A direct numerical method for a class of queueing problems

A Direct Numerical Method for a Class of Queueing Problems

J. Wijngaard

Management Science, Vol. 24, No. 13. (Sep., 1978), pp. 1441-1447.

Stable URL:
http://links.jstor.org/sici?sici=0025-1909%28197809%2924%3A13%3C1441%3AADNMFA%3E2.0.CO%3B2-D

Management Science is currently published by INFORMS.

MANAGEMENT SCIENCE
Vol. 24, No. 13, September 1978
Printed in U.S.A.

A DIRECT NUMERICAL METHOD FOR A CLASS OF QUEUEING PROBLEMS*

J. WIJNGAARD†

In this paper an efficient numerical method is given for determining stationary probabilities and average cost in queueing systems where the customers arrive singly and the service mechanism is negative exponential. State dependent service and arrival rates and batch service are allowed. The method is based on the property that the matrix of the embedded Markov process which describes the number of customers at the moments of arrival is almost triangular.
(QUEUES–ALGORITHMIC METHODS; QUEUES–APPROXIMATIONS)

1. Introduction

In recent years a lot of attention has been given to queueing problems with state dependent service and/or service in batches (see Rosenshine [4]). An important class of models with state dependent service are the controlled queues. Only for some of these problems is it possible to derive explicit expressions for the stationary probabilities or other interesting quantities. In this paper a numerical procedure is given for a rather general class of these problems. The most important restrictions are that the arrivals have to occur singly and that the service mechanism is negative exponential. State dependent service and arrival rates and batch service, or other types of nonindividual service are allowed. In these cases the stochastic process $X(t)$ of the number of customers at time t is a regenerative process (see Ross [5]). The cycle is the epoch between two successive times that an arriving customer finds the system empty. The ratio of the expected cycle cost, EC, to the expected cycle time, ET, is in general equal to the average cost. Let $c(n)$ be the expected cost per time in state n. If we set $c(i) = 0$ for $i \neq k$ and $c(k)$ equal to the expected time in state k (until the first transition), then the average "cost" EC/ET, is equal to the stationary probability that the system is in state k. The cycle cost can be interpreted as the total costs in a Markov chain with cost which is stopped as soon as state 0 is reached. In the next section we shall develop an algorithm for determining the expected cycle cost. Since the arrivals occur individually the embedded Markov process which describes the number of customers at the moments of arrival can only jump one step to the right. This almost triangularity of the (embedded) Markov matrix is used in the construction of the algorithm. If there is some integer q such that the jumps to the left can not be larger than q it is easy to give an upper bound for the error which can be made if the algorithm is stopped after n steps.

A special case where analytical results are known is that of a queue with Poisson arrivals, state dependent batch service, infinitely many servers, and negative exponential service time. This queue is equivalent to an inventory system with Poisson arrival of customers, an (s, q) order strategy and negative exponentially distributed lead time (see also Wijngaard and van Winkel [6]). To illustrate the algorithm, the calculation of the expected length of the busy period in this model is given as a numerical example. Galliher, Morse and Simond [1] derive explicit expressions for the stationary probabilities. The direct calculation of the expected length of the busy period, using the algorithm, developed here appears to be as fast as the calculation based on these explicit expressions. The advantage of the direct method is that it can be used for arbitrary state dependent cases.

* Accepted by Marcel F. Neuts; received September 1, 1976. This paper has been with the author 10 months for 2 revisions.

† Eindhoven University of Technology.

0025-1909/78/2413/1441$01.25

2. The Model

Let (P, c) be a Markov chain with cost on the nonnegative integers, denoted by $[0, \infty)$ (P is a Markov chain and c a nonnegative function on $[0, \infty)$. The transition probabilities are denoted by P_{ij}. Let f be an arbitrary real valued function on $[0, \infty)$, the function Pf is defined by $(Pf)(n) = \sum_{i=0}^{\infty} P_{ni} f(i)$, if this sum exists. If c is interpreted as the expected cost in the first period, then $P^{k-1}c$ gives the expected cost in the kth period and $\sum_{k=0}^{\infty} P^k c$ the total expected cost. The following assumptions are made:

(i) $P_{ni} = 0$ for $i > n + 1$ and for all $n \in [0, \infty)$.
$P_{nn+1} > 0$ for all $n \geqslant 1$.
$P_{00} = 1$, $c(0) = 0$.

