

# Finite element method implementation of complex envelope displacement analysis

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# Finite Element Method implementation of Complex Envelope Displacement Analysis

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# Summary

The Finite Element Method has become a widely used numerical tool in the field of mechanical engineering. The analysis of linear dynamic systems with deterministic loads, boundary conditions and material parameters has become a routine job. However, the analyses are restricted in the maximum wave number that can be taken into account. Despite the ever increasing computer power, high wave number solutions demand an unacceptably large number of degrees of freedom. Above that, the responses of dynamic systems at high frequencies are strongly dependent on, for example, the position of the excitation and the material parameters. This makes any deterministic calculation meaningless.

As a solution to these problems the applicability of envelope methods is under investigation. Most of the research on envelope methods has been carried out by Carcaterra and Sestieri. The last and final model developed is the *Complex Envelope Displacement Analysis* model that transforms a differential equation in a way that it solves for a smooth low wave number variable. The aim of this research project is to bring CEDA closer to application as an engineering tool.

In engineering practice the Finite Element Method is an appropriate tool for the analysis of vibration problems. Therefore, this study concentrates as a first step on the implementation of CEDA in a Finite Element Method formulation. Implementation of CEDA in a FEM formulation results in a tool that is named FEM-CEDA. At this point it is restricted to one-dimensional systems. In FEM-CEDA physical variables are transformed to CEDA field variables, theoretically over an infinite domain. To be able to solve the problem, this implies that the governing equations have to be discretised on a domain that extends beyond the physical domain. Application of the boundary conditions causes extra problems for the same reason. This problem is solved by a non standard inclusion of the boundary conditions for the displacements, as well as the boundary forces.

A numerical implementation of both damped and undamped systems is derived and tested. The element matrices of the transformed problem can be determined in the same way as in classical FEM. With a sufficiently fine discretisation the solution that is obtained with FEM-CEDA exactly equals the solution that is obtained with classical FEM techniques. It appears that the possibility of element reduction is not as good as expected since field variables on a finite domain are not sufficiently band-limited. To improve the possibility of element reduction, the effect of filtering and the application of non equidistant meshes have been investigated but did not lead to a satisfactory improvement.



# Chapter 1

## Introduction

One of the most important numerical tools in the field of mechanical engineering of the last decades is the Finite Element Method. Nowadays numerous FEM-based software packages are available. The principle of the Finite Element Method can be explained with the following three steps.

1. The domain that is considered is subdivided in a finite number of elements.
2. The (partial) differential equations that are valid on the domain are rewritten to linear algebraic equations that describe the solution on element level.
3. The elements are assembled to result in a set of linear algebraic equations that can be solved numerically.

### 1.1 Problem description

The use of these packages has also become common practice in engineering dynamics. The analysis of linear dynamic systems with deterministic loads, boundary conditions and material parameters has become a routine job. However, in high frequency dynamics two problems can be distinguished.

First, the analyses are restricted in the maximum wave number (spatial frequency) that can be taken into account. Despite the ever increasing computer power, high wave number solutions demand an unacceptably large number of degrees of freedom (the computational cost of solving the equations is of the order of  $N^3$  floating point operations, where  $N$  is the number of degrees of freedom). A solution may be found in techniques that reduce the number of degrees of freedom. Reduction techniques however often imply a loss of information.

Second, the responses of dynamic systems at high frequencies are strongly dependent on, for example, the position of the excitation and the material parameters, making calculation of a deterministic response meaningless. Stochastic methods become necessary, but some stochastic methods, like the Monte-Carlo method, use a combination of deterministic calculations and therefore imply high computational cost.

A solution that meets both problems would be a substantial reduction in the number of elements or degrees of freedom. Possible solutions are summarised in a literature survey by Raaymakers (1995a). A frequently applied method is Statistic Energy Analysis (SEA). In this method the systems' properties are averaged over a frequency or wave number band, where the assumption is made that the systems' parameters and boundary conditions are random variables. This method implies a loss of information since averaging techniques are used. Above that, there is a lower frequency limit for



significant SEA results. SEA can be used in vibro-acoustics when one is only interested in the average energy or displacement on the domain, represented by a single scalar quantity. It does however not contain information on how the response is distributed over the domain.

Other attempts have been made to deal with high frequencies using 'thermal analogies'. The analogy is based on the energy flow and can be used for longitudinal vibrations (under the assumption of weak damping). For flexural vibrations the thermal analogies do not hold but an estimate can be obtained by neglecting the near field solution. These methods are possibly covering the medium frequency gap between deterministic and statistical approaches but are not investigated in this research project.

Other methods that may cover the mid and high frequency range are envelope methods. A major advantage of these methods is that they are based on the same physical laws and properties that are used in low wave number deterministic calculations. To deal with the stochastic properties of the system, a stochastic method has to be used in addition. An important property of this method is that the distribution of the response on the domain is not averaged to one scalar measure or quantity.

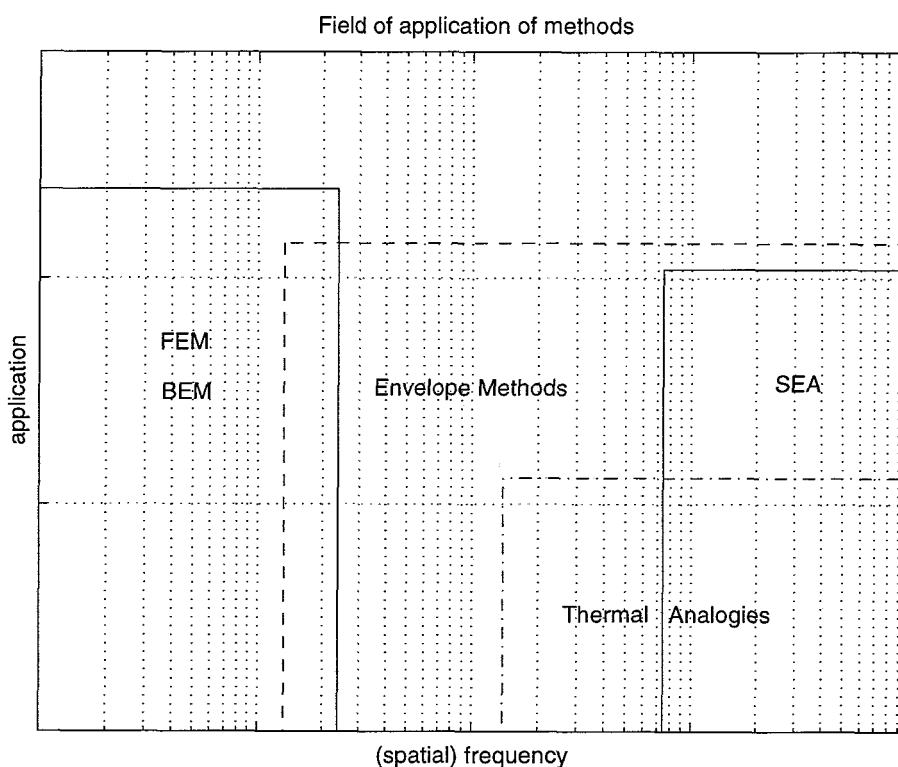


Figure 1.1: Overview of the field of application of methods. Finite and Boundary Element Method (FEM, BEM) techniques in the lower frequency band, Statistic Energy Analysis (SEA) in the upper frequency band, and Thermal analogies and Envelope methods covering the medium and high frequency range.

In figure 1.1 a schematic overview of the field of application of the methods mentioned above is given. FEM and BEM techniques cover the lower frequency band, and SEA covers the upper frequency band. Thermal analogies possibly cover the medium frequency gap. Envelope methods are the most promising alternative in the medium and high frequency band.

## 1.2 Envelope methods

Most of the research on envelope methods has been carried out by Carcaterra and Sestieri. A complete reference list is given in the bibliography. In the envelope models attempts are made to describe the envelope trend of some field variable (energy or displacement). The envelope is obtained by an appropriate use of the Hilbert transform. Three different envelope models have successively been developed:

*The Envelope Energy Model (EEM)*, where the kinetic energy was used. This model however was not capable of describing energy jumps at discontinuities. See Carcaterra & Sestieri (1994b, 1995e).

*The Envelope-Phase Energy Model (EPHEM)*, that considered the envelope energy and the displacement phase. This model solved the problems related with the energy jumps. However the model was rejected because of the complicated model equations. See Carcaterra (1994); Carcaterra & Sestieri (1995f).

*Complex Envelope Displacement Analysis (CEDA)* is the last and final model that is introduced by Carcaterra and Sestieri. The basic idea of this formulation consists of defining a new field variable, related to the physical displacement by a one-to-one correspondence, that has the property (under certain conditions) that most of its energy is concentrated in the low wave number region. See Carcaterra & Sestieri (1995b, 1997).

In a preceding research project the applicability of CEDA has also been investigated by Raaymakers (1995b).

## 1.3 Goal of this project

The state of the envelope methods presented by Carcaterra and Sestieri has not the level of an engineering tool. The goal of the research project described in this report is to make a new contribution to CEDA, that brings it closer to application as an engineering tool. With this desired engineering tool it will be possible to estimate the stochastic properties of the response of linear dynamic systems.

To reach this goal, two steps have to be taken. The first one is to implement the CEDA theory in a discrete formulation for deterministic calculations. The Finite Element Method (FEM) is chosen as discretisation technique and the result will be called FEM-CEDA. The second step is to add a stochastic method to it. In the research project that is described in this report only the first step is dealt with. Towards the application of CEDA as an engineering tool, several problems must still be solved:

- implementation of CEDA in a FEM formulation (or another discretisation technique);
- the problems with the unknown CEDA boundary conditions;
- application to damped systems;
- the occurrence of spurious high wave number solutions, mentioned by Carcaterra and Sestieri;
- application to higher order and more dimensional systems;
- extension to statistical systems, as to-date, the method is limited to deterministic problems.

The new developments, which are reported here, concern the first four items of the above list. Discussion on the fifth item is also included in this report. The final item is subject to future research.

## 1.4 Overview of the report

For implementation of the CEDA theory in a FEM formulation a thorough understanding of the CEDA concept is necessary. The CEDA theory is explained in chapter 2. First, the signal transformations are explained and second, the application of the transformations to differential equations is addressed. In chapter 3 the application of the Finite Element Method to the CEDA transformed differential equations is discussed and illustrated with the derivation of element matrices for a longitudinal vibrating beam. Special attention is paid to the boundary conditions. In chapter 4, numerical examples of the application of FEM-CEDA to damped and undamped longitudinal vibrating beam problems are given. From the results can be concluded that FEM-CEDA can be used to analyse one-dimensional high frequency dynamics problems, but the anticipated computational reduction was not achieved. To improve the results, the application of filtering and non-equidistant meshes is investigated. A general discussion on the applicability of FEM-CEDA as an engineering tool is presented in chapter 5. Chapter 6 contains conclusions and recommendations for future research.

## Chapter 2

# The CEDA concept

The complex envelope displacement theory is based on a variable transformation of high wave number variables into low wave number variables. The difference between high and low wave number variables is not given by a low or high wave number band limit. High and low are only used relative to each other <sup>1</sup>. The CEDA transformations can be performed on a variable if it meets certain criteria.

1. The variable has to be real, because the Hilbert transformation and the analytic signal are only defined for real signals. When complex signals (for instance when damping is present) are taken into account, the variables can be split in real and imaginary components and the transformations can be performed on both separately.
2. The variable has to be band limited. Most of its energy has to be concentrated in the high wave number region. The transformation shifts this energy contents to the low wave number region.

Signals that meet these criteria are sine or cosine wave forms. These are real signals with all energy concentrated in one wave number. With the use of the Fourier transform practically all signals can be decomposed in a combination of sine and cosine waves and thus CEDA can be applied to practically all signals. The second criterion can be met by dividing up signals that have a wide frequency range in small frequency bands as suggested in Carcaterra & Sestieri (1997).

A typical solution of a vibration problem is given in the upper left figure of figure 2.1. It is a longitudinal standing wave pattern in a beam that is clamped at both ends and excited with an harmonic oscillating force at  $x = 0.41$  m. It has a wave number spectrum that resembles that of a sine wave but it is not concentrated in one wave number.

The analytic form of this signal can be defined as a complex signal, whose real part is the original signal and whose imaginary part is the Hilbert transform of the original signal. The resulting analytic signal has a wave number spectrum that only has non-zero components for positive wave numbers, since for negative wave numbers the spectrum of the Hilbert transform is opposite to that of the original signal. Most of the energy of the signal is concentrated in the high wave number region.

When a wave number shift is performed most of the energy is concentrated in the low wave number region. The overall trend of the complex envelope displacement is much smoother and can be described by significantly fewer samples. The original signal can be reconstructed easily after the inverse wave number shift.

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<sup>1</sup>In high frequency dynamics, computer simulations will however be limited by an upper wave number limit, but this limit will be dependent on the problem formulation and the floating point machine accuracy.

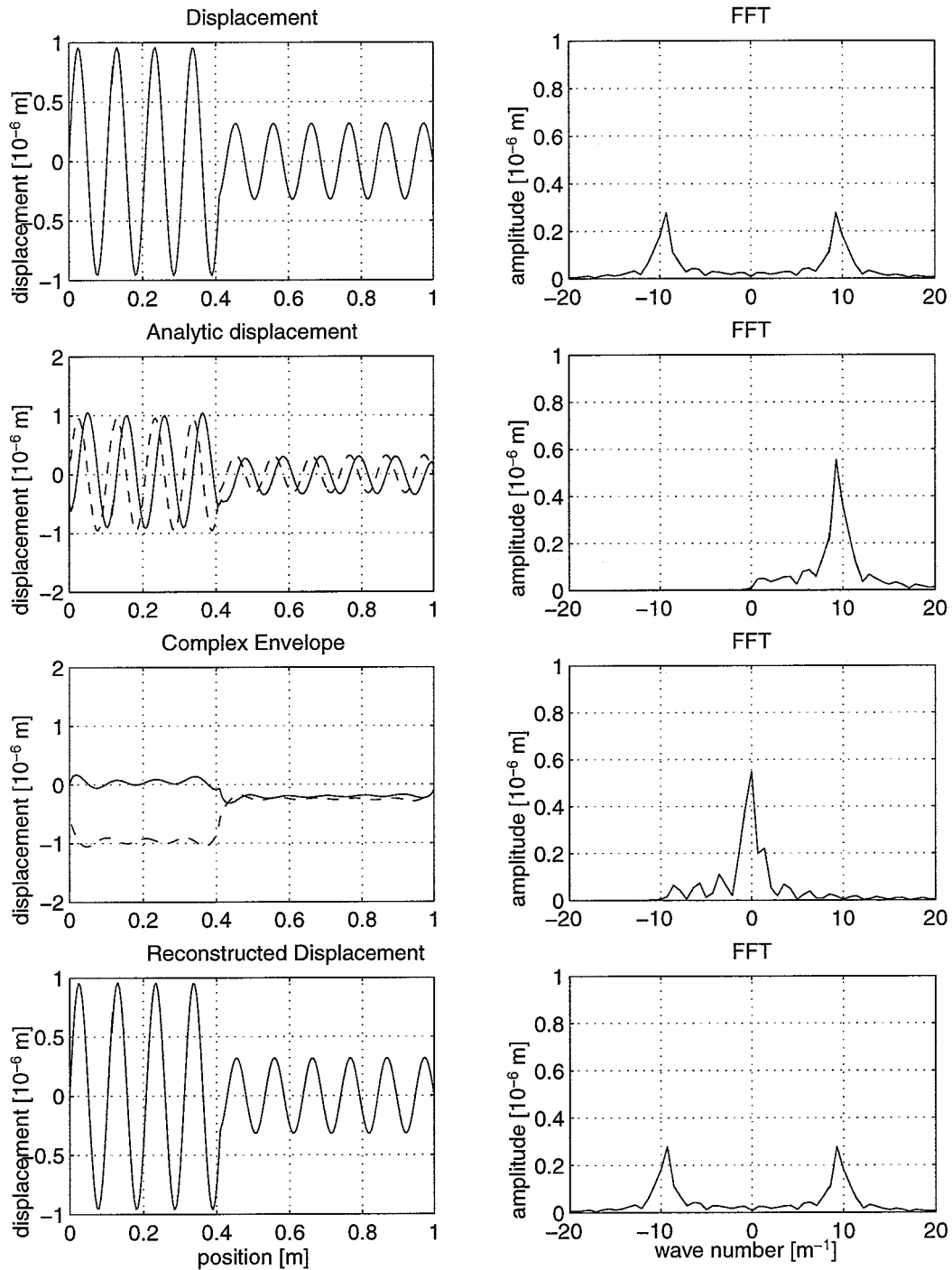


Figure 2.1: Concept of the CEDA transformations. Subfigures (a) to (d) show the displacement variables on the left and the corresponding wave number spectra on the right. (a) A high wave number signal. (b) The analytic signal consists of the original signal (dashed) and its Hilbert transform. (c) The complex envelope displacement. Real (-) and imaginary (- -) parts. (d) The reconstructed signal.

