A trade off between emergency repair and inventory investment

Department of Operations Planning and Control -- Working Paper Series

A TRADE OFF BETWEEN EMERGENCY REPAIR AND INVENTORY INVESTMENT

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Research Report TUE/BDK/LBS/95-05
May, 1995

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This research is sponsored by the Netherlands Organization for Scientific Research (NWO).

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Abstract: In this paper we investigate the effect of using emergency repair on the service performance of a repair shop. Failed parts arrive at the shop according to a Poisson process. If stock on hand of serviceable spare parts is positive, the failed part is exchanged for a spare part and sent into normal repair. If stock on hand is zero, the failed part is sent into an alternative emergency repair channel. The backorder is filled with the first part that becomes available from either one of the repair processes. Both repair processes are assumed to be exponentially distributed. This paper considers the trade off between using emergency repair and investment in spare parts, such that a predetermined service target for the repair shop is realized. Two service measures are considered, that is, the fraction of demand satisfied from stock on hand and the expected duration of a backorder.

Keywords: emergency, spare part, service, repair

1. Introduction

In many situations managers of stocking facilities resort to expediting or emergency procedures when they run out of stock. Especially in after sales organizations which are responsible for the supply of spare parts, emergency shipments are very important to assure a high service level to customers. When a customer request for a spare part cannot be met from stock on hand, an emergency order is issued to minimize the customers down time. In case of a repairable part, which we consider in this paper, the defective part can be sent into emergency repair instead of normal repair. The contribution of this paper is twofold. First, we present an exact analysis of an emergency repair model. Second, we use this model to make a trade off between emergency repair and inventory investment. The trade off is intended to achieve a certain service performance at minimum cost. Two service measures are considered, that is, the fraction of demand satisfied from stock on hand and the expected duration of a backorder.

The paper is organised as follows. In section 2 we briefly discuss the literature on the subject of emergency shipments. Special attention is given to a model by Hausman and Erkip(1994) which also allows for emergency shipments in stockout situations. In section 3 the emergency repair model is presented. An analysis of the emergency repair model is given in section 4. In section 5 some numerical results are given. The results of the Hausman-Erkip model will be compared with the results of our emergency repair model. Finally, in section 6 we present conclusions and suggestions for further research.

2. Literature review

A number of papers consider the possibility of using alternative supply modes for stock replenishment. Some early models describe periodic review base stock policies with two alternative lead times: one-period lead time for normal supply and zero-period lead time for emergency supply (see e.g. Barankin(1961), Daniel(1962), Fukuda(1964)). Whittmore and Saunders(1977) allow for lead times of arbitrary length. A model by Rosenshine and Obee(1976) compares a standing-order inventory system (i.e. regular order arrivals with the possibility of emergency orders and sell-offs) with a traditional periodic review base stock policy. Moinzadeh and Nahmias(1988) present a continuous review (s,S)-inventory policy with different reorder points for normal and emergency replenishments. Cohen *et al.*(1988) also consider an (s,S)-inventory policy with two priority classes of customers: normal replenishment orders and emergency replenishment orders with higher priority. Ernst and Cohen(1993) extend this model to a situation where the classification of customers is a decision variable.

The inventory models used for controlling spare part inventories are usually $(S\text{-}1,S)$-policies with Poisson demand. Such models are appropriate for low demand rates and expensive items such that ordering cost is negligible compared to holding and shortage cost. Sherbrooke(1968) presents a multi-echelon model called METRIC (Multi-Echelon Technique for Recoverable Item Control) for the control of spare parts. The METRIC model however does not allow for emergency shipments in stock-out situations. Hausman and Erkip(1994) present a METRIC-like multi-echelon model (first described by Muckstadt and Thomas(1980)) where emergency shipments are used when local warehouses run out of stock. They demonstrate the difference between using multi-echelon models vs. single-echelon models in a multi-echelon environment. However, they do not explicitly investigate the trade off between emergency shipments and inventory investment. We will compare the numerical results of their model with the results of our model. Finally, Moinzadeh and Schmidt(1988) consider a single-echelon $(S\text{-}1,S)$ inventory model with two modes of resupply. Initiation of an emergency resupply depends on the current stock level and on the remaining lead times of outstanding orders.

The emergency repair model presented in this paper also considers an $(S\text{-}1,S)$ inventory policy. We assume that a customer waiting for a spare part receives the part that first becomes available from the repair process, either normal or emergency repair. In the Muckstadt/Hausman model, however, it is

assumed that a customer initiating an emergency repair order receives that particular part resulting from the emergency repair, even if a normal repair order is completed before.
The model by Moinzadeh and Schmidt(1988) assumes constant normal and emergency resupply times. However, the repair process is usually a complex structure of activities. We model the repair process as an infinite server queue with exponentially distributed repair times.

3. The emergency repair model

We consider a stocking location for spare parts with stock level S where failed parts arrive according to a Poisson process with rate λ. An $(S\text{-}1,S)$ inventory policy is applied. If stock on hand is positive, the failed part is exchanged for a spare part and the failed part is sent to the repair shop for repair. If stock on hand is zero, the failed part is sent into emergency repair (which is faster than normal repair) and a backorder is created. This alternative emergency repair process is independent of the normal repair process. The backorder is filled with the first spare part that becomes available, either from the normal repair process or the emergency repair process. Backorders are filled according to the FCFS-principle (First Come First Serve). The repair times (both normal and emergency) are assumed to be exponentially distributed. The normal repair process can be seen as an $M|M|S$ queue and the emergency repair process can be seen as an $M|M|\infty$ queue. This model of the repair process implies that a failed part is immediately taken into normal repair (if stock on hand is positive or, equivalently, less than S parts are in repair) or emergency repair (if stock on hand is zero or, equivalently, S or more parts are in repair) and no waiting time occurs. The same modelling assumption of the repair process is used in the well-known METRIC model by Sherbrooke(1968). The model is depicted in figure 1.

