}{r} = 0
\]

\[
\frac{\partial \tau_{\theta \phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2 \tau_{r \theta}}{r} = 0 \quad (1-6)
\]

\[
\frac{\partial \tau_{rz}}{\partial z} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rz}}{r} = 0.
\]

Finally, for spherical coordinates, we find (Fig. 1-4)

\[
\text{Fig. 1-4. Equilibrium of stresses near P. Spherical coordinates.}
\]
\[
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta \phi}}{\partial \phi} + \frac{1}{r} (2\sigma_r - \sigma_\theta - \sigma_\phi + \tau_{r\theta} \cot \theta) = 0
\]

\[
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta \phi}}{\partial \phi} + \frac{1}{r} (\sigma_\theta - \sigma_\phi) \cot \theta + 3\tau_{r\theta} = 0
\]  
(1-7)

\[
\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\phi \theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta \phi}}{\partial \phi} + \frac{1}{r} (3\tau_{r\phi} + 2\tau_{\phi \theta} \cot \theta) = 0.
\]

1.3. Principal directions. Invariants

If the stresses in \( P \) on three perpendicular planes are given, we can calculate the stress vector \( \vec{p} \) in any plane through \( P \). We call the direction cosines of the unity vector \( \hat{n} \) on the plane 1, m and n. The components of \( \vec{p} \) are then given by (Fig. 1-5)

\[
\begin{align*}
 p_x &= \sigma_x l + \tau_{yx} m + \tau_{zx} n \\
p_y &= \tau_{xy} l + \sigma_y m + \tau_{zy} n \\
p_z &= \tau_{xz} l + \tau_{yz} m + \sigma_z n
\end{align*}
\]
(1-8)

Fig. 1-5. Stresses on a plane near \( P \).

Proof of (1-8). Consider OABC. If the area of \( \Delta ABC = 1 \), then area

area \( \Delta OCB = 1 \),
area \( \Delta OCA = m \),
area \( \Delta OAB = n \).

The equations (1-8) then follow from the equilibrium of forces in directions \( X, Y \) and \( Z \).

\[
|\vec{p}| \text{ follows from } \\
|\vec{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2} \]
(1-9)
The normal and shear stresses on the plane are then found by

\[ \sigma = p_x \cdot l + p_y \cdot m + p_z \cdot n \]

\[ \tau = \sqrt{p^2 - \sigma^2} \]  \hspace{1cm} (1-10)

A principal direction is defined as a direction in which all shear stresses are zero.

In the case of Fig. 1-5, \( \vec{p} \) and \( \vec{n} \) have then the same direction. Thus

\[ p_x = \sigma \cdot l \]
\[ p_y = \sigma \cdot m \]
\[ p_z = \sigma \cdot n \]  \hspace{1cm} (1-11)

Substitution of (1-11) in (1-8) gives

\[ (\sigma_x - \sigma)l + \tau_{yx} \cdot m + \tau_{zx} \cdot n = 0 \]
\[ \tau_{xy} \cdot l + (\sigma_y - \sigma) m + \tau_{zy} \cdot n = 0 \]  \hspace{1cm} (1-12)
\[ \tau_{xz} \cdot l + \tau_{yz} \cdot m + (\sigma_z - \sigma) n = 0. \]

Using, in addition,

\[ l^2 + m^2 + n^2 = 1, \]  \hspace{1cm} (1-13)

We may solve \( \sigma, l, m, \) and \( n \) from (1-12) and (1-13).

The homogenous linear set of equations (1-12) only has solutions if

\[ \begin{vmatrix} \sigma_x - \sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0 \]  \hspace{1cm} (1-14)

The solution of this 3rd-power equation in \( \sigma \) always gives three real solutions. The matching principal direction are orthogonal.

For (1-14) we write

\[ \sigma^3 - I_1 \cdot \sigma^2 + I_2 \cdot \sigma - I_3 = 0. \]  \hspace{1cm} (1-15)
As the principal stresses in a point are the same, independent of the coordinate system, chosen \( I_1, I_2 \) and \( I_3 \) must be invariant for rotations of the coordinate system. They are the invariants of the stress tensor. We find

\[
I_1 = \sigma_x + \sigma_y + \sigma_z
\]

\[
I_2 = \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2
\]

\[
I_3 = \sigma_x \sigma_y \sigma_z + 2 \tau_{xy} \tau_{yz} \tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{zx}^2 - \sigma_z \tau_{xy}^2.
\]  

(1-16)

Switching to principal stresses, we write 1, 2 and 3 instead of \( x, y \) and \( z \), while all shear stresses disappear. From (1-16) we then find

\[
I_1 = \sigma_1 + \sigma_2 + \sigma_3
\]

\[
I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1
\]

\[
I_3 = \sigma_1 \sigma_2 \sigma_3
\]  

(1-17)

\( I_1 \) depends on the hydrostatic pressure \( p \).

We define this quantity as

\[
\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = -p.
\]  

(1-18)

If one main stress is zero, we have a plane state of stress, if two main stresses are zero, we have a linear state of stress.

Let \( \sigma_1 = 0 \). We choose the \( X \) and \( Y \) axes perpendicular to \( \sigma_3 \), thus \( \sigma_1 \) and \( \sigma_2 \) lying in the \( XOY \) plane. Then \( \sigma_3 = \sigma_2 = 0 \).
As \( \sigma_z = \sigma_2 \), the XOY plane is a principal one. Thus \( T_{zx} = T_{zy} = 0 \).

As in the equations (1-16) and (1-17) respectively, the magnitude of \( I_1 \) and \( I_2 \) respectively is the same. We find:

\[
\begin{align*}
\sigma_1 &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + T_{xy}^2} \\
\sigma_2 &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + T_{xy}^2}
\end{align*}
\]

(1-19)

Fig. 1-6 shows the Mohr-circle for the plane state of stress. We see that its geometrical properties match (1-19).

We can extend this figure, by also giving the directions of the planes on which the stresses work. If an element is loaded with a known magnitude of \( \sigma_x, \sigma_y \) and \( \sigma_z \), we will call \( \Omega \) the angle between the X-axis and the direction of \( \sigma_1 \) (Fig. 1-7). The pole \( P \) represents the point of intersection of the planes on which the stresses work. The vectorial sum of \( \sigma_x \) and \( T_{xy} \) gives the magnitude (not the direction!) of \( P_x \), the total stress on the plane parallel to the YOZ plane.
Fig. 1-7. Determination of the directions of the principal planes.

Using $\Omega$, we find:

centre: $\frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_1 + \sigma_2}{2}$

radius: $\tau_{\text{max}} = \frac{\sigma_1 - \sigma_2}{2} = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2} = \frac{\tau_{xy}}{\sin 2\Omega} = \frac{\sigma_x - \sigma_y}{2 \cos 2\Omega}$

Principal stresses:

$$\begin{cases} \sigma_1 \\ \sigma_2 \end{cases} = \frac{\sigma_x + \sigma_y}{2} \pm \tau_{\text{max}}$$

Using principal stresses, we find

$$\tau_{xy} = \tau_{\text{max}} \sin 2\Omega = \frac{\sigma_1 - \sigma_2}{2} \sin 2\Omega .$$

$$\begin{cases} \sigma_x \\ \sigma_y \end{cases} = \frac{\sigma_1 + \sigma_2}{2} \pm \frac{\sigma_1 - \sigma_2}{2} \cos 2\Omega .$$

Fig. 1-8 gives the formulae for a coordinate transformation.
Fig. 1-8. Rotation of coordinate system over an angle $\varphi$.

\[
\sigma_x = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\varphi \\
\sigma_y = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\varphi \\
\tau_{xy} = \frac{\sigma_1 - \sigma_2}{2} \sin 2\varphi \\
\sigma_{x'} = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2(\varphi - \varphi') \\
\sigma_{y'} = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2(\varphi - \varphi') \\
\tau_{x'y'} = \frac{\sigma_1 - \sigma_2}{2} \sin 2(\varphi - \varphi')
\]

Fig. 1-9 gives another picture of the stresses as they work on different planes near the same point $P$. 

Fig. 1-9. Stresses near $P$
Remark. When deriving the equilibrium of moments (1-3) the sign convention of $T_{xy}$ was as in Fig. 1-10.

Fig. 1-10. Sign convention of $T_{xy}$ according to equilibrium of moments.

Fig. 1-11. Sign convention of $T_{xy}$ according to Mohr circle.

For the circle of Mohr, however, we have a convention as in Fig. 1-11. Our calculations will always contain the sign convention as used in Fig. 1-10.

For the general state of stress it is also possible to draw circles of Mohr. We will, however, not deal with these here.

1.4. The stress deviator. [1]

We may interpret the stress tensor as the sum of two other tensors: the deviator stress tensor and the hydrostatic stress tensor.

$$
\begin{pmatrix}
\sigma_x & \tau_{yx} & \tau_{zx} \\
\tau_{xy} & \sigma_y & \tau_{zy} \\
\tau_{xz} & \tau_{yz} & \sigma_z
\end{pmatrix} = \begin{pmatrix}
\sigma_x - \sigma_m & \tau_{yx} & \tau_{zx} \\
\tau_{xy} & \sigma_y - \sigma_m & \tau_{zy} \\
\tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m
\end{pmatrix} + \begin{pmatrix}
\sigma_m & 0 & 0 \\
0 & \sigma_m & 0 \\
0 & 0 & \sigma_m
\end{pmatrix}
$$

(1-20)

deviator stress tensor

hydrostatic stress tensor
We write
\[ G^{-1}_x = \frac{G_x}{G_h}, \text{ etc.} \] (1-21)

In a deformable medium the hydrostatic stress tensor causes a change of volume while the shape remains the same; the deviator stress tensor causes changes of shape at a constant volume.

The invariants of the deviator stress tensor are:

\[ I'_1 = 0 \]
\[ I'_2 = \sigma_y \sigma' + \sigma_z \sigma' - \tau_{xy} - \tau_{yz} - \tau_{zx} \]
\[ I'_3 = \sigma_x \sigma' + 2 \tau_{xy} \tau_{yz} \tau_{zx} - \sigma_y \tau^2 - \sigma_z \tau^2 - \sigma_x \tau^2 \]

Substituting (1-21) and (1-18), it follows for \[ I'_2 \]

\[ I'_2 = -\frac{1}{3} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x) - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \] (1-23)

1.5. The yield criterion.

Experiments have shown that certain combinations of stresses cause a remaining deformation. It has been found that a function
\[ \bar{\sigma} = \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx} \] then reaches a definite value.

The magnitude of \( \bar{\sigma} \) is determined by material properties.

If plastic flow occurs in an isotropic medium, we assume that the material remains isotropic during the flow process. As in this case \( \bar{\sigma} \) is invariant with respect to coordinate transformation, we can also write:
\[ \bar{\sigma} = \sigma (I'_1, I'_2, I'_3, t) \] (t = time).

Generally, the dependence on time (creep) is neglected. Moreover, experiments have proved that the volume remains approximately constant during the deformation process. This means that \( \bar{\sigma} \) is independent of \( I'_1 \). Thus \( \bar{\sigma} = \bar{\sigma} (I'_2, I'_3) \), or rather (as changes of volume play no part)
\[ \bar{\sigma} = \sigma (I'_2, I'_3). \]
Next we also assume that no Bausinger effect occurs. This means that the tensile stress-deformation curve is congruent with the pressure-deformation curve. Then \( \sigma \) can only depend on even powers of the stresses. Therefore we cancel the dependence of \( \sigma \) on \( I_3' \). Thus \( \sigma = \sigma (I_2') \).

Actually, it appears that we can assume a simple relation between \( \sigma \) and \( I_2' \), viz. \( \sigma = -3I_2' \). Thus

\[
\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x + 3 I_{xy}^2 + 3 I_{yz}^2 + 3 I_{zx}^2
\]

(1-24)

Or, in principal stresses,

\[
\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1 \sigma_2 - \sigma_2 \sigma_3 - \sigma_3 \sigma_1
\]

(1-25)

This is the Von Mises flow-condition. For ductile materials it is found to describe reality pretty well. In a future chapter we shall see that \( \sigma \) is connected with specific work.

The physical background of this flow condition is the hypothesis that flow occurs as soon as a definite amount of specific work is reached, its magnitude depending on the kind of material.

For any combination of stresses for which \( \sigma \) is attained, plastic flow occurs.

In other literature we often find \( \sigma = 3k^2 \), in which \( k \) is called the "plasticity constant".

If \( \sigma \) is constant, not depending on the deformation, the material is called ideal plastic; if \( \sigma \) depends on the deformations, we have a strain-hardening material. Most technical materials show strain-hardening during the deformation process. Exceptions are lead and mercury.

When \( \sigma_3 = 0 \), we find from (1-25)

\[
\sigma^2 = \sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2 = 3k^2
\]

(1-26)

Fig. 1-12. The yield ellipse (plane state of stress).
Formula (1-26) is the equation of an ellipse (fig. 1-12). For any point \((G_1, G_2)\) inside the ellipse, the deformation will be elastic. As soon as such a stress combination is attained that a point reaches the boundary and remains on it, plastic deformation occurs. For strain-hardening materials, \(G\) increases with increasing deformation. The ellipse then "grows" during the process, while the point remains on the boundary and can never go outside the ellipse.

For ideal plastic material, the value of \(G\) remains constant. The point remains on the boundary of the ellipse, while the size of the ellipse does not increase.

Finally, for a linear state of stress, we have according to (1-26)

\[
\bar{G} = \bar{G}_1
\]  

(1-27)

This means that \(\bar{G}\) can be determined by a simple tensile test, by dividing the force through the momentary area of the section. In that case, a strain-hardening material will show an increase of \(\bar{G}\), while its value will remain constant for ideal plastic material.
Fig. 1-8

Fig. 1-9
Fig. 1-10

Fig. 1-11

Fig. 1-12
2. Strains [2].

2.1. Introduction.

We consider in an undeformed medium the infinitesimal line element $P_0Q_0$ (Fig. 2-1) with coordinates:

$$P_0 = (x_0, y_0, z_0).$$

$$Q_0 = (x_0 + dx_0, y_0 + dy_0, z_0 + dz_0).$$

Thus, with length $dr_0$, for which

$$dr_0^2 = dx_0^2 + dy_0^2 + dz_0^2.$$  \[(2-1)\]

The medium is now subjected to a deformation process. The point $P_0$ then moves over distances $u$, $v$ and $w$ in directions $X$, $Y$ and $Z$. The point $Q_0$ moves over distances $u + du$, $v + dv$, $w + dw$. We call the deformed line element $PQ$. So the original line element $P_0Q_0$ with length $dr_0$ is deformed to PQ, with length

$$dr^2 = dx^2 + dy^2 + dz^2.$$  \[(2-2)\]

For $du$, $dv$ and $dw$ near $(x_0, y_0, z_0)$ we obtain

$$du = \frac{\partial u}{\partial x} dx_0 + \frac{\partial u}{\partial y} dy_0 + \frac{\partial u}{\partial z} dz_0 = dx - dx_0,$$

$$dv = \frac{\partial v}{\partial x} dx_0 + \frac{\partial v}{\partial y} dy_0 + \frac{\partial v}{\partial z} dz_0 = dy - dy_0,$$

$$dw = \frac{\partial w}{\partial x} dx_0 + \frac{\partial w}{\partial y} dy_0 + \frac{\partial w}{\partial z} dz_0 = dz - dz_0.$$  \[(2-3)\]

Fig. 2-2 gives an illustration of this deformation for a plane state of strain.

