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## Six-Bar Cognates of a Stephenson Mechanism

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### Abstract

The paper treats the analogy of Roberts' Law for Stephenson-1 and Stephenson-2 mechanisms. Besides the investigation of cognates producing identical six-bar coupler curves of genus 7, the plural generation of identically moving coupler-planes by alternative six-bar linkages is observed.

**Zusammenfassung** — Verwandte Sechsgliedrige Gelenkgetriebe der Stephensonschen Mechanismen: E. A. Dijkman.

Die Arbeit behandelt ein Analogon des Robertsschen Satzes für Stephenson-Getriebe 1. und 2. Art. Neben der Untersuchung von verwandten Getrieben, welche identische Koppelkurven vom Geschlecht 7 erzeugen, wird auch die mehrfache Erzeugung der Bewegung einer Koppelene durch austauschbare Sechstabgetriebe betrachtet.

**Резюме** — Родственные шестизвенники механизмов Стефенсона: Е. Дийксман.

В работе рассматривается аналогия закона Робертса для механизмов Стефенсона 1 и 2 типа.

Кроме исследования родственных механизмов с идентичными так называемыми высшими шатунными кривыми шестого рода, также рассматривается многочисленное образование идентичных шатунных плоскостей альтернативными шестизвенными шарнирными механизмами.

### Summary

SIX-BAR linkages with the kinematic configuration of the mechanism of Stephenson have been investigated regarding the generation of identical moving coupler-planes and the generation of coupler-curves. The six-bars considered are those in which the fixed link is a binary one. It is shown that the *knee*-curve may generally be generated by a coinciding turning-joint of *three* equivalent six-bars of the Stephenson-1 type. Moreover, there is a *two*-fold generation of each of the two moving links attached to this turning-joint. Special cases are observed and an application on the well-known straight-line linkwork of Hart is given.

It is further shown that each of the binary links connecting the two moving triangles of a Stephenson-2 six-bar mechanism may be generated by two different six-bars of that type. The curve generated by a coupler-point attached to such a link will be produced by *four* different Stephenson-2 mechanisms. The cognates are also investigated in special cases such as the case where the two moving triangles of the Stephenson-2 mechanism are similar. One may then obtain  $\infty^2$  different cognates of Watt's form.

In addition, it will be shown that the moving triangle of a Stephenson-2 six-bar, not directly attached to the fixed link, may be generated by *three* cognates including the

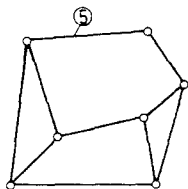
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initial one. And the turning joint between this triangle and one of the two binary links referred to above in a Stephenson-2 mechanism, may generally be generated by *five* different cognates of that type.

Finally, it will be shown that in case this triangle degenerates into a binary link, (at the same time turning a five-sided sub-chain into a four-sided one), a *five-fold* generation of a curve produced by any coupler-point attached to the binary link will occur.

## 1. Introduction

In 1846 Robert Stephenson designed a link reversing gear in order to control the motion of a steam engine. At the time, the mechanism was everywhere used as a valve gear for locomotives. Although the actual mechanism of Stephenson had very specific dimensions, Burmester[1] in 1888 associated the name of Stephenson to any linkage of this kind. The kinematic configuration of such a linkage, named after Stephenson, is shown in Fig. 1. It is a six-bar linkage with a five-sided loop and a four sided loop as two *independent* sub-chains.