(ii) There is integer N and positive numbers a and b such that $P_{nn+1} \leqslant a$ and $P_{nn} \leqslant 1 - 2a - b$ for $n \geqslant N$.

(iii) There is an $n_0 \geqslant N$ such that, starting in n_0, it is possible to reach state 0 without visiting $[n_0, \infty)$. That means, there is an integer k and a sequence $n_0, n_1, n_2, \ldots, n_k = 0$ with $n_i < n_0$ for $i = 1, \ldots, k-1$ and $P_{n_i n_{i+1}} > 0$ for $i = 0, \ldots, k-1$.

(iv) There is an r, $a/(a+b) < r < 1$, such that $r^n c(n)$ is bounded on $[0, \infty)$.

The assumption (ii) implies that for n large enough the probability of a jump to the right is smaller than the probability of a jump to the left. The difference is at least b.

Condition (iii) is satisfied if the state 0 is reachable from the state N (if there is some path from N to 0).

The state 0 is absorbing and the cost here is 0. We are interested in the total expected cost starting in some state n, $\sum_{l=0}^{\infty} (P^l c)(n)$.

In Lemma 1, §5 the existence of this sum is proved and it is shown that $r^n \cdot \sum_{l=0}^{\infty} (P^l c)(n)$ is bounded in n.

For convenience we shall denote the space of all real valued functions f on $[0, \infty)$ such that $r^n \cdot f(n)$ is bounded on $[k, \infty)$ by B_r^k and $\sup_{n \geqslant k} |r^n f(n)|$ by $\|f\|_k$.

3. Approximation of the Total Costs

Let $v := \sum_{l=0}^{\infty} P^l c$, then for $n \in [0, \infty)$

$$v(n) = c(n) + \sum_{i=0}^{n+1} P_{ni} v(i) = c(n) + P_{nn+1} v(n+1) + \sum_{i=0}^{n} P_{ni} v(i), \tag{1}$$

or

$$v(n+1) - v(n) = \frac{1}{P_{nn+1}} \left\{ \sum_{i=0}^{n-1} P_{ni} (v(n) - v(i)) - c(n) \right\}. \tag{2}$$

Since 0 is an absorbing state and $c(0) = 0$, $v(0) = 0$. Let $u(n) := v(n+1) - v(n)$, then for $n \geqslant 1$

$$u(n) = \frac{1}{P_{nn+1}} \sum_{k=0}^{n-1} u(k) \sum_{i=0}^{k} P_{ni} - \frac{c(n)}{P_{nn+1}}. \tag{3}$$

Hence $u(n)$ satisfies the following equation in x,

$$x(n) = \frac{1}{P_{nn+1}} \sum_{k=0}^{n-1} x(k) \sum_{i=0}^{k} P_{ni} - \frac{c(n)}{P_{nn+1}}, \qquad n \geqslant 1. \tag{4}$$

Each solution of this equation is determined by $x(0)$. Let f be the solution with

$f(0) = 0$ and let g be the solution of the homogeneous equation

$$x(n) = \frac{1}{P_{nn+1}} \sum_{k=0}^{n-1} x(k) \sum_{i=0}^{k} P_{ni}$$

with $g(0) = 1$. Then the general solution of (4) is $x = f + dg$ with d a constant. Since u is a solution of (4), $u = f + d_u g$ with $d_u = v(1)$. For g we have

$$g(n) = \frac{1}{P_{nn+1}} \sum_{k=0}^{n-1} g(k) \sum_{i=0}^{k} P_{ni} \geqslant \frac{g(n-1)}{P_{nn+1}} \sum_{i=0}^{n-1} P_{ni}. \tag{5}$$

Hence, for $n \geqslant N$, $g(n) \geqslant g(n-1)\,(a+b)/a$.

