An analogue of Grace's apolarity theorem

AN ANALOGUE OF GRACE'S APOLARITY THEOREM

BY

N. G. DE BRUIJN

(Mathematisch Instituut der Technische Hogeschool, Delft).

1. A special case of GRACE's Apolarity Theorem [1]) reads: If all the roots of the polynomials

$$A(x) = \binom{n}{0} a_0 + \binom{n}{1} a_1 x + \ldots + \binom{n}{n} a_n x^n$$

$$B(x) = \binom{n}{0} b_0 + \binom{n}{1} b_1 x + \ldots + \binom{n}{n} b_n x^n$$

lie on the unit circle $|x| = 1$, then the same holds for the roots of the polynomial

$$AB(x) = \binom{n}{0} a_0 b_0 + \binom{n}{1} a_1 b_1 x + \ldots + \binom{n}{n} a_n b_n x^n.$$

By putting $x = e^{iz}$ we can obtain a composition-theorem for trigonometric polynomials: If

$$S(z) = \sum_{-N}^{N} \binom{2N}{N+\nu} s_\nu e^{\nu i z}, \quad T(z) = \sum_{-N}^{N} \binom{2N}{N+\nu} t_\nu e^{\nu i z} \qquad (1.1)$$

have real roots only, then the same holds for

$$ST(z) = \sum_{-N}^{N} \binom{2N}{N+\nu} s_\nu t_\nu e^{\nu i z}.$$

If the function $f(y)$ of the real variable y is continuous in

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[1]) J. H. GRACE, Proc. Cambridge Phil. Soc. **11**, 352—357 (1900—'02); G. SZEGÖ, Math. Zeitschr. **13**, 28—55 (1922).

$(-\infty, \infty)$ and $f(y) = O(e^{\mu y^2})$ $(\mu < \frac{1}{2})$, then the trigonometric integral

$$F(z) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f(y)\, e^{izy}\, dy \tag{1.2}$$

can be approximated by functions of the type (1.1) [2]):

$$F(z) = \lim_{N\to\infty} 2^{-2N} \pi^{\frac{1}{2}} \sum_{-N}^{N} \binom{2N}{N+\nu} f(\nu 2^{\frac{1}{2}} \cdot N^{-\frac{1}{2}})\, e^{i\nu z\sqrt{(2/N)}} \tag{1.3}$$

From this formula we observe an analogy between the theorem just stated and Theorem 1 below.

An entire function $F(z)$ will be called a *function of the type R* if it is the limit, uniformly in any bounded region of the complex plane, of a sequence of polynomials with real roots only. It is known [3]) that this condition is equivalent to the fact that $F(z)$ is the product of a factor $e^{-\frac{1}{2}\alpha z^2}$ $(\alpha \geq 0)$ and a function of genus (Geschlecht) ≤ 1 with real roots:

$$F(z) = A e^{-\frac{1}{2}\alpha z^2 + bz} z^k \prod_{\nu=1}^{\infty} (1 + c_\nu z)\, e^{-c_\nu z} \tag{1.4}$$

where $\alpha \geqslant 0$, b and c_ν real, $\Sigma c_\nu^2 < \infty$, $k = 0, 1, 2 \ldots$

If necessary we shall call $F(z)$, given by (1.4), a function of the type $R(\alpha)$.

Theorem 1. *Let $f(y)$ and $g(y)$ be continuous in $-\infty < y < \infty$, and*

$$f(y) = O(e^{\frac{1}{2}\rho y^2}),\ g(y) = O(e^{\frac{1}{2}\sigma y^2}) \tag{1.5}$$

where $0 \leq \rho < 1$, $0 \leq \sigma < 1$. Suppose that the functions

$$F(z) = \frac{e^{\frac{1}{2}z^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f(y)\, e^{iyz}\, dy,\quad G(z) = \frac{e^{\frac{1}{2}z^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} g(y) e^{iyz}\, dy \tag{1.6}$$

are of the type R. Then, if furthermore $\rho + \sigma < 1$, the composition

$$FG(z) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f(y)\, g(y)\, e^{iyz}\, dy \tag{1.7}$$

is also of the type R.

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[2]) We do not actually use this formula; it is given here to show analogy only.

[3]) G. Pòlya and I. Schur, Journ. f. Math. **144**, 89—113 (1914).

