

## Coverings by rook domains

***Citation for published version (APA):***

Blokhuis, A. (1982). *Coverings by rook domains*. (Eindhoven University of Technology : Dept of Mathematics : memorandum; Vol. 8204). Technische Hogeschool Eindhoven.

***Document status and date:***

Published: 01/01/1982

***Document Version:***

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

***Please check the document version of this publication:***

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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Memorandum 1982-04

March 1982

COVERINGS BY ROOK DOMAINS

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# COVERINGS BY ROOK DOMAINS

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## 0. Abstract

The following inequalities and values for coverings by rook domains are proved:

$$(i) \quad \sigma\left(1 + t \frac{q^{r-1} - 1}{q - 1}, q\right) \leq (q - t + 1)q^{k-r};$$

here  $q$  is a prime and  $k = 1 + t \frac{q^{r-1} - 1}{q - 1}$ .

$$(ii) \quad \sigma(n, kt) \leq \sigma(n, k)t^{n-1} \quad \text{for any } n, k \text{ and } t.$$

$$(iii) \quad \sigma(q+1, qt) = q^{q-1} t^q \quad \text{for any prime power } q \text{ and any } t.$$

## 1. Introduction

Let  $V = (V_k^n, d)$  denote the metric space of all  $n$ -tuples  $(a_1, a_2, \dots, a_n)$  with  $a_i \in \{1, 2, \dots, k\}$  provided with the Hamming distance:

$d(\underline{a}, \underline{b}) = |\{i \mid a_i \neq b_i\}|$ . A subset  $W$  of  $V$  is called a covering (by rook-domains) if each point of  $V$  is at distance  $\leq 1$  from some point in  $W$ .

We are interested in bounds on the number of points in a minimal covering of  $V$ , to be denoted by  $\sigma(n, k)$ . Points of  $W$  will be called rooks, the sphere of radius 1 around a rook a rook-domain. Since each rook-domain contains  $1 + n(k-1)$  points we get  $\sigma(n, k) \geq \frac{k^n}{1 + n(k-1)}$ . Equality can be

obtained if  $k$  is a prime power and  $1 + n(k-1) \mid k$ . E. Rodemich [1] proved that this bound can be improved to  $\sigma(n,k) \geq \frac{k^{n-1}}{n-1}$  in the case  $k \geq n$ .

2. A generalization of the bounds of van Lint and Kamps

A trivial observation is that  $\sigma(n+1,k) \leq k\sigma(n,k)$ . This observation, combined with  $\sigma(4,3) = 3^2$  yields  $\sigma(13,3) \leq 3^{11}$ , but actually  $\sigma(13,3) = 3^{10}$ . It is natural therefore to study the behaviour of  $\sigma(n,k)$  in between. In [2] J.H.van Lint and H.J.L. Kamps proved  $\sigma(9,3) \leq 2 \cdot 3^6$ . We will now demonstrate a technique which generalizes their construction.

Let  $A = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k)$  be a matrix with  $k$  columns and  $r$  linearly independent rows, with  $\underline{a}_i \in \mathbb{F}_q^r$  where  $q$  is a prime. Let  $S$  be a set of points in  $\mathbb{F}_q^r$  such that  $\{\underline{s} + \alpha \underline{a}_i \mid \underline{s} \in S, \alpha \in \mathbb{F}_q, 1 \leq i \leq k\} = \mathbb{F}_q^r$ .

Lemma.  $W := \{\underline{w} \in \mathbb{F}_q^k \mid A\underline{w} \in S\}$  is a covering of  $V_q^k = \mathbb{F}_q^k$  and  $|W| = |S| \cdot q^{k-r}$ .

Proof. Take  $\underline{x} \in \mathbb{F}_q^k$ , then  $A\underline{x} \in \mathbb{F}_q^r$ , so we may write  $A\underline{x} = \underline{s} + \alpha \underline{a}_i$ . Let  $e_i = (0, 0, \dots, 1, 0, \dots, 0)$  denote the  $i^{\text{th}}$  unit vector in  $\mathbb{F}_q^k$ , then  $A(\underline{x} - \alpha e_i) = \underline{s} \in S$  hence  $\underline{x} - \alpha e_i \in W$ , and  $d(\underline{x}, W) \leq 1$ .

Application

$$A = \begin{pmatrix} 0 & 0 & 0 & & 1 & 1 \\ 0 & 0 & 0 & & \vdots & \vdots \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 1 & 1 & & \cdot & \cdot \\ \hline 1 & 1 & t & & 1 & \dots & t \end{pmatrix} \quad S = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q-t \end{pmatrix} \right\}$$

the columns of A are all projective (r-1)-vectors over  $\mathbb{F}_q$ , repeated t times, with last coordinates 1,2,...,t together with the vector  $(0\ 0, \dots, 0\ 1)^T$ , so  $k = 1 + t \frac{q^{r-1} - 1}{q - 1}$ .

It is easily checked, using the pigeonhole principle, that the pair A,S satisfies the conditions, hence

$$\sigma(k,q) \leq (q - t + 1)q^{k-r}.$$

#### 4. A sequence of cases meeting the Rodemich bound

Theorem.  $\sigma(n,kt) \leq \sigma(n,k)t^{n-1}$ .

Proof. Let W be a covering of  $V_k^n$ . Regard  $V_{kt}^n$  as obtained from  $V_k^n$  by replacing each point by  $V_t^n$  and give  $V_{kt}^n$  coordinates as follows:

For  $a = (a_1, a_2, \dots, a_n) \in V_k^n$  and  $b = (b_1, b_2, \dots, b_n) \in V_t^n$  the point in position b of the set  $V_t^n$  replacing a gets coordinates

$$((a_1 - 1)t + b_1, (a_2 - 1)t + b_2, \dots, (a_n - 1)t + b_n).$$

Now for each rook in W fill the corresponding set  $V_t^n$  with  $t^{n-1}$  rooks placed at the points  $(x_1, x_2, \dots, x_n)$  satisfying  $x_1 + x_2 + \dots + x_n \equiv 0 \pmod{t}$ . It is easy to verify that the set of rooks thus defined covers  $V_{kt}^n$ .

Corollary. If q is a prime power then  $\sigma(q+1,qt) = q^{q-1}t^q$ .

Proof. Since  $\sigma(q+1,q) = q^{q-1}$  by the Hamming bound we get

$\sigma(q+1,qt) \leq q^{q-1}t^q$ . Rodemich's equality however, gives  $\sigma(q+1,qt) \geq q^{q-1}t^q$ .

References

- [ 1 ] E. Rodemich, Covering by Rook Domains. Journal of Comb. Theory 9  
(1970), 117 - 128 .
- [ 2 ] J.H. van Lint & H.J.L. Kamps, A covering Problem. Coll. Math. Societas János Bolyai 4, Comb. Theory and it's Applications,  
Balatonfüred 1969 (Hungary).