**Figure 1.** Stephenson's form.

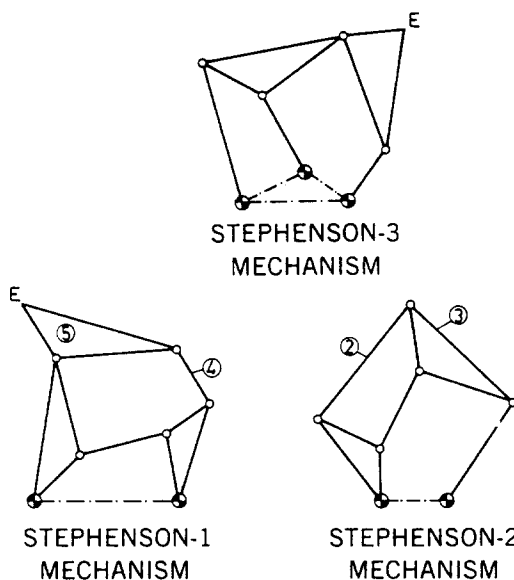
Nowadays, only historical value can be attached to the fact that Stephenson made bar 5 the body (frame-work) of the locomotive. (The mechanism obtained by this choice of frame is now commonly called a Stephenson-2 mechanism.) Successively choosing other bars as frame, one obtains no more than three essentially different mechanisms (see Fig. 2). Of these, the one with three pivot centers on the frame has already been investigated by Rischen[2], who found *six* mechanisms of this type with the same six-bar curve, generated by point *E* of the mechanism. All six mechanisms having the same configuration of Stephenson and all having three pivot centers on the frame are called *cognates*.

The name *cognate* was first introduced by Hartenberg and Denavit in their paper[10, 11] on "*cognate linkages*," with respect to alternative four-bar mechanisms, coupler-points of which trace identical coupler curves. The name was coined in connection with the three alternative linkages appearing in the well-known theorem of Roberts–Chebyshev. Hartenberg and Denavit also extended the meaning of cognation to special forms of six-bar linkages. It is for this reason, that the writer of the present paper took the liberty to do likewise, even in those cases where there is no restriction on the link lengths of the sixbars of the linkage.

Since Roberts[12] discovered the existence of alternative linkages in the case of four-bars, many authors, including Cayley[13], Schor[14], Meyer zur Capellen[15] and de Jonge[16] have occupied themselves with this theorem. Each of them has found another proof of Roberts' Law, and accordingly the number of proofs has steadily increased and a situation has been arising that is similar to the one which developed around the proposition of Pythagoras at one time.

According to the spirit of these times, however, (with the ever-growing attention given to the practical use of the computers and so on) most of these efforts have been in the range of *analytical* treatment in contrast to the *geometrical* way of finding four-bar cognates, as originally practised by Roberts!

It was, maybe, for this reason, that nobody paid much attention to a new geometrical proof of Roberts' Law, as given by H. Pflieger–Haertel[17] during the 2nd World War. Although he was the first to use the



**Figure 2.** Stephenson-1, -2 and -3 types of mechanism.

principle of *stretch-rotation* in this matter, he also hesitated to rely on it entirely and therefore added a proof based on a known derivation in complex numbers.

Also, in a paper [18] about parallel moving bars a proof of Robert's Law, based on the principle of stretch-rotation, has been given.

In the paper presented here, the *geometrical* principle of stretch-rotation is entirely relied on, and the application of this principle has been extended to six-bar linkages of Stephenson's form.

It may be emphasized, therefore, that in spite of the existence of computers, the quickest way to find the cognates of such forms follows the classic treatment of geometry, although I am sure that, once we know of their existence, other, *analytical*, proofs may be found with which the designer will be enabled to calculate the dimensions of the cognates with the help of those same computers.

Finally, it is thought necessary to emphasize a fact which is probably not recognized, that the proofs of the *existence* of the six-bar cognates of Stephenson's form are also given in this paper by means of the principle of stretch-rotation: the presentation given here provides not only the way of finding cognates, but also simultaneously the proof, that the given statements are true.

In addition to the investigation of Rischen, we are interested in the number of cognates with regard to the remaining types, viz. the Stephenson-1 and the Stephenson-2 mechanism. Further, we want to know how to design such alternative mechanisms. Thus, from the Stephenson-1 mechanism, for example, we investigate the multi-generation of the curve produced by an arbitrarily chosen point  $E$  in the moving plane 5. Each mechanism of the Stephenson-1 type which is able to generate the identical six-bar curve, will then be called a cognate of the initial mechanisms. A similar assumption will hold for cognates of the Stephenson-2 type. It is, however, important to include in such denominations of cognates an indication of the link to which the generating point  $E$  is going to be attached; for, generally, it is not possible to generate the same curve by different Stephenson-2 (or -1) mechanisms if the generating point  $E$  is not attached to similarly situated links in the mechanisms.