Remarks. 1. It will be shown in Theorem 2 that if $F(z)$ is of the type R, a continuous function $f(y)$ can be found which satisfies (1.6) and (1.5) for a certain $\varrho < 1$. This shows that the conditions imposed on $f(y)$ and $g(y)$ are no essential restrictions.

2. If $f(y)$ is a polynomial in y, then $F(z)$ is a polynomial in z; the converse is also true.

3. We cannot derive Theorem 1 from the analogous composition-theorem for trigonometric polynomials stated above. Namely, the fact that $F(z)$ is of the type R does by no means imply that the trigonometric polynomials occurring on the right of (1.3) have real roots only.

Therefore we follow a different method: we first prove that Theorem 1 is true if $F(z)$ is a polynomial (Lemma 2), the general case will be derived from this special case.

4. Theorem 1 remains true if the class of functions of the type R is replaced by the larger class of functions which are limits of sequences of polynomials $F_n(z)$ whose roots lie in a fixed strip $|\, Im\, z\,| \leq \Delta$. We do not go into details of the proof; cf. Theorem 3 of the preceding paper.

5. We give a single application of Theorem 1. Let $P(u)$ be a polynomial with real coefficients, whose roots lie in the strip $|\, Im\, u\,| \leq 1$. Then all the roots of the polynomial

$$F(z) = (2\pi)^{-\frac{1}{2}}\, e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\, P(iy)\, e^{izy}\, dy$$

are real. Owing to our theorem it is sufficient to show this for the cases $P(iy) = iy - a$ (a real) and $P(iy) = (iy - a)^2 + b^2$ (a and b real, $0 < b \leq 1$). We find $F(z) = -z - a$ and $F(z) = (z + a)^2 - (1 - b^2)$ respectively.

Other applications can be made to trigonometric polynomials. Namely, if $f(y)$ is of the form

$$f(y) = \sum_{-M}^{M} a_\nu\, e^{\lambda\nu y}$$

then we find for the expression (1.5)

$$F(z) = \sum_{-M}^{M} a_\nu\, e^{\frac{1}{2}\lambda^2\nu^2}\, e^{i\lambda\nu z}.$$

2. Lemma 1. *If $F(z)$ is a polynomial of degree M, with real roots, and if $G(z)$ is a function of the type R, then the function*

$$\sum_{\nu=0}^{M} (-1)^{\nu} F^{(\nu)}(z)\, G^{(\nu)}(z)/\nu! \tag{2.1}$$

is of the type R.

This was proved in the preceding paper for the case that G is also a polynomial. The present lemma directly follows, for if the polynomial sequence $G_n(z)$ converges to $G(z)$ uniformly in any bounded region then we have $G_n^{(\nu)} \to G^{(\nu)}$ also uniformly.

Lemma 2. *If the conditions of theorem 1 are satisfied and if $f(y)$ is a polynomial, then $FG(z)$ is of the type R.*

Proof. We shall prove that the expressions (1.7) and (2.1) are equal (cf. Remark 2 above); after that Lemma 1 immediately gives the result. We have

$$FG(z) = (2\pi)^{-\frac{1}{2}}\, e^{\frac{1}{2}z^2} f\left(-i\frac{d}{dz}\right) \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} g(y)\, e^{izy}\, dy =$$

$$= (2\pi)^{-\frac{1}{2}}\, e^{\frac{1}{2}z^2} f\left(-i\frac{d}{dz}\right) \{e^{-\frac{1}{2}z^2} G(z)\}.$$

If M is the degree of $f(y)$ we find that this expression can be written in the form

$$FG(z) = \sum_{\nu=0}^{M} c_{\nu}(z)\, G^{(\nu)}(z), \tag{2.2}$$

where the coefficients $c_{\nu}(z)$ are independent of G. We can determine the $c_{\nu}(z)$ by applying (2.2) to the case

$$g(y) = e^{-ipy+\frac{1}{2}p^2},\quad G(z) = e^{pz},$$

where p is an arbitrary parameter. We obtain

$$FG(z) = (2\pi)^{-\frac{1}{2}}\, e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2 - ipy + \frac{1}{2}p^2 + izy} f(y)\, dy =$$

$$= (2\pi)^{-\frac{1}{2}}\, e^{pz+\frac{1}{2}(z-p)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2 + i(z-p)y} f(y)\, dy = e^{pz} F(z-p).$$

Consequently, by (2.2), we have for any p

$$e^{pz} \sum_{\nu=0}^{M} (-1)^\nu p^\nu F^{(\nu)}(z)/\nu! = e^{pz} \sum_{\nu=0}^{M} c_\nu(z)\, p^\nu.$$

It follows that $c_\nu(z) = (-1)^\nu F^{(\nu)}(z)/\nu!$, and so $FG(z)$ equals the expression (2.1).