The boundedness of $v(n)r^n$ implies $v(n) \leqslant \|v\|_0 / r^n$, hence

$$|u(n)| \leqslant \max(v(n+1), v(n)) \leqslant \|v\|_0 r^{n+1}.$$

Since $r > a/(a+b)$ and $g(n)$ is positive for $n \geqslant n_0$ (see (5) and condition (iii)) this implies

$$\lim_{n\to\infty} \frac{u(n)}{g(n)} = 0. \tag{6}$$

This result can be used in the construction of an algorithm to determine v.

We have $v(n) = \sum_{i=1}^{n-1} u(i) + v(1)$ and $u(n) = f(n) + v(1) \cdot g(n)$. The functions $f(\cdot)$ and $g(\cdot)$ can be easily calculated recursively. The only difficulty is the determination of $v(1)$. But (6) implies $f(n)/g(n) + v(1) = u(n)/g(n) \to 0$, hence

$$v(1) = \lim_{n\to\infty} \frac{-f(n)}{g(n)}. \tag{7}$$

That means that for n large enough the quantity $-f(n)/g(n)$ can be used as approximation for $v(1)$. The error which is made is $|u(n)|/g(n)$ which is less than or equal to $\|v\|_0 / g(n) r^{n+1}$. It is rather difficult in general to evaluate $\|v\|_0$. But if the following extra condition is satisfied it is possible to calculate simultaneously $f(n)$, $g(n)$ and an upper bound for $|f(n)/g(n) + v(1)|$, the error of $-f(n)/g(n)$ as approximation for $v(1)$.

(v) There is an integer $q > 0$ such that for all n the probability $P_{ni} = 0$ for all $i < n - q$.

In the rest of this section we assume that condition (v) is satisfied.

Define $f^*(n) := f(n)/g(n)$ for all $n \geqslant n_0$. Let for each $k \geqslant n_0 + q$ the integers k_h and k_l be chosen such that $f^*(n)$ attains its maximum on $[k-q, k-1]$ in k_h and its minimum on k_l.

In Lemma 2, §5 it is shown that $v(1)$ satisfies the following inequality for all $k \geqslant n_0 + q$.

$$-f^*(k_h) \leqslant v(1) \leqslant -f^*(k_l) + \frac{\|c\|_0}{g(k-1)r^{k-1}} \cdot \frac{1}{(a+b)r - a}. \tag{8}$$

This yields the following algorithm for the determination of $v(1)$.

Choose an $\epsilon > 0$. Calculate $f(k)$ and $g(k)$ for $k = 1, 2, 3, \ldots$ recursively. For $k \geqslant n_0 + q$ one has to calculate simultaneously $f^*(k_h)$ and $f^*(k_l)$. As soon as

$$f^*(k_h) - f^*(k_l) + \frac{\|c\|_0}{g(k-1)r^{k-1}} \cdot \frac{1}{(a+b)r - a} \leqslant \epsilon$$

one stops calculation and takes the actual value of $-f^*(k)$ as approximation for $v(1)$. The error is not larger than ϵ.

4. Numerical Example

To illustrate the method it is applied to the determination of the "busy period" in the inventory model mentioned in the introduction. This inventory model consists of the following assumptions:

—customers arrive according to a Poisson process with parameter λ, the demand per customer is one.

—unfulfilled demand is backlogged.

—orders for replenishing the inventory are placed as soon as the inventory on hand plus on order minus backorders reaches a level s. The lead-time is exponentially distributed with mean $1/\mu$.

Let n be the difference between $s + q$ and the net inventory (inventory on hand minus backorders), n represents the state of the system.

