

## An analogue of Grace's apolarity theorem

**Citation for published version (APA):**

Bruijn, de, N. G. (1949). An analogue of Grace's apolarity theorem. *Nieuw Archief voor Wiskunde, serie 2*, 23, 69-76.

**Document status and date:**

Published: 01/01/1949

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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## AN ANALOGUE OF GRACE'S APOLARITY THEOREM

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1. A special case of GRACE'S Apolarity Theorem<sup>1)</sup> reads:  
If all the roots of the polynomials

$$A(x) = \binom{n}{0} a_0 + \binom{n}{1} a_1 x + \dots + \binom{n}{n} a_n x^n$$

$$B(x) = \binom{n}{0} b_0 + \binom{n}{1} b_1 x + \dots + \binom{n}{n} b_n x^n$$

lie on the unit circle  $|x| = 1$ , then the same holds for the roots of the polynomial

$$AB(x) = \binom{n}{0} a_0 b_0 + \binom{n}{1} a_1 b_1 x + \dots + \binom{n}{n} a_n b_n x^n.$$

By putting  $x = e^{iz}$  we can obtain a composition-theorem for trigonometric polynomials: If

$$S(z) = \sum_{-N}^N \binom{2N}{N+\nu} s_\nu e^{\nu iz}, \quad T(z) = \sum_{-N}^N \binom{2N}{N+\nu} t_\nu e^{\nu iz} \quad (1.1)$$

have real roots only, then the same holds for

$$ST(z) = \sum_{-N}^N \binom{2N}{N+\nu} s_\nu t_\nu e^{\nu iz}.$$

If the function  $f(\nu)$  of the real variable  $\nu$  is continuous in

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<sup>1)</sup> J. H. GRACE, Proc. Cambridge Phil. Soc. **11**, 352—357 (1900—'02);  
G. SZEGÖ, Math. Zeitschr. **13**, 28—55 (1922).

$(-\infty, \infty)$  and  $f(y) = O(e^{\mu y^2})$  ( $\mu < \frac{1}{2}$ ), then the trigonometric integral

$$F(z) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f(y) e^{izy} dy \quad (1.2)$$

can be approximated by functions of the type (1.1)<sup>2)</sup>:

$$F(z) = \lim_{N \rightarrow \infty} 2^{-2N} \pi^{\frac{1}{2}} \sum_{-N}^N \binom{2N}{N+y} f(y 2^{\frac{1}{2}} N^{-\frac{1}{2}}) e^{iyz \sqrt{(2/N)}} \quad (1.3)$$

From this formula we observe an analogy between the theorem just stated and Theorem 1 below.

An entire function  $F(z)$  will be called a *function of the type R* if it is the limit, uniformly in any bounded region of the complex plane, of a sequence of polynomials with real roots only. It is known<sup>3)</sup> that this condition is equivalent to the fact that  $F(z)$  is the product of a factor  $e^{-\frac{1}{2}\alpha z^2}$  ( $\alpha \geq 0$ ) and a function of genus (Geschlecht)  $\leq 1$  with real roots:

$$F(z) = A e^{-\frac{1}{2}\alpha z^2 + b z} \prod_{\nu=1}^{\infty} (1 + c_{\nu} z) e^{-c_{\nu} z} \quad (1.4)$$

where  $\alpha \geq 0$ ,  $b$  and  $c_{\nu}$  real,  $\sum c_{\nu}^2 < \infty$ ,  $k = 0, 1, 2, \dots$

If necessary we shall call  $F(z)$ , given by (1.4), a function of the type  $R(\alpha)$ .

**Theorem 1.** *Let  $f(y)$  and  $g(y)$  be continuous in  $-\infty < y < \infty$ , and*

$$f(y) = O(e^{\frac{1}{2}\rho y^2}), \quad g(y) = O(e^{\frac{1}{2}\sigma y^2}) \quad (1.5)$$

where  $0 \leq \rho < 1$ ,  $0 \leq \sigma < 1$ . Suppose that the functions

$$F(z) = \frac{e^{\frac{1}{2}z^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f(y) e^{iyz} dy, \quad G(z) = \frac{e^{\frac{1}{2}z^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} g(y) e^{iyz} dy \quad (1.6)$$

are of the type  $R$ . Then, if furthermore  $\rho + \sigma < 1$ , the composition

$$FG(z) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f(y) g(y) e^{iyz} dy \quad (1.7)$$

is also of the type  $R$ .