In order to investigate essentially different cases, it must be clear that only the plane 5 of a Stephenson-1 mechanism is of interest (The investigation of cognates generating the identical six-bar curve produced by a point of plane 4 may be done in a similar way). A Stephenson-2 mechanism, however, has two links, viz. the links 2 and 3, which play an interesting part in our search for cognates.

## 2. The Two-Fold Generation of the Moving Plane 5 of a Stephenson-1 Mechanism and the Three-Fold Generation of the Knee-Curve\*

The initial Stephenson-1 mechanism consists of the five-bars<sup>†</sup>  $\square A_0FDCB_0$ ,  $\square FABCD$ , the four-bar  $\square A_0ABB_0$  and the rigid triangles  $\triangle A_0AF$ ,  $\triangle B_0BC$  and  $\triangle FDE$  (see Fig. 3). The curve described by point  $D$  of the mechanism is called the *knee-curve* which is an *algebraic curve of order 14* and *genus 7*[3]. It will be shown that this curve may be generated by *three* different Stephenson-1 mechanisms. The curve generated

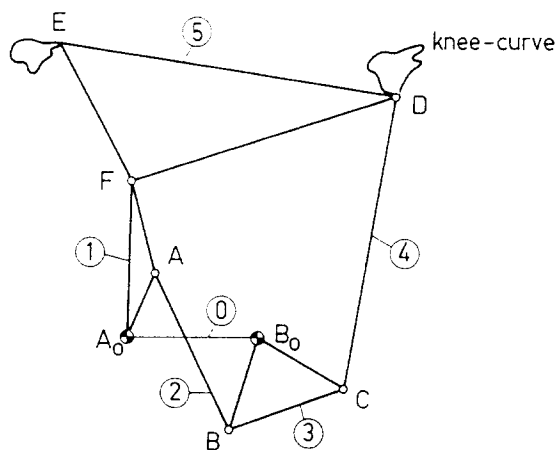


Figure 3. Initial Stephenson-1 mechanism.

by a point  $E$  of the moving plane 5 may be generated by not more than *two* different Stephenson-1 mechanisms. It will also be shown that the two last mentioned cognates generate the *entire* moving plane 5. According to Fig. 4 the design for such an alternative six-bar mechanism may be obtained in the following way.

(a) Form the linkage parallelograms  $ABCB'$  and  $ABB_0B'$ .

(b) Turn the four-bar  $A_0AB'B_0$  about  $A$  through the angle.

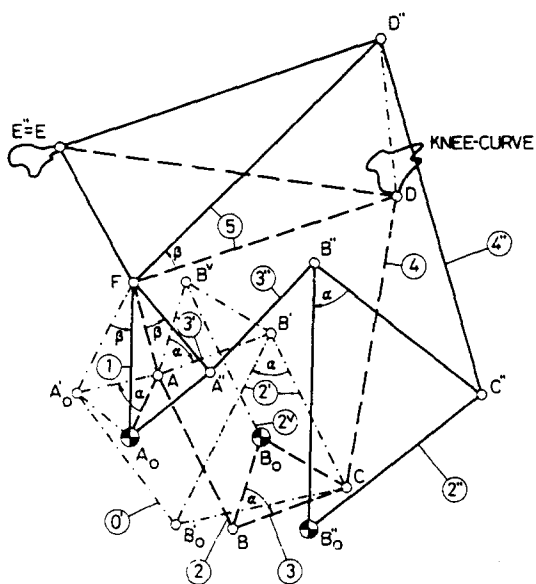
$\alpha = \sphericalangle B'AB' = \sphericalangle B_0BC$ , and multiply the four bars in size simultaneously from  $A$  by the factor  $f_\alpha = B'A/B'A = CB/B'B$ .

(c) One obtains the four-bar  $\square A_0AB'B'_0 \sim \square A_0AB'B_0$ .

(d) One now forms the rigid quadrilateral  $FAA_0A'_0$  and the rigid triangle  $\triangle B'B'_0C$  which is similar to  $\triangle BCB_0$  (The corresponding links of the four-bars  $A'_0AB'B'_0$  and  $A_0AB'B_0$  move at the same angular velocity at any time. The opposite sides of the parallelogram  $AB'CB$  move also at identical angular velocities. Therefore, so do the links  $B'B'_0$  and  $B'C$ . Since they also have a turning-joint  $B'$  in common, both links belong to the same moving plane 2' and  $\triangle B'B'_0C$  is a rigid triangle. The same may be said of  $\triangle AB'B'$ . By agreement we further state that: If at any time two non-identical moving planes have the same angular velocity, such planes will bear the same figure, and the non-identity will be indicated by different numbers of primes attached to the figure).

