A formal definition of derivation trees in systems of natural deduction

696975

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computing Science

Memorandum 82-20

November 1982

A formal definition of derivation trees in systems of natural deduction

by

H. Balsters

P.O. Box 513
5600 MB Eindhoven
The Netherlands

A FORMAL DEFINITION OF DERIVATION TREES IN SYSTEMS OF NATURAL DEDUCTION

by

H. Balsters

Introduction

The motivation behind this note was a certain dissatisfaction with regard to the lack of formal rigour in familiar natural deduction presentations of first order logic. Usually such presentations are only in part formal and often use is made of intuitive pictures of so-called "derivation trees", without actually giving precise rules of their construction. The technique of "cancelling of hypotheses", for example, ususally lacks a proper formal treatment. Which hypotheses are to be cancelled and what cancelling formally amounts to is not made clear. Sometimes one has the impression that the cancelling technique is governed only by some pragmatic rule of thumb.

In this paper a formal recursive definition will be given of the set of derivation trees in a classical first order system of predicate logic, based on natural deduction. For an exposition of the intuitive background of our subject, see [1].

I. Preliminaries

0. Language

0.0. Alphabet

Our alphabet consists of the following symbols:

(i) k-ary predicate symbols : $p_0^k, p_1^k, p_2^k, \ldots, p_n^k, \ldots$

(ii) k-ary function symbols : $f_0^k, f_1^k, f_2^k, \ldots, f_n^k, \ldots$

(iii) constant symbols : $c_0, c_1, c_2, \ldots \ldots, c_n, \ldots$

(iv) variables : $x_0, x_1, x_2, \ldots \ldots, x_n, \ldots$

(v) connectives : $\wedge, \rightarrow, \bot, \forall$

(vi) the cancellation symbol : \$ (from cancelled \$entence)

(vii) auxiliary symbols : (,), ,

where $k, n \in \mathbb{N}$ and $\mathbb{N} = \{0,1,2,\ldots\}$.

The set of variables is denoted by VAR. Furthermore, we shall use meta-variables c (for constant symbols) and x,y (for variables).

0.1. We introduce three syntactical categories: TERM (the set of *terms*), FORM (the set of *formulas*) and CANCEL (the set of *cancelled formulas*).

0.1.0. Definition

TERM is the smallest set X such that

(i) $VAR \subset X$

(ii) $c_n \in X$

(iii) if $t_1,\ldots,t_k \in X$ then $f_n^k(t_1,\ldots,t_k) \in X$

where $k,n \in \mathbb{N}$.

0.1.1. Definition

FORM is the smallest set X such that

(i) $\perp \in X$

(ii) if $t_1,\ldots,t_k \in \text{TERM}$ then $p_n^k(t_1,\ldots,t_k) \in X$

(iii) if $\varphi,\psi \in X$ and $\Box \in \{\wedge, \rightarrow\}$ then $(\varphi \Box \psi) \in X$

(iv) if $\varphi \in X$ then $\forall x_n \varphi \in X$

where $k,n \in \mathbb{N}$.

The formulas in (i) and (ii) are called *atoms*. In what follows we shall use $\neg\varphi$ as an abbreviation for $(\varphi \rightarrow \perp)$.

0.1.2. Definition

$\text{CANCEL} := \{\$(\varphi) \mid \varphi \in \text{FORM}\}$.

Our language now consists of FORM $\cup$ CANCEL.

0.1.3. Remark. From now on we shall omit indices and just write an f or p instead of f_n^k or p_n^k to avoid messy notation.

1. Free variables

1.0. Definition

Let $t \in \text{TERM}$, then the set $FV(t)$ of *free variables of* t is defined by

(i) $FV(x) := \{x\}$

(ii) $FV(c_n) := \emptyset$

(iii) $FV(f(t_1,\ldots,t_k)) := \bigcup_{j=1}^{k} FV(t_j)$

where $k,n \in \mathbb{N}$.

1.1. Definition

Let $\varphi \in$ FORM, then the set $FV(\varphi)$ of *free variables of* φ is defined by

(i) $FV(\perp) := \emptyset$

(ii) $FV(p(t_1,\ldots,t_k)) := \bigcup_{j=1}^{k} FV(t_j)$

(iii) $FV(\varphi \,\Box\, \psi) := FV(\varphi) \cup FV(\psi)$

(iv) $FV(\forall x\, \varphi) := FV(\varphi) - \{x\}$

where $k \in \mathbb{N}$ and $\Box \in \{\wedge, \rightarrow\}$.

2. The substitution operator

2.0. Definition

Let $s,t \in$ TERM, then $[x:= t]s$ is defined by

(i) $[x:= t]y := \begin{cases} t, & \text{if } x \equiv y \\ y, & \text{otherwise} \end{cases}$

(ii) $[x:= t]c := c$

(iii) $[x:= t]\, f(t_1,\ldots,t_k) = f([x:= t]t_1,\ldots,[x:= t]t_k)$

where $k \in \mathbb{N}$. By $\equiv$ we denote syntactical equality.