Using (2-3) we find for (2-2)

$$dr^2 = (du + dx_0)^2 + (dv + dy_0)^2 + (dw + dz_0)^2.$$  \[(2-4)\]

Once more using (2-3), we eliminate $du$, $dv$ and $dw$ from the last expression.
We find:
\[
\text{dr}^2 = \left\{ \frac{\partial u}{\partial x} \text{dx}_o + \frac{\partial u}{\partial y} \text{dy}_o + \frac{\partial u}{\partial z} \text{dz}_o + \text{dx}_o \right\}^2 + \left\{ \frac{\partial v}{\partial x} \text{dx}_o + \frac{\partial v}{\partial y} \text{dy}_o + \frac{\partial v}{\partial z} \text{dz}_o + \text{dy}_o \right\}^2 + \left\{ \frac{\partial w}{\partial x} \text{dx}_o + \frac{\partial w}{\partial y} \text{dy}_o + \frac{\partial w}{\partial z} \text{dz}_o + \text{dz}_o \right\}^2.
\]

(2-4)

For infinitesimal strains we know that
\[
\left\{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x} \right\} \ll \left\{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x} \right\}.
\]

From (2-4) we now derive
\[
\text{dr}^2 = \text{dx}_o^2 + \left( 1 + \frac{\partial u}{\partial x} \right) \text{dx}_o \text{dy}_o + \left( 1 + \frac{\partial v}{\partial y} \right) \text{dy}_o \text{dz}_o + \left( 1 + \frac{\partial w}{\partial z} \right) \text{dz}_o \text{dx}_o + \text{dy}_o^2 + \left( 1 + \frac{\partial u}{\partial x} \right) \text{dy}_o \text{dz}_o + \left( 1 + \frac{\partial v}{\partial y} \right) \text{dz}_o \text{dx}_o + \text{dz}_o^2 + \left( 1 + \frac{\partial u}{\partial x} \right) \text{dz}_o \text{dx}_o + \left( 1 + \frac{\partial v}{\partial y} \right) \text{dz}_o \text{dy}_o.
\]

Hence,
\[
\text{dr}^2 = \left( 1 + \frac{\partial u}{\partial x} \right) \text{dx}_o^2 + \left( 1 + \frac{\partial v}{\partial y} \right) \text{dy}_o^2 + \left( 1 + \frac{\partial w}{\partial z} \right) \text{dz}_o^2 + \frac{\partial u}{\partial x} \text{dx}_o \text{dy}_o + \frac{\partial v}{\partial y} \text{dy}_o \text{dz}_o + \frac{\partial w}{\partial z} \text{dz}_o \text{dx}_o.
\]

Next we introduce direction cosines for \(\text{dr}_o\):
\[
\frac{\text{dx}_o}{\text{dr}_o} = 1 ; \quad \frac{\text{dy}_o}{\text{dr}_o} = m ; \quad \frac{\text{dz}_o}{\text{dr}_o} = n.
\]

Then substitution of (2-5) gives:
\[
\left( \frac{\text{dr}}{\text{dr}_o} \right)^2 = \left( 1 + \frac{\partial u}{\partial x} \right) \text{dx}_o^2 + \left( 1 + \frac{\partial v}{\partial y} \right) \text{dy}_o^2 + \left( 1 + \frac{\partial w}{\partial z} \right) \text{dz}_o^2 + \frac{\partial u}{\partial x} \text{dx}_o \text{dy}_o + \frac{\partial v}{\partial y} \text{dy}_o \text{dz}_o + \frac{\partial w}{\partial z} \text{dz}_o \text{dx}_o.
\]

Using \(l^2 + m^2 + n^2 = 1\), we find
\[
\left( \frac{\text{dr}}{\text{dr}_o} \right)^2 = 1 + 2 \frac{\partial u}{\partial x} l^2 + 2 \frac{\partial v}{\partial y} m^2 + 2 \frac{\partial w}{\partial z} n^2 + \frac{1}{2} \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) lm + \frac{1}{2} \left( \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} \right) mn + \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) nl.
\]

(2-6)

2.2. Infinitesimal strains.

We define an infinitesimal strain as:
\[ \frac{d\varepsilon_r}{dt} = \frac{dr}{dr_o} = \frac{dr}{dr_o} - 1. \] (2-7)

So
\[ \left( \frac{dr}{dr_o} \right)^2 = (d\varepsilon_r + 1)^2 \approx 2d\varepsilon_r + 1. \] (2-8)

With (2-6) we find from (2-8)
\[ d\varepsilon_r = \frac{\partial u}{\partial x} l^2 + \frac{\partial v}{\partial y} m^2 + \frac{\partial w}{\partial z} n^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) lm + \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) mn + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) nl \] (2-9)

Based on (2-9) we define tensile strains as \( d\varepsilon_x = \frac{\partial u}{\partial x} \), etc., and shear strains as \( d\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \), etc.

From the definition of \( d\gamma_{xy} \) it follows by changing \((x, y)\) and \((u, v)\)
respectively that \( d\gamma_{xy} = d\gamma_{yx} \), etc.

For (2-9) we write
\[ d\varepsilon_r = d\varepsilon_x l^2 + d\varepsilon_y m^2 + d\varepsilon_z n^2 + d\gamma_{xy} lm + d\gamma_{yx} ml + \] (2-10)
\[ + d\gamma_{yz} mn + d\gamma_{zy} mn + d\gamma_{zx} nl + d\gamma_{xz} ln. \]

From (2-10) it follows that the strains are components of the (symmetrical) strain tensor
\[ \begin{pmatrix}
\frac{d\varepsilon_x}{2} & \frac{d\gamma_{yx}}{2} & \frac{d\gamma_{zx}}{2} \\
\frac{d\gamma_{yx}}{2} & \frac{d\varepsilon_y}{2} & \frac{d\gamma_{zy}}{2} \\
\frac{d\gamma_{zx}}{2} & \frac{d\gamma_{zy}}{2} & \frac{d\varepsilon_z}{2}
\end{pmatrix} \] (2-11)

It is obvious that \( d\varepsilon_x \) represents an infinitesimal tensile strain in \( X \)-direction. The physical meaning of \( d\gamma_{xy} \) is shown in Fig. 2-3: \( d\gamma_{xy} \) is the infinitesimal change of the originally perpendicular angle between \( X \) and \( Y \).

**Fig. 2-3.**

Meaning of \( d\gamma_{xy} \):
\[ d\varepsilon_1 = \frac{\partial v}{\partial x}; \quad d\varepsilon_2 = \frac{\partial u}{\partial y}; \quad d\gamma_{xy} = d\varepsilon_1 + d\varepsilon_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \]

2.3. Strain rates.

In plastic flow problems, strain rates are often more important than strains. We define
\[ \dot{\varepsilon}_r = \frac{d\varepsilon_r}{dt} \] (2-12)
Now from (2-11) we find the components of the strain rate tensor.

\[
\begin{pmatrix}
\dot{\varepsilon}_x & \dot{\gamma}_{yx} / 2 & \dot{\gamma}_{zx} / 2 \\
\dot{\gamma}_{xy} / 2 & \dot{\varepsilon}_y & \dot{\gamma}_{zy} / 2 \\
\dot{\gamma}_{xz} / 2 & \dot{\gamma}_{yz} / 2 & \dot{\varepsilon}_z
\end{pmatrix}
\]  

(2-13)

2.4. Small strains.

The small strain is defined as the sum of infinitesimal strains, divided by the finite length of material $s$ (Fig. 2-4).

\[
\varepsilon_x = \sum \frac{\begin{array}{c}
\partial u^i \\
\partial x_i
\end{array}}{s} dx_i = \frac{\Delta s_x}{s} = \sum \frac{\{ \delta \varepsilon_x \}_i dx_i}{\sum dx_i}
\]

(2-14)

Fig. 2-4. Definition of "small strains".

Remark. Generally, the small strain is not the finite sum of infinitesimal strains

\[
\sum \frac{\delta \varepsilon_i dx_i}{\sum dx_i} \neq \sum \frac{\delta \varepsilon_i}{\sum dx_i}.
\]

This is only the case when $\sum_i \frac{\delta \varepsilon_i dx_i}{\sum dx_i} = \sum_i \frac{\delta \varepsilon_i}{\sum dx_j} dx_j$, that is, when $\varepsilon_i$ does not depend on $x_j$, thus when the strain is the same everywhere. This situation is called uniform strain.

We define the small shear strains $\gamma_{xy}$ etc., analogously to the infinitesimal shear strains, with $\sum \gamma_{xy} \neq \gamma_{xy}$, etc.

For a strain in direction $r$ we find again

\[
\varepsilon_r = \varepsilon_x m^2 + \varepsilon_y n^2 + \varepsilon_z l^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zn} nl
\]

(2-15)
2.5 Principal directions and invariants

As in Part. 1.3, we can calculate principal directions and invariants for the infinitesimal and small strain tensor. A Mohr-circle can also be constructed in the same way.

For the first invariant we find

$$J_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$  (2-16)

Its physical meaning can be seen as follows:

Consider a rectangular block with dimensions $s_1$, $s_2$ and $s_3$ parallel to the principal directions. After deformation the lengths are $s_1(1 + \varepsilon_1)$, $s_2(1 + \varepsilon_2)$ and $s_3(1 + \varepsilon_3)$.

The change of volume is:

$$V = s_1 s_2 s_3 (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) - s_1 s_2 s_3,$$

$$V \approx s_1 s_2 s_3 (\varepsilon_1 + \varepsilon_2 + \varepsilon_3), \text{ as } \varepsilon \text{ is small.}$$

Since plastic flow occurs at a constant volume, we have for the plastic part of the deformation:

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0.$$  (2-17)

The shear strains do not influence the change of volume.

Note: When the deformation is elastic, the change of volume is generally not zero:

$$\frac{\Delta V}{V} = \varepsilon_x^{el} + \varepsilon_y^{el} + \varepsilon_z^{el} = \varepsilon_1^{el} + \varepsilon_2^{el} + \varepsilon_3^{el}.$$

2.6 The linear strain

The linear strain $\Delta$ is the (finite) elongation of the finite length of material $s_0$, divided by the original length (Fig. 2.5).

Fig. 2-5.

Definition of "linear strain"
In this way we introduce tensile strains only, not shear strains. So we do not consider \( \Delta_x \) as a tensor component.

When in a medium shear plays a part as well, we can introduce the finite tensile strains in two different ways (Fig. 2-6),

(a) as a quantity that indicates a change of length in a fixed direction (\( \Delta_a \));

(b) as a quantity that indicates a change of distance between two points moving with the material. (\( \Delta_b \)).

Fig. 2-6.
Interpretation of \( \Delta \) in a material subjected to shear.

We shall define the linear tensile strain as mentioned in (b). The index refers to the original direction of the line element.

For infinitesimal deformations the difference between \( \Delta_a \) and \( \Delta_b \) disappears. Because of the incompressibility, we find for the linear strain, but only for principal directions:

\[
(1 + \Delta_1) (1 + \Delta_2) (1 + \Delta_3) - 1 = 0.
\]

As the linear strain is defined with respect to the moving material, (2-19) cannot be applied to directions in which shear strains occur. So, \( (1 + \Delta_x) (1 + \Delta_y) (1 + \Delta_z) - 1 \neq 0 \). See also the example in Part 2.7.

For a number of reasons, which will be explained later, we finally introduce a fourth definition of strain, viz.

2.7. The logarithmic strain (natural strain).

The logarithmic strain is the finite sum of small strains for which the linear strain is \( \Delta \).

\[
\delta = \frac{\Delta}{1} = \int \frac{ds}{s} = \ln \frac{s}{s_0} . \quad (2-20)
\]

With (2-18) we find

\[
\delta = \ln (1 + \Delta) \quad (2-21)
\]
For this type of strain as well, we only introduce tensile strain components.

For main directions, we find with (2-21) and (2-19):

\[ \delta_1 + \delta_2 + \delta_3 = \ln (1 + \Delta_1) + \ln(1 + \Delta_2) + \ln(1 + \Delta_3), \]
\[ \delta_1 + \delta_2 + \delta_3 = 0 \]  \hspace{1cm} (2-22)

Note (1). For finite deformation we have again \( \delta_x + \delta_y + \delta_z \neq 0 \).

Note (2). For \( \Delta \to 0 \) we find \( \delta \approx \Delta \approx \varepsilon \approx \frac{d\varepsilon}{d\varepsilon} \).

As we have not defined the shear strains for finite deformations, we first have to find the principal directions for any problem to be solved. We may do this as follows:

For a certain state of deformation caused by tensile strain and shear strain, we formulate then tensile strain of an arbitrarily chosen element of line. Then, by means of differentiation, we determine in what directions the tensile strain is extreme. These are then assumed to be principal directions.

Example. A finite, rectangular element \( ABCD \) is deformed to \( ABCD \) according to Fig. 2-7.

The arbitrary line \( AE_0 \) in the undeformed element is stretched to \( AE \). As in direction \( X \) no change of length occurs, we have \( D_0E_0 = DE \).

The deformation is entirely determined by the angle \( \gamma = \angle D_0AD \).

Fig. 2-7.

Deformation of an element \( ABCD_0 \) to \( ABCD \).

For the tensile strains in coordinate directions we have

\[ \delta_x = 0 \]
\[ \delta_y = \ln \frac{AD}{AD_0} = \ln \cos^{-1} \gamma \]  \hspace{1cm} (2-23)
\[ \delta_z = 0 \]

So, we actually see that
\[ \delta_x + \delta_y + \delta_z \neq 0. \]

For the line AE the tensile strain is:

\[ \delta_r = \ln \frac{AE}{AE_0} = \ln \frac{\cos \theta}{\cos \psi}. \]

As \( D_1 E_1 = D_0 E - D_0 D \), we have

\[ \tan \theta = \tan \psi - \tan \psi. \] (2-24)

Using \( \cos \psi = (1 + \tan^2 \psi)^{-\frac{1}{2}} \), we find for \( \delta_r \)

\[ \delta_r = \frac{1}{2} \ln \frac{1 + \tan^2 \psi}{1 + \tan^2 \theta}. \] Using (2-24), this expression becomes:

\[ \delta_r = \frac{1}{4} \ln \frac{1 + \tan^2 \psi}{1 + (\tan \psi - \tan \psi)^2} \] (2-25)

We have now expressed the tensile strain in an arbitrary direction, which is determined by \( \gamma \), in \( \gamma \) itself and in the fixed angle \( \psi \) determining the deformation. So, \( \delta_r \) is extreme, when:

\[ \frac{d\delta_r}{d\gamma} = 0 \] (2-26)

From (2-26) we derive:

\[ \tan^2 \gamma - \tan \gamma \tan \psi - 1 = 0. \] (2-27)

Substitution of \( \tan \psi = 2q \), gives for the principal directions

\[ \tan \gamma_1 = q + \sqrt{1 + q^2} \] (2-28)

\[ \tan \gamma_2 = q - \sqrt{1 + q^2} \]

So the principal directions are fixed by \( (\gamma_1, \gamma_2) = (\psi_1(q(\gamma)), \psi_2(q(\gamma))) \). Unlike the case of small strains, the angle \( \psi \) is no tensor component. The magnitude and direction of the principal strains, however, are unambiguously determined by \( \psi \).

Substitution of (2-28) in (2-25) gives
\[ \delta_1 = \frac{1}{2} \ln \frac{1 + (q + \sqrt{1 + q^2})^2}{1 + (q - \sqrt{1 + q^2})^2}, \]

\[ \delta_2 = \frac{1}{2} \ln \frac{1 + (q - \sqrt{1 + q^2})^2}{1 + (q + \sqrt{1 + q^2})^2}. \]

(2-29)

Thus, as \( \delta_3 = \delta_z = 0 \), we find that actually \( \delta_1 + \delta_2 + \delta_3 = 0 \), as was expected.

So: In the rectangular block \( ABCD \), deformed to the parallelogram \( ABCD \), there exist rectangular elements \( PQRS \) with sides parallel to the principal directions changing to elements \( PQRS \), which are rectangular as well. (Fig. 2-8).

2.8. Elastic and plastic deformation.

When dealing with elasto-plastic problems mathematically exactly, we must superimpose the elastic and plastic strains.

\[ \varepsilon_{\text{tot}} = \varepsilon_{\text{el}} + \varepsilon_{\text{pl}}. \]

(2-30)

For finite strains, however, we have \( \varepsilon_{\text{el}} \ll \varepsilon_{\text{pl}} \). Therefore we shall often approximate

\[ \varepsilon_{\text{tot}} \approx \varepsilon_{\text{pl}}. \]

Example. For steel we have (approximately)

\[ E = 21 \times 10^{10} \text{ N/m}^2, \text{ and e.g.} \]

\[ \sigma_{\text{flow}} = 21 \times 10^7 \text{ N/m}^2 \]

So \( \varepsilon_{\text{el}} \max = \frac{\sigma_{\text{flow}}}{E} = 10^{-3} \).

\( \varepsilon_{\text{pl}} \) on the contrary, may have a magnitude as high as 2 to 4, so that in many cases the approximation is admissible.
3. Constitutive equations [6], [2].

3.1. The incremental stress-strain relations

Both the strain tensor $e_{ij}$ and the stress tensor $\sigma_{ij}$ have 9 components. We consider each of these as a component of a vector in a 9-dimensional space ($R^9$). The yield condition: $f(\sigma_x, \sigma_y, \sigma_z, \tau_{xy} \ldots) = \psi$ may then be represented as a curved surface in $R^9$. (In $R_2$ this "surface" is e.g. the yield ellipse, Fig. 1-12).

For each combination of stresses for which $f = \psi$, plastic flow takes place. Then generally the magnitude of $\psi$ increases (strain hardening). The surface "expands".

Any state of stress for which $f < \psi$ causes elastic deformation only. Consider an external load $(X^{(1)}_K, F^{(1)}_L)$ where $X_K = \text{volume forces}$ and $F_L = \text{surface forces}$, causing an elastic state of stress $\sigma_{ij}^{(1)}$. We can represent this state of stress as a point $P(\sigma_{ij}^{(1)})$ in $R^9$ (Fig. 3-1). We call this point the image-point of this state of stress. We will now change the external conditions to $(X^{(2)}_K, F^{(2)}_L)$, so that plastic flow takes place, starting in $Q$. In the case of a strain-hardening material, $Q$ generally moves along the surface, while at the same time the surface "expands", e.g. until $Q'$ is reached. When we now restore the original external conditions $(X^{(1)}_K, F^{(1)}_L)$ we find that the state of stress does not return to $P$, but to $P'$.

Fig. 3-1. Generation of residual stresses.

This means that for one external load more than one state of stress exist. This phenomenon is called the generation of residual stresses.

When a change of the external load does not bring the image-point to the surface, then a return to the original state of load also restores the states of stress and strain to their original situation. The process is then reversible and there is no dissipation of energy. In the theory of elasticity the latter theorem is known as the Theorem of Uniqueness of Solution of Kirchhoff, viz. for a given state of load there exists only one state of strain.
Now we carry out an experiment of thought: Suppose, it is possible to pass through a cycle in such a way that (infinitesimal) plastic deformation yet occurs. The original, elastic stresses we call $\sigma_{ij}^*$ (Fig. 3-2).

Fig. 3-2. Cyclic process with infinitesimal plastic deformation.

We now change the stresses until the yield surface is reached ($\sigma_{ij}^*$). Then the external load adds an infinitesimal stress $d\sigma_{ij}$, causing a plastic deformation $de_{ij}^{pl}$, and an infinitesimal elastic deformation as well. After this we change the external load so that $\sigma_{ij}^*$ is reached again. As an unproved hypothesis, which is, however, essential for the entire theory of plasticity, we will now suppose that plastic deformation always entails dissipation of energy.