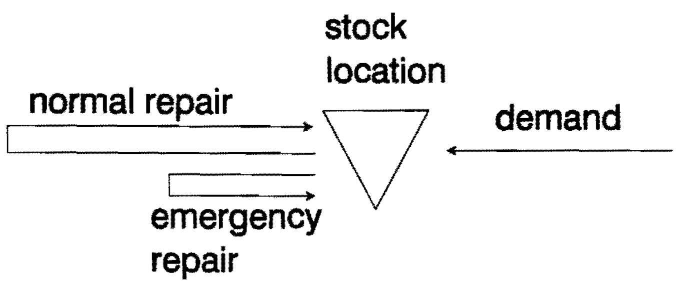

Figure 1: the emergency repair model

We define the following variables:

A : interarrival time of failed parts (exponentially distributed with rate λ)
R_n : normal repair time (exponentially distributed with rate μ)
R_e : emergency repair time (exponentially distributed with rate τ)
S : stock level of spare parts
(i,j) : system state with i parts in normal repair $(0 \leq i \leq S)$ and j parts in emergency repair $(j \geq 0)$
$p_{i,j}$: steady state probability associated with state (i,j)

In section 4 we show how to calculate these probabilities $p_{i,j}$ using Markov chain analysis. These probabilities enable us to compute a number of performance measures. For example, we can determine the expected number of parts in normal repair, the expected number of parts in emergency repair, the expected number of parts in repair (either normal or emergency), and the expected number of backorders. Define:

L^e : expected number of parts in emergency repair
L^n : expected number of parts in normal repair

L	: expected number of parts in repair
B	: expected number of backorders

Then:

$$L^e = \sum_{i=0}^{S}\sum_{j=1}^{\infty} j\ p_{i,j}$$

$$L^n = \sum_{i=1}^{S}\sum_{j=0}^{\infty} i\ p_{i,j}$$

$$L = L^e + L^n = \sum_{k=1}^{\infty} k\ r(k)\ , \qquad r(k) = \sum_{i+j=k} p_{i,j}$$

$$B = \sum_{k=S+1}^{\infty} (k-S)\ r(k) = L - S + \sum_{k=0}^{S} (S-k)\ r(k)$$

In this paper we concentrate on two performance measures: the **expected fill rate (EFR)** and the **expected duration of a backorder (W^*)**. The EFR is defined as the fraction of demand that can be satisfied from stock on hand, i.e. by the PASTA property (see Wolff(1982)) this is the same as the fraction of time there are at most S-1 parts in repair:

$$EFR = \sum_{i+j<S} p_{i,j} \tag{1}$$

Let W represent the expected time it takes to exchange an arbitrary failed part for a spare part. The expected duration of a backorder W^* is then computed as follows:

$$\begin{aligned} W^* &= \frac{W}{Pr\{S \text{ or more parts are in repair}\}} = \frac{W}{1 - EFR} \\ W &= \frac{B}{\lambda} \qquad (Little) \end{aligned} \tag{2}$$

An important aspect to be considered when discussing the trade off between inventory investment and emergency repair is the cost structure of the model. The total costs (TC) consist of two components: **inventory costs (IC)** and **repair costs (RC)**:

$$\text{TC} = \text{IC} + \text{RC} \tag{3}$$

The inventory costs are calculated as follows:

$$\text{IC} = K\ S\ h \tag{4}$$

The total repair costs consist of normal repair costs and emergency repair costs:

$$\begin{aligned} \text{RC} &= Pr\{\text{ emergency repair }\}\ \lambda f^e K + Pr\{\text{ normal repair }\}\ \lambda f^n K \\ &= Pr\{\text{ S or more parts in repair }\}\ \lambda f^e K + Pr\{\text{ less than S parts in repair }\}\ \lambda f^n K \\ &= (1\text{-}EFR)\ \lambda f^e K + EFR\ \lambda f^n K \end{aligned} \tag{5}$$

with

K : unit price of one spare part
h : inventory cost for one spare part (as fraction of K)
f^e : emergency repair cost for one spare part (as fraction of K)
f^n : normal repair cost for one spare part (as fraction of K)

The cost of emergency repair (f^e) depends on the speed of the emergency repair (τ). We assume a linear relationship between f^e and τ (see figure 2). However, it is possible to model any kind of relationship.

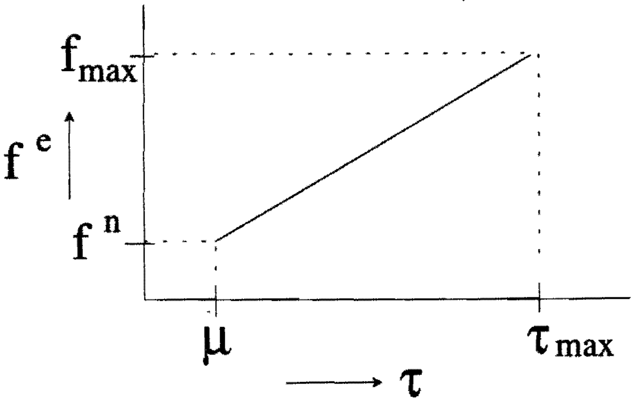

Figure 2: f^e as function of τ

For $\tau = \mu$ the emergency repair costs are equal to the normal repair costs ($f^e = f^n$). For $\tau = \tau_{max}$ the emergency repair costs are equal to a maximum value ($f^e = f_{max}$). τ_{max} represents a maximum emergency repair rate. Note that the value of f_{max} is an indication of the cost of using emergency repair. The emergency repair cost for any value of τ ($\mu<\tau<\tau_{max}$) can be calculated as follows:

$$f^e = \frac{f_{max} - f^n}{\tau_{max} - \mu}\tau + f^n - \mu\frac{f_{max} - f^n}{\tau_{max} - \mu} \tag{6}$$