The number of outstanding orders in state n is $[n/q]$ ($[x]$ is the largest integer smaller than or equal to x). Hence the probability that the first event, starting in state n, is the arrival of a customer is $\lambda/(\lambda + [n/q]\mu)$ and $[n/q]\mu/(\lambda + [n/q]\mu)$ is the probability that the first event is the arrival of an order. That means that the behaviour of n from event to event between two visits to state 0 can be described by a Markov chain with

$$P_{nn-q} = \frac{[n/q]\mu}{\lambda + [n/q]\mu} \quad \text{and} \quad P_{nn+1} = \frac{\lambda}{\lambda + [n/q]\mu} \quad \text{for } n = 1, 2, 3, \ldots .$$

Let $c(n)$ be equal to the expected time in state n until the first event, $c(n) = 1/(\lambda + [n/q]\mu)$ for $n = 1, 2, \ldots$. If we add $P_{00} = 1$ and $c(0) = 0$ then $\sum_{l=0}^{\infty}(P^l c)(n)$ is equal to the expected time until the first visit to state 0, starting in state n. It is easy to see that all conditions (i), . . . , (v) are satisfied in this case.

In Table 1 for $\lambda = 2$, $\mu = 1$ the values of $g(n)$, $f(n)$, $-f^*(n)$ are given for $n = 9, \ldots, 19$, together with an upper bound of $|f^*(n) + v(1)|$.

TABLE 1

n	$g(n)$	$f(n)$	$-f^*(n)$	upper bound
9	9.19	− 46.25	+ 5.034014	1.467643
10	21.28	− 108.13	+ 5.080764	0.380952
11	50.58	− 256.81	+ 5.077541	0.164224
12	162.09	− 822.88	+ 5.076537	0.076407
13	467.91	− 2376.13	+ 5.078207	0.013480
14	1361.16	− 6912.13	+ 5.078128	0.004876
15	4977.89	− 25278.31	+ 5.078117	0.002527
16	17017.38	− 86416.91	+ 5.078155	0.000325
17	58391.07	− 296518.86	+ 5.078154	0.000106
18	241159.04	− 1224642.73	+ 5.078154	0.000055
19	949702.50	− 4822736.00	+ 5.078154	0.000005

5. Proofs

In this section the results used in the preceding sections are proved.

LEMMA 1. *Let the conditions* (i), . . . , (iv) *be satisfied. Then the total expected cost, starting at n, $\sum_{l=0}^{\infty}(P^l c)(n)$ exists and the function $\sum_{l=n_0}^{\infty} P^l c$ is an element of B_r^0.*

PROOF. First the expected cost until the first visit to $[0, n_0 - 1]$ is considered. Let $\tilde{P}f$ for f an arbitrary function on $[n_0, \infty)$ be given by $(\tilde{P}f)(n) = \sum_{i=n_0}^{\infty} P_{ni} f(i)$.

Then $\sum_{l=0}^{\infty}(\tilde{P}^l c)(n)$, if existing, is equal to the expected cost starting in n until the first visit to $[0, n_0 - 1]$. By assumption (iv) the function c is an element of $B_r^{n_0}$. Let f be an arbitrary element of $B_r^{n_0}$, then for $n \geqslant n_0$

$$\begin{aligned} r^n|(\tilde{P}f)(n)| &= r^n\Big|\sum_{i=n_0}^{\infty} P_{ni}f(i)\Big| \leqslant r^n \sum_{i=n_0}^{\infty} P_{ni}\frac{\|f\|n_0}{r^i} \\ &= \|f\|_{n_0}\left(\frac{P_{nn+1}}{r} + P_{nn} + \sum_{i=n_0}^{n-1} P_{ni}r^{n-i}\right) \\ &\leqslant \|f\|_{n_0}\left\{r + (1-r)\left(P_{nn} + \frac{1+r}{r}P_{nn+1}\right)\right\} \\ &\leqslant \left\{r + (1-r)\left(1 - 2a - b + \frac{1+r}{r}a\right)\right\}\|f\|_{n_0}. \end{aligned}$$