## 2.1 The CEDA Transformations

The CEDA transformations are explained here in more detail. The Hilbert transform  $\tilde{u}(x)$  of a variable  $u(x)$  is defined in Bracewell (1965) as the convolution of  $u(x)$  with  $1/\pi x$ .

$$\tilde{u}(x) = \frac{1}{\pi x} * u(x) = \int_{-\infty}^{\infty} \frac{u(\xi)}{\pi(x - \xi)} d\xi. \quad (2.1)$$

In the wave number domain the Hilbert transform can be written as

$$\tilde{U}(s) = j \operatorname{sign}(s) U(s), \quad (2.2)$$

and corresponds to a filter operation where the amplitudes are unchanged but the phase is increased by  $\pi/2$  for positive wave numbers and decreased by  $\pi/2$  for negative wave numbers. The analytic signal  $\hat{u}(x)$  can now be defined as:

$$\hat{u}(x) = u(x) + j\tilde{u}(x). \quad (2.3)$$

In the wave number domain this results in a wave number spectrum that equals zero for negative wave numbers. For positive wave numbers the spectrum is related to the spectrum of the original variable by a factor two. In a numerical implementation of the analytic signal this property can be used for a fast computation of the analytic signal and the Hilbert transform. For example in MATLAB the analytic signal is computed by:

1. Fast Fourier Transformation of the variable,
2. multiplication of the positive wave number spectrum by a factor two and setting the negative wave number spectrum to zero,
3. Inverse Fourier Transformation.

The Complex Envelope  $\overleftarrow{u}$  is derived by multiplying the analytic signal  $\hat{u}$  by  $e^{-jk_s x}$ . This operation can be interpreted as a frequency modulation <sup>2</sup>.

$$\overleftarrow{u}(x) = \hat{u}(x) e^{-jk_s x}. \quad (2.4)$$

The notation with the left arrow is chosen according to Carcaterra and Sestieri, to represent the shifting of the wave number pattern to the left towards the origin. When  $u(x)$  is a cosine wave with wave number  $k$  then the Complex Envelope  $\overleftarrow{u}$  will equal:

$$\overleftarrow{u}(x) = e^{j(k-k_s)x}. \quad (2.5)$$

This signal has an amplitude equal to 1 and the phase has an angular frequency  $(k - k_s)$ , which is a smooth signal with low wave number when  $k_s$  is chosen close to  $k$ . The property of the low wave number can be very useful in more complex cases where very high sample frequencies are needed. A transformation to the complex envelope would reduce the amount of samples significantly. From the Complex Envelope the original signal can be reconstructed by the inverse transformation:

$$u(x) = \operatorname{Re} \left\{ \overleftarrow{u}(x) e^{jk_s x} \right\}. \quad (2.6)$$

---

<sup>2</sup>Frequency modulation, as used in radio communication, increases the frequency of waves in the audio range (kHz-band) to frequency variations in the (ultra-)short wave range (MHz-band). In CEDA the modulation decreases the wave number contents towards the origin.

The multiplication by  $e^{jk_s x}$  can be interpreted as a frequency demodulation of  $u(x)$  and  $u(x)$  will have higher frequency contents than  $\tilde{u}(x)$ . When working with discrete signals it is therefore necessary to resample  $\tilde{u}(x)$  by an interpolation routine. In appendix A the signal transformations are illustrated in more detail.

## 2.2 Application of CEDA to differential equations

Transformation of variables does not yet solve the problem of the high computational cost of solving equations. The aim is to rewrite the equations in a way that it solves for a low wave number variable. This can be done when the transformations can be applied to the differential equations. Since the CEDA transformations are linear they can be added to linear differential equations of any order easily. This process will be illustrated for the second order differential equation:

$$au''(x) + bu'(x) + cu(x) = f(x) \quad (2.7)$$

where  $u'(x)$  and  $u''(x)$  are the first and second derivatives of  $u$  to  $x$  and  $a$ ,  $b$  and  $c$  are real constants. Since the Hilbert transformation is a linear operation equation (2.7) also holds for the Hilbert transform  $\tilde{u}(x)$  and for the analytic signal  $\hat{u}(x)$ :

$$a\hat{u}''(x) + b\hat{u}'(x) + c\hat{u}(x) = \hat{f}(x). \quad (2.8)$$

When equation (2.4) is written inversely and the derivatives to  $x$  are determined:

$$\hat{u}(x) = \tilde{u}(x) \quad (2.9)$$

$$\hat{u}'(x) = \left[ \tilde{u}'(x) + jk_s \tilde{u} \right] e^{jk_s x} \quad (2.10)$$

$$\hat{u}''(x) = \left[ \tilde{u}''(x) + 2jk_s \tilde{u}' - k_s^2 \tilde{u} \right] e^{jk_s x}, \quad (2.11)$$

equation (2.8) can be rewritten in the complex envelope displacement  $\tilde{u}$ :

$$a\tilde{u}''(x) + [b + 2ajk_s]\tilde{u}'(x) + [c + bjk_s - ak_s^2]\tilde{u}(x) = \tilde{f}(x). \quad (2.12)$$

Note that the exponential term  $e^{jk_s x}$  is factored out at both sides. The differential equation is forced by a complex input signal  $\tilde{f}(x)$  and therefore the solution  $\tilde{u}(x)$  will be complex as well. The parameters of the differential equations are complex too. The equations can be rewritten to a twice as long set of differential equations in real variables with real coefficients by:

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \tilde{u}_{re}'' \\ \tilde{u}_{im}'' \end{bmatrix} + \begin{bmatrix} b & -2ak_s \\ 2ak_s & b \end{bmatrix} \begin{bmatrix} \tilde{u}_{re}' \\ \tilde{u}_{im}' \end{bmatrix} + \begin{bmatrix} c - ak_s^2 & -bk_s \\ bk_s & c - ak_s^2 \end{bmatrix} \begin{bmatrix} \tilde{u}_{re} \\ \tilde{u}_{im} \end{bmatrix} = \begin{bmatrix} \tilde{f}_{re} \\ \tilde{f}_{im} \end{bmatrix}. \quad (2.13)$$

The complex envelope  $\tilde{f}(x)$  of the input  $f(x)$  can be determined, since  $f(x)$  is a known input signal. The solution to these equations is the complex envelope displacement  $\tilde{u}(x)$ . The real displacement  $u(x)$  can be reconstructed with the aid of equation (2.6).

## Boundary conditions

A second order differential equation in real variables needs two real valued boundary conditions, e.g.:

$$\begin{aligned} u(0) &= u_0 \\ u(L) &= u_L. \end{aligned} \quad (2.14)$$

Note that this is a two point boundary condition and not an initial value boundary condition. In CEDA notation two complex boundary conditions (2 real terms plus 2 imaginary terms) are needed. The relation with the boundary conditions of the original problem is given by:

$$\begin{aligned} \operatorname{Re} \left\{ \overleftarrow{u} e^{jk_s x} \right\}_{x=0} &= u_0 \\ \operatorname{Re} \left\{ \overleftarrow{u} e^{jk_s x} \right\}_{x=L} &= u_L \end{aligned} \quad (2.15)$$

Note that there are no restrictions on the imaginary parts.

## 2.3 General CEDA formulation for differential equations

In the previous section the CEDA transformations were illustrated for a second order differential equation. In this section a more generalised description of the transformations is given. The equation of motion for a general one-dimensional undamped structure can be written as:

$$m\omega_0^2 u(x) + \mathbf{L}(u(x)) = p(x), \quad (2.16)$$

where  $\mathbf{L}$  is a differential operator,  $u(x)$  is the displacement and  $p(x)$  is the external load, assumed to be harmonic in time with frequency  $\omega_0$ . The CEDA transformations can be written as:

$$\overleftarrow{u}(x) = [u(x) + j\mathbf{H}(u(x))]e^{-jk_s x}, \quad (2.17)$$

where  $\mathbf{H}$  is the Hilbert operator. The complex envelope operator  $\mathbf{E}$  is now defined as:

$$\mathbf{E}(\cdot) = [\mathbf{I}(\cdot) + j\mathbf{H}(\cdot)]e^{-jk_s x}, \quad (2.18)$$

where  $\mathbf{I}$  is the identity operator. The inverse operator  $\mathbf{E}^{-1}$  is:

$$\mathbf{E}^{-1}(\cdot) = \operatorname{Re} \left\{ (\cdot) e^{jk_s x} \right\}. \quad (2.19)$$

The CEDA transformations can now be applied to the general differential equation (2.16), which leads to:

$$m\omega_0^2 \overleftarrow{u}(x) + \mathbf{E} \left\{ \mathbf{L}[\mathbf{E}^{-1}(\overleftarrow{u}(x))] \right\} = \overleftarrow{p}(x), \quad (2.20)$$

or:

$$m\omega_0^2 \overleftarrow{u}(x) + \overleftarrow{\mathbf{L}}[\overleftarrow{u}(x)] = \overleftarrow{p}(x), \quad (2.21)$$

where a new complex envelope operator

$$\overleftarrow{\mathbf{L}} = \mathbf{E}\mathbf{L}\mathbf{E}^{-1} \quad (2.22)$$

is introduced.

When the displacement spectrum is not narrow banded, a bandwidth decomposition technique can be developed to obtain an envelope solution that still offers a significant numerical benefit Carcaterra & Sestieri (1997)).





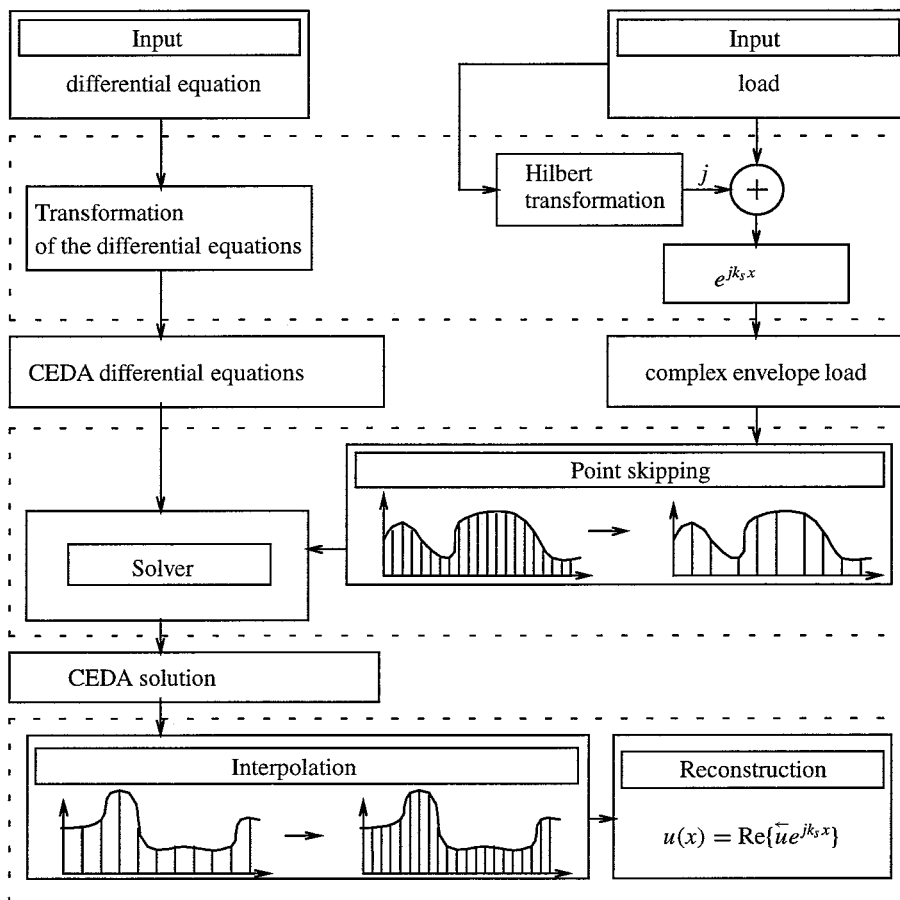


Figure 2.2: Schematic overview of CEDA. The three dashed blocks represent: the transformation of the differential equations, the solution procedure and the reconstruction of the solution.



## Chapter 3

# Discretisation of the CEDA equations

Two point boundary condition problems can be solved with shooting techniques, Finite Difference (FD) techniques or with the Finite Element Method (FEM). FEM is the most general applicable method and is therefore chosen to solve the CEDA equations. The steps involved to solve a problem with FEM techniques are:

1. Subdivide the domain that is considered into a finite number of elements. Dependent on the dimensions of the domain and the type of problem these can be one-, two- or three-dimensional elements with rectangular or triangular shape.
2. In combination with the choice for an element mesh, the type of elements are chosen. Elements can have linear or quadratic interpolation of variables. This results in a different number of nodes on the element.
3. The partial differential equations that are valid on the domain are written in a weighted residual formulation. A Galerkin approach is chosen which means that the interpolation of the weighing functions is chosen equal to the interpolation of the solution.
4. The weighted residual formulation is integrated (with partial integration) and is rewritten to linear algebraic equations that describe the solution on element level.
5. The elements are assembled to result in a set of linear algebraic equations. These equations can be written in a matrix notation as:  $\mathbf{Cu} = \mathbf{F}$ , where  $\mathbf{C}$  can be rearranged to a banded matrix.
6. Boundary conditions are applied to the equations by rearranging the known and unknown variables to have all known variables on the right hand side of the equations and all unknown variables in column  $\mathbf{u}$ .
7. The set of equations is solved numerically. This produces a least squares solution to the equations.