In practice one often can choose between three or four emergency rates. In section 5 we give some numerical results for the performance measures using this cost structure.

4. Model analysis

In this section we describe how to calculate the steady state probabilties $p_{i,j}$ that are needed to compute the performance measures of the emergency repair model. First we consider the Markov process embedded on the states (i,j) with $0 \le i \le S$, $0 \le j \le S$ and compute the associated steady state probabilities $\tilde{p}_{i,j}$ (section 4.1). For this we need to compute the transition rates for the states (i,S) (section 4.2). Finally, we compute the steady state probabilities of the original Markov process (section 4.3).

4.1 The embedded Markov process

The transition-rate diagram of the original model is depicted in figure 3.

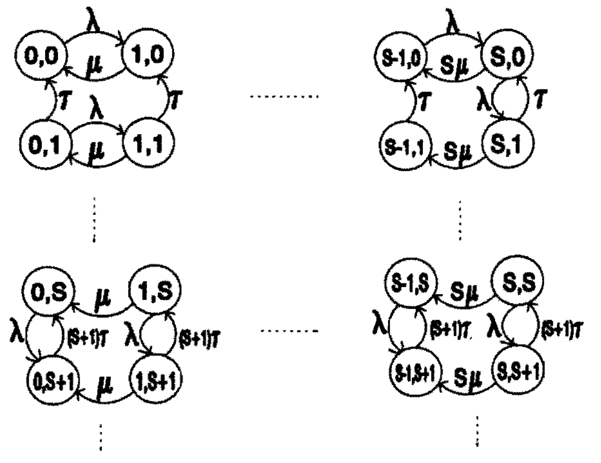

Figure 3: transition-rate diagram of the original model

We define level j as the set of all states with j parts in emergency repair:

$$level\ j := \{\ (i,j)\ \mid\ 0 \leq i \leq S\ \}$$

The state space for this system is infinite, since the number of parts in emergency repair is not restricted $(j \geq 0)$. We first consider the process embedded on the states (i,j) with $0 \leq i \leq S$, $0 \leq j \leq S$ (i.e. excursions to levels higher than S are not considered). To calculate the transition rates for the states (i,S) in the embedded Markov process, we must first calculate for the original process, the probability that, starting in state $(i,S+1)$ at level $S+1$, one returns to level S in state (k,S). Define

$$\pi_{ik} := Pr\ \{\ \textit{first return to level } S \textit{ in state } (k,S)\ \mid\ \textit{start in state } (i,S+1)\ \}$$

with:

$$\sum_{k=0}^{i} \pi_{ik} = 1 \tag{7}$$

Note that $\pi_{i,k} = 0$ for $k>i$. The transition-rate diagram of the embedded Markov process is depicted in figure 4. Let $\tilde{p}_{ij}$ be the steady state probabilities of the embedded Markov process.

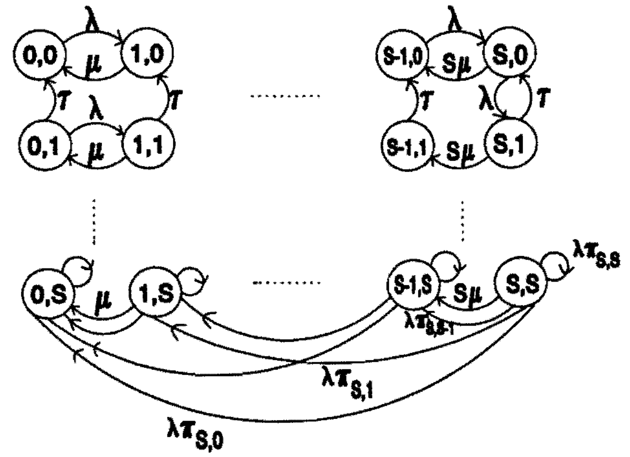

Figure 4: transition-rate diagram of the embedded model

From figure 4 we can derive the equilibrium equations for the embedded model:

$$i=0,\ j=0..S-1 \ : \quad \tilde{p}_{0,j}\{\lambda + j\tau\} = \tilde{p}_{0,j+1}(j+1)\tau + \tilde{p}_{1,j}\mu$$

$$i=0,\ j=S \ : \quad \tilde{p}_{0,S}S\tau = \tilde{p}_{1,S}\mu + \lambda\sum_{k=1}^{S}\pi_{k,0}\tilde{p}_{k,S}\mu$$

$$i=1..S-1,\ j=0 \ : \quad \tilde{p}_{i,0}\{\lambda + i\mu\} = \tilde{p}_{i-1,0}\lambda + \tilde{p}_{i+1,0}(i+1)\mu + \tilde{p}_{i,1}\tau$$

$$i=S,\ j=0 \ : \quad \tilde{p}_{S,0}\{\lambda + S\mu\} = \tilde{p}_{S-1,0}\lambda + \tilde{p}_{S,1}\tau$$