Then, for an infinitesimal amount of energy, we have:

$$d\sigma_{ij}^* \cdot d\varepsilon_{ij} > 0 \quad (3-1)$$

$$(\sigma_{ij}^* - \sigma_{ij}^*) \cdot de_{ij}^{pl} > 0. \quad (3-2)$$

(We use the summation convention). Superimposing plastic strain coordinates on the stress coordinates, (3-1) and (3-2) represent scalar products of stress and strain vectors. A positive scalar product requires a sharp angle between the vectors. As this condition must be satisfied for all combinations of $\sigma_{ij}^* - \sigma_{ij}^*$ and $de_{ij}^{pl}$, we can draw the following conclusions (Fig. 3-3):

(a) Inadmissible direction of $de_{ij}^{pl}$.
(b) Inadmissible shape of yield surface.
(c) $de_{ij}^{pl}$ has to be perpendicular to the convex surface.

(a) The vector $de_{ij}^{pl}$ is perpendicular to the yield surface, If not we can always find a situation which gives a negative scalar product.
(b) The yield surface must be convex.
(c) The most important conclusion, which is, in fact, the mathematical formulation of the condition mentioned in (a), is:
\[
\frac{d\epsilon^p_{ij}}{d\lambda} = d\lambda \cdot \frac{\partial f}{\partial \sigma_{ij}} \tag{3-3}
\]

These are the incremental stress-strain relations, associated with the yield condition.

The factor of proportionality \(d\lambda\) appears to be not a constant, but a quantity whose magnitude depends on the local stress and strain.

On the basis of (1-25), we write

\[
f = B \sigma^2 \tag{3-4}
\]

in which \(B\) is an arbitrary constant. Then, using (1-25), we find

\[
\begin{align*}
\frac{d\epsilon_{11}}{d\lambda} &= B \sigma (2\sigma_1 - \sigma_2 - \sigma_3) \\
\frac{d\epsilon_{22}}{d\lambda} &= B \sigma (2\sigma_2 - \sigma_3 - \sigma_1) \\
\frac{d\epsilon_{33}}{d\lambda} &= B \sigma (2\sigma_3 - \sigma_1 - \sigma_2).
\end{align*} \tag{3-5}
\]

Adding the squares of (3-5) we obtain

\[
\left(\frac{d\epsilon_{11}}{d\lambda}\right)^2 + \left(\frac{d\epsilon_{22}}{d\lambda}\right)^2 + \left(\frac{d\epsilon_{33}}{d\lambda}\right)^2 = 6B^2 \sigma^2 \tag{3-6}
\]

We now introduce the effective strain

\[
\bar{\epsilon} = \frac{1}{\sqrt{\frac{2}{3}} \left\{ \left(\frac{d\epsilon_{11}}{d\lambda}\right)^2 + \left(\frac{d\epsilon_{22}}{d\lambda}\right)^2 + \left(\frac{d\epsilon_{33}}{d\lambda}\right)^2 \right\} } \tag{3-7}
\]

Then, substitution of (3-7) in (3-6) gives

\[
\frac{d\lambda}{d\bar{\epsilon}} = \frac{\bar{\epsilon}}{2B\sigma} \tag{3-8}
\]

Substitution of (3-8) into (3-5) gives

\[
\begin{align*}
\frac{d\epsilon_{11}}{d\bar{\epsilon}} &= \frac{\bar{\epsilon}}{\sigma} \left(\sigma_1 - \frac{\sigma_2 + \sigma_3}{2} \right) \\
\frac{d\epsilon_{22}}{d\bar{\epsilon}} &= \frac{\bar{\epsilon}}{\sigma} \left(\sigma_2 - \frac{\sigma_3 + \sigma_1}{2} \right) \\
\frac{d\epsilon_{33}}{d\bar{\epsilon}} &= \frac{\bar{\epsilon}}{\sigma} \left(\sigma_3 - \frac{\sigma_1 + \sigma_2}{2} \right).
\end{align*} \tag{3-9}
\]
These are the Lévy-Von Mises equations. Of course, in these the arbitrary factor $B$ has disappeared.

In literature, we often find $B = \frac{1}{3}$. Then

$$f = \frac{1}{3} \sigma^2 = k^2 \quad (3-10)$$

and

$$d\lambda = \frac{3}{2} \cdot \frac{d\sigma}{\sigma}. \quad (3-11)$$

For all strains as defined in chapter 2 we may now write

$$d\delta_1 = \frac{d \sigma}{\sigma} \left( \sigma_1 - \frac{\sigma_2 + \sigma_3}{2} \right); \quad d\Delta_1 = \frac{d \sigma}{\sigma} \left( \sigma_1 - \frac{\sigma_2 + \sigma_3}{2} \right), \text{ etc.}$$

3.2. Comparison of elastic and plastic deformation.

The equations (3-9) hold exclusively for the plastic part of the deformations. For the elastic part, we have Hooke's Law

$$\varepsilon_1 = \frac{1}{E} \left\{ \sigma_1 - \nu (\sigma_2 + \sigma_3) \right\}, \text{ cyclic} \quad (3-12)$$

Comparison of (3-9) and (3-12) leads to the following table

<table>
<thead>
<tr>
<th>elastic state</th>
<th>plastic state</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $E$ is a material constant</td>
<td>$\frac{d\sigma}{\sigma}$ depends on the local stress and strain</td>
</tr>
<tr>
<td>(b) The total elastic strain is proportional to the stress</td>
<td>The infinitesimal increase of the plastic strain is proportional to the stress</td>
</tr>
<tr>
<td>(c) The constant of Poisson $\nu$ is a material constant</td>
<td>The factor $\frac{1}{2}$ is the same for all metals and alloys</td>
</tr>
</tbody>
</table>
Adding the equations (3-12), we find

\[ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1}{\rho} (1 - 2\nu) (\sigma_1 + \sigma_2 + \sigma_3). \]

For the boundary case elastic-plastic it actually appears that:

\[ \lim_{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \to 0} \nu = \frac{1}{2} \]

3.3. Relations between stress and strain rate.

From (3-9), dividing by \( dt \), we find immediately

\[ \dot{\varepsilon}_{11} = \frac{\sigma_1}{\rho} (\sigma_1 - \frac{\sigma_2 + \sigma_3}{2}) , \text{ cyclic} \]  \hspace{1cm} (3-13)

with \( \sigma = \sqrt{\frac{2}{3} (\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{22}^2 + \dot{\varepsilon}_{33}^2)} \).

\hspace{1cm} (3-14)

3.4. Incremental stress-strain relations for linear stress.

From (3-9), when \( \sigma_2 = \sigma_3 = 0 \), it follows that

\[ d\delta_1 = \frac{d\bar{\delta}}{\rho} \cdot \sigma_1 \]  \hspace{1cm} (3-15)

As \( \bar{\sigma} = \bar{\sigma}_1 \),

\[ d\delta_1 = d\bar{\delta} \text{, or} \]

\[ \delta_1 = \bar{\delta} \]  \hspace{1cm} (3-16)

Apparently, the effective stress-strain curve for a linear state of stress is identical with the true stress-strain curve of the material.
3.5. Specific work.

Consider a prismatic bar, length $l_0$, area of the section $F$, stretched to a length $l$ as a result of an external load $P$, so that plastic deformation occurs. We neglect the elastic deformation.

The infinitesimal (plastic) work consumed by this deformation is

$$dW = P \, dl = \sigma_1 \, F \, l \cdot \frac{dl}{l} = V \, \sigma_1 \, d\delta_1.$$  \hspace{1cm} (3-17)

The infinitesimal specific work, that is, the infinitesimal work per unit of volume, is:

$$dA = \sigma_1 \, d\delta_1.$$  \hspace{1cm} (3-18)

For a general state of stress, we may superimpose

$$dA = \sigma_1 d\delta_1 + \sigma_2 d\delta_2 + \sigma_3 d\delta_3.$$  \hspace{1cm} (3-19)

With (3-9) we write for (3-19)

$$dA = \sigma_1 \left( \frac{d\delta_1}{\delta} \right) \left( \sigma_1 - \frac{\sigma_2 + \sigma_3}{2} \right) + \sigma_2 \left( \frac{d\delta_2}{\delta} \right) \left( \sigma_2 - \frac{\sigma_3 + \sigma_1}{2} \right) + \sigma_3 \left( \frac{d\delta_3}{\delta} \right) \left( \sigma_3 - \frac{\sigma_1 + \sigma_2}{2} \right).$$  \hspace{1cm} (3-20)

The specific work is

$$A = \int \frac{dA}{\delta} = \frac{dA}{\delta_1}.$$  \hspace{1cm} (3-21)

in which $\delta_1$, $\delta_2$ and $\delta_3$ are logarithmic strains.

From (3-20) it appears that we may write for (3-11):

$$d\lambda = \frac{3d\delta}{2\delta} = \frac{dA}{2k^2} = \frac{dA}{2k^2}. $$  \hspace{1cm} (3-22)

In case of plastic deformation we may assume that the energy is almost entirely converted into heat [2].
3.6. The deformation equation [2].

Experiments have shown that for many ductile materials a relation exists between $\bar{\sigma}$ and $\bar{J}$, which can be approximated by (Fig. 3-4):

$$\bar{\sigma} = c \cdot \bar{J}^m.$$  \hfill (3-23)

Drawn on a log-log scale, this relation is represented by a straight line.

Fig. 3-4 Experimental relation between $\bar{\sigma}$ and $\bar{J}$ for several values of $m$.

In (3-23), $c$ and $m$ are material constants. $c$ is the value of the effective stress if $\bar{J} = 1$. $m$ is the strain-hardening exponent. For $m = 0$ we have an ideal plastic material. With the help of this relation, we can express the specific work either in $\bar{\sigma}$ or in $\bar{J}$ and material constants.

$$A = \int_{\bar{J}_1}^{\bar{J}_2} \bar{\sigma} d\bar{J} = \int_{\bar{J}_1}^{\bar{J}_2} c \cdot \bar{J}^m d\bar{J} = \frac{c}{m+1} \left( \bar{J}_2^{m+1} - \bar{J}_1^{m+1} \right)$$  \hfill (3-24)

and also, from (3-24)

$$A = \frac{c}{m+1} \left\{ \frac{m+1}{\bar{\sigma}_2} - \frac{m+1}{\bar{\sigma}_1} \right\}.$$  \hfill (3-25)

Remark 1. When an isotropic material is subjected to a non-uniform deformation, it will become anisotropic.

Remark 2. The quantities $c$ and $m$ are characteristic of the behaviour of the material during the plastic deformation. Their magnitude, however, largely depends on $\bar{J}$. Moreover, they depend on temperature and structure of the material. In practice this means that when examining a deformation process, they have to be determined from the process-data themselves.
The following table gives some compression values of $c$ and $m$ ($m$ determined at $\Delta = 1$, $\Delta = 0$, room temperature) [2].

<table>
<thead>
<tr>
<th>Material</th>
<th>$c$ ($N/m^2$)</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Al 6061 - O</td>
<td>$21 \times 10^7$</td>
<td>0.2</td>
</tr>
<tr>
<td>Al 6061 - T6</td>
<td>$42 \times 10^7$</td>
<td>0.05</td>
</tr>
<tr>
<td>Al 2024 - T4</td>
<td>$72 \times 10^7$</td>
<td>0.09</td>
</tr>
<tr>
<td>$\alpha$-Brass, C.R.</td>
<td>$58 \times 10^7$</td>
<td>0.34</td>
</tr>
<tr>
<td>SAE 1112, Ann.</td>
<td>$77 \times 10^7$</td>
<td>0.19</td>
</tr>
<tr>
<td>SAE 1112, C.R.</td>
<td>$77 \times 10^7$</td>
<td>0.08</td>
</tr>
<tr>
<td>SAE 4135, Ann</td>
<td>$102 \times 10^7$</td>
<td>0.17</td>
</tr>
<tr>
<td>SAE 4135, H.T.- R.C. 18</td>
<td>$110 \times 10^7$</td>
<td>0.14</td>
</tr>
<tr>
<td>SAE 4135, H.T.- R.C. 26</td>
<td>$140 \times 10^7$</td>
<td>0.09</td>
</tr>
<tr>
<td>SAE 4135, H.T.- R.C. 35</td>
<td>$168 \times 10^7$</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Table 1.
Some static compression test data (From: Thomsen, Yang, Kobayashi, Mechanics of plastic deformation in metals).

3.7. Integration of the plasticity equations [2].

For infinitesimal deformations we have

$$d\delta_1 = \frac{d\bar{\sigma}}{\delta} \left( \sigma_1 - \frac{\sigma_2 + \sigma_3}{2} \right), \text{cyclic} \quad (3-9)$$

To find the relations between stresses and finite deformations, (3-9) has to be integrated. This means that the quantities in the right-hand side have to be known as functions of $\delta$ or as functions of an arbitrary parameter $\mu$.

Then, during the entire deformation process, all stresses have to be known. In the yield space the locus of the consecutive places of all image points is a line called the stress path. So the ultimate strain suiting a stress condition does not only depend on the stress condition itself, but also on the stress path.
Example 1. Let the stresses be known as functions of a parameter \( \mu \).

\[
\sigma_1 = f_1(\mu) \\
\sigma_2 = f_2(\mu) \\
\sigma_3 = f_3(\mu). 
\]  

Then we can find:

\[
\bar{\delta} = \varepsilon_1 (\mu) \tag{3-27}
\]

\[
\bar{\delta} = \varepsilon_2 (\mu) \text{ if strain hardening takes place, that is if } m \neq 0. \tag{3-28}
\]

Using (3-9), we find:

\[
\delta_1(\mu) = \int \frac{\overline{\delta}}{\delta_1} d\bar{\delta} (\sigma_1 - \frac{\sigma_2 + \sigma_3}{2}) = \int \frac{\overline{\delta}_2}{\delta_1} \frac{F_1(\mu)}{\varepsilon_1(\mu)} d\overline{\delta} = \int \frac{\overline{\delta}_2}{\delta_1} \frac{F_2(\mu)}{\varepsilon_2(\mu)} d\mu
\]

and similar integrals for \( \delta_2 \) and \( \delta_3 \). In many cases these integrals can only be solved numerically.

Example 2. We assume that during the entire deformation process a fixed ratio between the principal stresses exists. The stress path then is a straight line:

\[
\frac{\sigma_2}{\sigma_1} = \alpha \; ; \; \frac{\sigma_3}{\sigma_1} = \beta \tag{3-28}
\]

Substitution of (1-25) and (3-28) in (3-9) gives

\[
\delta_1 = \frac{d\overline{\delta}}{\sigma_1} \frac{(1 - \frac{\alpha}{2} - \frac{\beta}{2})}{\sqrt{1 + \alpha^2 + \beta^2 - \alpha \beta}} = K_1 \overline{\delta} \tag{3-29}
\]

We find:

\[
\delta_1 = K_1 \overline{\delta} \\
\delta_2 = K_2 \overline{\delta} \\
\delta_3 = K_3 \overline{\delta} \tag{3-30}
\]
where

\[ \delta = \sqrt{\frac{2}{3} (\delta_1^2 + \delta_2^2 + \delta_3^2)} \] (3-31)

\[ K_1 = \frac{1 - \alpha - \beta}{2} \left( \frac{\beta^2}{1 + \alpha^2 + \beta^2 - \alpha - \alpha \beta - \beta} \right)^{\frac{1}{2}} \]

\[ K_2 = \frac{\alpha - \beta - 1}{2} \left( \frac{\alpha^2}{1 + \alpha^2 + \beta^2 - \alpha - \alpha \beta - \beta} \right)^{\frac{1}{2}} \] (3-32)

\[ K_3 = \frac{\beta - \alpha - 1}{2} \left( \frac{\beta^2}{1 + \alpha^2 + \beta^2 - \alpha - \alpha \beta - \beta} \right)^{\frac{1}{2}} \]

Elimination of \( \alpha \) and \( \beta \) from (3-30) and (3-31) leads to:

\[ \delta_1 = \frac{\delta}{6} (\sigma_1 - \frac{\sigma_2 + \sigma_3}{2}), \text{ cyclic} \] (3-33)

We now introduce the Modulus of Plasticity:

\[ P = \frac{\sigma}{\delta} . \] (3-34)

Then, similar to Hooke's Law, we write for the integrated Lévy- Von Mises equations:

\[ \delta_1 = \frac{1}{P} (\sigma_1 - \frac{\sigma_2 + \sigma_3}{2}) \]

\[ \delta_2 = \frac{1}{P} (\sigma_2 - \frac{\sigma_3 + \sigma_1}{2}) \] (3-35)

\[ \delta_3 = \frac{1}{P} (\sigma_3 - \frac{\sigma_1 + \sigma_2}{2}) \]

In (3-35) \( P \) is not a constant, as can be seen from:
We would emphasise that (3-35) should only be applied if during the entire deformation process a fixed ratio between the principal stresses or the principal strains is maintained. For the proof of the latter statement, see [5].

**Example 3.** For an ideal plastic material we cannot calculate the strain from the stresses, as $\overline{\sigma} = c$, thus $\overline{\delta} = g_2(\mu)$ cannot be found from $\overline{\sigma}$; $\overline{\delta}$ then has to be given separately.

**Example 4.**
(a) A strain hardening material is subjected to a linear stress until it has yielded. Then a second stress is applied, while the first is kept constant, until renewed plastic flow starts to occur (Fig. 3-6).