The following example of the identity just proved leads to a well-known formula for Hermitean polynomials. If $H_k(z)$ is defined by

$$H_k(z) = (-1)^k e^{\frac{1}{2}z^2} \left(\frac{d}{dz}\right)^k e^{-\frac{1}{2}z^2} = \frac{e^{\frac{1}{2}z^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} (-iy)^k e^{izy}\, dy,$$

then the composition $H_k H_l(z)$ equals $H_{k+l}(z)$, and so we have

$$H_{k+l}(z) = \sum_{\nu=0}^{\infty} (-1)^\nu H_k^{(\nu)}(z)\, H_l^{(\nu)}(z)/\nu!$$

3. Lemma 3. *If $\varphi(z)$ is of the type $R(0)$ then there exists a sequence of polynomials $\varphi_n(z)$, with real roots only, such that*

a) *$\varphi_n(z) \to \varphi(z)$ uniformly in any bounded region,* (3.1)

b) *for any $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$, independent of n, such that throughout the z-plane we have*

$$|\varphi_n(z)| < C(\varepsilon)\, e^{\varepsilon|z|^2}, \quad |\varphi(z)| < C(\varepsilon)\, e^{\varepsilon|z|^2}. \tag{3.2}$$

We omit the proof since it requires no ideas which are not frequently used in the theory of entire functions. One can take

$$\varphi_n(z) = A(1 + bz/n)^n z^k \prod_{\nu=1}^{n} (1 + c_\nu z)(1 - c_\nu z/n)^n,$$

if $\varphi(z)$ is given by (1.4) (with $\alpha = 0$, φ being of the type $R(0)$).

Lemma 4. *If $F(z)$ ($\not\equiv 0$) is of the type $R(\alpha)$, then there exists a positive constant C such that*

$$|F(\pm it)| > C\, e^{\frac{1}{2}\alpha t^2} \tag{3.3}$$

for t real, $t > 1$.

Namely, we directly infer from (1.4) that

$$| F(it) | = Ae^{\frac{1}{2}at^2} | t |^k \Pi | 1 + c_\nu it | \geq Ae^{\frac{1}{2}at^2} | t |^k.$$

4. Theorem 2. *If $F(z)$ is of the type $R(\alpha)$ $(\alpha \geq 0)$ then a uniquely determined continuous function $f(y)$ exists such that*

$$F(z) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f(y) e^{izy} dy \tag{4.1}$$

and

$$f(y) = O(e^{\frac{1}{2}\rho y^2}) \qquad (y \to \pm \infty) \tag{4.2}$$

for any number $\varrho > \alpha/(1+\alpha)$, but for no number $\varrho < \alpha/(1+\alpha)$.

Proof. We have $f(z) = e^{-\frac{1}{2}\alpha z^2} \varphi(z)$, where $\varphi(z)$ is of the type R(0). It follows by (3.2) that application of the Fourier inversion formula is legitimate. i.e. the function

$$f(y) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+\alpha)z^2} \varphi(z) e^{-izy} dz \tag{4.3}$$

satisfies (4.1). We find, by a shift of the integration path,

$$f(y) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+\alpha)\{z+iy/(1+\alpha)\}^2 - \frac{1}{2}y^2/(1+\alpha)} \varphi(z) dz =$$

$$= (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}\alpha y^2/(1+\alpha)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+\alpha)x^2} \varphi\{x - iy/(1+\alpha)\} dx.$$

If $\varepsilon > 0$ we have for real values of x and y, by (3.2),

$$| \varphi\{x - iy/(1+\alpha)\} | < C(\varepsilon) e^{\epsilon x^2 + \epsilon y^2 (1+\alpha)^{-2}}.$$

Consequently (4.2) is true for $\varrho > \alpha/(1+\alpha)$. Conversely, if $0 \leq \varrho < 1$ and if (4.2) is true, we infer from (4.1)

$$| F(-it) | < (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2 + ty} Ce^{\frac{1}{2}\rho y^2} dy = C_1 e^{\frac{1}{2}\rho t^2/(1-\rho)}.$$

Now by Lemma 4 we have $\varrho/(1-\varrho) \geq \alpha$, and so $\varrho \geq \alpha/(1+\alpha)$.