$$i=S,\ j=1..S-1 \ : \quad \tilde{p}_{S,j}\{\lambda + S\mu + j\tau\} = \tilde{p}_{S,j-1}\lambda + \tilde{p}_{S,j+1}(j+1)\tau$$

$$i=S,\ j=S \ : \quad \tilde{p}_{S,S}\{\lambda\sum_{k=0}^{S-1}\pi_{S,k} + S\mu + S\tau\} = \tilde{p}_{S,S-1}\lambda$$

$$i=1..S-1,\ j=S \ : \quad \tilde{p}_{i,S}\{\lambda\sum_{k=0}^{i-1}\pi_{i,k} + i\mu + S\tau\} = \tilde{p}_{i,S-1}\lambda + \lambda\sum_{k=i+1}^{S}\pi_{k,i}\tilde{p}_{k,S} + \tilde{p}_{i+1,S}(i+1)\mu$$

$$1 \le i,j \le S-1,\ i+j \le S \ : \quad \tilde{p}_{i,j}\{\lambda + i\mu + j\tau\} = \tilde{p}_{i-1,j}\lambda + \tilde{p}_{i+1,j}(i+1)\mu + \tilde{p}_{i,j+1}(j+1)\tau$$

$$1 \le i,j \le S-1,\ i+j \le S \ : \quad \tilde{p}_{i,j}\{\lambda + i\mu + j\ \tau\} = \tilde{p}_{i,j-1}\lambda + \tilde{p}_{i+1,j}(i+1)\mu + \tilde{p}_{i,j+1}(j+1)\tau$$

Once the probabilities $\pi_{i,k}$ are known, we can solve the equilibrium probabilities $\tilde{p}_{i,j}$ for the embedded system from the equilibrium equations above, together with the normalization equation:

$$\sum_{i=0}^{S}\sum_{j=0}^{S}\tilde{p}_{i,j} = 1 \qquad (8)$$

4.2 Calculation of the transition rates $\pi_{i,k}$

Suppose that at time t=0 the system is in level S+1 and i parts are in normal repair. Then $\pi_{i,i-k}$ represents the probability that k of these parts ($k \le i$) are repaired in time T_{S+1}, where T_{S+1} represents the stochastic time it takes to go from level S+1 to level S:

$$\pi_{i,i-k} = \int_0^{\infty} r_{k,i}(t)\ dPr\{T_{S+1}=t\} \tag{9}$$

with

T_n := length of time interval to return in level n-1when starting in level n
$r_{k,i}\ (t)$:= Pr{k parts are repaired in the normal repair channel in time t | i parts are in normal repair at time $t=0$ }

Using the fact that the normal repair times are exponentially distributed with parameter μ and application of the binomium of Newton gives:

$$\begin{aligned} r_{k,i}\ (t) &= \binom{i}{k} (1-e^{-\mu t})^k\ (e^{-\mu t})^{i-k} \\ &= \binom{i}{k} e^{-\mu(i-k)t} \sum_{l=0}^{k} \binom{k}{l} (-1)^l\ e^{-\mu l t} \\ &= \binom{i}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^l\ e^{-\mu(i-k+l)t} \\ &= \binom{i}{k} \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n}\ e^{-\mu(i-n)t} \end{aligned}$$

Substitution of this relation in (9) gives

$$\begin{aligned} \pi_{i,i-k} &= \binom{i}{k} (-1)^k \sum_{n=0}^{k} \binom{k}{n} (-1)^n \int_0^{\infty} e^{-\mu(i-n)t}\ dPr\{T_{S+1}=t\} \\ &= \binom{i}{k} (-1)^k \sum_{n=0}^{k} \binom{k}{n} (-1)^n\ \varphi_{S+1}(\mu(i-n)) \end{aligned}$$

where $\varphi_n(s)$ represents the Laplace-Stieltjes transform of T_n. The Laplace-Stieltjes transform $\varphi_n(s)$ can be calculated from the following expressions (see Appendix A for a detailed analysis):

$$\varphi_1(s) = \frac{1}{\lambda}\left\{\lambda + s - \frac{s}{1 + \sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{-\lambda}{s+i\tau}}\right\} ,$$

$$\varphi_{n+1}(s) = \frac{(\lambda + n\tau + s)\ \varphi_n(s) - n\tau}{\lambda\ \varphi_n(s)} , \qquad n=1,2,...$$

4.3 Calculation of $p_{i,j}$

Once we have found the steady state probabilities $\tilde{p}_{i,j}$ of the embedded Markov process, we can determine the steady state probabilities $p_{i,j}$ of the original process, using the following relation:

$$p_{i,j} = \tilde{p}_{i,j} \cdot C \ , \quad 0 \le i \le S,\ 0 \le j \le S$$

C represents the probability that the original process is in the set of states (i,j) with $0 \le i \le S$ and $0 \le j \le S$:

$$C = \sum_{i=0}^{S}\sum_{j=0}^{S} p_{i,j}$$

Define:

$$q(j) := \sum_{i=0}^{S} p_{i,j} , \qquad j \ge 0$$

$$\tilde{q}(j) := \sum_{i=0}^{S} \tilde{p}_{i,j} , \qquad 0 \le j \le S$$

Note that $q(j)$ resp. $\tilde{q}(j)$ represents the probability that j parts are in emergency repair (level j) in the original process resp. the embedded process. From the transition-rate diagram of the original process we can derive the following equation (general balance principle):

$$\lambda\, q(S) = (S+1)\tau q(S+1)$$

$$q(S+1) = \frac{\lambda}{\tau}\frac{1}{S+1}\, q(S)$$

It then follows for $j > S$:

$$q(j) = \left(\frac{\lambda}{\tau}\right)^{j-S} \frac{1}{j(j-1)..(S+1)}\, q(S)$$

$$= \left(\frac{\lambda}{\tau}\right)^{j} \frac{1}{j!}\, S! \left(\frac{\tau}{\lambda}\right)^{S} q(S)$$

$$= \left(\frac{\lambda}{\tau}\right)^{j} \frac{1}{j!}\, S! \left(\frac{\tau}{\lambda}\right)^{S} C\, \tilde{q}(S)$$

