Coverings by rook domains

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0. Abstract

The following inequalities and values for coverings by rook domains are proved:

(i) $\sigma(1 + t\,\frac{q^{r-1} - 1}{q - 1}, q) \le (q - t + 1)q^{k-r}$;

here q is a prime and $k = 1 + t\,\frac{q^{r-1} - 1}{q - 1}$.

(ii) $\sigma(n,kt) \le \sigma(n,k)t^{n-1}$ for any n,k and t .

(iii) $\sigma(q + 1,qt) = q^{q-1}\, t^{q}$ for any prime power q and any t.

1. Introduction

Let $V = (V_k^n,d)$ denote the metric space of all n-tuples $(a_1,a_2,\dots,a_n)$ with $a_i \in \{1,2,\dots,k\}$ provided with the Hamming distance: $d(\underline{a},\underline{b}) = |\{i \mid a_i \ne b_i\}|$. A subset W of V is called a <u>covering</u> (<u>by rook-domains</u>) if each point of V is at distance ≤ 1 from some point in W. We are interested in bounds on the number of points in a minimal covering of V, to be denoted by $\sigma(n,k)$. Points of W will be called <u>rooks</u>, the sphere of radius 1 around a rook a <u>rook-domain</u>. Since each rook-domain contains $1 + n(k - 1)$ points we get $\sigma(n,k) \ge \frac{k^n}{1 + n(k - 1)}$. Equality can be obtained if k is a prime power and $1 + n(k-1) | k$. E. Rodemich [1] proved that this bound can be improved to $\sigma(n,k) \geq \frac{k^{n-1}}{n-1}$ in the case $k \geq n$.

2. A generalization of the bounds of van Lint and Kamps

A trivial observation is that $\sigma(n+1,k) \leq k\sigma(n,k)$. This observation, combined with $\sigma(4,3) = 3^2$ yields $\sigma(13,3) \leq 3^{11}$, but actually $\sigma(13,3) = 3^{10}$. It is natural therefore to study the behaviour of $\sigma(n,k)$ in between. In [2] J.H. van Lint and H.J.L. Kamps proved $\sigma(9,3) \leq 2 \cdot 3^6$. We will now demonstrate a technique which generalizes their construction.

Let $A = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k)$ be a matrix with k columns and r linearly independent rows, with $\underline{a}_i \in \mathbb{F}_q^r$ where q is a prime. Let S be a set of points in $\mathbb{F}_q^r$ such that $\{\underline{s} + \alpha\underline{a}_i \mid \underline{s} \in S, \alpha \in \mathbb{F}_q, 1 \leq i \leq k\} = \mathbb{F}_q^r$.

<u>Lemma</u>. $W := \{\underline{w} \in \mathbb{F}_q^k \mid A\underline{w} \in S\}$ is a covering of $V_q^k = \mathbb{F}_q^k$ and $|W| = |S| \cdot q^{k-r}$.

<u>Proof</u>. Take $\underline{x} \in \mathbb{F}_q^k$, then $A\underline{x} \in \mathbb{F}_q^r$, so we may write $A\underline{x} = \underline{s} + \alpha\underline{a}_i$. Let $e_i = (0,0,\ldots,1,0\ldots0)$ denote the i^{th} unit vector in $\mathbb{F}_q^k$, then $A(\underline{x} - \alpha\underline{e}_i) = \underline{s} \in S$ hence $\underline{x} - \alpha\underline{e}_i \in W$, and $d(x,W) \leq 1$.

<u>Application</u>

$$A = \left(\begin{array}{c|cc|c|cc} 0 & 0 & 0 & & 1 & 1 \\ & & & & \vdots & \vdots \\ 0 & 0 & 0 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 1 & 1 & & \vdots & \vdots \\ \hline 1 & 1 & t & & 1 & \ldots\ t \end{array}\right) \qquad S = \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q-t \end{pmatrix} \right\}$$

the columns of A are <u>all</u> projectieve (r-1)-vectors over $\mathbb{F}_q$, repeated t times, with last coordinates 1,2,...,t together with the vector $(0\ 0,\dots,0\ 1)^T$, so $k = 1 + t\,\frac{q^{r-1}-1}{q-1}$.

It is easily checked, using the pigeonhole principle, that the pair A,S satisfies the conditions, hence

$$\sigma(k,q) \le (q - t + 1)q^{k-r} .$$

4. A sequence of cases meeting the Rodemich bound

<u>Theorem</u>. $\sigma(n,kt) \le \sigma(n,k)t^{n-1}$.

<u>Proof</u>. Let W be a covering of V_k^n. Regard V_{kt}^n as obtained from V_k^n by replacing each point by V_t^n and give V_{kt}^n coordinates as follows:
For $a = (a_1,a_2,\dots,a_n) \in V_{kt}^n$ and $b = (b_1,b_2,\dots,b_n) \in V_t^n$ the point in position b of the set V_t^n replacing a gets coordinates

$$((a_1 - 1)t + b_1,\ (a_2 - 1)t + b_2,\dots,(a_n - 1)t + b_n) .$$

Now for each rook in W fill the corresponding set V_t^n with t^{n-1} rooks placed at the points $(x_1,x_2,\dots,x_n)$ satisfying $x_1 + x_2 + \dots + x_n \equiv 0 \pmod t$. It is easy to verify that the set of rooks thus defined covers V_{kt}^n .

<u>Corollary</u>. If q is a prime power then $\sigma(q+1,qt) = q^{q-1}t^q$.

<u>Proof</u>. Since $\sigma(q+1,q) = q^{q-1}$ by the Hamming bound we get $\sigma(q+1,qt) \le q^{q-1}\,t^q$. Rodemich's equality however, gives $\sigma(q+1,qt) \ge q^{q-1}t^q$.

References

[1] E. Rodemich,Covering by Rook Domains. Journal of Comb. Theory 9 (1970), 117 - 128 .

[2] J.H. van Lint & H.J.L. Kamps, A covering Problem. Coll. Math. Societas János Bolyai 4, Comb. Theory and it's Applications, Balatonfüred 1969 (Hungary).