(1) deformation OA, elastic
(2) deformation AA', plastic:

$$\delta_1 = \overline{\delta} = c \frac{1}{m} \bar{\sigma}_{II}^{\frac{1}{m}} \quad (\text{as OA'} = \overline{\sigma}_{II})$$

$$\delta_2 = \overline{\delta}_3 = -ic \frac{1}{m} \bar{\sigma}_{II}^{\frac{1}{m}}$$

(3) deformation A'B, elastic.

(b) We will now attain the same state of stress via OCB

(1) deformation OC, elastic.

(2) deformation CB, plastic with $\sigma_1 = \sigma_2 = \bar{\sigma}_{II}$

$$\delta_1 = \frac{\overline{\delta}}{\bar{\sigma}_{II}} \left( \sigma_1 - \frac{\bar{\sigma}_2}{2} \right) = \frac{\overline{\delta}}{2} = \frac{1}{m} \frac{1}{\bar{\sigma}_{II}}$$

$$\delta_2 = -\frac{1}{2c} \frac{1}{m} \bar{\sigma}_{II}^{\frac{1}{m}}$$

$$\delta_3 = 0$$

So we actually see that a different stress path gives a different ultimate strain in spite of the same ultimate state of stress.
Example 5. As Example 3. In this case, however, we let the material yield at $B$, keeping $\sigma_1$ constant (Fig. 3-7). We consider plastic deformation only:

(1) deformation $AA'$, as in Example 3.
(2) deformation $BB'$, $\sigma_1 = \sigma_{11}$, constant.

We calculate $\sigma_{2B'}$ from

$$\sigma_{III}^2 = \sigma_{II}^2 + \sigma_{2B'}^2 - \sigma_{II} \sigma_{2B'}$$

and find:

$$\sigma_{2B'} = \frac{\sigma_{II}}{2} \pm \frac{1}{2} \sqrt{4\sigma_{III}^2 - 3\sigma_{II}^2}$$

in which obviously only the $+$ sign is significant.

The strains are:

$$d\delta_1 = d\delta \left\{ \frac{3}{4} \frac{\sigma_{II}}{\sigma_{III}} - \sqrt{\frac{1}{4} - \frac{3}{16} \left( \frac{\sigma_{III}}{\sigma_{III}} \right)^2} \right\}$$

with $\sigma_{III} = c \delta^{-m}$. Substitution in (3-37) gives:

for $\delta_1$:

$$\delta_1 = \int_{S_B} S_{BB'} \left\{ \frac{3\sigma_{II}}{4c} \delta^{-m} - \sqrt{\frac{1}{4} - \frac{3\sigma_{II}^2}{16c^2} \delta^{-2m}} \right\} d\delta$$

(3-38)

For $\delta_2$ and $\delta_3$ we find similar integrals. They can only be solved numerically.

The total plastic deformation is then found by superimposing the calculated strains of (1) and (2).

Example 6. We continue the example of part 2-7 and calculate the stresses that caused the strain as given in Fig. 2-7.
From \( \gamma \) we calculated \( \delta_1 \), \( \delta_2 \) and \( \delta_3 \). We found \( \delta_3 = 0 \), so \( \delta_1 = -\delta_2 \).

We will now assume that the process took place while the ratio between the stresses was kept constant. Then, (3-31) gives:

\[
\delta = \sqrt[4]{\frac{1}{3}} \delta_1 = -\sqrt[4]{\frac{1}{3}} \delta_2 \tag{3-39}
\]

Next, we apply (3-34), for which \( P \) is first calculated from (3-35).

Let \( \gamma = 60^\circ \), then \( \theta = \frac{1}{2} \tan \gamma = 0.866 \), leading to (2-29): \( \delta_1 = 0.796 \) and \( \delta_2 = -0.796 \).

So \( \delta = 0.92 \)

Let the material constants be: \( c = 150 \times 10^7 \text{N/m}^2 \)

and \( m = 1.2 \)

Now \( P = 150 \times 10^7 \times 0.92^0 = 147.5 \times 10^7 \text{N/m}^2 \)

\[
2\delta_1 \cdot P = 314 \times 10^7 = 2\sigma_1 - \sigma_2 - \sigma_3
\]

\[
2\delta_2 \cdot P = -314 \times 10^7 = -\sigma_1 + 2\sigma_2 - \sigma_3
\]

From which:

\[
\sigma_1 = 105 \times 10^7 \text{N/m}^2
\]

\[
\sigma_2 = 105 \times 10^7 \text{N/m}^2
\]

\[
\sigma_3 = 0
\]

Thus the state of deformation was caused by a state of pure shear.

For the principal directions we have, according to (2-28),

\[
\tan \psi_1 = 2.188 \quad \text{or} \quad \psi_1 = 65^\circ 30'\ \tag{2-28}
\]

\[
\tan \psi_2 = -0.456
\]

For the stresses in coordinate directions we find, according to Fig. 3-8.

Fig. 3-8. Mohr circle and state of strain according to Example 6.
\[
\tan y_1 = \frac{T_{xy}}{\sqrt{\sigma_1^2 + \tau_{xy}^2}}
\]

\[
\sigma_y^2 + \tau_{xy}^2 = \sigma_1^2
\]

\[
\sigma_x = -\sigma_y
\]

From which we solve:

\[
\sigma_x = 67 \times 10^7 \text{ N/m}^2
\]

\[
\sigma_y = 67 \times 10^7 \text{ N/m}^2
\]

\[
T_{xy} = 81 \times 10^7 \text{ N/m}^2
\]

* Here we actually use a hypothesis:

The principal directions of stress and strain are identical.

Theoretically, this assumption is incorrect; however for technical processes it describes reality pretty well.

In Chapter 5 this will be discussed in more detail.
Fig. 3-4

Fig. 3-5

Fig. 3-6
4. Applications of the preceding theory.

4.1. Pure bending.

A rectangular bar or a sheet of metal (thickness \( h \), width \( b \)) is loaded by a bending moment \( M \). Near the neutral zone, elastic deformation will take place, until the yield stress \( \sigma_f \) has been reached. The outer layer will become plastic (Fig. 4-1).

![Fig. 4-1. Rectangular bar, loaded by a bending moment \( M \).]

As in the plastic region the strain increases, the stress will increase as well owing to the strain hardening. If the bar is bent to a radius \( r \) the strain will be

\[
\Delta = \frac{y}{r} \tag{4-1}
\]

In the elastic region, we have Hooke's Law:

\[
\sigma = \frac{E\varepsilon}{r}, \quad 0 \leq y \leq \frac{h}{2} \tag{4-2}
\]

(We consider part \( y \geq 0 \) only).

If the value \( \sigma_f \) is reached, we have, according to (4-2),

\[
\frac{h}{2} = \frac{\sigma_f r}{E} \tag{4-3}
\]

In the plastic region the stress increases with increasing \( y \), as

\[
\sigma' = c\Delta^m = c \left( \frac{y}{r} \right)^m \tag{4-4}
\]

Thus we can calculate the bending moment
\[ M = 2b \left[ G \int_0^{h_e/2} y \, dy + 2b \int_{h_e/2}^{h/2} G' y \, dy \right] \quad (4-5) \]

With (4-2), (4-3) and (4-4), we find for (4-5):

\[ M = 2b \frac{G_f^3 r^2}{3E^2} + \frac{c}{(m+2)r^m} \left[ \left( \frac{h}{2} \right)^{m+2} - \left( \frac{G_f r}{E} \right)^{m+2} \right] \quad (4-6) \]

We may simplify (4-6) on the assumption that \( G_f \) is continuous for \( y = h_e/2 \).

Thus

\[ G_f = \frac{G_f^m}{h^2} \quad (4-7) \]

\[ c = G_f^{-1-m} E^m \quad (4-8) \]

\[ M = \frac{2bc}{(m+2)} \frac{r^m}{r^2} \left( \frac{h}{2} \right)^{m+2} + \frac{2(m-1)}{3(m+2)} \frac{bG_f^3 r^2}{E^2} \quad (4-9) \]

For small values of \( r \), the elastic region may be neglected. We then omit the last term.

\[ M \approx \frac{2bc}{(m+2)} \frac{r^m}{r^2} \left( \frac{h}{2} \right)^{m+2} \quad (4-10) \]

We now remove the external moment. Then the radius increases from \( r \) to \( r_o \). As soon as the moment decreases, all stresses decrease. Therefore, the entire section is immediately elastic again. In the new situation the strain is

\[ \Delta_o = \frac{\gamma}{r_o} \quad (4-11) \]

The decrease of strain is

\[ \Delta - \Delta_o = \gamma \left( \frac{1}{r} - \frac{1}{r_o} \right) \quad (4-12) \]
Hence, for the stresses $\sigma_o$ and $\sigma'_o$ indicated in Fig. 4-1, we find

$$\sigma_o = \sigma - \frac{E}{r_0} \left( \frac{1}{r} - \frac{1}{r_0} \right) = \frac{E}{r_0} ; \ 0 \leq y \leq h/2$$

$$(4-13)$$

$$\sigma'_o = \sigma' - \frac{E}{r_0} \left( \frac{1}{r} - \frac{1}{r_0} \right) = c \left( \frac{y}{r} \right)^m - \frac{E}{r_0} \left( \frac{1}{r} - \frac{1}{r_0} \right) ; \ h/2 < y < h/2$$

The equilibrium of moments now requires

$$0 = \int_0^{h/2} \sigma_o y \ dy + \int_{h/2}^{h} \sigma'_o y \ dy \quad (4-14)$$

Substitution of (4-3) and (4-13) in (4-14) and integration gives

$$\frac{Eh^3}{24} \left( \frac{1}{r_0} - \frac{1}{r} \right) + \frac{ch^{m+2}}{4(2r)^m(m+2)} - \frac{cG_r^{m+2} r^2}{(m+2)E^{m+2}} + \frac{G_r^3 r^2}{3E^2} = 0 \quad (4-15)$$

Assuming that $r$ is small we neglect the last two terms and then find

$$\frac{1}{r} - \frac{1}{r_0} = \frac{3c}{(m+2)Er} \left( \frac{h}{2r} \right)^{m-1} \quad (4-16)$$

We now define

$$\frac{h}{2r} = \Delta_{\text{max}} \quad (4-17)$$

$\Delta_{\text{max}}$ is the strain in the outer layer of the bar or plate, if it is bent to a radius $r$. Then

$$\frac{1}{r} - \frac{1}{r_0} = \frac{3c\Delta_{\text{max}}^{m-1}}{(m+2)Er} \quad (4-18)$$

from which, as $r_0/r \approx 1$, we derive (using $1/(1-\epsilon) \approx 1 + \epsilon$ if $\epsilon \ll 1$)
When we consider a bar, of which the angle between the legs increases from $\varphi$ to $\varphi_0$, (Fig.4-2), we find

$$
\frac{r_0}{r} = 1 + \frac{3c}{(m+2)E\Delta^{1-m}_{\text{max}}}
$$

(4-19)

Fig.4-2. Relation between angle and radius.

$$
r(\pi - \varphi) = r_0(\pi - \varphi_0).
$$

(4-20)

Substitution of (4-20) in (4-19) gives:

$$
\varphi \approx \varphi_0 - \frac{3c(\pi - \varphi_0)}{(m+2)E\Delta^{1-m}_{\text{max}}}
$$

(4-21)

Example:

Suppose a bar has the following data:

$h = 20$ mm.

$r = 100$ mm.

$\varphi_0 = 45^\circ$

$E = 21,000 \times 10^7$ N/m$^2$

$c = 81.5 \times 10^7$ N/m$^2$

$m = 0.22$

Then we find $\Delta_{\text{max}} = h/2r = 0.1$ and $\varphi = 40^\circ 50'$. This is the angle to be given to the plate during bending.

Remark. Owing to the generation of a new, residual stress distribution in some cases renewed (secondary) plastic flow may occur. This has to be checked afterwards. In the case under consideration, no secondary plastic flow takes place, if

$$
\left| G_o \left( \frac{h}{2} \right) \right| < \left| G \frac{h}{2} \right|
$$

(4-22)
or

\[ \frac{Eh}{2} \left( \frac{1}{r} - \frac{1}{r_0} \right) < 2c \Delta_{\text{max}}^{1-m} \]  (4-23)

Using (4-18) we find from (4-23)

\[ m > -0.5 \]  (4-24)

This condition will always be satisfied.
Hence, secondary plastic flow will not take place.

4.2. Bending by shear-forces.

4.2.1. Shear stresses.

It is known that bending by shear forces causes shear stresses in the longitudinal direction of a bar. We shall calculate these stresses for a bar with a rectangular cross-section made of exponentially strain-hardening material. (Fig. 4-3). Owing to the moment distribution \( M = M(x) \), we have the following situation:

We consider the equilibrium in direction \( X \) of the upper part of the bar, as shown in Fig. 4-3. For the infinitesimal longitudinal force

\[ dL = \tau_{xy} b \, dx, \]  we have

\[ dL = \int_c^{h/2} c \left\{ \left( \frac{x}{r+dr} \right)^m - \left( \frac{x}{r} \right)^m \right\} dy \]  (4-25)
\[
I_{xy} \, dx = - \int_{y}^{h/2} \frac{cm}{r} (\frac{y}{r})^m \, dr \, dy
\]  
(4-26)

As \( I_{xy} = I_{yx} \), we find for \( S \):

\[
S = 2b \int_{0}^{h/2} I_{xy} \, dy
\]  
(4-28)

We now simplify Fig. 4-3 by the assumption that the exponential stress curve is also valid for the elastic region. In other words, we consider the entire section to be plastic. Then (4-27) holds for the entire section. Hence,

\[
S = -2b \int_{0}^{h/2} \frac{cm}{(m+1)r^{m+1}} \frac{dr}{dx} \left[ \left(\frac{h}{2} \right)^{m+1} - y \right] \, dy
\]  
(4-29)

\[
S = \frac{bmch}{m+2} \left(\frac{h}{2r}\right)^{m+1} \frac{dr}{dx}
\]  
(4-30)

We would have attained the same result by differentiation of (4-10) with respect to \( x \).

Apparently, just as in the linear theory of elasticity, we have

\[
S = \frac{dM}{dx}
\]  
(4-31)

Of course, this also follows from the equilibrium of forces (Fig 4-3).

4.2.2. The shape of the neutral phase.

With the help of the preceding theory, we can find the differential equation of the neutral phase. We will illustrate this for the case of a bar fixed at one end and loaded with an evenly distributed load, as shown in Fig. 4-4.

The moment is found to be

\[
M = \frac{1}{2}q(1-x)^2
\]  
(4-32)
We will assume that the strains are small. For that case we have already found for $M$ the expression (4-9). We write this equation as

$$M = c_1r^{-m} + c_2r^2$$

(4-33)

Substitution of (4-32) gives

$$\frac{1}{2}q(l-x)^2 = c_1r^{-m} + c_2r^2$$

(4-34)

or, since the strains are small,

$$r \approx (y'')^{-1}$$

(4-35)

$$\frac{1}{2}q(l-x)^2 = c_1(y'')^m + c_2(y'')^{-2}$$

(4-36)

with boundary conditions $y(0) = 0$ and $y'(0) = 0$.

This differential equation can only be solved numerically. (4-36) holds for the plastic part of the bar. The maximum value of $x$ for which plastic flow is found, follows from

$$M_{\text{max.el.}} = \frac{\sigma_f}{E} \frac{1}{6} bh^2 = \frac{1}{2}q(l-x_{\text{max.pl.}})^2$$

(4-37)

The elastic part has to be "fixed" with an angle $y'(x_{\text{max.pl.}})$; then its shape may be calculated, e.g. with the theorem of Castigliano.

To determine the region of incipient plastic flow, we generalise (4-3) into

$$y^* = \frac{h}{2} = \frac{\sigma_f}{E} r(x)$$

(4-38)

where $r(x)$ has to be calculated from (4-34) numerically. As a check on this calculation we have

$$y_{\text{max}}^* = y^*(x_{\text{max.pl.}}) = \frac{h}{2}$$

(4-39)
We apply this theory to an ideal plastic material \((m = 0)\), and find from (4-33)

\[
M = \frac{\sigma_f}{1 - \nu}  bh^2 - \frac{\sigma_f \beta r^2}{3E} = \frac{1}{2} q (1-x)^2
\]  

(4-40)

from which we derive

\[
r = \frac{E}{G_f} \sqrt{\frac{1}{h^2} - \frac{3q (1-x)^2}{2b\sigma_f}}
\]  

(4-41)

Using (4-38), we find for the region of incipient plastic flow

\[
y^* = \sqrt{\frac{1}{h^2} - \frac{3q (1-x)^2}{2b\sigma_f}}
\]  

(4-42)

Substitution of (4-37) in (4-42) actually leads to \(y_{\text{max}}^* = h/2\)

If we wish to solve the problem for large strains, we may write instead of (4-33)

\[
M = c_1 r^{-m}
\]  

(4-43)

with

\[
r = \left\{ \frac{1+(y')^2}{y''} \right\}^{3/2}
\]  

(4-44)

from which we find a differential equation of the neutral zone with the help of (4-32).

A complete solution both for small and large strains follows, of course, by combining (4-34) and (4-44).

Remark. For the sake of simplicity, the calculations in Parts 4.1 and 4.2 were carried out for rectangular cross sections. We could easily extend the theory for symmetrical sections with respect to the Y-axis. In that case we have \(b = b(y)\) which has to be integrated as well.
4.2.3. Collapse load analysis. [8]

The plastic behaviour of a material is of importance in two fields of mechanical engineering:

a) Metal processing. In this case we are mainly interested in forces and deformations during the process. The occurrence of plastic flow is the purpose of the process.

b) Design of constructions. In this case we are interested in the state of load, for which plastic flow begins. The occurrence of plastic flow now is an undesirable limit-load phenomenon.