The normalizing constant C can now be computed as follows:

$$
\begin{aligned}
1 &= \sum_{j=0}^{\infty} q(j) \\
&= \sum_{j=0}^{S} q(j) + \sum_{j=S+1}^{\infty} q(j) \\
&= C \sum_{j=0}^{S} \tilde{q}(j) + \sum_{j=S+1}^{\infty} \left(\frac{\lambda}{\tau}\right)^{j} \frac{1}{j!}\, S! \left(\frac{\tau}{\lambda}\right)^{S} C\, \tilde{q}(S) \\
&= C \left\{ 1 + S! \left(\frac{\tau}{\lambda}\right)^{S} \tilde{q}(S) \left\{ e^{\lambda/\tau} - \sum_{j=0}^{S} \left(\frac{\lambda}{\tau}\right)^{j} \frac{1}{j!} \right\}\right\}
\end{aligned}
$$

$$
C = \left\{ 1 + S! \left(\frac{\tau}{\lambda}\right)^{S} \tilde{q}(S) \left\{ e^{\lambda/\tau} - \sum_{j=0}^{S} \left(\frac{\lambda}{\tau}\right)^{j} \frac{1}{j!} \right\}\right\}^{-1} \qquad (10)
$$

The steady state probabilities $p_{i,j}$ for $j>S$ can now be computed as follows. The equilibrium equation for state (i,S) in the original Markov process can be used to calculate $p_{i,S+1}$:

$$
\begin{aligned}
p_{0,S+1} &= \frac{p_{0,S}(S\tau+\lambda) - \mu p_{1,S}}{(S+1)\tau} , \\
p_{i,S+1} &= \frac{p_{i,S}(S\tau+\lambda+i\mu) - (i+1)\mu p_{i+1,S} - \lambda p_{i,S-1}}{(S+1)\tau} , \qquad 0<i<S , \\
p_{S,S+1} &= \frac{p_{S,S}(S\tau+\lambda+S\mu) - \lambda p_{S,S-1}}{(S+1)\tau} .
\end{aligned}
$$

Finally we can calculate $p_{i,j}$ recursively for all $j>S$:

$$
\begin{aligned}
p_{i,j+1} &= \frac{p_{i,j}(j\tau+\lambda+i\mu) - (i+1)\mu p_{i+1,j} - \lambda p_{i,j-1}}{(j+1)\tau} , \qquad 0\le i<S , \\
p_{S,j+1} &= \frac{p_{i,j}(j\tau+\lambda+S\mu) - \lambda p_{S,j-1}}{(j+1)\tau} .
\end{aligned}
$$

5. Numerical Results

In this section we give some numerical results for the performance measures of the emergency repair model described in section 3. In section 5.1 we examine the effect of using emergency repair on the expected fill rate (EFR). In section 5.2 we do the same for the expected duration of backorders (W^*). The trade off between using emergency repair and investing in inventory is examined in section 5.3, using the cost structure described in section 3. Finally, in section 5.4 we compare the results of our model with the ones of the model of Hausman and Erkip.

5.1 Expected fill rate (EFR)

Three parameters determine the value of EFR: the stock level S, the relative emergency repair rate

(=τ/μ), and the occupation rate ρ (=λ/μ). Figures 5 to 7 show the EFR as a function of τ/μ for different stock levels S and different values of ρ. The relative emergency repair rate τ/μ varies from 1 (i.e. no emergency repair) to 10 (i.e. $\tau_{max}= 10\mu$). The results show that for low values of ρ (figure 5 and 6) the EFR can hardly be improved by using emergency repair. The largest increase in EFR can be observed for low stock levels. For low values of ρ it is obvious that the stock level S is the dominant factor in realizing a certain target for the EFR. Using emergency repair only leads to a marginal increase in EFR, which is not sufficient to consider a trade off with stock level reduction. For high values of ρ the increase in EFR due to emergency repair can be very significant. In these cases a trade off between emergency repair and stock level is possible. See for example figure 7 (ρ=5). Suppose a target value for EFR of 0.30 is to be realized. This can be done for three different combinations of stock level and emergency repair rate:

1) $S = 2$, $\tau/\mu = 6.3$
2) $S = 3$, $\tau/\mu = 1.8$
3) $S = 4$, $\tau/\mu = 1.1$

In section 5.3 we consider the associated costs for this trade off. Note that this example is purely illustrative. In practice, the target value for EFR is usually much higher.