Application of FEM to the CEDA equations follows these steps but involves some special steps as well, which will be described in this chapter.

### 3.1 FEM-CEDA for an undamped longitudinal vibrating beam

A longitudinal vibrating clamped-clamped beam will be used to demonstrate the FEM formulation of the CEDA equations. In appendix B the differential equations and the standard FEM formulation of

this problem is described as a reference. The equation of motion of a longitudinal vibrating beam can be transformed to its CEDA equivalent as explained in the previous chapter. It reads:

$$-\omega^2 \rho A \tilde{u} - EA \left[ \frac{\partial^2 \tilde{u}}{\partial x^2} + 2jk_s \frac{\partial \tilde{u}}{\partial x} - k_s^2 \tilde{u} \right] = \tilde{p}(x, t). \quad (3.1)$$

The seven steps involved to solve a problem with FEM techniques that were mentioned on the previous page will now be dealt with subsequently.

## Mesh

In FEM analyses it is common to adjust the mesh to the problem that is analysed. In a classical FEM analysis of a longitudinal vibrating beam problem an equidistant mesh can be applied since the solution globally has the same shape on the domain. On the contrary, CEDA variables are smooth low wave number signals with discontinuity jumps at specific points like excitation and clamping positions. A non-equidistant mesh with small elements at these jumps and larger elements on the rest of the domain is preferable to reduce the total number of elements. Several strategies to produce a mesh are possible. The mesh can consist of small and large elements or it can have a continuously varying element length. The derivation of the element matrices however, is independent on the mesh.

## Elements

In this theoretical example linear elements are used. A linear beam element has two nodes and the displacement variable is dealt with as a linear interpolation of the displacements at the nodes. The local position coordinate  $\xi$  on an element is used to define the interpolation of  $\tilde{u}(x)$ , with  $-1 < \xi < 1$ :

$$\tilde{u}(x) = \left[ \frac{1}{2}(1 - \xi) \quad \frac{1}{2}(1 + \xi) \right] \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}. \quad (3.2)$$

A linear discretisation provides a variable which is continuous over the domain of an element and over element borders. The spatial derivatives of this variable are not continuous between adjacent elements. If a quadratic or higher order discretisation is used the derivatives are continuous too.

## Weighted residual formulation

The equation of motion can be rewritten in a weighted residual formulation by multiplying the equation of motion with a weighing function and integrating the product over the domain. The weighted residual formulation of the CEDA problem is:

$$\int_0^L \eta(x) \left\{ -\omega^2 \rho A \tilde{u} - EA \left[ \frac{\partial^2 \tilde{u}}{\partial x^2} + 2jk_s \frac{\partial \tilde{u}}{\partial x} - k_s^2 \tilde{u} \right] \right\} dx = \int_0^L \eta(x) \tilde{p}(x) dx. \quad (3.3)$$

A Galerkin approach is used (discretisation of  $\eta(x)$  analogous to discretisation of  $\tilde{u}(x)$ ). The terms of the integral can now be integrated on element level. Equation (3.3) can be rewritten as:

$$-\omega^2 \underbrace{\int_0^L \eta(x) \rho A \tilde{u} dx}_a - \underbrace{\int_0^L \eta(x) EA \frac{\partial^2 \tilde{u}}{\partial x^2} dx}_b \quad (3.4)$$

$$- \underbrace{\int_0^L \eta(x) 2jEAk_s \frac{\partial \tilde{u}}{\partial x} dx}_c + \underbrace{\int_0^L \eta(x) EAk_s^2 \tilde{u} dx}_d = \underbrace{\int_0^L \eta(x) \tilde{p}(x) dx}_e.$$

When the terms  $a$ ,  $b$  and  $e$  are written in  $u$  and  $p$ , instead of  $\tilde{u}$  and  $\tilde{p}$ , these three terms equal the weighted residual formulation for the longitudinal vibrating beam problem, as described in Appendix B. Integration of term  $a$  leads to a mass matrix which is the same mass matrix as in the classical FEM problem:

$$\mathbf{M}_e = \frac{\rho AL_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \quad (3.5)$$

### Integration by parts

To reduce the order of the derivatives integration by parts is used. Integration by parts of term  $b$  gives

$$\underbrace{\int_0^{L_e} \frac{\partial \eta}{\partial x} EA \frac{\partial \tilde{u}}{\partial x} dx}_f - \underbrace{\left[ \eta EA \frac{\partial \tilde{u}}{\partial x} \right]_0^{L_e}}_g. \quad (3.6)$$

The term  $f$  in this expression results in the standard stiffness matrix:

$$\mathbf{K}_e = \frac{EA}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (3.7)$$

In classical FEM the term  $g$  can be moved to the right hand side of the weighted residual formulation and describes the external forces on the nodes. In FEM-CEDA a correction has to be made since the constitutive relation also undergoes a transformation:

$$F = EA \frac{\partial u}{\partial x} \quad \Rightarrow \quad \tilde{F} = EA \left[ \frac{\partial \tilde{u}}{\partial x} + jk_s \tilde{u} \right]. \quad (3.8)$$

Term  $g$  of equation (3.6) can be dealt with by writing the external forces on the nodes on the right hand side of the equations as in classical FEM and adding an extra term to the left hand side of the equation, which gives rise to matrix  $\mathbf{C}_{eB}$ :

$$\left[ \eta jk_s EA \tilde{u} \right]_0^{L_e} \quad \Rightarrow \quad \mathbf{C}_{eB} = k_s EA \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (3.9)$$

More attention is paid to the new terms  $c$  and  $d$  that are introduced in the CEDA weighted residual formulation:

$$\underbrace{\int_0^L \eta(x) 2jEAk_s \frac{\partial \tilde{u}}{\partial x} dx}_c + \underbrace{\int_0^L \eta(x) EAk_s^2 \tilde{u} dx}_d. \quad (3.10)$$

The real term gives a symmetric matrix  $\mathbf{C}_{eR}$  that is of the same structure as the element mass matrix  $\mathbf{M}_e$ , and the imaginary term leads to a skew-symmetric matrix  $\mathbf{C}_{eI}$ :

$$\mathbf{C}_{eR} = \frac{EAk_s^2 L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}; \quad \mathbf{C}_{eI} = k_s EA \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}. \quad (3.11)$$

It can simply be seen that if  $k_s$  is chosen equal to  $\omega\sqrt{\rho/E}$  matrix  $\mathbf{C}_{eR}$  cancels out  $-\omega^2\mathbf{M}_e$  and the problem is reduced from a high frequency dynamic problem to a quasi-static problem. The full complex algebraic equations for one CEDA element that result are:

$$\left\{ -\omega^2 \frac{\rho A L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{EA}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{EAk_s^2 L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \right. \quad (3.12)$$

$$\left. jk_s EA \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + jk_s EA \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tilde{\mathbf{u}}_e = \tilde{\mathbf{F}}_e + \int_0^{L_e} N^T(\xi) \tilde{p}(x) dx,$$

or:

$$[-\omega^2 \mathbf{M}_e + \mathbf{K}_e + \mathbf{C}_{eR} + j(\mathbf{C}_{eI} + \mathbf{C}_{eB})] \tilde{\mathbf{u}}_e = \tilde{\mathbf{F}}_e + \int_0^{L_e} N^T(x) \tilde{p}(x) dx. \quad (3.13)$$

### Matrix assembly

The element matrices can be derived analytically and most of them are dependent on the element length  $L_e$ . When a mesh is used with various element lengths the element matrices for all element lengths must be computed. After that the element matrices can be assembled. After assembly of the element matrices the equations can be written in the form:

$$\mathbf{C} \tilde{\mathbf{u}} = \tilde{\mathbf{F}} + \sum_e \int_0^L N^T(\xi) \tilde{p}(x) dx. \quad (3.14)$$

The matrix  $\mathbf{C}$  can be a block-diagonal matrix if the equations are rearranged. The right hand side of the equations can be determined by CEDA transformation of the physical load.

### Boundary conditions

For this type of problems two boundary conditions are necessary. The boundary conditions can be kinematic or dynamic. Kinematic boundary conditions are prescribed displacements or (first) derivatives of the displacements. Dynamic boundary conditions are prescribed forces or torques (for flexural vibrations; not for longitudinal vibrations). The boundary conditions of a clamped-clamped beam are the prescribed displacements at the clamps. The clamping forces are the yet unknown reaction forces. The physical boundary conditions are:

$$u(0) = u_0, \quad u(L) = u_L. \quad (3.15)$$

In CEDA notation boundary conditions on the complex envelope displacement  $\tilde{u}$  are needed. The relation to the physical boundary conditions is given by the back transformation and can be written as:

$$\operatorname{Re} \left\{ \tilde{u} e^{jk_s x} \right\}_{x=0} = u_0, \quad \operatorname{Re} \left\{ \tilde{u} e^{jk_s x} \right\}_{x=L} = u_L. \quad (3.16)$$

In the FEM-CEDA formulation of this problem the boundary conditions can not be applied in the way they are stated in equation (3.16), since the ‘Real part of’- operator is not an analytic function and can not be represented in a linear algebraic equation. The equations have to be split up in real and imaginary coefficients and real and imaginary variables to be able to apply the boundary conditions. Equation (3.16) can be rewritten as two boundary condition equations after splitting up in real and imaginary parts:

$$\left[ \overleftarrow{u}_r \cos(k_s x) - \overleftarrow{u}_i \sin(k_s x) \right]_{x=0} = u_0, \quad \left[ \overleftarrow{u}_r \cos(k_s x) - \overleftarrow{u}_i \sin(k_s x) \right]_{x=L} = u_L. \quad (3.17)$$

To apply these boundary conditions the  $N$  complex equations have to be rewritten as  $2N$  real equations with real coefficients and variables. There are however only 2 boundary condition equations present, where 4 boundary conditions are necessary.

In classical FEM the matrix equations can be solved by partitioning to a matrix notation where the unknown boundary forces can be computed by postprocessing (see Appendix C). However, in CEDA this can not be done because the physical boundary forces will lead to a CEDA force field, theoretically, over an infinite domain as illustrated in figure 3.1. This is an essential point in the CEDA

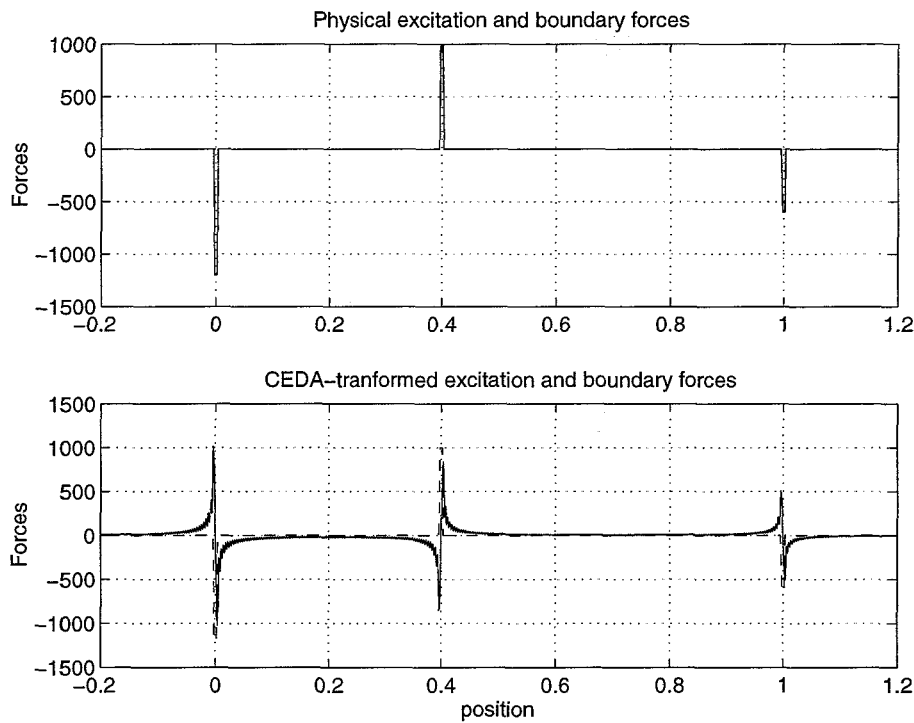


Figure 3.1: (a) Physical excitation and boundary forces and (b) the distributed loads after CEDA transformation. No shift is applied ( $k_s = 0$ ), so the lower figure is the analytic form of the physical forces. Real (--) and imaginary (-) parts. The small oscillations in the imaginary part are caused by the numerical implementation.

theory that is never mentioned by Carcaterra and Sestieri or by Raaymakers<sup>1</sup>. It implies that the domain on which a solution is calculated must be extended beyond the physical domain.

<sup>1</sup>Raaymakers determines a homogeneous solution on the physical domain and makes a correction for the boundary conditions afterwards



Now the known and unknown variables can not be partitioned as in classical FEM, because the nodes close to the clamps have unknown displacements *and* an unknown load. This problem can be dealt with as follows. The distributed loads that correspond with the boundary forces can be represented by the unknown physical amplitudes and their corresponding analytically known functions  $\overleftarrow{\delta}(x - x_0)$  corresponding to delta functions  $\delta(x - x_0)$ . For example the physical delta distribution boundary force  $F(L) = F_L \delta(x - L)$  is transformed to its CEDA equivalent:

$$\overleftarrow{p}(L) = F_L \overleftarrow{\delta}(x - L) = F_L \left( \delta(x - L) + j \frac{1}{\pi(x - L)} \right) e^{-jk_s x}. \quad (3.18)$$

This has two important consequences.

- The discretised domain has to be extended beyond the clamping points of the bar to a point where the clamping force distribution is negligible.
- The physical boundary forces have to be solved simultaneously with the displacement.

The CEDA transformed force fields at the boundaries are anti-symmetric with respect to the boundaries. Unfortunately this property of anti-symmetry can not be used because the displacement variable is nor symmetric, neither anti-symmetric with respect to the boundaries.