Since in the second case the limit load is more important than the plastic behaviour itself, we may as well consider the material as ideally plastic. This strongly simplifies the calculations and still gives the necessary information regarding incipient flow.

Consider a bar loaded with a moment distribution, \( M = M(x) \). We increase \( M \) until the bar breaks down, owing to the moment locally reaching a critical value. As the material is ideally plastic, this maximum moment does not increase further during deformation. On that spot, we have thus generated a plastic-hinge, which means a hinge transmitting a constant moment \( M^* \).

By applying the virtual work principle, we can easily determine the force \( P^* \) which causes the moment \( M^* \). This principle simply states:

The energy consumed by the load is the same as the energy needed to cause the deformation.

Example 1. Bar and load according to Fig. 4-5.

In the centre the moment \( M \) is maximum and a plastic hinge is generated. At the moment of collapse, we have

\[
P^* \cdot 1 \delta \varphi = M^* \cdot 2 \delta \varphi , \text{ thus } P^* = 2M^*/1.
\]

(4-45)

Example 2. Bar and load according to Fig. 4-6.

\[
P^* \cdot 31 \delta \varphi = M^* \cdot 4 \delta \varphi .
\]

\[
P^* = 4M^*/31
\]

(4-46)
Example 3. Bar and load according to Fig. 4-7.
\[ p^* = M^* / l \]

Fig. 4-7.

Example 4. Bar and load according to Fig. 4-8.
At first \( P \) increases to \( P_1 \), the value for which the fixed end becomes a plastic hinge. Then \( P_1 \) increases to \( P^* \), the value for which the centre of the bar becomes a plastic hinge as well. In that case we have

\[ M^* \left( \frac{P}{l} \right) + M^* \cdot 2 \delta \varphi = P \cdot 1 \delta \varphi \]
\[ P = 3M^* / l \]

Fig. 4-8.

Example 5. Bar and load of Fig. 4-9.
There are two possibilities.
(b) \( 2P^* \frac{1}{2} \delta \varphi = 2M^* \delta \varphi + M^* \delta \varphi \), hence, \( P^* = 3M^* / l \)

(c) \( P^* \cdot 1 \frac{1}{2} \delta \varphi = M^* \cdot 3 \frac{1}{2} \delta \varphi + M^* \delta \varphi \), hence, \( P^* = 5M^* / l \)

So, in reality, case (b) will occur.

4.3 Torsion of a circular cylindrical bar.

In order to find an approximative solution of the problem we will suppose:
(a) Small strains
(b) Perpendicular cross-sections remain flat; there are no strains in direction \( X \) (Fig. 4-10)
(c) In perpendicular cross-sections only shear stresses are generated. These are directed perpendicularly to the radius.
(d) The cross-section will approximately become fully plastic.

Fig. 4-10. Torsion of a cylindried bar.
From Fig. 4-10 it follows that

\[ M = 2\pi \int_0^\rho \rho^2 \tau(\rho) \, d\rho = 2\pi \int_0^\rho 2\tau(\rho) \, d\rho \quad (4-51) \]

We now consider an infinitesimal disc (Fig. 4-10). Section 1 has rotated over an angle \( d\varphi \) with respect to section 2. It follows that:

\[ \gamma = \rho \frac{d\varphi}{dx} \quad (4-52) \]

As in these sections we have shear stress only, it can be understood that \( \gamma_1 = -\tau_1 = \tau \). This can be easily proved by drawing the Mohr circles.

It also follows that for small strains \( \varepsilon_1 = -\varepsilon_2 = \gamma \).

The principal directions always make an angle of 45° with respect to the X-axis. We will now calculate \( I = I(\rho) \). From (1-24) we find

\[ \bar{\varepsilon} = I \sqrt{3} \quad (4-53) \]

From (3-31) we find

\[ \bar{\varepsilon} = \sqrt{\frac{4}{3}} \rho \frac{d\varphi}{dx} \quad (4-54) \]

((3-31) can be used in this case, as \( \varepsilon_2/\varepsilon_1 = -1 \), hence, constant).

Using \( \bar{\varepsilon} = c \varepsilon^m \), we find from (4-53) and (4-54):

\[ I = \frac{2^{m+1}}{3} 2^{m+1} a \left( \frac{d\varphi}{dx} \right)^m \rho^m \quad (4-55) \]

Substitution of (4-51) and integration yields

\[ M = \frac{2^{m+1}}{3} \frac{2 \pi \rho^2}{m+3} \left( \frac{d\varphi}{dx} \right)^m \rho^{m+3} \quad (4-56) \]
or, as \( \gamma_{\max} = r \frac{d \varphi}{dx} \),

\[
M = \left( \frac{2}{\pi} \right)^{m+1} \frac{2 \pi c}{m+3} \gamma_{\max}^m r^3
\]

4.4. Instability [2].

From practice it is known that plastic deformation cannot be continued unrestrictedly. At a certain moment, the process becomes unstable. Then, for instance, buckling or necking occurs, followed by fracture. For some of these cases, we will give an example of calculation.

4.4.1. Instability in tension.

If a circular cylindrical bar is loaded with tension, the strain will at the beginning be uniform. During the deformation, the area decreases. Owing to strain hardening, however, at the same time the material becomes stronger. For a certain critical degree of strain the load will be maximum. At that moment, the necking process begins, and thus the strain is no longer uniform. Continuation of the necking process soon leads to fracture.

The criterion for instability is

\[
\frac{dP}{d\delta} = 0
\]

(4-58)

where \( P \) is the outside tensile force. If the momentary area of the cross-section is \( A \), we have

\[
P = \overline{G}_1 A = \overline{G}_1 A = c(\delta)^m A
\]

(4-59)

However, we have as well

\[
\delta_1 = \delta = \ln \frac{1}{\overline{A}} = \ln \frac{\overline{A}}{A}, \text{ hence}
\]

\[
A = \overline{A} e^{-\delta}
\]

(4-60)

Thus
\[ P = P(\delta) = A_0 c(\delta)^m e^{-\delta} \]  
(4-61)

\[ \frac{dP}{d\delta} = A_0 c \left[ e^{-\delta} \left\{ m(\delta)^{m-1} - (\delta)^m \right\} \right] = 0 \]
(4-62)

or:

\[ \delta = m \]
(4-63)

In the case of compression, no instability occurs, as the area increases in that case.

\[ P = G_A = -G_A = cA(\delta)^m \]
(4-64)

\[ |\delta| = \delta = \ln \frac{A}{A_0}, \text{ hence} \]

\[ A = A_0 e^{\delta} \]
(4-65)

\[ P = -cA_0 (\delta)^m e^{\delta} \]
(4-66)

\[ \frac{dP}{d\delta} = -cA_0 \left[ e^{\delta} \left\{ m(\delta)^{m-1} + (\delta)^m \right\} \right] \neq 0 \]
(4-67)

4.4.2. Buckling [2]

From the theory of elasticity we know that for a compressed bar the buckling load is

\[ P_e = \frac{\pi^2 EI}{l^2} \]
(4-68)

In (4-68) \( I \) is the linear moment of inertia, \( E \) the modulus of elasticity, \( P_e \) the (elastic) buckling load and \( l \) the buckling length. We introduce the radius of gyration \( i \), so that \( I = Ai^2 \) (\( A \)=area). Then (4-68) yields
\( \sqrt{\text{buckling}} = \frac{P_a}{A} \frac{\mu^2 E}{(\lambda/2)^2} = \frac{\mu^2 E}{\lambda^2} \)  

(4-69)

in which \( \lambda = 1/i \) represents the slenderness of the bar. Now, if the bar is fully plastic, we replace \( E \) by the local slope of the \( \sigma - \varepsilon \) curve.

\[
E' = \frac{d\bar{\sigma}}{d\bar{\varepsilon}} = \frac{d(c\varepsilon^m)}{d\varepsilon} = cm(\varepsilon)^{m-1}
\]

(4-70)

Then \( \bar{\sigma}_{\text{buckling}} \) has the value \( \bar{\sigma} = c \varepsilon^m \). For (4-69) we find in that case

\[
c \varepsilon^m = \frac{\mu^2 cm(\varepsilon)^{m-1}}{\lambda^2}
\]

(4-71)

Hence, plastic buckling does not occur if

\[
\frac{\mu^2 m}{\varepsilon \lambda^2} < 1
\]

(4-72)

Example. For a certain material we have

\[
E = 21000 \times 10^7 \text{ N/m}^2, \quad m = 0.22
\]

\[
c = 81.5 \times 10^7 \text{ N/m}^2, \quad \sigma_f = 30 \times 10^7 \text{ N/m}^2
\]

A bar of this material, \( \varnothing 20 \text{ mm} \), is subjected to a pressure-force of 50,000 N. What is the admissible length.

(a) to prevent elastic buckling,

(b) to prevent plastic buckling?

(a) From (4-68) we have

\[
I \approx \frac{1}{20} d^4 = 2.3 \times 10^{-9} \text{ m}^4, \text{ hence}
\]

\[
\lambda = 3.14 \sqrt{\frac{21.10^7 \times 2.3 \times 10^{-9}}{5.10^4}} = 0.584 \text{ m}.
\]

*This value is not quite the same as would be found from (4-8). The reason is that the deformation equation is not quite applicable to small strains.*
(b) $|\varepsilon| = |\varepsilon_1| = \sigma_p/E = 1.43 \times 10^{-3}$. This is the smallest strain for which the bar becomes fully plastic.

From (4-72) we find

$$\lambda_p = \pi \sqrt{m/\varepsilon} = 39 = 1/i,$n which $i = \frac{20}{4} = 5 \text{ mm}.$

Hence,

$$l_p = 5 \times 10^{-3} \times 39 = 0.195 \text{ m}.$$

Conclusion. If $l > 0.584 \text{ m}$, elastic buckling occurs. If $0.195 < l < 0.584 \text{ m}$, plastic buckling occurs after incipient plastic flow.

If $l < 0.195 \text{ m}$, no buckling occurs at all.

Note. The quantity $l$ is not necessarily the real length, but the buckling length.

### 4.4.3. Instability of a thin-walled sphere under internal pressure. [2]

The criterion of instability is

$$\frac{dp}{d\delta} = 0 \quad (4-73)$$

in which $p$ is the relative pressure. From symmetry considerations, we have (Fig. 1-4)

$$\sigma_\theta = \sigma_\varphi \quad (4-74)$$

If $r$ is the radius and $t$ the shell thickness, it follows that from the equilibrium of a semi-sphere $p \pi r^2 = 2 \pi r t \sigma_\varphi$, hence

$$\sigma_\varphi = \rho r/2t \quad (4-75)$$

Inside we have $\sigma_r = -p$, outside $\sigma_r = 0$. As $r > 2t$, $\sigma_\theta > \sigma_r$.

Hence, we approximate
\[ \sigma_r = 0 \]  
\[ \overline{\sigma} = \sqrt{\sigma_{yy}^2 + \sigma_{\theta\theta}^2 - \sigma_{y\theta}^2} = \overline{\sigma}_{\theta} \]  
\[ (4-76) \]
\[ (4-77) \]

As the strain path is straight (this follows from (4-74) and (4-76), we have

\[ \delta_r = \frac{\partial \overline{\sigma}}{\partial \sigma_r} \left( \sigma_r - \frac{\sigma_{yy} + \sigma_{\theta\theta}}{2} \right) = -\bar{\delta} \]
\[ (4-78) \]

\[ \delta_\theta = -\frac{\bar{\delta}}{2} = \Delta \theta \]

Also \( \delta_r = \ln \frac{r}{r_0} \), hence

\[ t = t_0 e^{-\bar{\delta}} \]
\[ (4-79) \]

\[ \Delta \theta = \ln \frac{2mr}{2mr_0} = \ln \frac{r}{r_0} \], hence,

\[ r = r_0 e^{-\bar{\delta}/2} \]
\[ (4-80) \]

Using (4-75), (4-77), (3-23), (4-79) and (4-80), we find

\[ p = \frac{2t_0 \sigma_{yy}}{r} = \frac{2t_0 e^{-\bar{\delta}}}{r_0 e^{\frac{1}{2} \bar{\delta}}} = \frac{2t_0 e^{-\frac{1}{2} \bar{\delta}}}{r_0} \sigma_{yy} \]  
\[ (4-81) \]

\[ \frac{dp}{d\delta} = \frac{2t_0 c}{r_0} \left[ e^{\frac{3}{2} \bar{\delta}} \left\{ -\frac{3}{2} (\bar{\delta} + m (\bar{\delta})^m) \right\} \right] = 0 \]
\[ (4-82) \]

or

\[ \bar{\delta} = \frac{2}{3} m \]
\[ (4-83) \]
4.5. The tensile test [10].

Up to now we have supposed that a linear state of stress was generated near the neck. However, this is not entirely true. Bridgman gave an approximative solution to this problem.

He calculated a correction factor by considering the equilibrium of an infinitesimal element of a torus near to the minimum diameter of the neck. In Fig. 4-11, EFGH is a segment of a sphere with its centre in O. ADHE and BCGF are segments of toruses, of which MN is the radius of the main circle.

ABFE and DCGH are flat planes through OP. ABCD is a flat plane, perpendicular to OP and MN.

We will assume that only principal stresses are working on such an element.

From the equilibrium of forces in direction r, we find that

\[
\left( \sigma_z + h \frac{\partial \sigma_z}{\partial z} \right) \sin \theta' (r + \frac{dr}{2}) \, d\theta \, dr + \left( \sigma_r + \frac{\partial \sigma_r}{\partial r} \right) (r + dr) \, d\theta \, dh' +
\]

\[-\sigma_r \sin (d\theta) \, h \, dr - \sigma_r h \, r \, d\theta = 0 \quad (4-84)\]

Fig. 4-11 shows that from geometrical considerations we find:

\[
\begin{align*}
    h &= R \theta + \frac{a}{\theta} (\cos \theta' - \cos \theta) \\
    h' &= R \theta + \frac{a}{\theta} \cos (\theta' + d\theta') - \cos \theta
\end{align*} \quad (4-85)
\]

With (4-85) we find for (4-84), if we omit all higher order terms

\[
\sigma_z \frac{r^2}{a} = \sigma_r \left( \frac{3r^2}{2a} - \frac{a}{2} - R \right) - r \frac{d\sigma_r}{dr} \left[ R + \frac{a^2 - r^2}{2a} \right] + \sigma_{\theta} \left[ R - \frac{1}{2} \left( \frac{a^2 - r^2}{a} \right) \right] \quad (4-86)
\]
From circular symmetry we find that

\[ \frac{d\varepsilon_\theta}{dr} = \frac{d\varepsilon_r}{r} \quad (4-87) \]

while the incompressibility requires

\[ d\varepsilon_\theta = -\frac{2}{3} d\varepsilon_z \quad (4-88) \]

Using (3-9) and (4-87) we find

\[ \bar{\varepsilon}_\theta = \bar{\varepsilon}_r \quad (4-89) \]

Using (1-25) we find

\[ \bar{\varepsilon} = \sqrt{\bar{\varepsilon}_r^2 + \bar{\varepsilon}_z^2 + \bar{\varepsilon}_\theta^2 - \bar{\varepsilon}_r \bar{\varepsilon}_z - \bar{\varepsilon}_r \bar{\varepsilon}_\theta - \bar{\varepsilon}_\theta \bar{\varepsilon}_z} = \bar{\varepsilon}_z - \bar{\varepsilon}_r \quad (4-90) \]

From the experiments of Bridgman it appeared that \( \bar{\varepsilon} \) in the minimum section was constant. Hence, \( \bar{\varepsilon} \) is constant there as well. Substitution of (4-90) in (4-86) then yields.

\[ \frac{d\bar{\varepsilon}_r}{dr} \left[ R + \frac{a^2 - r^2}{2a} \right] + \frac{r\bar{\varepsilon}}{a} = 0 \quad (4-91) \]

With boundary condition \( \bar{\varepsilon}_{r(r=a)} = 0 \), hence, \( \bar{\varepsilon}_{(r=a)} = \bar{\varepsilon}_z \). Separation of variables and integration gives

\[ \bar{\varepsilon}_r = \bar{\varepsilon}_0 \ln \left( \frac{a^2 + 2ar - r^2}{2ar} \right) \quad (4-92) \]

\[ \bar{\varepsilon}_z = \bar{\varepsilon}_0 \left[ 1 + \ln \frac{a^2 + 2ar - r^2}{2ar} \right] \]

The load \( P \) has to satisfy

\[ P = 2\pi \int_0^a \bar{\varepsilon}_z dr = \pi a^2 \bar{\varepsilon}_{z\text{ave}} \quad (4-93) \]

from which we find

\[ \bar{\varepsilon}_0 = \frac{\bar{\varepsilon}_{z\text{ave}}}{\ln \left( 1 + (2R/a) \right)} \quad (4-94) \]
or

\[ \bar{G} = \bar{G}_\text{ave} \]  \hspace{1cm} (4-95)

in which

\[ C = \left[ \frac{(1+2R/a)}{\ln(1+a/2R)} \right]^{-1} \]  \hspace{1cm} (4-96)

In order to find the real value of \( \bar{G} \), we must multiply the value of \( \bar{G}_\text{ave} \) with \( C \).

Fig. 4-12 gives its numerical value.

**Fig. 4-12. Correction factor C according to Bridgman.**

### 4.6. Friction.

In most metal working processes, friction plays a part. This friction generally takes the character of metallic friction without a lubricant, as the temperature and the pressure are too high for a lubricant.

For a moderately loaded slider, we have a reasonable approximation according to Amontons' Laws.