The value of EFR for τ/μ=1 (i.e. no emergency repair) is equal to the EFR for an $M|M|\infty$ queue with repair rate μ. From Palm's theorem(1938) we know that the number of units in repair is Poisson distributed with mean $\rho=\lambda/\mu$ and is independent of the distribution of the repair time. The expected fill rate can then be expressed as follows:

$$EFR = \sum_{k=0}^{S-1} e^{-\rho} \frac{\rho^k}{k!} \tag{11}$$

The results show that the expected fill rate has a limit when increasing the relative emergency repair rate. In fact this limit is equal to one minus the blocking probability in an $M|M|S|S$ queue. For $\tau/\mu=\infty$, a customer arriving in a stockout situation is served in zero time (i.e. instant repair). The EFR-limit can be expressed as follows:

$$EFR = 1 - \frac{\frac{\rho^S}{S!}}{\sum_{k=0}^{S} \frac{\rho^k}{k!}} \tag{12}$$

Note that the EFR-limit is also independent of the repair time distribution.

Expressions (11) and (12) represent upper and lower bounds for the expected fill rate when using emergency repair. Expression (11) represents the EFR when no emergency repair is used and expression (12) represents the EFR when emergency repair is instantaneously. Figures 8, 9, and 10 show these upper and lower bounds for different stock levels and different values of ρ. Again it is obvious that the largest increase is realized for high values of ρ.

5.2 Expected duration of backorders (W^*)

The expected duration of backorders can be reduced significantly when using emergency repair. Figures 11, 12, and 13 show the normalized duration of backorders as a function of the relative emergency repair rate. $W^*\mu$ represents the expected duration of a backorder as a fraction of the normal repair time for a given emergency repair rate. The results show that in all cases the reduction in the duration of backorders is significant. A trade off between emergency repair and stock level is possible. See for example figure 12 (ρ=1). When aiming for a backorder duration of 30% of the normal repair time, three alternatives are possible:

1) $S = 0,\quad \tau/\mu = 3.3$
2) $S = 1,\quad \tau/\mu = 2.3$
3) $S = 2,\quad \tau/\mu = 1.5$

In order to make a trade off between these alternatives, we must calculate the associated total cost.

5.3 Cost trade off

The example in section 5.1 (with $\rho = 5$) demonstrated that an EFR of 30% can be realized in three ways. The optimal solution (i.e. minimal cost) depends on the parameter setting. Five parameters influence the trade off between stock level S and emergency repair rate τ: demand rate λ, normal repair rate μ, annual holding cost fraction h, normal repair cost fraction f^n, and the maximal emergency repair cost fraction f_{max} for $\tau = \tau_{max}$. Notice that the unit price K of one part does not affect the trade off! In the following example we assume a direct proportional relationship between τ and f^e (i.e. $f_{max} = 10 f^n$ for $\tau_{max} = 10\ \mu$) and unit price K=100.

options	EFR	IC	(%)	RC	(%)	TC
S=2, τ/μ=6.3	0.300	100	(37)	172	(63)	272
S=3, τ/μ=1.8	0.304	150	(72)	57	(28)	207
S=4, τ/μ=1.1	0.307	200	(84)	39	(16)	239

Table 1: cost calculations with λ=0.01, μ=0.002, h=0.5, and f^n=0.1

options	EFR	IC	(%)	RC	(%)	TC
S=2, τ/μ=6.3	0.300	20	(0.2)	8596	(99.8)	8616
S=3, τ/μ=1.8	0.304	30	(1)	2841	(99)	2871
S=4, τ/μ=1.1	0.307	40	(2)	1951	(98)	1991

Table 2: cost calculations with λ=0.1, μ=0.02, h=0.1, and f^n=0.5

Table 1 shows the cost calculations for very slow moving items (λ=0.01) with high inventory costs and low repair costs. The results show that option two is the cheapest option. Increasing inventory costs are balanced by decreasing repair costs, resulting in an optimal solution for S=3 and τ/μ=1.8. However, when dealing with fast moving items (λ=0.1) with low inventory costs and high repair costs (see table 2), the minimal costs are realized for S=4 and τ/μ=1.1. The inventory costs represent a negligible fraction of the total costs. The emergency repair rate is the dominant parameter, resulting in an optimal solution with the lowest emergency repair rate (τ/μ=1.1). It is important to observe that in the cases under consideration the EFR is 30%. This implies that 70% of the total demand is sent into emergency repair. When dealing with high fill rates (e.g.>90%), the trade off could yield other results.

A similar cost trade off can be made for the example in section 5.2 when aiming for a normalized backorder duration of 30%.