<sup>2)</sup> We do not actually use this formula; it is given here to show analogy only.

<sup>3)</sup> G. PÓLYA and I. SCHUR, Journ. f. Math. **144**, 89—113 (1914).

*Remarks.* 1. It will be shown in Theorem 2 that if  $F(z)$  is of the type R, a continuous function  $f(y)$  can be found which satisfies (1.6) and (1.5) for a certain  $\varrho < 1$ . This shows that the conditions imposed on  $f(y)$  and  $g(y)$  are no essential restrictions.

2. If  $f(y)$  is a polynomial in  $y$ , then  $F(z)$  is a polynomial in  $z$ ; the converse is also true.

3. We cannot derive Theorem 1 from the analogous composition-theorem for trigonometric polynomials stated above. Namely, the fact that  $F(z)$  is of the type R does by no means imply that the trigonometric polynomials occurring on the right of (1.3) have real roots only.

Therefore we follow a different method: we first prove that Theorem 1 is true if  $F(z)$  is a polynomial (Lemma 2), the general case will be derived from this special case.

4. Theorem 1 remains true if the class of functions of the type R is replaced by the larger class of functions which are limits of sequences of polynomials  $F_n(z)$  whose roots lie in a fixed strip  $|Im z| \leq \Delta$ . We do not go into details of the proof; cf. Theorem 3 of the preceding paper.

5. We give a single application of Theorem 1. Let  $P(u)$  be a polynomial with real coefficients, whose roots lie in the strip  $|Im u| \leq 1$ . Then all the roots of the polynomial

$$F(z) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} P(iy) e^{izy} dy$$

are real. Owing to our theorem it is sufficient to show this for the cases  $P(iy) = iy - a$  ( $a$  real) and  $P(iy) = (iy - a)^2 + b^2$  ( $a$  and  $b$  real,  $0 < b \leq 1$ ). We find  $F(z) = -z - a$  and  $F(z) = (z + a)^2 - (1 - b^2)$  respectively.

Other applications can be made to trigonometric polynomials. Namely, if  $f(y)$  is of the form

$$f(y) = \sum_{-M}^M a_\nu e^{\lambda\nu y}$$

then we find for the expression (1.5)

$$F(z) = \sum_{-M}^M a_\nu e^{\frac{1}{2}\lambda^2\nu^2} e^{i\lambda\nu z}.$$

2. Lemma 1. If  $F(z)$  is a polynomial of degree  $M$ , with real roots, and if  $G(z)$  is a function of the type  $R$ , then the function

$$\sum_{\nu=0}^M (-1)^\nu F^{(\nu)}(z) G^{(\nu)}(z)/\nu! \quad (2.1)$$

is of the type  $R$ .

This was proved in the preceding paper for the case that  $G$  is also a polynomial. The present lemma directly follows, for if the polynomial sequence  $G_n(z)$  converges to  $G(z)$  uniformly in any bounded region then we have  $G_n^{(\nu)} \rightarrow G^{(\nu)}$  also uniformly.

Lemma 2. If the conditions of theorem 1 are satisfied and if  $f(y)$  is a polynomial, then  $FG(z)$  is of the type  $R$ .

Proof. We shall prove that the expressions (1.7) and (2.1) are equal (cf. Remark 2 above); after that Lemma 1 immediately gives the result. We have

$$\begin{aligned} FG(z) &= (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} f\left(-i \frac{d}{dz}\right) \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} g(y) e^{izy} dy = \\ &= (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} f\left(-i \frac{d}{dz}\right) \{e^{-\frac{1}{2}z^2} G(z)\}. \end{aligned}$$

If  $M$  is the degree of  $f(y)$  we find that this expression can be written in the form

$$FG(z) = \sum_{\nu=0}^M c_\nu(z) G^{(\nu)}(z), \quad (2.2)$$

where the coefficients  $c_\nu(z)$  are independent of  $G$ . We can determine the  $c_\nu(z)$  by applying (2.2) to the case

$$g(y) = e^{-ipy + \frac{1}{2}p^2}, \quad G(z) = e^{pz},$$

where  $p$  is an arbitrary parameter. We obtain

$$\begin{aligned} FG(z) &= (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2 - ipy + \frac{1}{2}p^2 + izy} f(y) dy = \\ &= (2\pi)^{-\frac{1}{2}} e^{pz + \frac{1}{2}(z-p)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2 + i(z-p)y} f(y) dy = e^{pz} F(z - p). \end{aligned}$$

Consequently, by (2.2), we have for any  $p$

$$e^{pz} \sum_{\nu=0}^M (-1)^\nu p^\nu F^{(\nu)}(z)/\nu! = e^{pz} \sum_{\nu=0}^M c_\nu(z) p^\nu.$$

It follows that  $c_\nu(z) = (-1)^\nu F^{(\nu)}(z)/\nu!$ , and so  $FG(z)$  equals the expression (2.1).