With equation (3.18) that represents the CEDA transformed boundary forces as a column with known components and the unknown force amplitude at the boundary, the discretised equations can be reorganised to have the same number of equations as unknown variables. It reads:

$$\begin{bmatrix} \mathbf{C}^{bc} & \overleftarrow{\delta} \end{bmatrix} \begin{bmatrix} \overleftarrow{\mathbf{u}} \\ \mathbf{F}_{bc} \end{bmatrix} = \overleftarrow{\mathbf{F}} - \mathbf{C}_{bc} \mathbf{u}_{bc}, \quad (3.19)$$

where  $\mathbf{C}^{bc}$  is the system matrix where the columns  $\mathbf{C}_{bc}$  corresponding to the known boundary conditions  $\mathbf{u}_{bc}$  are eliminated. For each boundary condition a column  $\overleftarrow{\delta}$  is added to the system matrix. This column describes the relation of the transformed boundary force field to the unknown boundary force  $\mathbf{F}_{bc}$  analytically. The unknown displacement  $\overleftarrow{\mathbf{u}}$  and the unknown boundary forces  $\mathbf{F}_{bc}$  are solved simultaneous from this set of equations. In appendix C this nonstandard application of the boundary conditions in a numerical implementation of FEM-CEDA is explained in more detail. The first two items as stated in section 1.3, concerning the discretisation of the CEDA equations and the application of the boundary conditions, are now theoretically solved. A numerical example will be given in the next chapter.

## Numerical solution

The set of linear algebraic equations that results after discretisation can be solved numerically. Solution of the equations is the main contributor to the computational cost. The assembly of the matrix equations in  $N$  degrees of freedom takes about  $N^2$  floating point operations. Decomposition of the matrix on the left hand side and the column on the right hand side takes about  $\frac{1}{3}N^3$  and  $\frac{1}{2}N^2$  flops respectively. Deriving the solution by backward substitution takes another  $\frac{1}{2}N^2$  flops. In a numerical implementation of FEM-CEDA in MATLAB the matrix solution can be made much more efficient by using the property of matrix  $\mathbf{C}$  that it is a sparse matrix. The extra computations that are necessary for the CEDA transformations are of the order of  $N$  or  $N^2$  flops and are therefore less important than the solution procedure.

### 3.2 Schematic overview of FEM-CEDA

In figure 3.2 a schematic overview of FEM-CEDA is given.

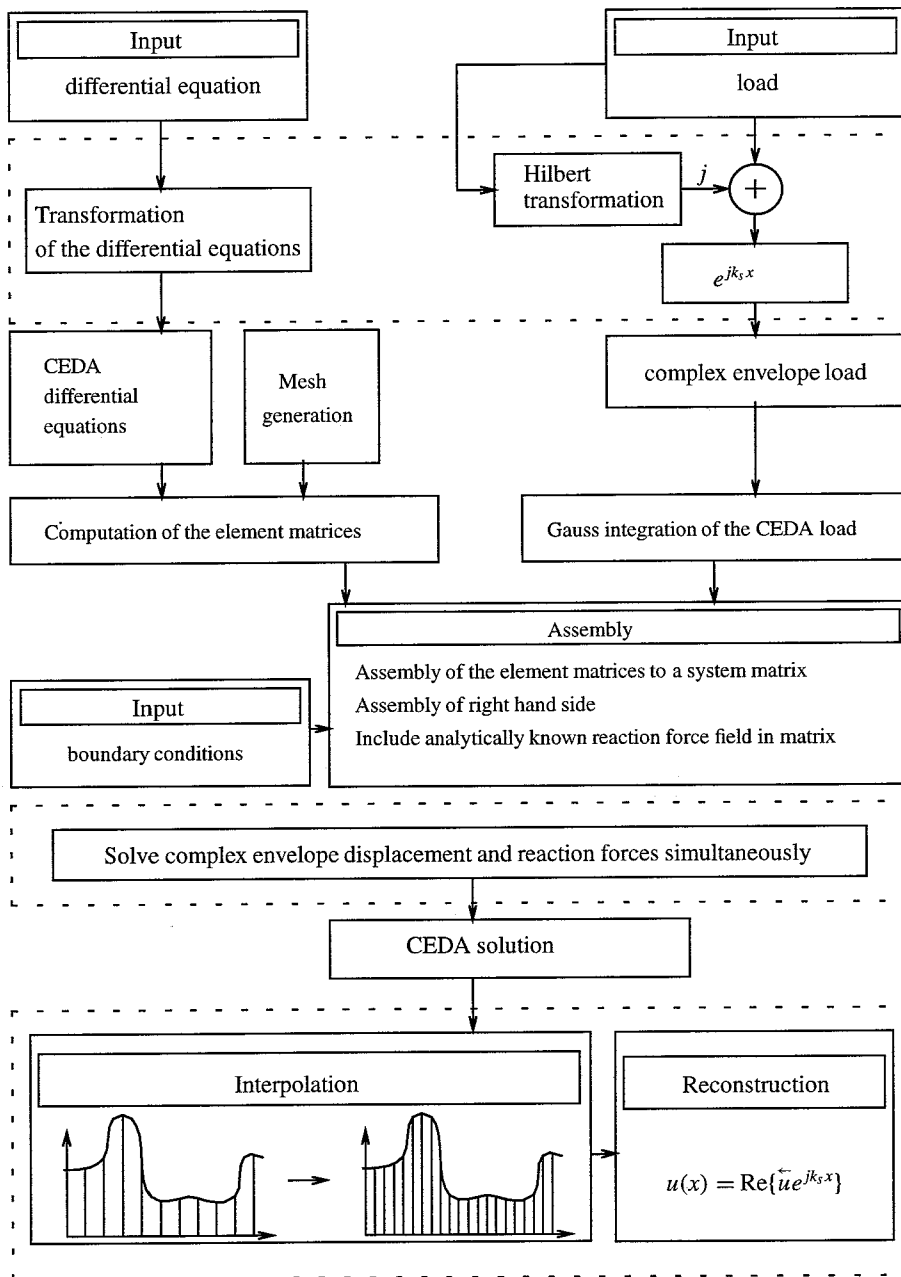


Figure 3.2: Schematic overview of FEM-CEDA.

### 3.3 FEM-CEDA for damped systems

As already stated in Carcaterra & Sestieri (1997) the introduction of damping in CEDA differential equations is not trivial. The solution to a damped problem is a complex displacement, and because the CEDA transformations are only applicable to real signals the complex displacement and the corresponding differential equation must be split up in real and imaginary components. For classical FEM this is explained in appendix B. The differential equations are rewritten in real components  $u_r$  and  $u_i$ .

After that, the real and imaginary parts of the physical variable can be both transformed to two complex envelope displacement variables. Application of the boundary conditions requires again that these complex envelope displacements are split in real and imaginary components:

$$\begin{aligned} u_r &\Rightarrow \overleftarrow{(u_r)} = \overleftarrow{(u_r)}_r + j\overleftarrow{(u_r)}_i \\ u_i &\Rightarrow \overleftarrow{(u_i)} = \overleftarrow{(u_i)}_r + j\overleftarrow{(u_i)}_i. \end{aligned} \quad (3.20)$$

It is possible to combine these four components to a notation in two components that are the real and imaginary part of the complex envelope displacement:

$$\overleftarrow{u} = [\overleftarrow{(u_r)}_r - \overleftarrow{(u_i)}_i] + j[\overleftarrow{(u_r)}_i + \overleftarrow{(u_i)}_r], \quad (3.21)$$

but from this representation the reconstruction cannot be performed unambiguously. The representation in four components does not lead to fundamental difficulties in the discretisation of the set of CEDA differential equations. The only disadvantage is that the number of equations is doubled twice, where in classical FEM the solution can be computed directly with complex variables.

## Chapter 4

# Numerical examples

In this chapter the numeric solutions of both damped and undamped longitudinal vibrating beam problems are determined with classical FEM techniques and with FEM-CEDA techniques. The computational effort and the accuracy of the solutions is compared.

A longitudinal vibrating beam with length  $L = 1$  m, area  $A = 10^{-4}$  m<sup>2</sup>, density  $\rho = 7800$  kg/m<sup>3</sup> and Young's modulus  $E = 210 \cdot 10^9$  N/m<sup>2</sup> is excited at  $x = 0.4$  m with a harmonic oscillating force with amplitude  $F_0 = 1000$  N and frequency  $f_{exc} = 50$  kHz. For the damped simulations the loss factor  $\eta = 0.05$ .

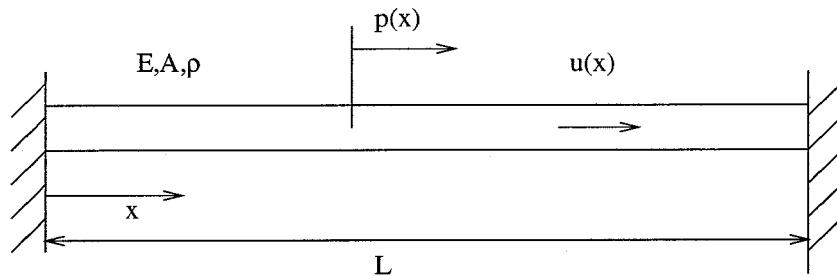


Figure 4.1: Longitudinal vibrating beam.

The response to this harmonic excitation in time will be an harmonic oscillation with the same frequency. The spatial solution will be a standing wave pattern. The spatial frequency or wave number is dependent on the wave speed in the beam and the oscillation frequency. The expected wave number according to equations (B.3) and (B.4) is 60.5 rad/m which corresponds to 9.6 wave lengths per meter.

In the rest of this chapter the simulations of table 4.1 will be discussed. First a reference solution

<i>nr.</i>	<i>Description</i>	<i># elements</i>	<i>damping</i>
1	Classical reference solution	100	undamped
2	CEDA solution	100+40	undamped
3	CEDA solution	25+10	undamped
4	Classical reference solution	100	damped
5	CEDA solution	43+18	damped

Table 4.1: Performed simulations

is computed with classical FEM techniques. Second a solution is computed with FEM-CEDA for the same element length as in the reference solution and the results of both simulations are analysed and compared. Subsequently, a damped longitudinal wave problem is solved with both classical FEM and FEM-CEDA techniques. Finally, the possibilities of element reduction with FEM-CEDA are under investigation.

#### 4.1 Classical FEM analysis of an undamped beam

To obtain a reference solution, a classical FEM analysis with 100 quadratic elements has been performed. It is common to discretise the domain with about ten nodal points per wavelength, but more nodes produce a smoother representation of the standing wave pattern. The elements are chosen to be equidistant, since the solution is a standing wave pattern over the whole length of the beam.

For the chosen element length a mass and stiffness matrix is computed and assembled into a  $201 \times 201$  system matrix. This is a sparse matrix and can be solved very efficiently. It therefore takes MATLAB far less than the theoretical  $\frac{1}{3}N^3$  flops to compute the solution. The results are given in figure 4.2. The computations are repeated for three different excitation positions to demonstrate the sensitivity of the solution with respect to the excitation position.

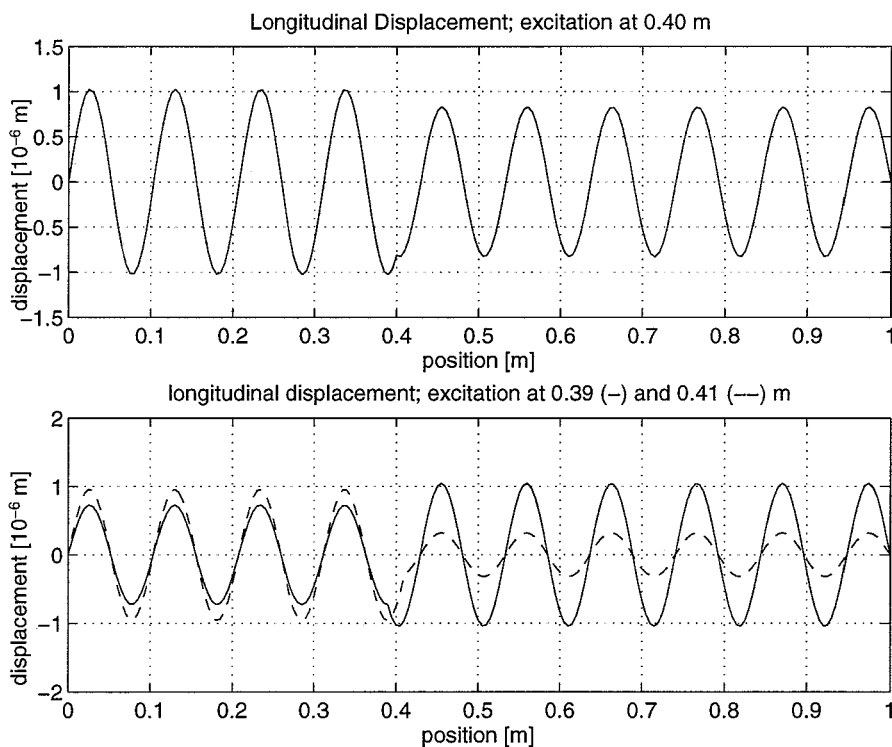


Figure 4.2: Classic FEM solution of vibrating beam problem for excitation positions 0.39, 0.40 and 0.41 meter.

## 4.2 FEM-CEDA analysis of an undamped beam

A FEM-CEDA analysis is performed with the same element length as in the classical FEM analysis. The wave number shift is chosen equal to the expected number of waves according to equation (B.4). The excitation position is 0.4 meter. This is the optimal wave number shift because it will shift the wave number contents towards zero. Shifting with other wave number shifts will also produce a low wave number complex envelope displacement.

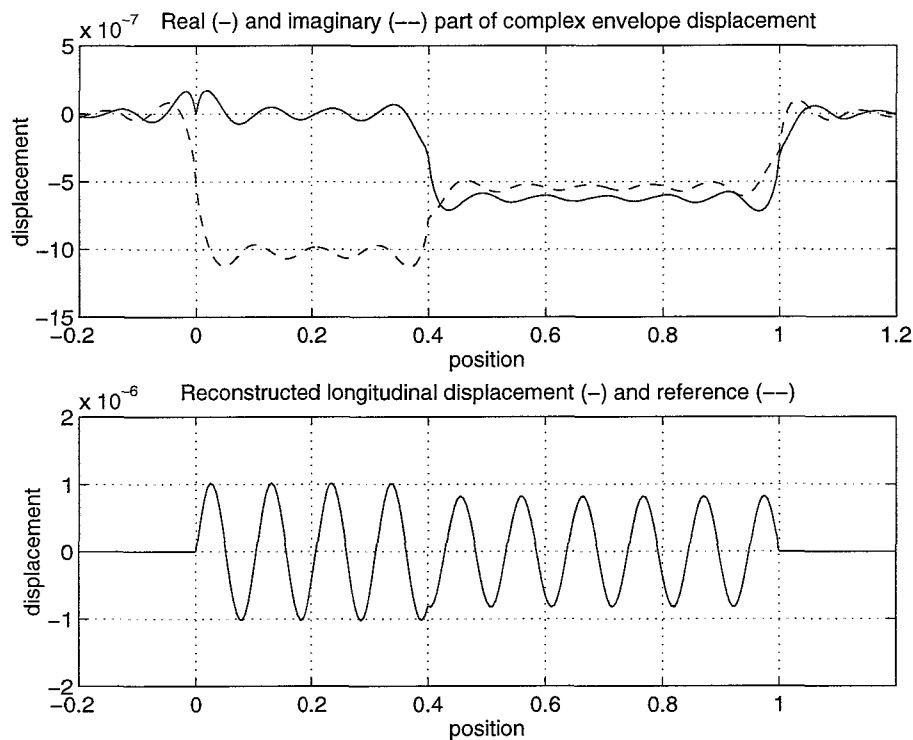


Figure 4.3: FEM CEDA solution of longitudinal vibrating beam. (a) Real (-) and imaginary (--) part of the complex envelope displacement. (b) Reconstructed displacement (-) and reference solution (--).