2.1. Definition

Let $t \in$ TERM and $\varphi \in$ FORM, then $[x:= t]\varphi$ is defined by

(i) $[x:= t]\perp := \perp$

(ii) $[x:= t]\, p(t_1,\ldots,t_k) := p([x:= t]t_1,\ldots,[x:= t]t_k)$

(iii) $[x:= t]\, (\varphi \,\Box\, \psi) := ([x:= t]\varphi \,\Box\, [x:= t]\psi)$

(iv) $[x:= t]\, \forall y\varphi := \begin{cases} \forall y\, \varphi & , \text{if } x \equiv y \\ \forall y[x:= t]\varphi & , \text{otherwise} \end{cases}$

where $k \in \mathbb{N}$.

2.2. Definition

Let $t \in \mathrm{TERM}$ and $\varphi \in \mathrm{FORM}$.

t *is free for* x *in* φ iff

(i) φ is an atom

(ii) $\varphi \equiv (\varphi_1 \square \varphi_2)$ and t is free for x in both φ_1 and φ_2 $(\square \in \{\wedge, \rightarrow\})$

(iii) $\varphi \equiv \forall y\, \psi$ and a) $x \equiv y$,

or b) $x \notin FV(\psi)$,

or c) $y \notin FV(t)$ and t is free for x in φ.

II. Derivation trees

In this section we define the set of *derivation trees* in a system of natural deduction in a completely formal way, without resorting to two-dimensional diagrams. Ordered pairs will be denoted by $\langle\alpha,\beta\rangle$ and ordered triples by $\langle\alpha,\beta,\gamma\rangle$.

0. Definition

The set of *derivation trees* $\mathcal{T}$ is the smallest set X such that

(i) $\mathrm{FORM} \cup \mathrm{CANCEL} \subset X$

(ii) if $T \in X$ and $\varphi \in \mathrm{FORM}$ then $\langle T,\varphi\rangle \in X$

(iii) if $T_1, T_2 \in X$ and $\varphi \in \mathrm{FORM}$ then $\langle T_1,T_2,\varphi\rangle \in X$.

1. Definition

Let $T \in \mathcal{T}$, then *the root* $R(T)$ *of tree* T is defined by

(i) if $q \in \mathrm{FORM} \cup \mathrm{CANCEL}$ then $R(q) := q$

(ii) $R(\langle T,\varphi\rangle) := \varphi$

(iii) $R(\langle T_1,T_2,\varphi\rangle) := \varphi$.

2. Definition (cancelling of leaves of a tree)

C is the function defined by $C \in \mathcal{T}^{\mathcal{T}\times \mathrm{FORM}}$ and

(i) if $q \in \mathrm{FORM} \cup \mathrm{CANCEL}$ then $C(q,\psi) := \begin{cases} \$(q) & \text{, if } q \equiv \psi \\ q & \text{, otherwise} \end{cases}$

(ii) $C(\langle T,\varphi\rangle,\psi) := \langle C(T,\psi),\varphi\rangle$

(iii) $C(\langle T_1,T_2,\varphi\rangle,\psi) := \langle C(T_1,\psi)\ ,\ C(T_2,\psi),\varphi\rangle$

C is called the *cancel function*.

3. Definition

Let $T \in \mathcal{T}$, then $L(T)$ the set of *uncancelled leaf labels of* T is defined by

(i) if $q \in \mathrm{FORM}$ then $L(q) := \{q\}$

if $q \in \mathrm{CANCEL}$ then $L(q) := \emptyset$

(ii) $L(\langle T,\varphi\rangle) := L(T)$

(iii) $L(\langle T_1,T_2,\varphi\rangle) := L(T_1) \cup L(T_2)$

4. Definition

The set of *derivation trees* $\mathcal{D} \subset \mathcal{T}$ is the smallest set X such that

(i) $\mathrm{FORM} \subset X$

(ii) if $D_1,D_2 \in X$ then $\langle D_1,D_2,(R(D_1) \wedge R(D_2))\rangle \in X$
($\wedge$-introduction)

(iii) if $D \in X$ and $R(D) = (\varphi \wedge \psi)$ then $\langle D,\varphi\rangle \in X$ and $\langle D,\psi\rangle \in X$
($\wedge$-elimination)

(iv) if $D_1,D_2 \in X$, $R(D_1) = \varphi$ and $R(D_2) = (\varphi \rightarrow \psi)$ then $\langle D_1,D_2,\psi\rangle \in X$
($\rightarrow$-elimination)

(v) if $D \in X$, $R(D) = \psi$ and $\varphi \in \mathrm{FORM}$ then $\langle C(D,\varphi),\ (\varphi \rightarrow \psi)\rangle \in X$
($\rightarrow$-introduction)

(vi) if $D \in X$, $R(D) = \neg\neg\varphi$ then $\langle D\ ,\ \varphi\rangle \in X$
(double negation)

(vii) if $D \in X$, $R(D) = \varphi$ and $x \in VAR - \cup \{FV(\psi) | \psi \in L(D)\}$

then $\langle D, \forall x\varphi \rangle \in X$ ($\forall$-introduction)

(viii) if $D \in X$, $R(D) = \forall x\varphi$, $t \in TERM$ and t is free for x in φ then

$\langle D, [x := t]\varphi \rangle \in X$ ($\forall$-elimination).

What we have presented here is a simple set-theoretic definition of the concept of derivation tree which is completely formal; it does not depend on any extra information or suggestive visual aids.

References

[1] Van Dalen, D., : Logic and Structure Springer Verlag, Berlin 1980.

[2] Prawitz, D.: Natural Deduction, Almqvist & Wiksell, Stockholm 1965.