According to these, the coefficient of friction is independent of
(a) the magnitude of the normal force,
(b) the magnitude of the real contact area,
(c) the speed of sliding.

We have

\[ \bar{f} = A_r \cdot \bar{T} \]  \hspace{1cm} (4-97)

in which \( \bar{f} \) = frictional force, \( A_r \) = real contact area, \( \bar{T} \) = average shear stress needed to overcome resistances caused by surface irregularities. For the normal force \( N \), we have

\[ N = A_r \cdot \bar{G} \]  \hspace{1cm} (4-98)

in which \( \bar{G} \) is the average normal pressure. Combination with (4-97) yields
\[ F = N \frac{T}{G} = \mu N \]  \hspace{1cm} (4-99)

in which \( \mu = \frac{T}{G} \) should be constant.

We will now discuss what happens if the normal pressure increases so strongly that plastic flow occurs. To this end, we consider a hard slider moving over a soft layer. Parts of the layer make contact with the slider and owing to the normal pressure, which may be very high locally, are "welded" together. Other parts may cause resistance by enclosing parts of the slider. Because of these effects, the toplayer of the softer material will move along with the hard slider. Under it, we have a layer which, consequently, is subjected to plastic deformation. The frictional force is the force needed to cause this plastic straining (Fig. 4-13)

![Fig. 4-13. Plastic strain by friction](image)

A reasonable solution to the analysis of this problem is not yet available. An approximation which is used frequently is that the stresses in the sublayer build up in such a way that the resistance against shear becomes maximum. In that case we have from (1-24)

\[ \tau_{xy} = \tau_{\text{max}} = \frac{G}{\sqrt{3}} \]  \hspace{1cm} (4-100)

If a coefficient of friction is introduced, we have

\[ \mu' = \frac{\tau_{\text{max}}}{\nu} = \frac{G}{\nu \sqrt{3}} \]  \hspace{1cm} (4-101)

which may differ locally, as \( \tau = \tau(\delta) \).

4.7. Thick-walled tube under internal pressure [1]

For a treatment of this problem we will assume:

(a) small strains,

(b) the length of the tube to be infinite. There are no boundary effects. Hence, \( \frac{\partial}{\partial z} = 0 \)
We introduce cylindrical coordinates $r$, $\theta$ and $z$.

Since we have circular symmetry, $\frac{\partial}{\partial \theta} = 0$.

From (1-6), we now find for the equilibrium in direction $r$:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

(4-102)

Fig. 4-14. Thick-walled tube under internal pressure.

4.7.1. Tube locked in direction $Z$. Ideally plastic material.

We now have

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0$$

(4-103)

First, we consider the stresses in an entirely elastic section. As an approximation, we will assume that we have also for elastic deformation $\varepsilon_z + \varepsilon_r + \varepsilon_\theta = 0$, hence, $\psi = \frac{1}{2}$. Using (3-12) we find

$$\varepsilon_z = \frac{1}{E} (\sigma_z - \frac{1}{2}(\sigma_r + \sigma_\theta)) = 0$$

(4-104)

$$2\sigma_z = \sigma_r + \sigma_\theta$$

(4-105)

For the strains, we find (Fig. 4-15)

$$\varepsilon_r = \frac{(u+du)-u}{dr} = \frac{\partial u}{\partial r}$$

(4-106)

$$= \frac{(r+u)d\theta - r \rho}{r \rho} = \frac{u}{r}$$

Hence, using (4-103) and the assumed incompressibility

$$\frac{\partial u}{\partial r} + \frac{u}{r} = 0$$

(4-107)

Integration of (4-107) and substitution in (4-106) yields
\[ \varepsilon_r = -\frac{c_1}{r^2} \]  \hfill (1-108)

\[ \varepsilon_\theta = +\frac{c_1}{r^2} \]

in which \( c_1 \) is an integration constant. Introducing \( G \) = modulus of rigidity, we write for (4-104)

\[ 2G_\varepsilon_z = \frac{2G}{E} \left( \sigma_z - \frac{1}{2}(\sigma_r + \sigma_\theta) \right) = \sigma_z' \]  \hfill (4-109)

as \( G/E = \frac{1}{3} \) for \( \gamma = \frac{1}{3} \). For \( \sigma_z' \) see (1-21).

Apparently, we have

\[ \sigma_z = \sigma_m + 2G_\varepsilon_z, \text{ cyclic} \]  \hfill (4-110)

With (4-110) and (4-108), we find

\[ \sigma_r - \sigma_\theta = -\frac{4Gc_1}{r^2} \]  \hfill (4-111)

Substitution in (4-102) yields

\[ \frac{dG_r}{dr} = \frac{4Gc_1}{r^3} \]  \hfill (4-112)

with boundary conditions \( G_r(a) = -p \) and \( G_r(b) = 0 \).

Integration of (4-112) yields

\[ c_1 = \frac{pa^2 b^2}{2G(b^2 - a^2)} \]  \hfill (4-113)

\[ \sigma_r = \frac{pa^2}{b^2 - a^2} \left( 1 - \frac{b^2}{r^2} \right) \]  \hfill (4-114)

using (4-111), (4-113) and (4-105)

\[ \sigma_\theta = \frac{pa^2}{b^2 - a^2} \left( 1 + \frac{b^2}{r^2} \right) \]  \hfill (4-114 a)
Up to now we have assumed that \( p \) was sufficiently small to prevent plastic flow. Assuming that the material is ideally plastic, we have

\[
\sigma_r^2 + \sigma_\theta^2 + \sigma_z^2 - \sigma_r \sigma_\theta - \sigma_\theta \sigma_z - \sigma_z \sigma_r = 3k^2
\]

or, using (4-105):

\[
|\sigma_\theta - \sigma_r| = 2k,
\]

\[
\sigma_\theta - \sigma_r = 2k \text{ if } \sigma_r < \sigma_\theta
\]

Hence, according to (4-114) and (4-115), no plastic deformation occurs, if

\[
\frac{pa^2b^2}{r(b^2-a^2)} < k, \ a \leq r \leq b
\]

The left-hand side of (4-116) is maximum if \( r=a \).

Apparently, plastic yielding begins at the inside of the tube. Substitution of \( r=a \) in (4-116) gives the value \( p^* \) of incipient plastic flow for the inside of the tube.

\[
p^* = (1-a^2/b^2)k
\]

Hence

\[0 < p < p^* \text{ complete elasticity}
\]

\[p = p^* \text{ incipient plastic flow}
\]

\[p > p^* \text{ increasing region of plastic flow.}
\]

The equilibrium in the plastic zone requires (substituting (4-115) in (4-102))
\[
\frac{\partial \sigma_r}{\partial r} = \frac{2k}{r} \quad p > p^*
\]  

With boundary condition \( \sigma_r(a) = -p \quad p > p^* \)

Using (4-118), (4-115\(^b\)) and (4-106), we find

\[
\begin{aligned}
\sigma_r & = 2k(\ln r/a) - p \\
\sigma_\theta & = 2k(1+\ln r/a) - p \\
\sigma_z & = k + 2k(\ln r/a) - p
\end{aligned}
\]  

In this stage of elasto-plasticity, the formulae (4-114) do not hold anymore, either for the plastic or for the elastic part of the tube. We will calculate the new elastic stresses from (4-114) by the assumption that on the elasto-plastic boundary (radius \( \rho \)), we have

\[ p^* = -\sigma_r(\rho). \]

Hence, we replace the tube \( a-b \) with pressure \( p \) by an entirely elastic tube \( \rho - b \) with pressure \( p^* \). From (4-114) we find

\[
\sigma_r = \frac{p^* \rho^2}{b^2 - \rho^2} \left( 1 - \frac{b^2}{r^2} \right) \quad \rho < r < b
\]  

in which, according to (4-117): \( p^* = (1 - \frac{\rho^2}{b^2}) k \), hence

\[
\sigma_r = \frac{k \rho^2}{b^2} \left( 1 - \frac{b^2}{r^2} \right) \quad \rho \rho < b
\]

\[
\sigma_\theta = \frac{k \rho^2}{b^2} \left( 1 + \frac{b^2}{r^2} \right)
\]

We calculate the magnitude of \( \rho \) by assuming continuity of \( \sigma_r \) for \( r = \rho \). From (4-119) and (4-122), we find

\[
1 + 2\ln \rho/a = \frac{p}{k} + \frac{\rho^2}{b^2}
\]  

From (4-123), \( \rho \) can be found numerically. The pressure \( p^{**} \) for which
the tube becomes fully plastic, can by found by substitution of
\[ \rho = b \] in (4-123). We find

\[ p^{**} = 2k \log \frac{b}{a} \]  \hspace{1cm} (4-124)

We will now consider what happens, if we load the tube with a pressure
\[ p \] for which \[ p^* \leq p < p^{**} \] and then release the pressure. It is clear that
in that case residual stresses will be generated. As the releasing
occurs entirely elastically, we must subtract (4-114) from stresses
according to (4-119) and (4-122), respectively. We find

\[ \frac{a < r < b}{r}: \]
\[ \sigma_r = 2k(\ln \frac{r}{a}) - p - \frac{pa^2}{b^2 - a^2} \left(1 - \frac{b^2}{r^2}\right) \]  \hspace{1cm} (4-125)

\[ \sigma_\theta = 2k(1+\ln \frac{r}{a}) - p - \frac{pa^2}{b^2 - a^2} \left(1 + \frac{b^2}{r^2}\right) \]

\[ \frac{a < r < b}{r}: \]
\[ \sigma_r = \left(\frac{k}{b^2} - \frac{pa^2}{b^2 - a^2}\right) \left(1 - \frac{b^2}{r^2}\right) \]  \hspace{1cm} (4-126)

\[ \sigma_\theta = \left(\frac{k}{b^2} - \frac{pa^2}{b^2 - a^2}\right)(1 + \frac{b^2}{r^2}) \]

Could renewed (secondary) plastic flow occur? It may be imagined that the
inside of the tube, which was mainly loaded with tensile stresses, will
after releasing be compressed so heavily that it becomes plastic again.
The criterion for the occurrence of this phenomenon follows from (4-115a)
and (4-125)

\[ \sigma_\theta - \sigma_r = 2k - \frac{2pa^2b^2}{r^2(b^2 - a^2)} \]  \hspace{1cm} (4-127)

with (4-117) we obtain for (4-127)
\[ \nabla_{\theta} - \nabla_{r} = 2k(1 - \frac{pa^2}{p^*r^2}) \]  
\[ (4-128) \]

Secondary plastic flow may occur, if

\[ 2k(1 - \frac{pa^2}{p^*r^2}) < - 2k \]  
\[ (4-129) \]

This will again be generated for \( r=a \). Then we find from (4-129)

\[ p > 2p^* \]  
\[ (4-130) \]

This means that we have to raise the pressure at least to double the value for incipient plastic flow in order to obtain secondary plastic flow. In that case, however, the tube must not collapse, in other words, \( p < p^{**} \). Therefore this will only occur if

\[ 2p^* < p^{**} \]  
\[ (4-131) \]

Using (4-117) and (4-124) we obtain from (4-131)

\[ 1 - \left( \frac{a^2}{b^2} \right) < \ln \frac{b}{a} \]  
\[ (4-132) \]

from which we derive \( b/a > 2.22 \). This is the condition under which secondary flow may be generated. If \( b/a < 2.22 \) it is fundamentally impossible to create this phenomenon. It is called the shake-down condition. In constructions we will generally take care that the latter condition is satisfied.

4.7.2. Tube locked in direction Z. Strain hardening material.

Since an ideally plastic material hardly exists in practice, the calculated shake-down condition has only a limited value. Hence, we shall repeat the calculation for an exponentially strain hardening material.

It is clear that formulae (4-102) to (4-114) remain unchanged. For the yield condition we now obtain, however,
(4-133)

\[ \sigma_\theta - \sigma_r = \frac{2G}{3} - \frac{2}{\sqrt{3}} \xi^m \]

From (4-108) we see that the strain path is straight. Hence,

\[ \xi = \sqrt{\frac{2}{3} (\xi_r^2 + \xi_\theta^2)} = \sqrt{\frac{4}{3} \frac{c_1}{r^2}} \]  

and, for (4-133), we find

\[ \sigma_\theta - \sigma_r = \frac{2G}{3} \left[ \frac{4}{3} \frac{c_1}{r^2} \right]^m \leq A \cdot r^{-2m} \]  

Hence, substituting (4-113)

\[ A = \frac{2c_{m+1}}{3^2} \left\{ \frac{p a b^2}{G (b^2 - a^2)} \right\}^m \]  

With (4-114) it follows that no plastic flow occurs, if

\[ \frac{2p a b^2}{r^2 (b^2 - a^2)} < A \cdot r^{-2m} \]  

Again, plastic flow begins for \( r = a \), if \( p = p^* \).

Using (4-137) and (4-136), we find:

\[ p^* = \left( \frac{c}{3^2 G} \right)^{1-m} \frac{1}{1-a^2} \left( \frac{a^2}{b^2} \right) \equiv B \left( \frac{a^2}{b^2} \right) \]  

Substitution of (4-135) in (4-102) gives the differential equation for the equilibrium in the plastic area.

\[ \frac{\partial \sigma_r}{\partial r} = A \cdot r^{-2m-1} \]  

With the boundary condition \( \sigma_r(a) = -p \). With the help of (4-135), integration gives
\[ G_r = A \left( \frac{a^{-2m}}{2m} - \frac{r^{-2m}}{2m} \right) - p \]

\[ \hat{G}_\theta = A \left( \frac{a^{-2m}}{2m} - \frac{2m-1}{2m} \cdot \frac{r^{-2m}}{2m} \right) - p \]

At the elasto-plastic boundary we have again (4-120) and (4-121). However, \( p^* \) is now given by (4-138). As in (4-122), we find

\[ \hat{G}_r = \frac{B\rho^2}{b^2} \left( 1 - \frac{b^2}{r^2} \right) \quad \rho < r < b \]  

\[ \hat{G}_\theta = \frac{B\rho^2}{b^2} \left( 1 + \frac{b^2}{r^2} \right) \]

The magnitude of \( \rho \) then follows from (4-140) and (4-141) for \( r = \rho \)

\[ \frac{A}{2m} \left( a^{-2m} - \rho^{-2m} \right) - p = B\left( \frac{\rho^2}{b^2} - 1 \right) \]  

(4-142)

from which \( \rho \) can be solved numerically.

\( p^{**} \) follows from (4-142) if \( \rho = b \)

\[ p^{**} = \frac{A}{2m} \left( a^{-2m} - b^{-2m} \right) \]

(4-143)

Again, we release the load, hence, superimpose (4-114) on (4-140):

\[ \hat{G}_r = A \left( \frac{a^{-2m}}{2m} - \frac{r^{-2m}}{2m} \right) - p - \frac{pa^2}{b^2 - a^2} \left( 1 - \frac{b^2}{r^2} \right) \]

(4-144)

\[ \hat{G}_\theta = A \left( \frac{a^{-2m}}{2m} - \frac{2m-1}{2m} \cdot \frac{r^{-2m}}{2m} \right) - p - \frac{pa^2}{b^2 - a^2} \left( 1 + \frac{b^2}{r^2} \right) \]
Secondary flow occurs only if

$$\sqrt{\nu} - \sqrt{r} = A r^{-2m} - \frac{2pa^2b^2}{r^2(b^2 - a^2)} < - Ar^{-2m} \quad (4-145)$$

This flow will begin for \( r = a \), hence, from (4-145)

$$p > A a^{-2m} \left(1 - \frac{a^2}{b^2}\right) \quad (4-146)$$

However, this is only possible, if \( p < p^{**} \), so

$$A a^{-2m} \left(1 - \frac{a^2}{b^2}\right) < \frac{A}{2m} \left(a^{-2m} - b^{-2m}\right) \quad (4-147)$$

Hence, for the shake-down condition, we derive

$$1 - \left(\frac{a}{b}\right)^2 = \frac{1}{2m} \left\{1 - \left(\frac{a}{b}\right)^{2m}\right\} \quad (4-148)$$

If \( m \to 0 \), the right-hand side becomes

$$\lim_{m \to 0} \frac{1 - \left(\frac{a}{b}\right)^{2m}}{2m} = 1 - e^{\frac{2mln a}{b}} = \frac{1 - 1 - 2mln \frac{a}{b} - \frac{(2mln a)^2}{2m}}{2m} = \ldots$$

$$= \ln \frac{b}{a} \quad (4-149)$$

Indeed, for this case we find again (4-132).

Numerical solution of (4-148) gives the picture of Fig. 4-16:

Fig. 4-16. The influence of \( M \) on the shake-down.

The explanation of this phenomenon is, of course, that strain hardening tends to prevent the occurrence of secondary plastic flow, as then a higher degree of deformation would be necessary to create this. We see from Fig. 4-16 and from (4-148) that, if \( m \to 0.5 \), we always have shake-down. Secondary plastic flow then cannot occur at all.
Fig. 4-3

\sigma = c \Delta^m

\sigma = c(\Delta + d\Delta)^m

\text{(S = shear force)}

dL = \tau_{xy} b \, dx

Fig. 4-4

Fig. 4-5

Fig. 4-6

Fig. 4-7
Fig. 4-8

Fig. 4-9
Fig. 4-12

Fig. 4-13

Fig. 4-14
5. Some special methods of solution.