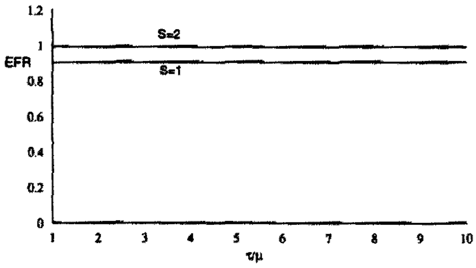

Figure 5: expected fill rate for ρ = 0.1

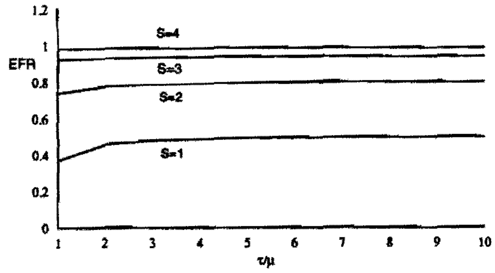

Figure 6: expected fill rate for ρ = 1

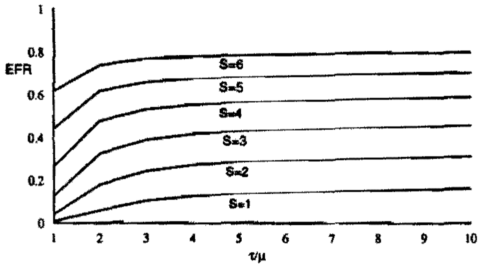

Figure 7: expected fill rate for ρ = 5

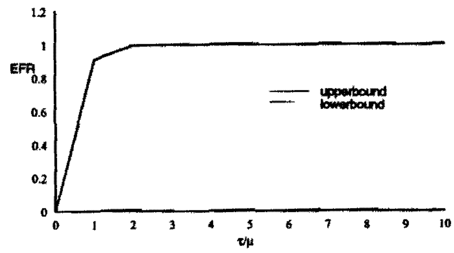

Figure 8: increase in EFR for ρ = 0.1

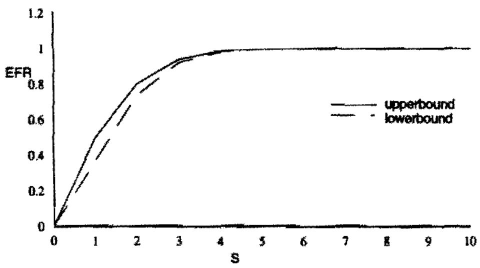

Figure 9: increase in EFR for ρ = 1

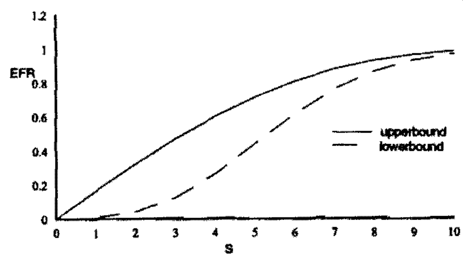

Figure 10: increase in EFR for ρ = 5

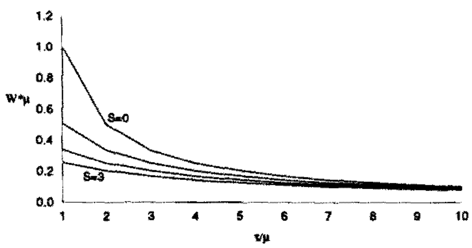

Figure 11: backorder lead time for ρ = 0.1

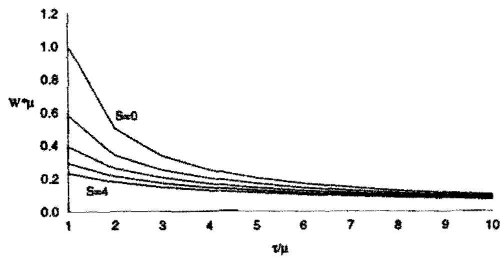

Figure 12: backorder lead time for ρ = 1

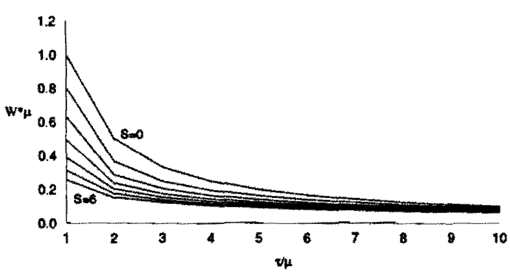

Figure 13: backorder lead time for ρ = 5

5.4 The Muckstadt-Hausman model

In order to compare the performance of a multi-echelon approach with that of a single-echelon approach, Muckstadt and Thomas (1980) developed an emergency resupply model with one central depot and a number of local warehouses. Hausman and Erkip (1994) used this model for further analysis. This model can be considered an extension of the METRIC model in that it allows for (faster) emergency shipments in case of a stockout situation at the local warehouses. We can also use this particular model to evaluate the performance of emergency supply (or emergency repair). In order to compare this multi-echelon model with our single-echelon model, we assume infinite stock at the central depot. Tables 3 and 4 show the results for both our emergency repair model (ERM) and the Muckstadt/Hausman model (MH).

τ / μ	S									
	0		1		2		3		4	
1	0.00	0.00	0.368	0.368	0.736	0.736	0.920	0.920	0.981	0.981
5	0.00	0.00	0.491	0.368	0.794	0.736	0.935	0.920	0.984	0.981
10	0.00	0.00	0.498	0.368	0.798	0.736	0.937	0.920	0.984	0.981
	ERM	MH	ERM	MH	ERM	MH	ERM	MH	ERM	MH

Table 3: effect of emergency repair on EFR for $\rho = 1$

τ / μ	S									
	0		1		2		3		4	
1	1.0	1.0	0.582	1.0	0.392	1.0	0.291	1.0	0.229	1.0
5	0.2	0.2	0.166	0.2	0.143	0.2	0.125	0.2	0.112	0.2
10	0.1	0.1	0.091	0.1	0.083	0.1	0.077	0.1	0.071	0.1
	ERM	MH	ERM	MH	ERM	MH	ERM	MH	ERM	MH

Table 4: effect of emergency repair on $W^*\mu$ for $\rho = 1$

Table 3 shows the expected fill rate for different values of the relative emergency repair rate and the stock level. The results show that the EFR can be increased significantly when using emergency repair, according to our ERM-model. According to the MH-model, however, the EFR is independent of the emergency repair rate! The reason for this lies in the fact that in the MH-model it is assumed that a customer arriving in a stockout situation and therefore triggering an emergency repair order, receives that particular emergency repair order. However, in our ERM-model, we assume that a customer in backlog receives the repair order that first arrives at the local warehouse. This can also be a normal repair order. This more realistic assumption explains the fact that in the ERM-model the EFR is dependent on the emergency repair rate.

