

Coverings by rook domains

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C O V E R I N G S B Y R O O K D O M A I N S

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COVERINGS BY ROOK DOMAINS

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0. Abstract

The following inequalities and values for coverings by rook domains are proved:

$$(i) \quad \sigma\left(1 + t \frac{q^{r-1} - 1}{q - 1}, q\right) \leq (q - t + 1)q^{k-r};$$

here q is a prime and $k = 1 + t \frac{q^{r-1} - 1}{q - 1}$.

$$(ii) \quad \sigma(n, kt) \leq \sigma(n, k)t^{n-1} \quad \text{for any } n, k \text{ and } t.$$

$$(iii) \quad \sigma(q+1, qt) = q^{q-1} t^q \quad \text{for any prime power } q \text{ and any } t.$$

1. Introduction

Let $V = (V_k^n, d)$ denote the metric space of all n -tuples (a_1, a_2, \dots, a_n) with $a_i \in \{1, 2, \dots, k\}$ provided with the Hamming distance:

$d(\underline{a}, \underline{b}) = |\{i \mid a_i \neq b_i\}|$. A subset W of V is called a covering (by rook-domains) if each point of V is at distance ≤ 1 from some point in W .

We are interested in bounds on the number of points in a minimal covering of V , to be denoted by $\sigma(n, k)$. Points of W will be called rooks, the sphere of radius 1 around a rook a rook-domain. Since each rook-domain contains $1 + n(k-1)$ points we get $\sigma(n, k) \geq \frac{k^n}{1 + n(k-1)}$. Equality can be

obtained if k is a prime power and $1 + n(k-1) \mid k$. E. Rodemich [1] proved that this bound can be improved to $\sigma(n,k) \geq \frac{k^{n-1}}{n-1}$ in the case $k \geq n$.

2. A generalization of the bounds of van Lint and Kamps

A trivial observation is that $\sigma(n+1,k) \leq k\sigma(n,k)$. This observation, combined with $\sigma(4,3) = 3^2$ yields $\sigma(13,3) \leq 3^{11}$, but actually $\sigma(13,3) = 3^{10}$. It is natural therefore to study the behaviour of $\sigma(n,k)$ in between. In [2] J.H.van Lint and H.J.L. Kamps proved $\sigma(9,3) \leq 2 \cdot 3^6$. We will now demonstrate a technique which generalizes their construction.

Let $A = (\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k)$ be a matrix with k columns and r linearly independent rows, with $\underline{a}_i \in \mathbb{F}_q^r$ where q is a prime. Let S be a set of points in \mathbb{F}_q^r such that $\{\underline{s} + \alpha \underline{a}_i \mid \underline{s} \in S, \alpha \in \mathbb{F}_q, 1 \leq i \leq k\} = \mathbb{F}_q^r$.

Lemma. $W := \{\underline{w} \in \mathbb{F}_q^k \mid A\underline{w} \in S\}$ is a covering of $V_q^k = \mathbb{F}_q^k$ and $|W| = |S| \cdot q^{k-r}$.

Proof. Take $\underline{x} \in \mathbb{F}_q^k$, then $A\underline{x} \in \mathbb{F}_q^r$, so we may write $A\underline{x} = \underline{s} + \alpha \underline{a}_i$. Let $\underline{e}_i = (0, 0, \dots, 1, 0, \dots, 0)$ denote the i^{th} unit vector in \mathbb{F}_q^k , then $A(\underline{x} - \alpha \underline{e}_i) = \underline{s} \in S$ hence $\underline{x} - \alpha \underline{e}_i \in W$, and $d(\underline{x}, W) \leq 1$.

Application

$$A = \begin{pmatrix} 0 & 0 & 0 & & 1 & 1 \\ 0 & 0 & 0 & & \vdots & \vdots \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 1 & 1 & & \cdot & \cdot \\ \hline 1 & 1 & t & & 1 & \dots & t \end{pmatrix} \quad S = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q-t \end{pmatrix} \right\}$$

the columns of A are all projectieve (r-1)-vectors over \mathbb{F}_q , repeated t times, with last coordinates 1,2,...,t together with the vector $(0\ 0, \dots, 0\ 1)^T$, so $k = 1 + t \frac{q^{r-1} - 1}{q - 1}$.

It is easily checked, using the pigeonhole principle, that the pair A,S satisfies the conditions, hence

$$\sigma(k,q) \leq (q - t + 1)q^{k-r}.$$

4. A sequence of cases meeting the Rodemich bound

Theorem. $\sigma(n,kt) \leq \sigma(n,k)t^{n-1}$.

Proof. Let W be a covering of V_k^n . Regard V_{kt}^n as obtained from V_k^n by replacing each point by V_t^n and give V_{kt}^n coordinates as follows:

For $a = (a_1, a_2, \dots, a_n) \in V_k^n$ and $b = (b_1, b_2, \dots, b_n) \in V_t^n$ the point in position b of the set V_t^n replacing a gets coordinates

$$((a_1 - 1)t + b_1, (a_2 - 1)t + b_2, \dots, (a_n - 1)t + b_n).$$

Now for each rook in W fill the corresponding set V_t^n with t^{n-1} rooks placed at the points (x_1, x_2, \dots, x_n) satisfying $x_1 + x_2 + \dots + x_n \equiv 0 \pmod{t}$. It is easy to verify that the set of rooks thus defined covers V_{kt}^n .

Corollary. If q is a prime power then $\sigma(q+1,qt) = q^{q-1}t^q$.

Proof. Since $\sigma(q+1,q) = q^{q-1}$ by the Hamming bound we get

$\sigma(q+1,qt) \leq q^{q-1}t^q$. Rodemich's equality however, gives $\sigma(q+1,qt) \geq q^{q-1}t^q$.

References

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