5.1. The general problem.

In chapters 1 to 3 we derived a mathematical model for finite straining with strain hardening. Though it would be possible to define shear components of natural strains, we have preferred not to do so, as a more direct approach is possible. Instead of the constitutive equations for finite shear, we use the following hypothesis:

In an isotropic medium the direction of extreme elongation is a principal direction both for strain and stress. (In fact, we have already used this hypothesis in Example 5, Part 3.7.) Actually, in theoretical respect, this hypothesis is incorrect. Hill [5], proved, however, that the direction of extreme strain-rate equals the direction of extreme stress. In technical processes, though, the principal directions of strain and strain-rate will generally be coincident.

In order to deform a material efficiently, the direction of deformation is generally kept constant during the entire process. Then the principal directions of strain and strain-rate are the same or become the same, if deformation continues.

If the direction of deformation changes abruptly, we may as well consider the deformed stage as a fresh initial one.

We may then calculate stresses and strains of the new process under the assumption that the new material properties can be determined from the preceding straining. If \( \Delta \) is small we may, in the second stage, consider the material as ideally plastic with \( \sigma' = c \Delta^m \) constant. Then is the preceding effective strain.

The following table gives a summary of equations and variables as they follow from the theory.
### Table: Equations and Variables

<table>
<thead>
<tr>
<th>Equations</th>
<th>New Variables</th>
<th>Number of New Variables</th>
<th>Number of Equations</th>
</tr>
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<tbody>
<tr>
<td>(a) eq. of equilibrium</td>
<td>$G_x, G_y, G_z$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>$T_{xy}, T_{yz}, T_{zx}$</td>
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<td></td>
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<tr>
<td>(b) transformation eq. for principal directions</td>
<td>$G_1, G_2, G_3$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>(c) yield condition</td>
<td>$\bar{Q}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(d) effective strain</td>
<td>$\delta, \delta_1, \delta_2, \delta_3$</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>(e) constitutive equations for principal directions</td>
<td>-</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td>(f) deformation equation</td>
<td>-</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>(g) strain = principal direction stress</td>
<td>-</td>
<td>-</td>
<td>2</td>
</tr>
<tr>
<td><strong>TOTAL:</strong></td>
<td><strong>14</strong></td>
<td><strong>14</strong></td>
<td></td>
</tr>
</tbody>
</table>

**Adstruction.**

Re: (b). From (1-15) we can solve $G_1, G_2$ and $G_3$. Hence, from (1-15) we derive 3 equations.

Re: (f). We suppose the quantities c and m to be known.

Re: (g). The orthogonal coordinate system $1' 2' 3'$ should be rotated in such a way that it coincides with $1'' 2'' 3''$. First rotate it so that $1'// 1''$. (1 equation). Then $2'// 2''$ (another equation) Then, automatically, $3'// 3''$. Hence, the third equation is dependent.

Though it is theoretically possible for any well-defined problem to solve these equations, we have no general adoptable procedure for this. Therefore, in practice we use special methods, which are sometimes approximate, or semi-experimental ones. This chapter deals with three of these methods, viz., virtual work, slab method of solution, and visioplasticity.
5.2. Virtual work [2].

We will suppose that boundary friction does not influence the stress distribution as this follows from the strains, but yet influences the work consumption of the system. Then we apply the virtual work principle (see Part 4.2.3).

5.2.1. Hollow sphere under internal pressure.

For a thick-walled hollow sphere under internal pressure \( p \), inside radius \( a \), outside radius \( b \), if the inside radius increases from \( a \) to \( a + da \), the incremental external work is:

\[
dW_E = p \cdot 4\pi a^2 da
\]  
(5-1)

The work consumed by the shell is:

\[
dW_I = V_{\text{shell}} \delta \delta = \frac{4}{3} \pi (b^3 - a^3) \delta \delta
\]  
(5-2)

As \( dW_E = dW_I \), we find

\[
3pa^{-1} \left( \frac{b^3}{a^3} - 1 \right)^{-1} da = \delta \delta
\]  
(5-3)

Using the kinematic relation between \( da \) and \( \delta \delta \) and the deformation equation, we may calculate \( p \).

We apply this procedure for a thin-walled sphere, thickness of wall \( t \). Equation (5-3) then simplifies to

\[
p \ da = t \ \delta \delta
\]  
(5-4)

in which \( t \) is the momentary wall-thickness.

The strains are

\[
\delta_r = \ln \frac{t}{t_0}
\]  
(5-5)

\[
\delta_\theta = \delta_\phi = \ln \frac{a}{a_0} = -\frac{\delta_r}{2}
\]
From (5-5) we see that the strain path is straight. Hence

\[ \bar{\delta} = \sqrt{\frac{2}{3}}(4+1)\ln \frac{a}{a_o} = 2 \ln \frac{a}{a_o} \]  

(5-6)

Differentiation of (5-6) with respect to \( a \) yields, after solution of \( da \)

\[ da = \frac{a_o}{2} e^{\frac{\bar{\delta}}{2}} \]  

(5-7)

Substitution of (5-7) in (5-4) then yields

\[ p = t \left( \frac{a_o}{a} \right)^2 \bar{\sigma} \left( 2 \frac{a}{a_o} \right) e^{-\frac{\bar{\delta}}{2}} \]  

(5-8)

or, using \( e^{\ln x} = x \) and (5-6)

\[ p = 2 \left( \frac{a_o}{a} \right)^2 \bar{\sigma} \]  

(5-9)

From (5-5) it follows that

\[ t/t_o = \left( \frac{a_o}{a} \right)^2 \]  

(5-10)

Hence,

\[ p = \left( \frac{2t}{a} \right) \bar{\sigma} \]  

(5-11)

in which \( \bar{\sigma} \) represents the effective stress caused by the pressure \( p \). Moreover, we have \( \bar{\sigma} = \sigma_y \) and \( \sigma_r = 0 \) (See part 4.4.3.)

5.2.2. Wire drawing.

We will simplify the process of wire drawing as follows:
(a) Plane sections remain plane during straining.
(b) Along the boundary acts a friction stress \( \tau_{\text{max}} = \bar{\sigma}/\sqrt{3} \), which does not influence the internal stress distribution. Fig. 5-1 gives the meaning of the symbols used.

Fig. 5-1. Wire drawing.
From geometrical considerations, we find

\[ r = x \tan \alpha \]
\[ r_o = r_0 \tan \alpha \]
\[ r_1 = r_1 \tan \alpha . \]

The equation of continuity requires

\[ r^2 \, dx = r_1^2 \, dl_1 = r_o^2 \, dl_o \]  \hspace{1cm} (5-13)

For the strains, we have

\[ \delta_x = \ln \frac{dx}{dl_0} = \frac{r_o}{r} \]
\[ \delta_r = -\ln \frac{r_o}{r} \]  \hspace{1cm} (5-14)
\[ \delta_\theta = -\ln \frac{r_o}{r} \]

Hence (compare (5-6)):

\[ \delta = 2\ln \frac{r_o}{r} = 2\ln \frac{x_o}{x} \]  \hspace{1cm} (5-15)
\[ d\delta = -2dx/x \hspace{0.5cm} (dx<0, \text{hence } d\delta>0) \]  \hspace{1cm} (5-16)

The force needed to draw the wire is \( P \). Then, the incremental external work is \( P \, dl_1 \). This is equal to the internal straining and boundary friction work required for a displacement of \( dl_1 \). If a disc at \( x \) moves over a distance \( dx \), the work required is (except for the friction).

\[ d^2w_I = \pi r^2 \, dx \, \delta \, d\delta \]  \hspace{1cm} (5-17)

Using (5-13), (3-23), (5-15) and (5-16) we write for this

\[ d^2w_I = -2\pi r_1^2 \, dl_1 \, c \left( 2 \ln \frac{x_o}{x} \right) \frac{dx}{x} \]  \hspace{1cm} (5-18)
The shear stress caused by boundary friction is

$$\tau_{\text{max}} = \frac{\mu}{\sqrt{3}} = \frac{c}{\sqrt{3}} \left(2\ln \frac{a}{x}\right)$$  \hspace{1cm} (5-19)

For the conical boundary of a disc, radius \(r\), height \(dx\) this becomes an infinitesimal force

$$dF = 2\pi r \frac{dx}{\cos \alpha} \tau_{\text{max}}$$  \hspace{1cm} (5-20)

If the disc moves over \(dx\), contracting over \(dr\), the boundary moves over \(dx/\cos \alpha\). The work then is

$$d\tilde{W} = \frac{dF}{\cos \alpha} \frac{dx}{\cos \alpha} = \frac{2\pi r}{\sqrt{3} \sin 2\alpha} r_1^2 \int c(2 \log \frac{x_m}{x}) \frac{dx}{x}$$  \hspace{1cm} (5-21)

\(d\tilde{W}_I\) has to be integrated from \(x_0\) to \(x\); \(d\tilde{W}_{II}\) from \(x_1\) to \(x_0\). Hence the total incremental work is

$$d\tilde{W} = \int_{x_1}^{x_0} \left( -d\tilde{W}_I + d\tilde{W}_{II} \right)$$  \hspace{1cm} (5-22)

or, using (5-18) and (5-21),

$$d\tilde{W} = \left(2 + \frac{4}{\sqrt{3} \sin 2\alpha}\right) r_1^2 \int \int \left(\ln \frac{x_0}{x}\right) \frac{dx}{x}$$  \hspace{1cm} (5-23)

Application of the virtual work principle then yields (substituting (5-12) in (5-23))

$$P = \left(2 + \frac{4}{\sqrt{3} \sin 2\alpha}\right) \pi r^2 c_m \int_{r_1}^{r_o} \left(\ln \frac{r_o}{r}\right) \frac{dr}{r}$$  \hspace{1cm} (5-24)

For the maximum allowable value of \(P\) we have

$$P^* = \pi r^2 \bar{c}_m \left(r=r_1\right) = r_1^2 c \left(\ln \frac{r_o}{r_1}\right)$$  \hspace{1cm} (5-25)
Thus, drawing is only possible if
\[
\left( \ln \frac{r}{r_0} \right)^m < \left( 2 + \frac{4}{3 \sin 2} \right) \int_{r_1}^{r_0} \left( \ln \frac{r}{r_0} \right)^m \frac{dr}{r} \hspace{1cm} (5-26)
\]

We can also treat the problem of wiredrawing under the assumption that plastic friction does not occur. In that case we may introduce a constant coefficient of friction $\mu$. If the force normal to the wall is $N$, the axial equilibrium requires $P = N \sin \alpha + N \cos \alpha$, or
\[
\mu N = \frac{\mu P}{\sin \alpha + \mu \cos \alpha} \hspace{1cm} (5-27)
\]

The area of the boundary is
\[
A = \frac{\pi (r_2^2 - r_1^2)}{\sin \alpha} \hspace{1cm} (5-28)
\]

Assuming that the friction force is equally distributed over the boundary of the cone, we have:
\[
\tau = \frac{\mu N}{A} = \frac{\mu P \sin \psi}{\pi (r_2^2 - r_1^2) \sin \alpha + \mu \cos \alpha} \hspace{1cm} (5-29)
\]

For an infinitesimal element from the boundary area, we find a force $dF$ from (5-29), viz.
\[
dF = \frac{2 \tau \rho \cos \alpha}{\cos \alpha} = \frac{2 \mu P \tan \alpha \cdot \rho \cos \alpha}{(r_0^2 - r_1^2) \sin \alpha + \mu \cos \alpha} \hspace{1cm} (5-30)
\]

If this element moves over a distance $dx$, then the energy dissipated by friction is
\[
d^2W_{\text{II}} = \frac{dF \cdot dx}{\cos \alpha} = \frac{2 \mu P \tan \alpha \cdot r^2 dx}{(r_0^2 - r_1^2) \sin \alpha + \mu \cos \alpha} \hspace{1cm} (5-31)
\]
Hence, if the entire cone moves over a distance $dx$, then

$$dW_{II} = \int_{r_1}^{r_0} dW_{II} = \frac{2\mu Fr_1^2 \log \frac{r_0}{r_1} dl_1}{(r_0^2 - r_1^2)(\sin \alpha + \mu \cos \alpha) \cos \alpha} = \phi_{II} dl_1 \quad (5-32)$$

By adding (5-32) and the deformation part of (5-24), we find the total incremental work. Application of the virtual work principle then gives a solution for $P$.

5.2.2. Deep drawing.

![Fig. 5-2. Deformation by deep-drawing.](image)

A circular blank with an initial radius $R_0$ is drawn in the shape of a cylindrical bowl with radius $R_1$. The wall thickness is $t$. The momentary depth of the bowl is $h$, the momentary outside radius of the blank is $R$, the rounding radius of the drawing ring is $\rho$. The radius $r$ is an arbitrary diameter on the part of the blank which is yet horizontal. We will assume that $t \ll R$, $r$, $R_1$ and $R_0$ and that moreover $\rho \ll R_1$.

Further, we will assume that $t$ remains constant during the entire process.

During the straining, work is consumed by:

(I) The contraction of the horizontal part of the blank;
(II) The bending and straightening, respectively, of the blank at a radius $\rho$;
(III) The friction of the blank on the radius $\rho$;
(IV) The friction of the blank-holder (This will be neglected);
(V) At the beginning of the deformation process:

The shaping of the lower radius of the product. This will be neglected as well.

From the continuity equation it follows that:

$$2R_1 h + r^2 = r_0^2 \quad (5-33)$$
if $r_0$ is the initial magnitude of $r$. Hence,

$$r_0 = \sqrt{r^2 + 2R_1 h}$$  \hspace{1cm} (5-34)

$$h = \frac{r_0^2 - r^2}{2R_1}$$  \hspace{1cm} (5-35)

If $r=R$, then $r_0 = R_0$, hence

$$R = \sqrt{R_0^2 - 2R_1 h}$$  \hspace{1cm} (5-36)

Further, by differentiation of (5-35), we obtain:

$$dr = -\frac{R_1}{r} \, dh$$  \hspace{1cm} (5-37)

We calculate the strains on radius $r$.

(We introduce cylindrical coordinates $r$, $\theta$ and $z$).

$$\delta_\theta = \ln \frac{r}{r_0} = -\ln \frac{r_0}{r}$$
$$\delta_r = -\delta_\theta$$ \hspace{1cm} \text{; } \delta_z = 0$$  \hspace{1cm} (5-38)

Apparently, we have

$$\bar{\delta} = \sqrt{\frac{4}{3} \ln \frac{r_0}{r}}$$  \hspace{1cm} (5-39)

or, using (5-34):

$$\bar{\delta} = \sqrt{\frac{1}{3} \ln \left(1 + \frac{2R_1 h}{r^2}\right)}$$  \hspace{1cm} (5-40)

Differentiation yields

$$d \bar{\delta} = -\sqrt{\frac{4R_1 h \, dr}{3 \, r(r^2 + 2R_1 h)}}$$ \hspace{1cm} (dr < 0)$$  \hspace{1cm} (5-41)

The effective stress on radius $r$ is
The incremental specific work on radius \( r \) is

\[
dA_i = \overline{\sigma} \, d\delta = -c \left( \frac{1}{3 \sqrt{3}} \ln \left( 1 + \frac{2R_i h}{r^2} \right) \right)^m \sqrt{\frac{1}{3} \left( \frac{4R_i h}{r^2 + 2R_i h} \right)}
\]

The incremental strain energy in a ring with radius \( r \), height \( t \) and width \( dr \) is

\[
d^2W_i = 2\sigma \, r \, dr \, dA_i
\]

With (5-37) and (5-42) we find for (5-44):

\[
d^2W_i = c \left( \frac{1}{3 \sqrt{3}} \ln \left( 1 + \frac{2R_i h}{r^2} \right) \right)^m \sqrt{\frac{1}{3} \left( \frac{4R_i h}{r^2 + 2R_i h} \right)}
\]

The incremental strain energy for the entire horizontal part of the blank is then

\[
dW_i = \int_{R_i}^{R} d^2W_i = \int_{r=R_i}^{r=R} f_i(h,r) \, dh \, dr = \varphi_i(h) \, dh
\]

Hence, \( \varphi_i \) can be calculated as a function of \( h \).

Next, we calculate the work, needed to bend the blank around the radius \( \rho \) and draw it straight again. (Fig.5-2). From the continuity it follows that

\[
2\pi r dr = -2(R_1 + \rho) dx_1 = -2\pi R_1 dx_2.
\]

Hence, using (5-37):

\[
dx_1 = \frac{R_1}{R_1 + \rho} \, dh
\]

\[
dx_2 = dh
\]
Fig. 5-2. Bending the blank according to the radius of the die.

Now we will consider the material as ideally plastic with yield stresses $\sigma_1$ and $\sigma_2$ at places 1 and 2, respectively. $\sigma_1$ and $\sigma_2$ are determined by the preceding strain hardening.