The following example of the identity just proved leads to a well-known formula for Hermitean polynomials. If  $H_k(z)$  is defined by

$$H_k(z) = (-1)^k e^{\frac{1}{2}z^2} \left( \frac{d}{dz} \right)^k e^{-\frac{1}{2}z^2} = \frac{e^{\frac{1}{2}z^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} (-iy)^k e^{izy} dy,$$

then the composition  $H_k H_l(z)$  equals  $H_{k+l}(z)$ , and so we have

$$H_{k+l}(z) = \sum_{\nu=0}^{\infty} (-1)^\nu H_k^{(\nu)}(z) H_l^{(\nu)}(z)/\nu!$$

**3. Lemma 3.** *If  $\varphi(z)$  is of the type  $R(0)$  then there exists a sequence of polynomials  $\varphi_n(z)$ , with real roots only, such that*

a)  $\varphi_n(z) \rightarrow \varphi(z)$  uniformly in any bounded region, (3.1)

b) for any  $\varepsilon > 0$  there exists a positive constant  $C(\varepsilon)$ , independent of  $n$ , such that throughout the  $z$ -plane we have

$$|\varphi_n(z)| < C(\varepsilon) e^{\varepsilon|z|^2}, \quad |\varphi(z)| < C(\varepsilon) e^{\varepsilon|z|^2}. \quad (3.2)$$

We omit the proof since it requires no ideas which are not frequently used in the theory of entire functions. One can take

$$\varphi_n(z) = A(1 + bz/n)^{n\alpha} \prod_{\nu=1}^n (1 + c_\nu z) (1 - c_\nu z/n)^n,$$

if  $\varphi(z)$  is given by (1.4) (with  $\alpha = 0$ ,  $\varphi$  being of the type  $R(0)$ ).

**Lemma 4.** *If  $F(z)$  ( $\neq 0$ ) is of the type  $R(\alpha)$ , then there exists a positive constant  $C$  such that*

$$|F(\pm it)| > C e^{\frac{1}{2}\alpha t^2} \quad (3.3)$$

for  $t$  real,  $t > 1$ .

Namely, we directly infer from (1.4) that

$$|F(it)| = Ae^{\frac{1}{2}at^2} |t|^k |H| |1 + c_v it| \geq Ae^{\frac{1}{2}at^2} |t|^k.$$

4. **Theorem 2.** *If  $F(z)$  is of the type  $R(a)$  ( $a \geq 0$ ) then a uniquely determined continuous function  $f(y)$  exists such that*

$$F(z) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f(y) e^{izy} dy \quad (4.1)$$

and

$$f(y) = O(e^{\frac{1}{2}\rho y^2}) \quad (y \rightarrow \pm \infty) \quad (4.2)$$

for any number  $\rho > a/(1+a)$ , but for no number  $\rho < a/(1+a)$ .

**Proof.** We have  $f(z) = e^{-\frac{1}{2}az^2} \varphi(z)$ , where  $\varphi(z)$  is of the type  $R(0)$ . It follows by (3.2) that application of the FOURIER inversion formula is legitimate. i.e. the function

$$f(y) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+a)z^2} \varphi(z) e^{-izy} dz \quad (4.3)$$

satisfies (4.1). We find, by a shift of the integration path,

$$\begin{aligned} f(y) &= (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+a)\{z+iy/(1+a)\}^2 - \frac{1}{2}y^2/(1+a)} \varphi(z) dz = \\ &= (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}ay^2/(1+a)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1+a)x^2} \varphi\{x - iy/(1+a)\} dx. \end{aligned}$$

If  $\varepsilon > 0$  we have for real values of  $x$  and  $y$ , by (3.2),

$$|\varphi\{x - iy/(1+a)\}| < C(\varepsilon) e^{\varepsilon x^2 + \varepsilon y^2/(1+a)^2}.$$

Consequently (4.2) is true for  $\rho > a/(1+a)$ . Conversely, if  $0 \leq \rho < 1$  and if (4.2) is true, we infer from (4.1)

$$|F(-it)| < (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2 + ty} C e^{\frac{1}{2}\rho y^2} dy = C_1 e^{\frac{1}{2}\rho t^2/(1-\rho)}.$$

Now by Lemma 4 we have  $\rho/(1-\rho) \geq a$ , and so  $\rho \geq a/(1+a)$ .