In figure 4.3 the results of the FEM-CEDA analysis of the longitudinal vibrating beam problem is given. Extra elements (20 at each side) are added outside the physical domain. The solution is reconstructed and equals the reference solution of figure 4.2 within an error of 1 percent. This numerical simulation shows that the FEM implementation of CEDA as described in Chapter 3 is implemented correctly. In this simulation however the expected advantages of CEDA are not yet used. On the contrary, the computational cost of the FEM-CEDA simulation is greater than with classical FEM because of the following reasons:

- Extra elements have to be added outside the physical domain because the transformed loads theoretically have a contribution on an infinite domain.
- The equations have to be rewritten in real variables with real coefficients by splitting up all complex variables and coefficients in real and imaginary components to be able to apply the boundary conditions.

The numerical implementation of this FEM-CEDA solution is explained in more detail in Appendix C. The FEM-CEDA matrix equations also contains a sparse matrix, which can be solved very efficiently. The computational cost of the MATLAB simulation of this FEM-CEDA analysis was approximately double the cost of the classical FEM simulation in MATLAB.

### Analysis of the results

To analyse the differences between the displacement variable and the complex envelope displacement variable, these field variables and the corresponding wave number spectra are printed in figure 4.4. In this figure it can be seen that the complex envelope displacement has most of its wave number contents in the low wave number region, where the reference displacement has most of its frequency contents at about ten waves per meter. The complex envelope displacement however also has some

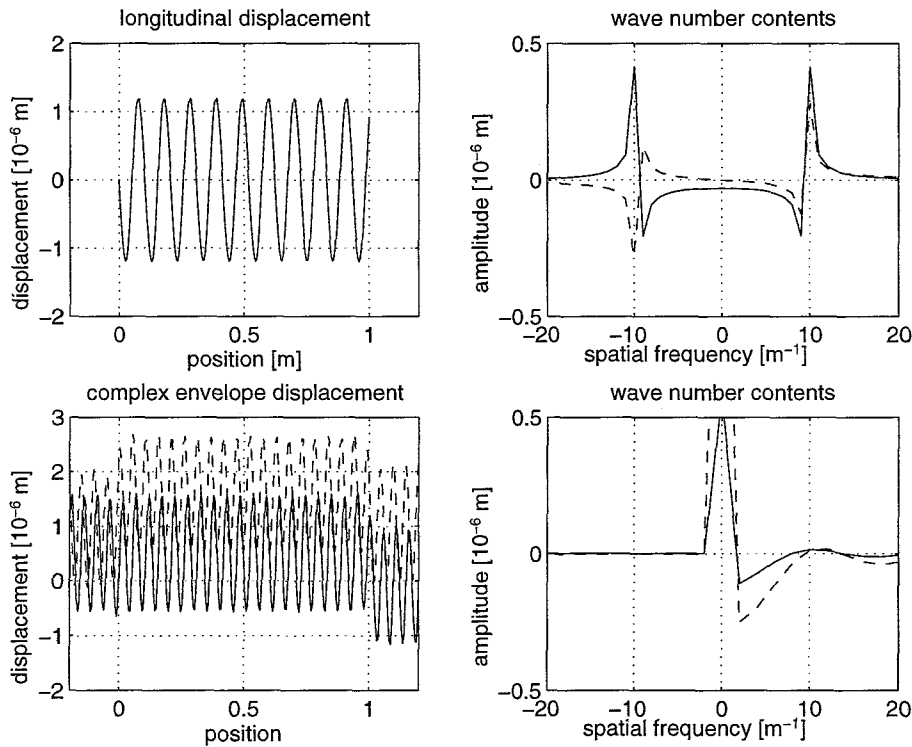


Figure 4.4: FFT analysis of reference displacement and complex envelope displacement.

frequency contents at ten waves per meter and higher. This corresponds to the small vibrations in the signal. These wave numbers are present in the complex envelope because the reference solution is not sufficiently band limited in the high wave number region, but also has frequency contents in the low wave number region. This part of the spectrum is shifted to the high wave number region by the CEDA transformations.

Because of this frequency contents the complex envelope displacement still has to be sampled by about the same number of samples as the original signal to represent it accurately. This conflicts with the aims of CEDA, being a reduction in the number of samples or elements by a transformation from a high wave number signal to a low wave number signal.

At first sight the complex envelope displacement variable is smoother than the original displacement, but the maximum frequency present in the variables is equal for both the displacement and the complex envelope displacement. The energy contents however are shifted and therefore the complex envelope displacement has most of its energy concentrated in the low wave number region, where the original displacement has most of its energy concentrated in the high wave number region.

The FEM-CEDA simulation is repeated with 25 elements on the interval  $[0, 1]$ . A classical FEM analysis with this number of elements produces a very erroneous solution, because it has only 2 or 3 elements per wave length. At both ends 5 extra elements were added to deal with the boundary forces. On the physical domain the reconstructed solution equals the reference solution sufficiently accurate.

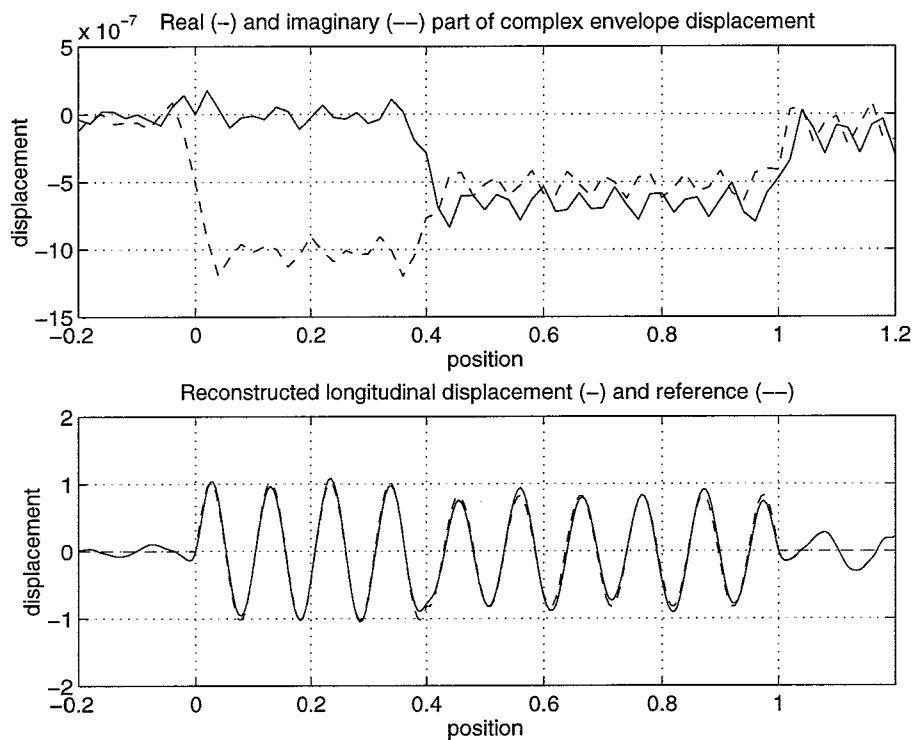


Figure 4.5: FEM CEDA solution of longitudinal vibrating beam. (a) Real (-) and imaginary (- -) part of the complex envelope displacement. (b) Reconstructed displacement (-) and reference solution (- -).

At this moment the numerical implementation of FEM-CEDA requires a node at the excitation position. In the simulation of figure 4.5 this requirement was met. However with this coarse mesh a simulation with excitation position at 0.39 or 0.41 can not be performed, unless the mesh is changed.

### 4.3 FEM and FEM-CEDA analyses of a damped longitudinal vibrating beam

The analysis of damped longitudinal vibrations in a beam with classic FEM techniques can be performed in a similar way as with undamped vibrations. In Appendix B is described how the damping matrix can be obtained and what set of algebraic equations results. The classical FEM solution for a damped vibrating beam are obtained with an element mesh of 100 quadratic elements. The loss factor



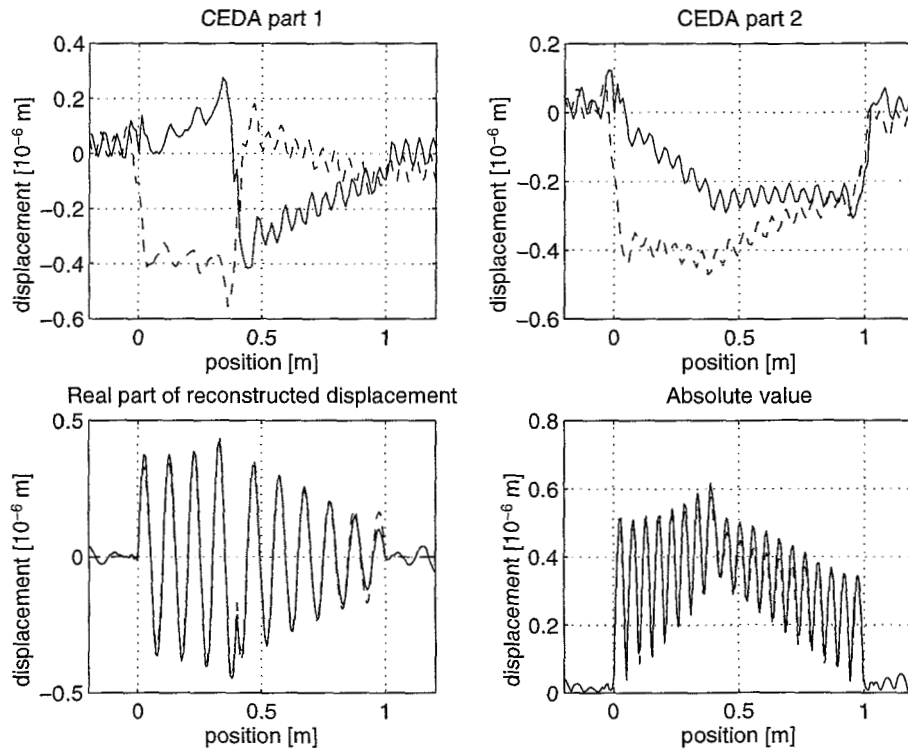


Figure 4.6: Simulation results of FEM-CEDA computations for a damped longitudinal vibrating beam. A mesh with 61 elements is used. (a) Real (-) and imaginary (- -) part of complex envelope displacement part 1, corresponding with the real part of the physical displacement. (b) Real (-) and imaginary (- -) part of complex envelope displacement part 2, corresponding with the imaginary part of the physical displacement. (c and d) Real part and absolute value of the reconstructed (-) and reference (- -) displacement.

that describes the damping effects equals  $\eta = 0.05$ . The results are used as a reference solution for the FEM-CEDA results in figure 4.6.

The discretisation of the equations and the application of boundary conditions of FEM-CEDA for damped systems follows the same recipe as in undamped FEM-CEDA. The damped FEM-CEDA procedure is programmed in MATLAB and the results for a beam with loss factor  $\eta = 0.05$  are given in figure 4.6. Note the decaying amplitude as a result of damping. The CPU-times required for the performed simulations are summarised in table 4.2.

<i>nr.</i>	<i>Description</i>	<i># elements</i>	<i>damping</i>	<i>Flops</i>
1	Classical reference solution	100	undamped	$1.5 \cdot 10^5$
2	CEDA solution	100+40	undamped	$2.9 \cdot 10^5$
3	CEDA solution	25+10	undamped	$1.2 \cdot 10^5$
4	Classical reference solution	100	damped	$1.7 \cdot 10^5$
5	CEDA solution	43+18	damped	$4.0 \cdot 10^5$

Table 4.2: CPU-times represented by the number of floating point operations (flops)

## 4.4 Possible improvements on FEM-CEDA

A numerical implementation of FEM-CEDA for both damped and undamped systems is successfully derived and produces very accurate solutions when fine discretisations are used. The anticipated element reduction is not as good as was expected and therefore some research has been devoted to possible improvements on FEM-CEDA.

Two methods have been under investigation based on the assumption that the complex envelope displacement is a smooth low wave number variable. The first is non-equidistant meshing. It uses a more coarse discretisation on intervals where the solution is expected to have a small gradient. The second is filtering, which can filter out high wave number components.

### Non-equidistant meshing

The complex envelope displacement is a smooth variable with larger gradients at the clamps and at the excitation position. In a Finite Element Method analysis such a variable can be solved very efficiently with a non-equidistant mesh. Several types of non-equidistant meshes can be used:

- A mesh with zones with different element lengths. Zones with small elements are used at the clamps and excitation and zones with large elements in the rest of the mesh.
- A mesh with a continuously varying element length with small elements at the clamps and excitation and large elements further away from these points.

Both non-equidistant mesh types are implemented and tested and allow a small reduction in the number of elements.

### Filtering

Attempts have also been made to deal with the high wave number spectra by application of low pass filters. The simplest way to reduce high wave number components is expected to be found in filtering the input. When the input (for the vibrating beam problem this is the applied load) is filtered with a low pass filter that has a cut off frequency just below the high wave numbers that are expected in the displacement variable, the system will not be excited enough and the response is close to zero. If the input does contain the wave number contents corresponding with the high wave number part of the response this wave number contents will be present in the computed response. Hence, filtering the input does not solve the problem of the high wave numbers.

Application of a filter on the response variable is not trivial since the response is only known after solution of the discretised equations and therefore a discretisation has to be chosen based on the wave number contents of the unfiltered response. Attempts have been made to implement an inverse filter operation in the matrix equations that directly solves the filtered signal. The high wave number contents of the response can be filtered out with only a small error in the reconstructed signal as can be seen in figure 4.7. Note that this filter operation is performed on a known displacement response, that is obtained from a classic FEM simulation. The filtered complex envelope displacement in figure 4.7 is significantly smoother and can be sampled by a significant smaller amount of samples. The filtering however can only be applied to a known variable and the complex envelope displacement is not yet known. It needs to be solved first from the matrix equations with a fine mesh. Implementation of an inverse filter in the matrix equations proved to be impossible. Therefore it is concluded that filtering can not be used to improve the efficiency of FEM-CEDA. More on filtering is described in Appendix C.