At 1 and 2, respectively, we may calculate the effective strain from (5-40), with $r=R_1+r'$ and $r=R_1$, respectively.

$$\delta_1 = \frac{1}{3} \ln \left( 1 + \frac{2R_1 h}{(R_1 + r')^2} \right)$$

$$\delta_2 = \frac{1}{3} \ln \left( 1 + \frac{2h}{R_1} \right)$$

(5-48)

From (5-48) follow the values of $\sigma_1$ and $\sigma_2$:

$$\sigma_1 = c \delta_1^m$$

$$\sigma_2 = c \delta_2^m$$

(5-49)

We consider the entire section to be plastic. The average effective strain in the section then is:

$$\bar{\delta} = \frac{t}{2\rho}$$

(5-50)

As $\sigma_1$ and $\sigma_2$ are constant, the specific work is

$$A_1 = \sigma_1 \frac{t}{2\rho}$$

$$A_2 = \sigma_2 \frac{t}{2\rho}$$

(5-51)

Hence, the straining-energy for the bending and straightening of the infinitesimal rings 1 and 2 is:
Using (5-47) we derive from (5-52)

\[ dW_{II} = dW_1 + dW_2 = (\bar{\sigma}_1 + \bar{\sigma}_2) \frac{R_i d\theta}{\rho} = \varphi(h) dh \]  \hspace{1cm} (5-53)

Finally, we calculate the contribution of the work consumed by the friction of the blank at the radius of the die. We consider the blank as a string drawn over a rough circular cylinder. The force at the "pulling part" is \( P \) and equals the force needed for deep-drawing. We know that the force at the other end then equals:

\[ P_0 = P e^{-\mu \theta} \]  \hspace{1cm} (5-54)

if \( \theta \) is the angle subtended by the string on the cylinder and \( \mu \) the coefficient of friction. In this case, \( \theta = \pi/4 \). If we displace \( P \) over a distance \( dh \), the work consumed by friction is:

\[ dW_{III} = (P - P_0) dh = P(1 - e^{-\frac{\mu \pi}{4}})dh. \]  \hspace{1cm} (5-55)

Now we may apply the virtual work principle.

\[ dW_1 + dW_{II} + dW_{III} = P dh \]  \hspace{1cm} (5-56)

Using (4-46), (4-53) and (4-55), we derive:

\[ P = (\varphi_1 + \varphi_{II}) e^{\frac{\mu \pi}{4}} \]  \hspace{1cm} (5-57)

Remark. Wire-drawing was an example of a steady-state process. A steady-state process is defined as one in which, in the fixed coordinate system, all strains are independent of time. If this condition is not satisfied, we have a non-steady state process. Examples of steady-state processes are:
chip-forming processes, wire extrusion, sheet rolling.
Non-steady state processes: punching, forging, bending.

5.3. The slab method of solution.

We simplify the process as follows:

(1) A plane perpendicular to the direction of flow is a principal plane.
(2) The stress in such a plane is invariant.
(3) Friction forces do not influence the internal stress-distribution.

Using these assumptions we choose a set of principal planes and consider these as slabs of infinitesimal thickness. A balance of forces yields an equation of equilibrium, which may be integrated analytically or numerically and together with the boundary conditions gives a(n) (average) solution.

Example. Thick-walled sphere (inside radius $a$, outside radius $b$) under internal pressure. We consider an infinitesimal spherical shell with a radius $r$ (spherical coordinates $r, \theta$ and $\phi$). In order to calculate infinitesimal strains, we consider $dr$ as a displacement. Then, analogously to (4-106) it follows that

$$\frac{d\varepsilon_\phi}{d\phi} = \frac{(r+dr) d\phi - rd\phi}{r} = \frac{dr}{r} \tag{5-58}$$

With (2-20) it follows that

$$\delta_\phi = \int \frac{dr}{r} = \ln \frac{r}{r_0} \tag{5-59}$$

$$\delta_\theta = \ln \frac{r}{r_0}$$

hence

$$\delta_r = -2\ln \frac{r}{r_0} \tag{5-59a}$$

As the ratio between the strains remains constant, it follows that
\[ \delta = 2 \ln \frac{r}{r_0} \]  

(5-60)

From (1-7) we derive, since \( \frac{\partial \psi}{\partial \varphi} = \frac{\partial \psi}{\partial \theta} = 0 \), that

\[ \frac{d\sigma_r}{dr} + \frac{2(\sigma_r - \sigma_\theta)}{r} = 0 \]  

(5-60a)

As \( \sigma_\theta = \sigma_\varphi \) and \( \sigma_\varphi > \sigma_r \), it follows with (1-25) that

\[ \sigma = \sigma_\varphi - \sigma_r \]  

(5-61)

Hence,

\[ \frac{d\sigma_r}{dr} - \frac{2 \sigma}{r} = 0 \]  

(5-62)

in which

\[ \sigma = c \left\{ 2 \ln \frac{r}{r_0} \right\}^m \]  

(5-63)

Since \( m \) is small we may assume an average value of \( \sigma \) and consider \( \sigma = \sigma_\varphi (r) \). Then, integration of (5-62) and substitution of the boundary condition \( \sigma_r (b) = 0 \), yields.

\[ -\sigma_r = p = 2 \sigma \ln \frac{b}{r} \]  

(5-64)

We divide a thick-walled sphere into a number of slabs. The ultimate pressure then follows from

\[ P_{\text{tot}} = \Delta p_1 + \Delta p_2 + \ldots + \Delta p_n = \sum_{i=1}^{n} \Delta p_i \]  

(5-65)

\[ = 2 \sigma \ln \frac{r_1}{r_0} + 2 \sigma \ln \frac{r_1}{r_2} + \ldots \]  

\[ P_{\text{tot}} = 2 \sum_{i=1}^{n} \sigma_{(i-1)i} \ln \frac{r_{i-1}}{r_i} \]  

(5-66)
In the case of a thin-walled sphere, it follows from (5-64), using $a \ll t$, that

$$ p = 2\sigma_1 \ln \left( \frac{a + t}{a} \right) \sigma \frac{2t}{a} $$

(5-67)

which is identical with (5-11).

5.4. Visioplasticity.

Visioplasticity is an experimental method for the determination of strains and/or strainrates.

On a test specimen a grid pattern is produced, e.g. by photographic methods. During the application of a known load the deformations of the grid pattern are photographed and measured afterwards. From these data strains and stresses are calculated.

Generally, this method is only applicable to plane states of strain, or states of strain for which the plane on which the grid is produced, is a principal plane.

Fig. 5-4. Successive stages of strain of a grid.

We will treat the case of a non-steady state process. With intervals of time $\Delta t$ we make photographs of the deforming grid and on successive photographs we see patterns as in Fig. 5-4.

We take care that each rectangle is so small that during the straining it can be approximated by a parallelogram. We will assume that the strain of such a parallelogram is uniform.

We also introduce a moving coordinate-system $\bar{X} - \bar{Y}$ and consider the strains with respect to this system. The angle $\gamma$ determines the rotation of the parallelogram as a whole.

The quantities which determine the straining are

$\gamma =$ the change of the originally right angle.

$a_1 =$ the new length in direction $\bar{X}$.

$b_1 =$ the new length in direction $\bar{Y}$. 

We shall calculate the principal directions for an arbitrary element. Fig. 5-5 gives a picture of the deformation. The area of the parallelogram does not have to remain constant, hence $\delta_3 \neq 0$ is allowed. In Fig. 5-6 we see the same straining process, split up into two stadia:

(I) $A_0 B_0 C_0 D_0 \rightarrow A_0 B_0' C_0' D_0'$,

(II) $A_0 B_0' C_0' D_0' \rightarrow A_0 BCD$.

We consider the straining of an arbitrary element of line $AE_o$ characterised by the angle $\gamma$, to $A_0 E'$ characterised by $\gamma'$ and to $A_0 E$ characterised by $\gamma$. Their lengths are called $r_o$, $r'$ and $r$, respectively.

For the first stage we have:

$$\delta r_1 = \ln \frac{r'}{r_o}$$

(5-68)

For the second stage

$$\delta r_{II} = \ln \frac{r}{r'}$$

(5-69)

The total strain follows from

$$\delta r = \ln \frac{r}{r_o} = \ln \left( \frac{r}{r'} \cdot \frac{r'}{r_o} \right) = \ln \frac{r}{r'} + \ln \frac{r'}{r_o}$$

(5-70)

$$\delta r = \delta r_1 + \delta r_{II}$$

(5-71)
Apparently — even in this non-linear case — we are allowed to superimpose these logarithmic strains. From Fig. 5.6 we find

\[
\begin{align*}
\frac{r'_2}{r_0^2} &= x_o^2 + y_o^2 \\
(1 + \Delta x_I)^2 + y_o^2(1 + \Delta y_I)^2
\end{align*}
\]

With \( \Delta x_I = \frac{a_i - a}{a} \) and \( \Delta y_I = \frac{b_i - b}{b} \)

We introduce:

\[
\begin{align*}
(1 + \Delta x_I)^2 &= u \\
(1 + \Delta y_I)^2 &= v
\end{align*}
\]

From (5.72) and (5.73) we also have:

\[
\begin{align*}
\left( \frac{r'}{r_0} \right)^2 &= u \sin^2 \gamma + v \cos^2 \gamma \\
\tan \gamma &= \frac{x_0}{y_0} ; \quad \tan \xi = \frac{x_0}{y_0} \sqrt{\frac{u}{v}} \\
\tan \gamma &= \sqrt{v/u} \tan \xi
\end{align*}
\]

Using

\[
\begin{align*}
\sin^2 \gamma &= \frac{\tan^2 \gamma}{1 + \tan^2 \gamma} \\
\cos^2 \gamma &= \frac{1}{1 + \tan^2 \gamma}
\end{align*}
\]

and (5.75), we find for (5.74)
\[
\left( \frac{r}{r_0} \right)^2 = \frac{uv (1+ \tan^2 \phi)}{u + v \tan^2 \phi} \tag{5-77}
\]

Hence

\[
\delta_{r_I} = \frac{1}{2} \ln u + \frac{1}{2} \ln v + \frac{1}{2} \ln (1 + \tan^2 \phi) - \frac{1}{2} \ln (u + \tan^2 \phi) \tag{5-78}
\]

For the second stage of strain we find

\[
\delta_{r_{II}} = \ln \frac{r}{r_I} = \ln \frac{\cos \phi}{\cos \psi} \tag{5-79}
\]

Using (5-76) it follows from (5-79) that

\[
\delta_{r_{II}} = - \ln \cos \psi - \frac{1}{2} \ln (1 + \tan^2 \phi) \tag{5-80}
\]

Hence, using (5-71) we find

\[
\delta_r = \frac{1}{2} \ln u + \frac{1}{2} \ln v - \ln \cos \psi - \frac{1}{2} \ln (u + v \tan^2 \phi) \tag{5-81}
\]

However, we are interested in \( \delta_r = \delta_r(u, v, \gamma, \psi) \) as \( u, v \) and \( \phi \) characterize the strain of the grid and \( \psi \) the ultimate position of \( r \).

From Fig. 5-6 we see: \( D'E = D'D + D'E' \), or

\[
\tan \phi = \tan \gamma - \tan \psi \tag{5-82}
\]

Introducing

\[
\tan \psi = w \tag{5-83}
\]

it follows with (5-82) that

\[
\delta_r = \frac{1}{2} \ln u v - \ln \cos \gamma - \frac{1}{2} \ln \left( u + v (\tan \gamma - w)^2 \right) \tag{5-84}
\]

If we wish to know \( \delta_r \) as a function of \( u, v \) and \( w \) and the angle \( \gamma \), which does not depend on time, we subsitute (5-82) and (5-75) in (5-81) and find
\[ \delta_r = \ln \cos \gamma + \frac{1}{2} \ln v + \frac{1}{2} \ln \left\{ 1 + \left( \frac{u}{v} \tan \gamma + w \right)^2 \right\} \] (5-85)

The principal strains and directions are found the most readily from (5-84) by requiring that

\[ \frac{d\delta_r}{d\psi} = 0 \] (5-86)

from which we derive

\[ \tan^2 \psi - \left( \frac{u}{vw} + w - \frac{1}{w} \right) \tan \psi - 1 = 0 \] (5-87)

Introduce

\[ \frac{u}{vw} + w - \frac{1}{w} = 2p \] (5-88)

then (compare with (2-28)) the principal directions are given by

\[ \tan \psi_1 = p + \sqrt{1 + p^2} \]
\[ \tan \psi_2 = p - \sqrt{1 + p^2} \] (5-89)

Substitution in (5-84) yields

\[ \delta_1 = \frac{1}{2} \ln \frac{uv \left\{ 1 + (p + \sqrt{1 + p^2})^2 \right\}}{u + v (p - w + \sqrt{1 + p^2})^2} \]
\[ \delta_2 = \frac{1}{2} \ln \frac{uv \left\{ 1 + (p - \sqrt{1 + p^2})^2 \right\}}{u + v (p - w - \sqrt{1 + p^2})^2} \] (5-90)
\[ \delta_3 = -(\delta_1 + \delta_2) = -\frac{1}{2} \ln uv \]

We are interested in the value of \( \delta_1 \) which permits of application in the constitutive equation. This quantity is not \( \frac{d}{dt} \delta_1 \) as it is of no importance how \( \delta_1 \) itself changes as a function of time, but how the element of line whose strain is momentarily a principal strain changes its length as a function of time.
Hence, we wish to know

\[
\frac{D\delta_1}{Dt} = \left( \frac{d\delta_r}{dt} \right)
\]

\[\delta_r = \delta_1 \tag{5-91}\]

\(\frac{D}{Dt}\) is an operator used for the material derivative. It is a symbol indicating how the distance between two material points changes as a function of time. Fig. 5-7 illustrates this. The material point E has the positions \(E', E''\) and \(E'''\) at \(t=0\), \(t=\Delta t\), and \(t=2\Delta t\) respectively. In position (2) the line \(AE''\) is parallel to a principal direction. The quotient giving the value of \(\delta_1\) is then

\[
\delta_1 \approx \frac{ln AE''/ AE'}{2 \Delta t} \tag{5-92}
\]

Only if \(AE'\) and \(AE'''\) indicated principal directions as well, could we find \(\delta_1\), by a straightforward differentiation of (5-90). The calculation pattern now is as follows: From consecutive straining patterns we measure \(a_1, b_1\) and \(\gamma\). From these data we graphically determine \(u(t), v(t)\) and \(w(t)\), and by measuring the slopes we find \(\dot{u}, \dot{v}\) and \(\dot{w}\).

As, according to (5-85),

\[
\delta_r = \delta_r (u(t), v(t), w(t), \gamma) \tag{5-93}
\]

it follows that

\[
\frac{d\delta_r}{dt} = \frac{\partial \delta_r}{\partial u} \dot{u} + \frac{\partial \delta_r}{\partial v} \dot{v} + \frac{\partial \delta_r}{\partial w} \dot{w} \tag{5-94}
\]

\[
\frac{\partial \delta_r}{\partial u} = \frac{\tan \gamma \left( \frac{u}{v} \tan \gamma + w \right)}{2 \sqrt{uv} \left[ 1 + \left( \frac{u}{v} \tan \gamma + w \right)^2 \right]} \tag{5-95}
\]

\[
\frac{\partial \delta_r}{\partial v} = \frac{1}{2v} \left[ 1 - \frac{u}{v} \tan \gamma \left( \frac{u}{v} \tan \gamma + w \right) \right]
\]

\[
\frac{\partial \delta_r}{\partial w} = \frac{\sqrt{u/v} \tan \gamma + w}{1 + \left( \sqrt{u/v} \tan \gamma + w \right)^2}
\]
Next we substitute the angle $\psi$ in (5-95) since the principal directions $\psi_1$ and $\psi_2$ are known. Using (5-75) and (5-82), we find

$$\frac{\partial \delta}{\partial u} = \frac{\tan \psi (\tan \psi - w)}{2v (1 + \tan^2 \psi)} = \frac{\sin^2 \psi (\tan \psi - w)}{2v}$$

$$\frac{\partial \delta}{\partial v} = \frac{1}{2v} \left[ 1 - \frac{u \tan \psi (\tan \psi - w)}{v(1 + \tan^2 \psi)} \right] = \frac{1}{2v} \left[ 1 - \frac{u \sin^2 \psi (\tan \psi - w)}{v} \right]$$

$$\frac{\partial \delta}{\partial w} = \frac{\tan \psi}{1 + \tan^2 \psi} = \sin^2 \psi$$

(5-96)

For $\psi = \psi_1$, we substitute (5-89), and with (5-76) find that

$$\sin^2 \psi_1 = \frac{2 + 2p^2 + 2p \sqrt{1 + p^2}}{3 + 2p^2 + 2p \sqrt{1 + p^2}}$$

(5-97)

Finally, substitution of (5-97) and (5-89) in (5-96) and substitution of (5-96) in (5-94), together with (5-88), gives a very complicated expression for

$$\dot{\delta}_1 = \dot{\delta}_1 (u, v, w, \dot{u}, \dot{v}, \dot{w})$$

Remark. The expression found in this way is not entirely exact. We have not considered the rotation rate of $\delta_1$. However, if $\delta_1$ does not change its direction rapidly, the effect will be small. In this calculation we neglect it.

We now assume that $\dot{\delta}_1$, $\dot{\delta}_2$ and $\dot{\delta}_3$ have been established in this way. ($\dot{\delta}_3$ follows from $\dot{\delta}_1 + \dot{\delta}_2 + \dot{\delta}_3 = 0$). With (3-4) we then calculate for each element of grit the value of $\delta$. The equations (3-13) and (1-25) are now 4 equations with 4 variables, viz. $\sigma_1$, $\sigma_2$, $\sigma_3$ and $\tau$. These can be solved. Rotating the local stresses over angles $(\pi/2 - \gamma)_i + \delta_i$ we find the values of $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ in the fixed coordinate system. The integrals of these stresses over the area of the section must yield the outside load.

For every calculation we find a value of $\delta$ and $\sigma$. If we plot these values we may find a deformation-rate equation, by supposing that, for
instance, a relation $\mathcal{F} = c' \frac{\delta}{\delta} m'$ exists. Experiments will have to show if $c'$ and $m'$ are better material constants than $c$ and $m$. If this is the case, it would be a strong argument for finding future solutions by dividing the equations (3-9) by $dt$ in stead of integration demonstrated in Part 3.7. The disadvantage of more difficult methods of measuring may then weigh less heavy than the advantage of the existence of actually invariant material constants.
Fig. 5-7

\[ \Delta t \]

\[ \delta r = \delta \gamma \]