A note on ordered bipartite graphs

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics

Memorandum 74-03

Issued, January 7,1974

A NOTE ON ORDERED BIPARTITE GRAPHS

by

N.G. de Bruijn

University of Technology
Department of Mathematics
PO Box 513, Eindhoven
The Netherlands

A note on ordered bipartite graphs

by N.G. de Bruijn.

The result explained in this note is implicit in a paper by M. Aigner [1].

Let (A,B,Z) be a bipartite graph, i.e. A and B are sets, and Z is a subset of $A \times B$. A matching is a subset M of Z with the property that if $(a,b) \in M$, $(c,d) \in M$ then $a \neq c$ and $b \neq d$. M is not necessarily maximal.

Now assume both A and B linearly ordered. By recursion we define an algorithm for obtaining a matching:

Algorithm 1. If $Z = \emptyset$, we take $M_1 = \emptyset$. If $Z \neq \emptyset$, let a_1 be the first (first in the sense of the linear order) element of A for which there exists $b \in B$ with $(a,b) \in Z$; let b_1 be the first b with $(a_1,b) \in Z$. Take $Z^* = Z \setminus \{(\bigcup_{a \in A} (a,b_1)) \cup (\bigcup_{b \in B}(a_1,b))\}$. Apply the algorithm to (A,B,Z^*); this produces a matching M^*. Now take

$$M = M^* \cup \{(a_1,b_1)\}.$$

We also consider

Algorithm 2. The definition is the same as the one of algorithm 1 but for the definition of (a_1,b_1). We now take b_1 to be the first element of B for which an $a \in A$ exists with $(a,b_1) \in Z$, and a_1 the first a with $(a,b_1) \in Z$.

Theorem. Algorithm 1 and algorithm 2 produce the same M.

Sketch of proof. Induction with respect to $|Z|$. The case $|Z| = 0$ is trivial. Take $|Z| > 0$. It is easy to show that there is a pair $(a_0,b_0) \in Z$ such that

$$(\forall_{b \in B}\ (a_0,b) \in Z \Rightarrow b \geq b_0) \wedge (\forall_{a \in A}(a,b_0) \in Z \Rightarrow a \geq a_0).$$

Take

$$\hat{Z} := Z \setminus \{(\bigcup_{a \in A} (a,b_0)) \cup (\bigcup_{b \in B} (a_0,b))\}.$$

Applying Algorithm 1 to $(A,B,\hat{Z})$ we get $\hat{M}_1$. It can be shown that application of Algorithm 1 to (A,B,Z) produces $M_1 = \hat{M}_1 \cup \{(a_0,b_0)\}$. And we can show a similar thing with algorithm 2 and $\hat{M}_2$, M_2. By induction hypothesis $\hat{M}_1 = \hat{M}_2$. Hence $M_1 = M_2$.

[1] M. Aigner, "Lexicographic matching in boolean algebras", Journal of Combinatorial Theory (B)14, 187 - 194 (1973).