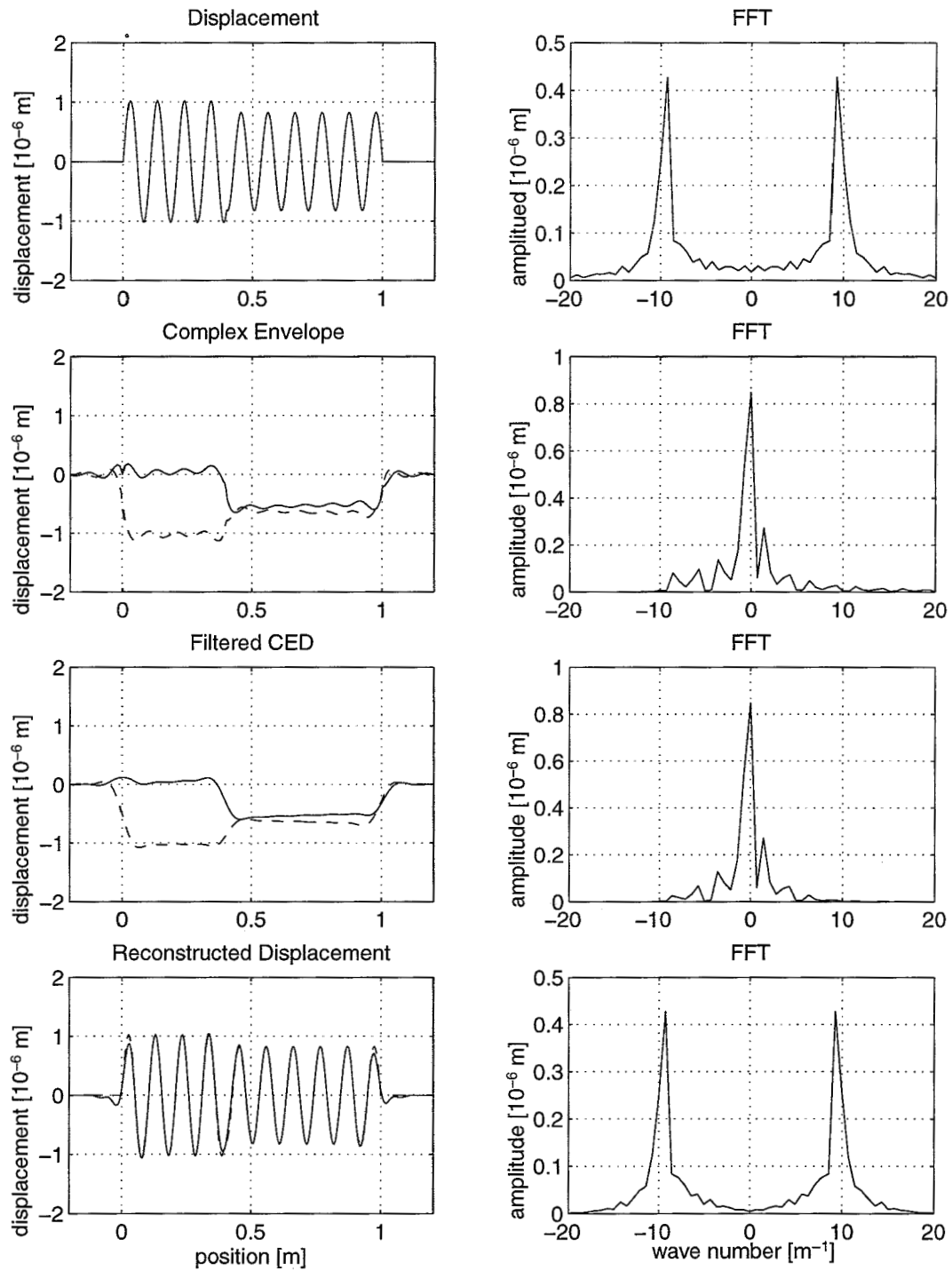


Figure 4.7: Effects of filtering on the complex envelope displacement and the reconstructed displacement. Sub-figures (a) to (d) show the displacement variable on the left and the corresponding wave number spectrum on the right. (a) Displacement variable. (b) Corresponding complex envelope displacement, real (-) and imaginary (- -) parts. (c) Filtered complex envelope displacement, real (-) and imaginary (- -) parts. (d) Reconstructed (-) and original (- -) displacement.

## 4.5 Discussion

A numerical implementation of FEM-CEDA is successfully derived and produces accurate solutions. Adding damping to the system equations is possible, but makes it necessary to double the number of equations. The spurious solutions mentioned by Carcaterra and Sestieri were not encountered. The problems described in item 3 and 4 of subsection 1.3 concerning damping and spurious solutions are therefore considered to be solved.



# Chapter 5

## Discussion

### 5.1 Summary of FEM-CEDA development

In the previous chapters a FEM-CEDA method has been derived and tested for both damped and undamped systems. To make discussion on FEM-CEDA easier, a brief summary is given here:

1. The CEDA transformations are based on the Hilbert transform, a signal transformation defined for real signals only. The transformations are linear and can be applied to signals and to linear differential equations.
2. The CEDA transformed differential equations can be translated to a Finite Element Method notation in the same way as in classical FEM. Mass, damping and stiffness matrices can be derived and assembled to a set of linear algebraic equations.
3. The application of the boundary conditions in FEM-CEDA involves a non standard procedure, because the boundary forces are transformed to force fields over an infinite domain. This force field can be written as the product of the physical boundary force and an analytically known function and makes simultaneous solution of the displacement and the boundary forces possible.
4. The application of the boundary conditions also makes it necessary to split up the equations in real and imaginary parameters and real and imaginary variables.
5. Since the force field that corresponds with the boundary conditions has a contribution outside the physical domain, the problem has to be discretised on an interval with extensions beyond the physical domain until a point where the contribution of these force fields is negligible.
6. The anticipated reduction in the number of elements is not as good as was expected, because the solutions in the examples are not sufficiently band limited in the high wave number domain. This problem is a structural problem for signals on a finite domain.
7. The addition of damping to the differential equations does not lead to fundamental problems. A major drawback is that the Hilbert transform can only be applied to real variables and therefore the equations have to be rewritten in real and imaginary components.
8. Attempts have been made to improve the numerical efficiency of FEM-CEDA by application of non-equidistant meshes and the application of filters. Non-equidistant meshing is possible to a certain degree. Filtering has proved to be inadequate to reduce the number of elements.

## 5.2 Discussion

The FEM-CEDA method gives a new contribution to the application of envelope methods in high frequency dynamics. The derivation of the FEM-CEDA implementation brings CEDA closer to an applicable tool in engineering mechanics. The advantages and disadvantages will be discussed here. FEM-CEDA is meant to be used in vibro-acoustics for computation of high frequency vibrations. The responses of these high frequency vibrations are strongly dependent on for example the material parameters, the position of the excitation and the boundary conditions. One deterministic calculation becomes meaningless in this case and stochastic properties of the material properties and excitation position have to be taken into account.

To the authors knowledge there is no stochastic FEM implementation available (although research has been done by Sun (1979a)). Monte Carlo techniques can be used to deal with the stochastic properties as demonstrated by Raaymakers (1995b). Monte Carlo techniques use a combination of deterministic calculations to deal with the stochastic properties. The effect of stochastic properties of the material parameters makes recalculation of the element and system matrices necessary whereas for the stochastic properties in the excitation only the right hand side of the FEM-CEDA equations has to be recomputed. The solution process of the matrix equations, which is the main contributor to computational cost, has to be performed in both cases. For numerical efficiency it is therefore important to minimise the number of elements used in the FEM-CEDA analyses. The necessity to double the number of equations as described in items 4 and 7 has negative consequences on this point, that unfortunately can not be overcome.

Another drawback is mentioned in item 5; the necessity to extend the discretisation outside the physical domain. Examples can be mentioned where this is not necessary, for instance with free ends (no reaction force or torque) and with rotationally symmetric structures where only one sector of the structure is modelled.

The major concern in the applicability of FEM-CEDA lies with item 6, since the property of field variables being band limited can not be guaranteed in most cases. The spatial point excitation in the example in this report and in the examples in Carcaterra's articles, does not result in a wave number band limited response, one of the basic assumptions of CEDA. A wave number band decomposition technique was proposed by Carcaterra but is not yet implemented in FEM formulation.

The analyses performed by Carcaterra were restricted to second order systems. The solution to a fourth order transversal vibrating beam problem is approximated by neglecting the near field solution and computing the far field solution only (Carcaterra & Sestieri (1997)). FEM-CEDA is not restricted in the order of the problems that can be analysed. Higher order differential equations can be translated to linear algebraic equations by standard FEM techniques.

No spurious solutions as reported by Carcaterra were found. The non standard inclusion of the CEDA boundary conditions as presented in this report is not mentioned in Carcaterra's work. The spurious solutions that Carcaterra and co-workers have found is ascribed to the solution method (shooting), they used. Small errors in the initial conditions will give odd homogenous (high frequency) solutions.

The extension of FEM-CEDA to more dimensional systems is not trivial. The Hilbert transform and the analytic signal are only defined for one-dimensional variables. The extension of CEDA to more dimensional systems is investigated by Sestieri and Adamo.

## Chapter 6

# Conclusions and recommendations

### 6.1 Conclusions

Based on the research described in this report the following conclusions can be drawn:

- The CEDA transformations can be applied to both signals and differential equations. The differential equations can be reformulated in CEDA notation to solve for a complex envelope displacement variable. The solution to the original problem can theoretically be reconstructed without any loss of information. With a proper choice of the frequency shift, dynamic problems can be transformed to quasi-static problems.
- It was shown to be possible to implement the CEDA theory in a Finite Element Method formulation. Element matrices can be computed and assembled to a system matrix as in classical FEM. The inclusion of the boundary conditions demands some special operations that are not part of standard FEM techniques.
- An important disadvantage in FEM-CEDA is that the system has to be discretised outside the physical domain (except for systems with free or periodic boundaries).
- The anticipated element reduction is less than expected as a result of the following reasons:
  - The extension of the discretisation outside the physical domain requires a greater amount of elements than a corresponding classical FEM analysis.
  - The solution on a finite interval is not sufficiently band limited that it can be described by a coarse mesh.
  - The complex envelope load has high wave number contents and the process of point-skipping is restricted by the wave number contents of this load.
- Filtering the complex envelope displacements makes it possible to describe the complex envelope displacement sufficiently accurate with significantly fewer samples. The inverse filtering could however not be implemented in the equations.
- Application of FEM-CEDA as an engineering tool is not yet recommended, since FEM-CEDA requires more insight of the user in the system that is analysed. The computational reduction that can be achieved is not significant in the examples described in this report.



## 6.2 Recommendations

Further development of FEM-CEDA towards application as an engineering tool requires the following steps:

- Application to higher order and more dimensional systems. As stated in the discussion the application to higher order systems is considered to cause no fundamental difficulties. Application to more dimensional systems demands a formulation of the Hilbert transformation for more dimensional variables. Sestieri and co-workers are investigating the latter.
- Extension to uncertain systems by addition of a stochastic method. FEM-CEDA in its current implementation is limited to deterministic problems and can be used in Monte Carlo techniques to deal with the random properties of systems. It would be computationally more efficient if the stochastic method could be added to the Finite Element Method formulation as suggested by Sun (1979a).
- Development of a band width decomposition technique that can be used in combination with FEM-CEDA.

Before effort is made to deal with the items above it must be considered that CEDA might not be the appropriate method to deal with the high wave number domain. As explained in this report CEDA works optimal for signals on infinite domains, and since most practical problems are restricted to finite domains, the applicability of CEDA is still dependent on future research.

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## Appendix A

# Variable transformations

In this appendix the variable transformations are illustrated. The transformations are not performed on 'signals' but on 'field variables'. The term signal should only be used for time-varying signals, where a field variable is a function of the position. When FFT analyses are performed the frequency spectrum of a signal is calculated. In terms of field variables one speaks of wave numbers. However, the transformations described here can be seen as numerical operations on a column of data, representing either a signal in time or a field variable as a function of position.

<i>signals</i>	<i>field variables</i>
time	position
frequency	wave number

Table A.1: Analogies in time and space

Figure A.1 shows a harmonic cosine wave, which is a real signal. The wave number is 10 waves per meter. The wave number spectrum is real. The Hilbert transform of this cosine wave is a sine wave. The sine wave is also a real signal, but the wave number spectrum is imaginary. This is shown in figure A.2.

The analytic signal is a complex signal that is composed of a real component and an imaginary component which is the Hilbert transform of the real component. Hence, the analytic signal that corresponds with a cosine wave is a helix, whose real part is a cosine wave and whose imaginary part is a sine wave. Figures A.3 and A.4 give a graphic representation of this field variable. Note that the wave number spectrum does only have a non-zero component for positive wave numbers.

When a wave number shift is performed on the analytic signal (or analytic field variable) the complex envelope displacement is found. This is a low wave number complex field variable (see figure A.5). The transformation from the analytic signal to the complex envelope can be seen as a representation in a rotating coordinate frame. This coordinate frame rotates in the complex plane (around the  $x$ -axis) with an angular frequency almost equal to the spatial frequency of the analytic signal.

The transformations do not result in a loss of information. Therefore the original cosine wave can be reconstructed from the complex envelope exactly. The reconstructed signal of figure A.6 equals the cosine wave of figure A.1 within the floating point machine accuracy.

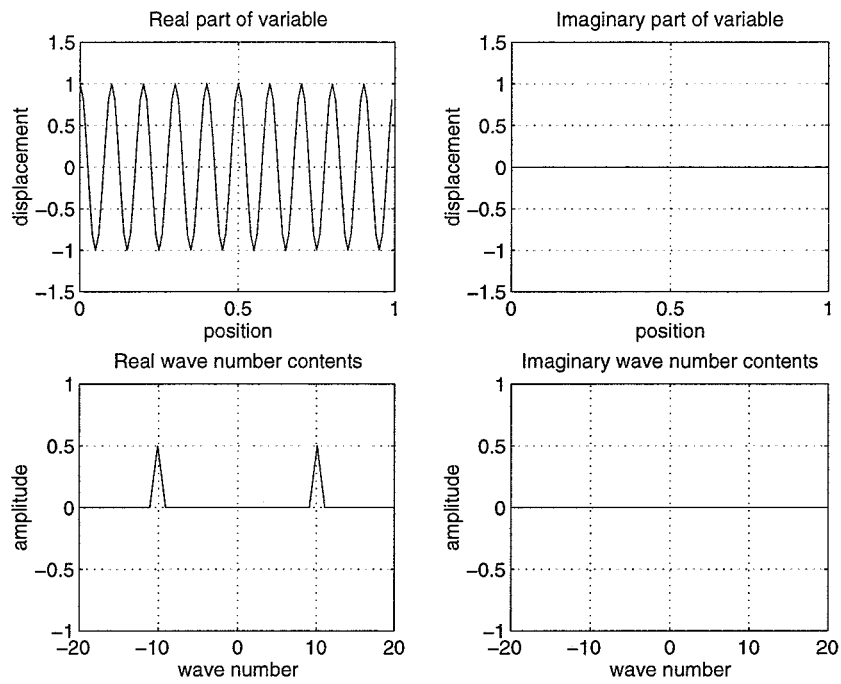


Figure A.1: The cosine wave and its wave number spectrum.