**Lemma 5.** *If  $F(z)$  is of the type  $R(a)$  with  $0 \leq a < 1$ , and if  $f(y)$  is the function (4.3) which satisfies (4.1), then a sequence of polynomials  $f_n(y)$  exist such that*

- a)  $f_n(y) \rightarrow f(y)$  uniformly in any finite (real)  $y$ -interval;  
 b) if  $\varepsilon > 0$ , a constant  $C(\varepsilon)$  exists, independent of  $n$ , such that

$$|f_n(y)| < C(\varepsilon) e^{\frac{1}{2}\alpha y^2/(1+\alpha) + \varepsilon y^2} \quad (-\infty < y < \infty); \quad (4.4)$$

- c) the polynomials

$$F_n(z) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}\alpha z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f_n(y) e^{izy} dy \quad (4.5)$$

have real roots only and satisfy  $F_n(z) \rightarrow F(z)$  uniformly in any bounded region of the  $z$ -plane.

**Proof.** Put  $F(z) = e^{-\frac{1}{2}\alpha z^2} \varphi(z)$ ,  $\varphi(z)$  of the type  $R(0)$ . Let  $\varphi_n(z)$  be the sequence of polynomials occurring in Lemma 3; take

$$F_n(z) = (1 - \alpha z^2/2n)^n \varphi_n(z) \quad (4.6)$$

and consider (4.5) as the definition of the polynomials  $f_n(y)$ . We have, just as in the proof of Theorem 2,

$$f_n(y) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} (1 - \frac{1}{2}\alpha z^2/n)^n \varphi_n(z) e^{-izy} dz. \quad (4.7)$$

Put  $z = x - iy/(1 + \alpha)$  and take  $-\infty < x < \infty$  as the integration path again. Using

$$|1 - \frac{1}{2}\alpha z^2/n|^n \leq (1 + \frac{1}{2}\alpha |z|^2/n)^n \leq e^{\frac{1}{2}\alpha |z|^2} \quad (4.8)$$

and (3.2)

$$|\varphi_n(z)| < C(\varepsilon) e^{\varepsilon x^2 + \varepsilon y^2(1+\alpha)^{-2}} \quad (\varepsilon > 0)$$

we find

$$|f_n(y)| \leq (2\pi)^{-\frac{1}{2}} C(\varepsilon) e^{\frac{1}{2}\alpha y^2/(1+\alpha) + \varepsilon y^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-\alpha)x^2 + \varepsilon x^2} dx.$$

This proves b); c) is immediate from (4.6); a) can be derived from (4.7) by using c), Lemma 3b), and the estimation (4.8) for real values of  $z$ .

**5. Proof of Theorem 1.** Since  $\varrho \geq 0$ ,  $\sigma \geq 0$ ,  $\varrho + \sigma < 1$  we may suppose  $0 \leq \varrho < \frac{1}{2}$ . It now follows from Theorem 2 that  $F(z)$  belongs to a type  $R(\alpha)$  with



$a/(1+a) \leq \varrho < \frac{1}{2}$ , and hence  $a < 1$ . Consequently we may apply Lemma 5 to  $F(z)$ . Let  $f_n(y)$  and  $F_n(z)$  be the functions occurring in that lemma. It follows from Lemma 2 that

$$F_n G(z) = (2\pi)^{-\frac{1}{2}} e^{\frac{1}{2}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} f_n(y) g(y) e^{izy} dy$$

is a function of the type R. Furthermore (1.5) and (4.4) lead to

$$|e^{-\frac{1}{2}y^2} f_n(y) g(y)| < C_1(\varepsilon) e^{-\varepsilon y^2}$$

for a certain  $\varepsilon > 0$ ,  $C_1(\varepsilon)$  independent of  $n$ . Finally Lemma 5, a) shows that  $F_n G(z) \rightarrow FG(z)$  uniformly in any bounded region. Since  $F_n G(z)$  is of the type R, the same holds now for  $FG(z)$ , which proves the theorem.

(Ingekomen 21.7.48).