Hence $\tilde{P}f$ is also an element of $B_r^{n_0}$ and $\|\tilde{P}f\|_{n_0} \leqslant \rho\|f\|_{n_0}$ with ρ defined by

$$\rho := r + (1-r)\left(1 - 2a - b + \frac{1+r}{r}a\right) \qquad \left(= 1 - (1-r)\left(a + b - \frac{a}{r}\right)\right).$$

Since $1 > r > a/(a+b)$ the constant ρ is between 0 and 1 and therefore $\sum_{l=0}^{\infty}\tilde{P}^l f$ is also an element of $B_r^{n_0}$ with

$$\Big\|\sum_{l=0}^{\infty}\tilde{P}^l f\Big\|_{n_0} \leqslant \frac{1}{1-\rho}\|f\|_{n_0}.$$

Hence

$$\sum_{l=0}^{\infty}(\tilde{P}^l c)(n) \leqslant \frac{1}{r^n}\,\frac{1}{1-\rho}\|c\|_{n_0} \quad \text{for } n \geqslant n_0.$$

Once in $[0, n_0 - 1]$ there is a positive probability of reaching state 0 without coming again in $[n_0, \infty)$. Therefore the total cost starting in n exist for each n and $\sum_{l=0}^{\infty}\tilde{P}^l c - \sum_{l=0}^{\infty}P^l c$ is bounded on $[n_0, \infty)$. This implies $\sum_{l=0}^{\infty}P^l c \in B_r^0$. Q.E.D. More general results of this type can be found in van Hee and Wessels [2].

LEMMA 2. *Let the conditions i . . . v be satisfied. Then for all* $k \geqslant n_0 + q$

$$-f^*(k_h) \leqslant v(1) \leqslant -f^*(k_l) + \frac{\|c\|_0}{g(k-1)r^{k-1}} \cdot \frac{1}{(a+b)r - a}.$$

PROOF. Define $c^*(n) = c(n)/P_{nn+1}g(n)$ for $n \geqslant n_0$. Then for $n \geqslant n_0 + q$ (by the definition of $f(n)$ and $f^*(n)$)

$$\begin{aligned} f^*(n) &= \frac{1}{g(n)P_{nn+1}} \sum_{1=0}^{n-1} f(l) \sum_{i=n-q}^{l} P_{ni} - c^*(n) \\ &= \frac{1}{g(n)P_{nn+1}} \sum_{l=0}^{n-1} \frac{f(l)}{g(l)} g(l) \sum_{i=n-1}^{l} P_{ni} - c^*(n). \end{aligned}$$

If P^*_{nl} is defined by

$$P^*_{nl} = \frac{g(l)\sum_{i=n-q}^{l} P_{ni}}{g(n)P_{nn+1}} \qquad \text{for } n, l \geqslant n_0,$$

then

$$\sum_{k=n-q}^{n-q} P^*_{nl} = \frac{\sum_{l=n-q}^{n-1} g(l) \sum_{i=n-q}^{l} P_{ni}}{g(n)P_{nn+1}} = 1$$

and

$$f^*(n) = \sum_{l=n-q}^{n-1} f^*(l)P^*_{nl} - c^*(n) \quad \text{for } n \geqslant n_0 + q.$$