Lemma 5. *If $F(z)$ is of the type $R(\alpha)$ with $0 \leq \alpha < 1$, and if $f(y)$ is the function* (4.3) *which satisfies* (4. 1), *then a sequence of polynomials $f_n(y)$ exist such that*

a) $f_n(y) \to f(y)$ *uniformly in any finite (real) y-interval;*

b) *if* $\varepsilon > 0$, *a constant* $C(\varepsilon)$ *exists, independent of n, such that*

$$| f_n(y) | < C(\varepsilon)\, e^{\frac{1}{2}ay^2/(1+a)+\varepsilon y^2} \quad (-\infty < y < \infty); \qquad (4.4)$$

c) *the polynomials*

$$F_n(z) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f_n(y)\, e^{izy}\, dy \qquad (4.5)$$

have real roots only and satisfy $F_n(z) \to F(z)$ *uniformly in any bounded region of the z-plane.*

Proof. Put $F(z) = e^{-\frac{1}{2}az^2}\, \varphi(z)$, $\varphi(z)$ of the type $R(0)$. Let $\varphi_n(z)$ be the sequence of polynomials occurring in Lemma 3; take

$$F_n(z) = (1 - az^2/2n)^n\, \varphi_n(z) \qquad (4.6)$$

and consider (4.5) as the definition of the polynomials $f_n(y)$. We have, just as in the proof of Theorem 2,

$$f_n(y) = (2\pi)^{-\frac{1}{2}}\, e^{\frac{1}{2}y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, (1 - \tfrac{1}{2}az^2/n)^n\, \varphi_n(z)\, e^{-izy}\, dz. \qquad (4.7)$$

Put $z = x - iy/(1 + a)$ and take $-\infty < x < \infty$ as the integration path again. Using

$$| 1 - \tfrac{1}{2}az^2/n |^n \leq (1 + \tfrac{1}{2}a\, | z |^2/n) \leq e^{\frac{1}{2}a|z|^2} \qquad (4.8)$$

and (3.2)

$$| \varphi_n(z) | < C(\varepsilon)\, e^{\varepsilon x^2 + \varepsilon y^2(1+a)^{-2}} \quad (\varepsilon > 0)$$

we find

$$| f_n(y) | \leq (2\pi)^{-\frac{1}{2}}\, C(\varepsilon)\, e^{\frac{1}{2}ay^2/(1+a)+\varepsilon y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-a)x^2+\varepsilon x^2}\, dx.$$

This proves b); c) is immediate from (4.6); a) can be derived from (4.7) by using c), Lemma 3b), and the estimation (4.8) for real values of z.

5. Proof of Theorem 1. Since $\varrho \geq 0$, $\sigma \geq 0$, $\varrho + \sigma < 1$ we may suppose $0 \leq \varrho < \frac{1}{2}$. It now follows from Theorem 2 that $F(z)$ belongs to a type $R(\alpha)$ with

$\alpha/(1+\alpha) \leq \varrho < \frac{1}{2}$, and hence $\alpha < 1$. Consequently we may apply Lemma 5 to $F(z)$. Let $f_n(y)$ and $F_n(z)$ be the functions occurring in that lemma. It follows from Lemma 2 that

$$F_nG(z) = (2\pi)^{-\frac{1}{2}}\, e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f_n(y)\, g(y)\, e^{izy}\, dy$$

is a function of the type R. Furthermore (1.5) and (4.4) lead to

$$| e^{-\frac{1}{2}y^2} f_n(y)\, g(y) \,| < C_1(\varepsilon)\, e^{-\varepsilon y^2}$$

for a certain $\varepsilon > 0$, $C_1(\varepsilon)$ independent of n. Finally Lemma 5, a) shows that $F_nG(z) \rightarrow FG(z)$ uniformly in any bounded region. Since $F_nG(z)$ is of the type R, the same holds now for $FG(z)$, which proves the theorem.

(Ingekomen 21.7.48).