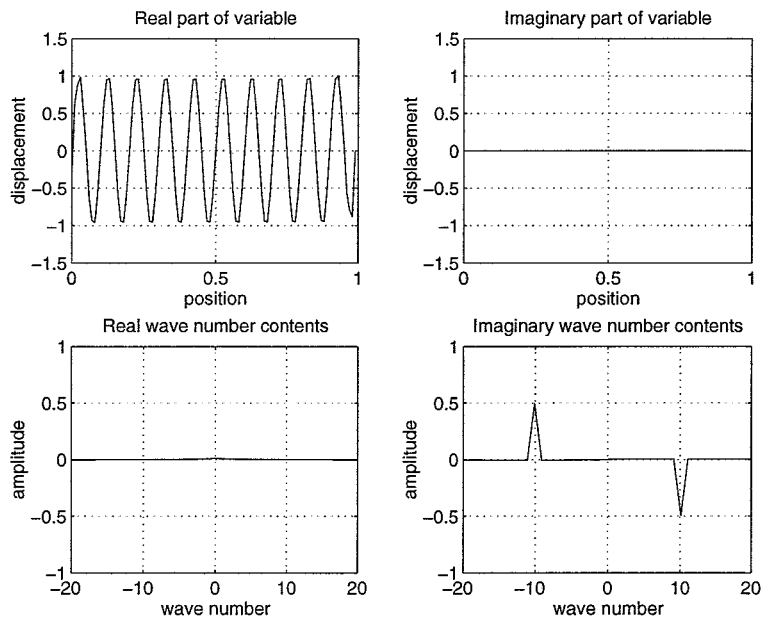


Figure A.2: The sine wave and its wave number spectrum. The sine wave is the Hilbert transform of the cosine wave.

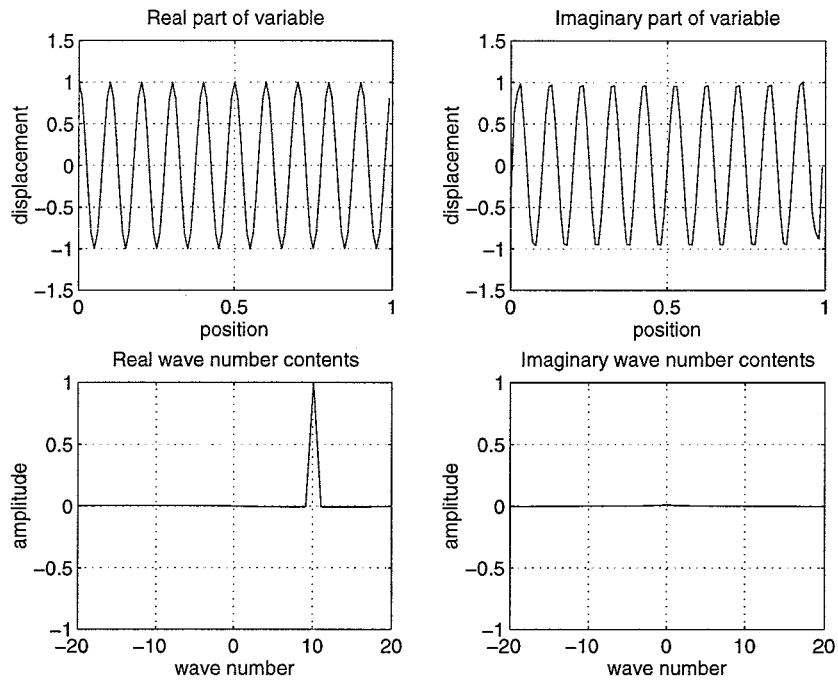


Figure A.3: The analytic signal and its wave number spectrum.

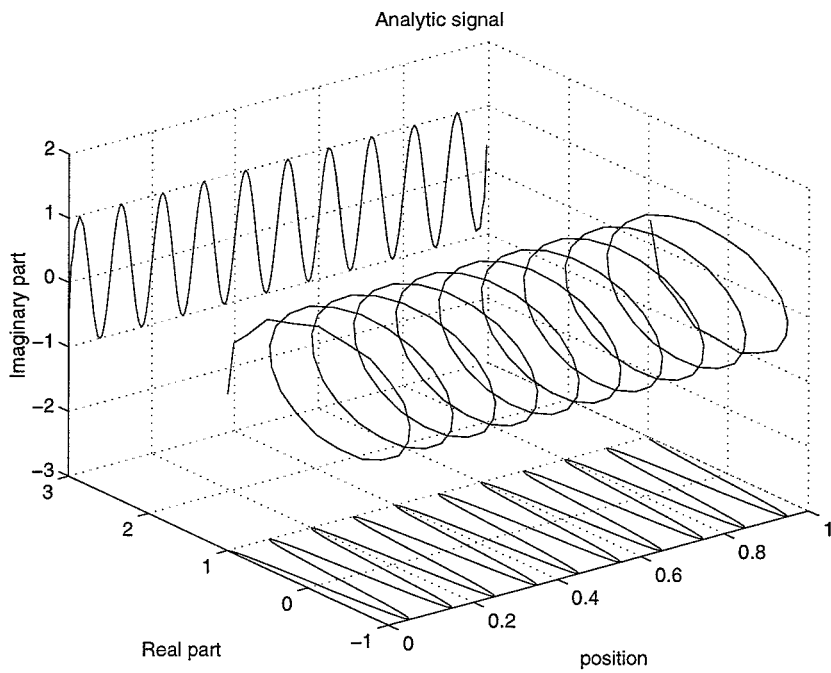


Figure A.4: The analytic signal is a helix in the complex space.

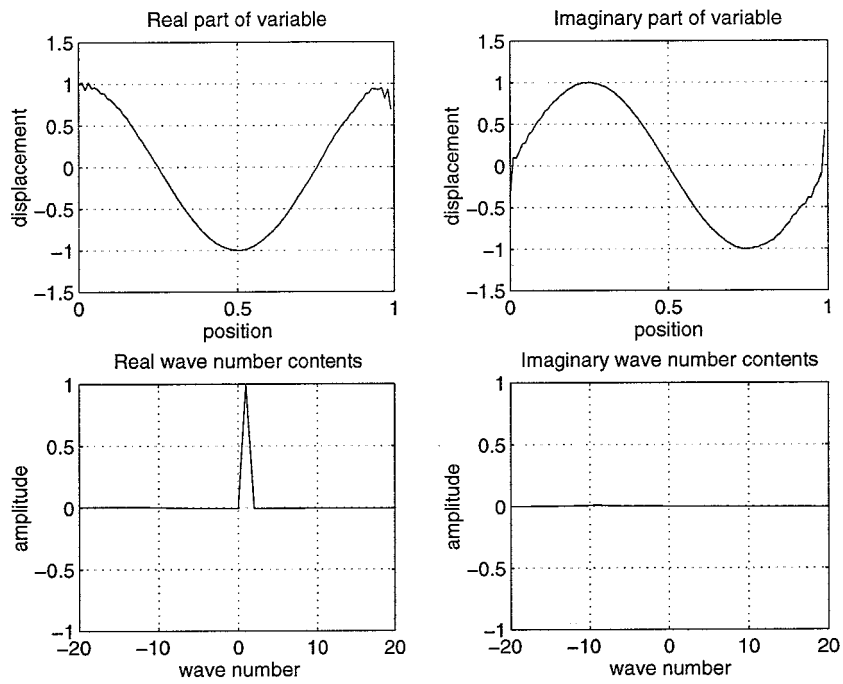


Figure A.5: The complex envelope and its wave number spectrum.

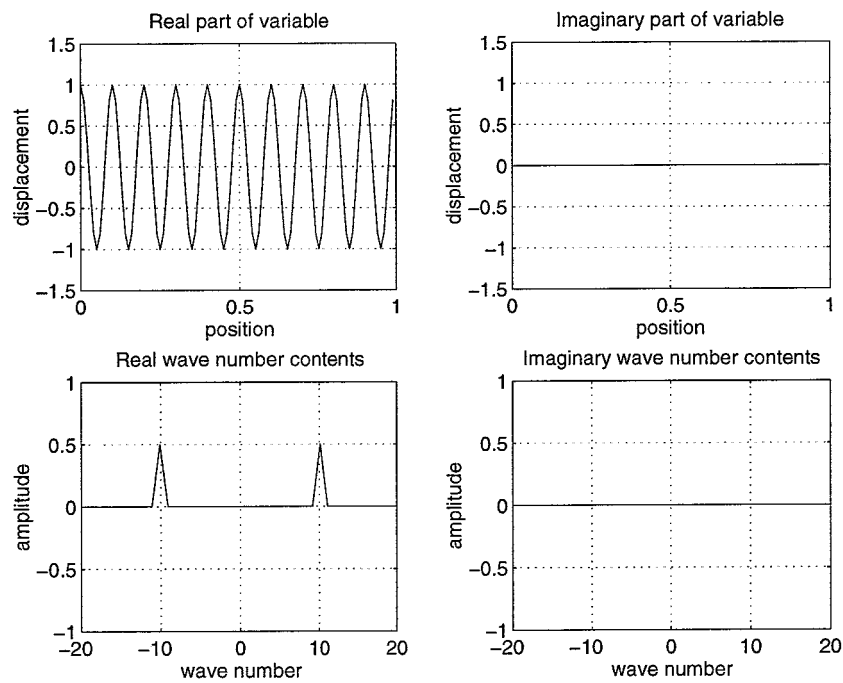


Figure A.6: The reconstructed signal and its wave number spectrum.

## Appendix B

# Longitudinal vibrations in a beam

### Equations of motion

To demonstrate the CEDA transformations a simple continuous dynamic system is chosen being a beam of length  $L$  which is clamped at both ends. In this appendix this system is analysed analytically and with classic FEM techniques. Only longitudinal harmonic vibrations are assumed.

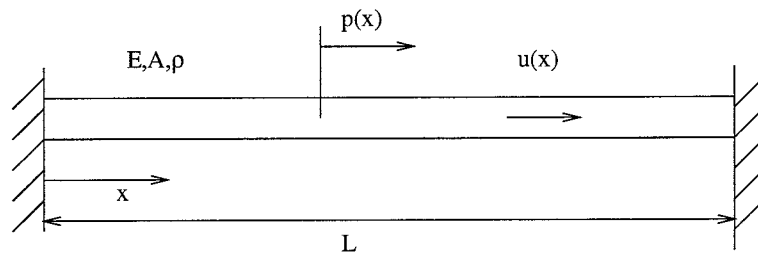


Figure B.1: Longitudinal vibrating beam.

The equation of motion for an undamped homogeneous beam of length  $L$ , with cross-sectional area  $A$ , density  $\rho$  and Young's modulus  $E$ , is:

$$\rho A \frac{\partial^2 u}{\partial t^2} - EA \frac{\partial^2 u}{\partial x^2} = p(x, t). \quad (\text{B.1})$$

The boundary conditions for this clamped-clamped beam are:  $u(0, t) = u(L, t) = 0$ . Equation (B.1) is often referred to as the *wave equation*. This equation can be rewritten (with the right hand side set to zero) as:

$$\frac{\partial^2 u}{\partial t^2} - c_L^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad (\text{B.2})$$

where  $c_L$  is the phase speed for longitudinal waves which equals:

$$c_L = \sqrt{\frac{E}{\rho}}. \quad (\text{B.3})$$



The free vibration solution to equation (B.2) will have a wave number  $k$ , which can be derived from:

$$k^2 = \frac{\omega^2}{c_L^2}, \quad k = \omega \sqrt{\frac{\rho}{E}} \quad (\text{B.4})$$

which corresponds with waves with wavelength  $\lambda = \frac{2\pi}{k}$ .

For an harmonic excitation  $p(x, t) = p(x)e^{j\omega t}$  the solution  $u$  will also be harmonic;  $u(x, t) = u(x)e^{j\omega t}$ . This reduces the equations of motion to a second order differential equation in  $x$ .

$$-\omega^2 \rho A u(x) - EA \frac{\partial^2 u}{\partial x^2} = p(x) \quad (\text{B.5})$$

## FEM formulation

The weighted residual formulation of problem (B.5) is:

$$-\omega^2 \underbrace{\int_0^L \eta(x) \rho A u(x) dx}_a - \underbrace{\int_0^L \eta(x) EA u''(x) dx}_b = \int_0^L \eta(x) p(x) dx. \quad (\text{B.6})$$

With linear discretisation of  $u(x)$ ,

$$u(x) = \left[ \frac{1}{2}(1 - \xi) \quad \frac{1}{2}(1 + \xi) \right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (\text{B.7})$$

and application of Galerkin<sup>1</sup>, the element mass and stiffness matrices can be derived. For one element of length  $L_e$  integration of term  $a$  leads to a mass matrix:

$$\mathbf{M}_e = \frac{\rho A L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \quad (\text{B.8})$$

Partial integration of term  $b$  gives the expression:

$$\int_0^{L_e} \eta'(x) EA u'(x) dx - [\eta EA u'(x)]_0^{L_e}. \quad (\text{B.9})$$

Integration of the term on the left in expression (B.9) results in a stiffness matrix:

$$\mathbf{K}_e = \frac{EA}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (\text{B.10})$$

The term on the right in expression (B.9) can be moved to the right hand side of equation (B.6) and describes the external forces on the nodes according to the constitutive relation:

$$\mathbf{F} = EA u''(x) \quad (\text{B.11})$$

Assembly of the element matrices leads to an  $(N + 1) \times (N + 1)$  matrix when the structure is divided up into  $N$  elements. The weighted residual formulation (B.6) of differential equation (B.1) is now discretised to a set of algebraic equations:

$$[-\omega^2 \mathbf{M} + \mathbf{K}] \mathbf{u} = \mathbf{F} + \sum_e \int_0^{L_e} N^T(\xi) p(x) dx \quad (\text{B.12})$$

and can be solved numerically.

<sup>1</sup>A Galerkin approach implies that the discretisation of  $\eta(x)$  is analogous to the discretisation of  $u(x)$ . See Steenhoven (1982); Brekelmans (1994)

## Damping

If damping is present the equation of motion changes to:

$$\rho A \frac{\partial^2 u}{\partial t^2} + c_d A \frac{\partial u}{\partial t} - EA \frac{\partial^2 u}{\partial x^2} = p(x, t). \quad (\text{B.13})$$

where  $c_d$  describes the damping. For harmonic vibrations the equation again reduces to a second order differential equation <sup>2</sup> in  $x$ :

$$-EAu''(x) + \underbrace{j\omega c_d A u(x)}_a - \omega^2 \rho A u(x) = p(x). \quad (\text{B.14})$$

When the equation is discretised with FEM techniques, the extra term  $a$  in equation (B.14) will result in element damping matrices  $\mathbf{D}_e$  that can be found by integration of:

$$\int_0^{L_e} \eta(x) c_d A u(x) dx, \quad (\text{B.15})$$

and read:

$$\mathbf{D}_e = \frac{c_d A L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \quad (\text{B.16})$$

After assembly the set of linear algebraic equations reads:

$$[-\omega^2 \mathbf{M} + j\omega \mathbf{D} + \mathbf{K}] \mathbf{u} = \mathbf{F} + \sum_e \int_0^{L_e} \mathbf{N}^T(\xi) p(x) dx \quad (\text{B.17})$$

The solution  $\mathbf{u}$  to these equations is a complex field variable. The equations can be rewritten in real variables and real parameters as:

$$\begin{bmatrix} \mathbf{K} - \omega^2 \mathbf{M} & -\mathbf{D} \\ \mathbf{D} & \mathbf{K} - \omega^2 \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}_r \\ \mathbf{u}_i \end{bmatrix} = \begin{bmatrix} \mathbf{F}_r(x) \\ \mathbf{F}_i(x) \end{bmatrix} \quad (\text{B.18})$$

where the complete right hand side of the equations is represented by  $\mathbf{F}$ .