Hence $f^*(n)$ for $n \geqslant n_0 + q$ can be interpreted as the total expected cost, starting in n, of a Markov process with transition probabilities P^*_{ij} which is stopped as soon as a state l with $l < n_0 + q$ is reached. In each state $l \geqslant n_0 + q$ the cost is $-c^*(l)$, the cost of stopping the process in $l < n_0 + q$ is $f^*(l)$. It is easy to see now that for $n > k \geqslant n_0 + q$ the cost $f^*(n)$ can be written as $f^*(n) = C_{nk} + R_{nk}$, where C_{nk} is the expected cost until a state $l < k$ is reached and R_{nk} is the rest of the cost. Let $P_l(n)$ be the probability that l is the first state reached in $[0, k-1]$, then

$$R_{nk} = \sum_{l=k-q}^{k-1} P_l(n)f^*(l).$$

Hence

$$f^*(n) = C_{nk} + \sum_{l=k-q}^{k-1} P_l(n)f^*(l) \quad \text{and} \quad \sum_{l=k-q}^{k-1} P_l(n)\{f^*(n) - f^*(l)\} = C_{nk}.$$

This implies

$$f^*(n) - f^*(k_h) \leqslant C_{nk} \quad \text{and} \quad f^*(n) - f^*(k_l) \geqslant C_{nk}.$$

The process moves in each transition at least one step to the left, hence

$$0 \geqslant C_{nk} \geqslant -\sum_{j=k}^{n} c^*(j).$$

Since $v(1) = -\lim_{n\to\infty} f^*(n)$ this gives the following bounds on $v(1)$

$$-f^*(k_h) \leqslant v(1) \leqslant -f^*(k_l) + \sum_{j=k}^{\infty} c^*(j). \tag{1}$$

It is not difficult to give an upper bound on $\sum_{j=k}^{\infty} c^*(j)$. We had

$$c^*(j) = \frac{c(j)}{P_{jj+1}g(j)} = \frac{c(j)}{\sum_{l=0}^{j-1} g(l) \sum_{i=0}^{l} P_{ji}} \leqslant \frac{c(j)}{g(j-1) \sum_{i=0}^{j-1} P_{ji}}.$$

Let k be chosen such that $k > n_0$ (see the proof of Lemma 1), then

$$c^*(j) \leqslant \frac{c(j)}{g(j-1)(a+b)} \quad \text{for } j \geqslant k$$

while

$$g(j) \geqslant g(k-1)\left(\frac{(a+b)}{a}\right)^{j+1-k} \quad \text{for } j \geqslant k.$$

Since $r^n c(n)$ is bounded in n and $r > a/(a+b)$ this implies

$$\sum_{j=k}^{\infty} c^*(j) \leqslant \frac{\|c\|_0}{g(k-1)r^{k-1}} \; \frac{1}{(a+b)r-a}\,.$$

Together with (1) this completes the proof. Q.E.D.

If $q = 1$ and $P_{nn-1} > 0$ for all $n > 0$ then $g(n) > 0$ for all n and $P^*_{nn-1} = 1$ for $n > 0$. Hence $f^*(n) = f^*(n-1) - c^*(n)$ for all $n > 0$ and

$$v(1) = \sum_{1}^{\infty} c^*(n) = \sum_{1}^{\infty} \frac{c(n)}{g(n)P_{nn+1}}\,,$$

where $g(\cdot)$ is given by $g(n) = P_{nn-1}g(n-1)/P_{nn+1}$. This corresponds of course with the results given by Karlin [3, p. 204].

References

1. Galliher, H. P., Morse, P. M. and Simond, H., "The Dynamics of Two Classes of Continuous-Review Inventory Systems," *Operations Res.*, Vol. 7 (1959), pp. 362–384.
2. van Hee, K. M. and Wessels, J., "Markov Decision Processes and Strongly Excessive Functions," *Stochastic Proc. Appl.* (to appear).
3. Karlin, S., *A First Course in Stochastic Processes*, Academic Press, London, 1966.
4. Rosenshine, M., "Queueing Theory: The State of the Art," *AIIE Transactions*, Vol. 7 (1975), pp. 257–267.
5. Ross, S. M., *Applied Probability Models with Optimization Applications*, Holden-Day, San Francisco, Calif., 1970.
6. Wijngaard, J. and van Winkel, E. G. F., "Average Costs in a Continuous Review (s, S) Inventory System with Exponentially Distributed Leadtime, *Operations Res.* (to appear).