A similar observation can be made when looking at the other performance measure $W^*\mu$ (see table 4). The ERM-results show a significant reduction in expected backorder duration time for increasing values of the emergency repair rate and the stock level. However, the MH-results show that the value of $W^*\mu$ is independent of the stock level! Because of the assumption that a customer in backlog receives an emergency repair order, the expected duration of the backorder in independent of the stock level.

6. Conclusions

In this paper we consider the trade off between using emergency repair and inventory investment in order to realize a certain target service performance. The service measures considered are the expected fill rate (fraction of demand satisfied from stock on hand) and the expected duration of a backorder. The problem is formulated as a Markov model, assuming exponential repair times and a Poisson demand process. Embedding the original infinite state space with the help of transition rates, we are able to analyze the emergency repair model exactly.

The numerical results show that a significant reduction in the expected duration of a backorder can be obtained when using emergency repair. The expected fill rate can also be increased significantly, especially for high values of the occupation rate. In order to make a trade off, we introduced a cost structure to evaluate several alternatives that lead to the same service performance. The values of the parameters (e.g. cost of emergency repair, inventory holding cost) determine the trade off between using emergency repair and inventory investment. An extended numerical analysis is needed to evaluate the exact influence of each parameter.

We also compared the numerical results of our model with the results of an emergency supply model by Muckstadt/Hausman. The numerical results of our model show a significant influence of the emergency repair rate and the stock level on the expected fill rate and the expected backorder lead time. The results of the Muckstadt/Hausman model show that the expected fill rate is independent of the emergency repair rate and the expected backorder lead time is independent of the stock level. The reason for this difference lies in the fact that in our model we assume that a customer in backlog receives the first part that becomes available (either from the normal or the emergency repair process) whereas the Muckstadt/Hausman model assumes that a backorder is filled with the emergency repair order that it triggered itself.

Subject for further research is the testing of the numerical results for different repair time distributions. We showed that the minimum EFR (i.e. no emergency repair) and the maximum EFR (i.e. instant emergency repair) are insensitive to the distribution of the repair times. A simulation study could be used to test this insensitivity result for any repair distribution. Finally, we assumed that an emergency repair order is issued when stock on hand is zero. It could also be worthwhile to investigate the situation where an emergency repair order is issued when stock on hand equals an integer value b with $b > 0$.

Appendix A

In this appendix we derive the expression given in section 4.2 for the Laplace Stieltjes transform $\varphi_n(s)$ of the return time T_n from level n to level n-1 ($n \geq 1$). Define:

A : interarrival time of failed parts (exponentially distributed with parameter λ)
C : cycle time between two consecutive arrivals of emergency repair orders that arrive in an empty system (i.e. no emergency repair orders present)

The cycle time C can then be written as follows:

$$C = T_1 + A \tag{A.1}$$

From Takács (1962) we can find the following expression for the Laplace-Stieltjes transform $\gamma(s)$ of the cycle time C in an $M(\lambda)|M(\tau)|\infty$ queue (page 211, expression 8):

$$\begin{aligned} \gamma(s) &= E[e^{-sC}] \\ &= 1 - \frac{s}{\lambda+s} \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{-\lambda}{s+i\tau}} \end{aligned} \tag{A.2}$$

Because T_1 and A are independent, we can write:

$$\begin{aligned} \gamma(s) &= E[e^{-sC}] \\ &= E[e^{-s(T_1+A)}] \\ &= E[e^{-sT_1}] \; E[e^{-sA}] \\ &= \varphi_1(s) \frac{\lambda}{\lambda+s} \end{aligned} \tag{A.3}$$

From expression (A.2) and (A.3) we can derive the following expression for the Laplace-Stieltjes transform $\varphi_1(s)$ of T_1 :

$$\varphi_1(s) = \frac{1}{\lambda} \left\{ \lambda + s - \frac{s}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{-\lambda}{s+i\tau}} \right\} \tag{A.4}$$

In order to calculate the Laplace-Stieltjes transform T_n for $n>1$, we use the following relation:

$$\begin{aligned} T_n &= X_n & &\textit{with probability } \frac{n\tau}{\lambda+n\tau} \\ &= X_n + T_{n+1} + T_n & &\textit{with probability } \frac{\lambda}{\lambda+n\tau} \end{aligned}$$

where X_n represents the sojourn time at level n. Note that X_n is exponentially distributed with parameter $\lambda+n\tau$. Since X_n , T_n , and T_{n+1} are independent, we can write:

$$\begin{aligned}
\varphi_n(s) &= E[e^{-sT_n}] \\
&= E[e^{-sX_n}] \left(\frac{\lambda}{\lambda + n\tau} E[e^{-s(T_{n+1}+T_n)}] + \frac{n\tau}{\lambda + n\tau} \cdot 1 \right) \\
&= E[e^{-sX_n}] \left(\frac{\lambda}{\lambda + n\tau} E[e^{-sT_{n+1}}] \; E[e^{-sT_n}] + \frac{n\tau}{\lambda + n\tau} \right) \\
&= \frac{\lambda + n\tau}{\lambda + n\tau + s} \left(\frac{\lambda}{\lambda + n\tau} \varphi_{n+1}(s) \; \varphi_n(s) + \frac{n\tau}{\lambda + n\tau} \right).
\end{aligned}$$

Finally, we find ($n \geq 0$):

$$\varphi_{n+1}(s) = \frac{(\lambda + n\tau + s) \; \varphi_n(s) - n\tau}{\lambda \; \varphi_n(s)} .$$

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