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<sup>2</sup>The equation of motion for harmonic vibrations can be rewritten as:  $u''(x) + \frac{\rho \omega^2}{E} (1 - j\eta) u(x) = -\frac{1}{E} p(x)$ , where  $\eta$  is the dimensionless loss-factor that satisfies:  $\eta = \frac{c_d}{\rho \omega}$ .



## Appendix C

# Numerical implementation of FEM-CEDA boundary conditions

This appendix describes the numerical implementation of the FEM-CEDA boundary conditions in more detail. The derivation of the element matrices is not discussed here because it is equivalent with classical FEM. The matrix assembly in FEM-CEDA is almost equal to matrix assembly in classical FEM. Element matrices are added to a system matrix at positions corresponding with the nodal displacements of that element. For a beam this will produce a block diagonal matrix:

$$\mathbf{C} = \begin{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} & & & \\ & \ddots & & \\ & & \begin{bmatrix} c_N \end{bmatrix} & \\ & & & \end{bmatrix}. \quad (\text{C.1})$$

On the right hand side of the equations a column  $\overleftarrow{\mathbf{F}}$  is assembled and the nodal displacements are in column  $\overleftarrow{\mathbf{u}}$ .

With classical FEM the equations are partitioned to have the known displacements (boundary conditions) in  $u_2$ , and the unknown forces in  $F_2$ :

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \quad (\text{C.2})$$

The reaction forces at the boundaries can be calculated afterwards or simultaneously when the equations are rewritten as:

$$\begin{bmatrix} C_{11} & 0 \\ C_{21} & -I \end{bmatrix} \begin{bmatrix} u_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} F_1 - C_{12}u_2 \\ -C_{22}u_2 \end{bmatrix}. \quad (\text{C.3})$$

This partitioning however is not possible in FEM-CEDA. As explained in Chapter 3 the CEDA boundary conditions can only be applied when the equations are split up in real and imaginary components. The FEM-CEDA matrix equation can be rewritten in real variables and coefficients as:

$$\begin{bmatrix} \text{Re}\{\mathbf{C}\} & -\text{Im}\{\mathbf{C}\} \\ \text{Im}\{\mathbf{C}\} & \text{Re}\{\mathbf{C}\} \end{bmatrix} \begin{bmatrix} \overleftarrow{\mathbf{u}}_r \\ \overleftarrow{\mathbf{u}}_i \end{bmatrix} = \begin{bmatrix} \overleftarrow{\mathbf{F}}_r \\ \overleftarrow{\mathbf{F}}_i \end{bmatrix}. \quad (\text{C.4})$$

Now the boundary conditions can be applied. A boundary condition  $u(a) = u_a$  at position  $a$  can be represented in CEDA variables as:

$$\overleftarrow{u}_r(a) \cos(k_s a) - \overleftarrow{u}_i(a) \sin(k_s a) = u_a. \quad (\text{C.5})$$

The real or imaginary part of the variable can be eliminated by substitution of:

$$\overleftarrow{u}_r(a) = u_a + \frac{\sin(k_s a)}{\cos(k_s a)} \overleftarrow{u}_i. \quad (\text{C.6})$$

If this equation is substituted in the column  $\overleftarrow{\mathbf{u}}$ , the columns of matrix  $\mathbf{C}$  that correspond with  $\overleftarrow{u}_r(a)$  and  $\overleftarrow{u}_i(a)$  can be combined to one column and one variable:

$$\begin{bmatrix} | & & | & | & & | \\ \mathbf{C}_{1_r} & \cdots & \mathbf{C}_{a_r} & \mathbf{C}_{a_i} & \cdots & \mathbf{C}_{N_i} \\ | & & | & | & & | \end{bmatrix} \begin{bmatrix} \overleftarrow{u}_{1_r} \\ \vdots \\ \overleftarrow{u}_{a_r} \\ \overleftarrow{u}_{a_i} \\ \vdots \\ \overleftarrow{u}_{N_i} \end{bmatrix} = \begin{bmatrix} | & & | & & & | \\ \mathbf{C}_{1_r} & \cdots & \mathbf{C}_{a_i} + \frac{\sin(k_s a)}{\cos(k_s a)} \mathbf{C}_{a_r} & \cdots & & \mathbf{C}_{N_i} \\ | & & | & & & | \end{bmatrix} \begin{bmatrix} \overleftarrow{u}_{1_r} \\ \vdots \\ \overleftarrow{u}_{a_i} \\ \vdots \\ \overleftarrow{u}_{N_i} \end{bmatrix}. \quad (\text{C.7})$$

This reduces the  $2N \times 2N$  matrix  $\mathbf{C}$  to a  $2N \times 2N - 1$  matrix  $\mathbf{C}^a$ , which is not square. The unknown reaction force  $F(a)$  at this boundary and the corresponding distributed force field  $F(a)\overleftarrow{\delta}_a$  can be written at the left hand side of the equation to have all unknown variables at the left hand side and all known variables at the right hand side:

$$[\mathbf{C}^a] \begin{bmatrix} \overleftarrow{\mathbf{u}} \\ F(a) \end{bmatrix} - \overleftarrow{\delta}_a F(a) = \overleftarrow{\mathbf{F}} - \mathbf{C}_{a_r} u(a). \quad (\text{C.8})$$

Now the equations can be rewritten as:

$$\begin{bmatrix} \mathbf{C}^a & -\overleftarrow{\delta}_a \end{bmatrix} \begin{bmatrix} \overleftarrow{\mathbf{u}} \\ F(a) \end{bmatrix} = \overleftarrow{\mathbf{F}} - \mathbf{C}_{a_r} u(a). \quad (\text{C.9})$$

Now the set of equation has a  $2N$  equations in  $2N$  unknowns. The application of boundary conditions can be repeated and always leads to a set of  $2N$  equations in  $2N$  unknowns, since for every known (fixed) displacement an unknown reaction force results.

## Appendix D

# Filtering as an alternative to CEDA

During the research that is done on the FEM-implementation of CEDA, the idea rose that filtering could be used to deal with high frequency signals. This will be illustrated by figure D.1.

Assume a real signal that has a wave number spectrum with most of the energy concentrated in the high wave number region. When this signal is multiplied with  $e^{-jk_s x}$ , a complex signal results. Its wave number spectrum is shifted to the left over a distance  $k_s$ . When this shifted signal is filtered the high wave number contents can be filtered out and a low wave number complex signal results. The filtered spectrum has to be doubled to have the same spectral power contents. The original real signal can be reconstructed by taking the real part of the backward shift of the low wave number signal, or  $\text{Re}\{(\cdot)e^{jk_s x}\}$ . The accuracy of the reconstructed signal depends on the filtering. It is exact if the filter would be an ideal filter which filters out all frequency contents above  $k_s$ .

### Filtering in differential equations

The operators that are used in the above signal transformations can be represented by linear operators. This implies that they can also be applied to linear differential equations. A simple differential equation:

$$au''(x) + bu'(x) + cu(x) = f(x) \quad (\text{D.1})$$

is transformed in a similar way as in CEDA (see chapter 2) to:

$$a\hat{u}''(x) + [b + 2ajk_s]\hat{u}'(x) + [c + bjk_s - ak^2]\hat{u}(x) = f(x). \quad (\text{D.2})$$

The solution  $\hat{u}(x)$  is filtered after it is solved from the discretised equations and the solution to the original problem can be reconstructed.

To make the solution process more efficient the low wave number signal as in figure D.1 (c) must be the signal that is solved in a coarse discretised mesh. Therefore it is necessary to implement the filtering in the differential equations. Two options were investigated:

- Apply an inverse discretised filter in the discretised equations.
- Filter the input signal to prevent the high wave number signals from being excited.

These options are explained in more detail in the subsections below:

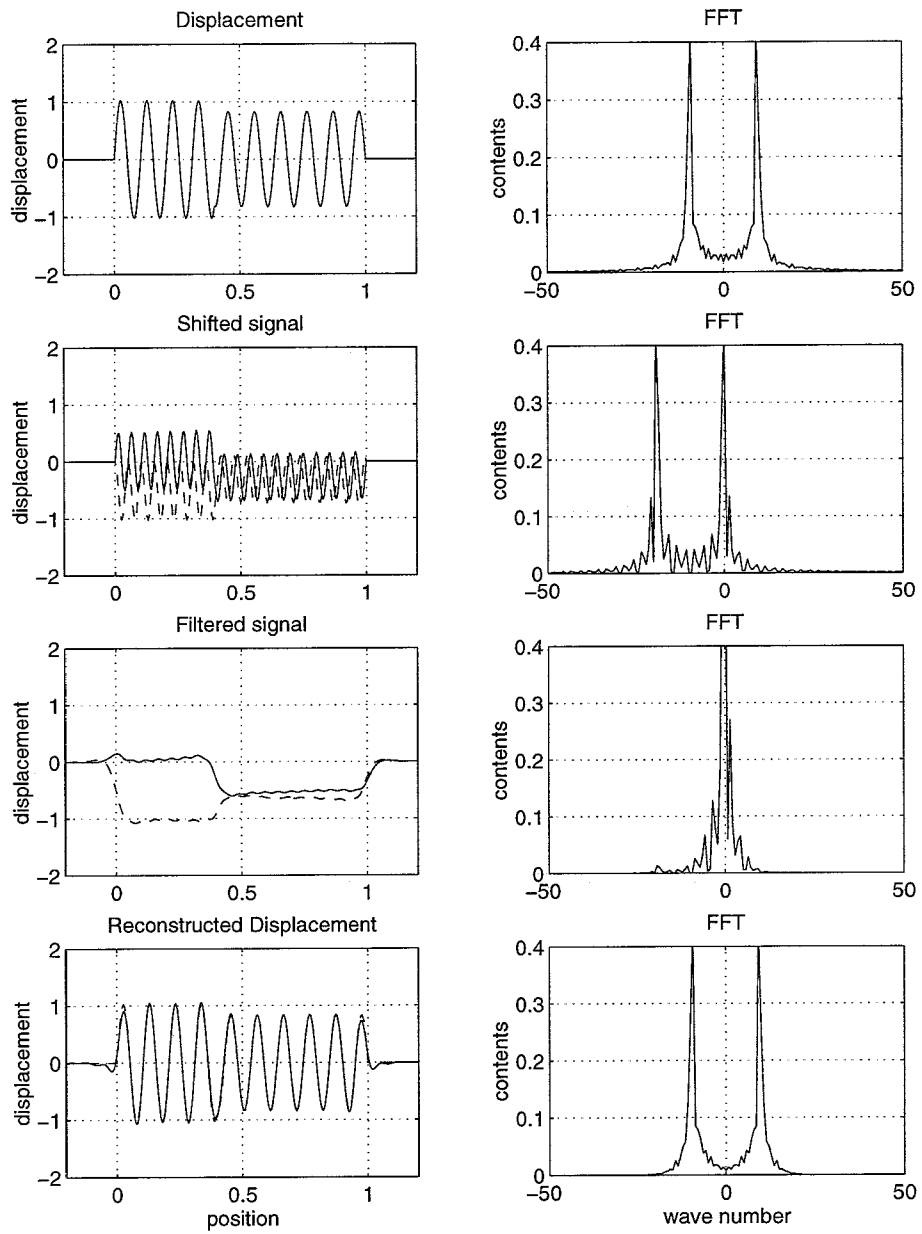


Figure D.1: Transformation to low wave number signals with filtering. Subfigures (a) to (d) show the displacement variable on the left and the corresponding wave number spectrum on the right. (a) Original displacement signal. (b) Shifted signal (real (-) and imaginary (- -)part). (c) Filtered signal (real (-) and imaginary (- -)part). (d) Reconstructed (-) and original (- -) signal.

## Inverse filter

The FEM discretised form of equation (D.2) solves the shifted signal  $\hat{u}(x)$  from the linear algebraic equation:

$$\mathbf{C}\hat{u}(x) = \mathbf{F}. \quad (\text{D.3})$$

The filtered low wave number signal  $\bar{u}(x)$  can be determined by a filtering with a discrete Butterworth filter designed according to the *transposed direct form II* structure in Oppenheim & Schaffer (1975). A digital filter of order  $N$  can be described by the difference equation:

$$y(n) = b_0x(n) + b_1x(n-1) + \dots + b_Nx(n-N) - a_1y(n-1) + \dots + a_Ny(n-N). \quad (\text{D.4})$$

The parameters of a filter can be calculated by a filter design procedure. The design procedure involves choosing the pass band and stop band specifications. This results in the choice for a filter order  $N$  and cut-off frequency  $\omega_N$  (natural frequency). When filtering digital signals the cut-off frequency  $\omega_N$  is often related to the sampling frequency of the signal, that determines the maximum frequency present in the signal. The dimensionless cut-off frequency can have a value between 0 (DC-pass filter) and 1 (all pass filter). When the filtered sequence is reversed and run back through the filter, the resulting sequence has precisely zero-phase distortion and the filter order is doubled.

The filtering of a signal can be performed by running a signal through the filter. With equation (D.4) the filter can be described in matrix notation as:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ a_1 & 1 & \ddots & \vdots \\ a_2 & a_1 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & a_2 & a_1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} b_0 & 0 & \dots & 0 \\ b_1 & b_0 & \ddots & \vdots \\ b_2 & b_1 & b_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & b_2 & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix}. \quad (\text{D.5})$$

When the matrices  $\mathbf{A}$  and  $\mathbf{B}$  can be inverted, the filter process can be described as:

$$\mathbf{y} = \mathbf{A}^{-1}\mathbf{B}\mathbf{x} = \mathbf{T}\mathbf{x}. \quad (\text{D.6})$$

Zero-phase distortion can be achieved by a backward filter  $\mathbf{T}_{\text{back}}$  that is the transposed of  $\mathbf{T}$  relative to the lower left to upper right diagonal. The filtering can now be applied as a matrix operation to the matrix equations of a FEM discretised problem.

$$\begin{aligned} \mathbf{C}\hat{u} &= \mathbf{F} \\ \bar{u} &= \mathbf{T}\hat{u} \\ \mathbf{C}\mathbf{T}^{-1}\bar{u} &= \mathbf{F} \end{aligned} \quad (\text{D.7})$$

This however does not work correctly when a coarse mesh is used. Due to undersampling large errors result in the solution. Filtering this solution is therefore meaningless.

## Filtering the input

It was assumed that the high wave number contents of the solution could also be eliminated from the solution when the input signal would not contain high wave number contents. Filtering the input signal however did not produce useful results.