\*The name of the curve has been introduced by R. Müller in 1895. ("Ueber eine gewisse Klasse von übergeschlossenen Mechanismen" *Z. f. Math. u. Phys.* Vol. 40 pp. 257–278.)

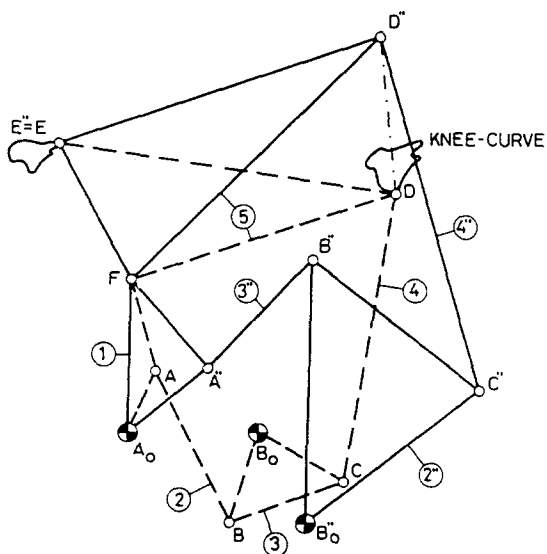
†(The author uses the term "five-bar" to mean what is more usually called "five-sided, or pentagonal, loop"; similarly "four-bar" signifies a "quadrilateral loop").



**Figure 4.** The two-fold generation of plane 5 of a Stephenson-1 mechanism.

(e) Next turn the two five-bars  $\triangle FAB'CD$  and  $\triangle FA'B'CD$  about  $F$  through the angle  $\beta = \angle A_0FA'_0$  and multiply the five-bars in size simultaneously by the factor  $f_\beta = FA_0/FA'_0$ .

(f) One thus obtains a new six-bar consisting of the five-bars  $\triangle FA''B''C''D''$  and  $\triangle FA_0B_0C''D''$ , the four-bar  $\square A_0A''B''B_0$  and the rigid triangles  $\triangle A_0FA''$ ,  $\triangle B_0B''C''$ , and  $\triangle FD''E$ . In this six-bar, point  $B_0$  is a second pivot center on the frame of the alternative mechanism (see also Fig. 5). Like the initial one, the six-bar thus found has one degree of freedom of movement. It has also a configuration similar to a Stephenson-1 mechanism. Therefore it may be called a Stephenson-1 cognate. The cognate



**Figure 5.** The two-fold generation of plane 5 of a Stephenson-1 mechanism.

thus found produces the same moving plane 5 as the initial one. That is to say, any generating point of plane 5 will be produced by both mechanisms and the generating points of the two mechanisms are coincident points at every point of time. So both points will generate the identical curve.

### 2.1 General account of the presented design

A linkage mechanism consists of several closed kinematic sub-chains or loops. For instance, the Stephenson-I mechanism has two five-bars and one four-bar.

These sub-chains, however, may be seen as closed vector polygons, which can be represented by equivalent vector equations.

Let the vector equation

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_0 = \mathbf{0} \quad (1)$$

represent the four-bar  $A_0ABB_0$ , and

$$\mathbf{b}_1 + \mathbf{b}_5 + \mathbf{b}_4 + \mathbf{b}_3 + \mathbf{a}_0 = \mathbf{0} \quad (2)$$

represent the five-bar  $A_0FDCB_0$  (see figs. 4 and 5.) (The order of these vectors may not be interchanged, otherwise other linkages are represented. For instance, the equation  $\mathbf{a}_1 + \mathbf{a}_3 + \mathbf{a}_2 + \mathbf{a}_0 = \mathbf{0}$  represents the four-bar  $A_0AB'B_0$ ). Then the equation

$$(\mathbf{b}_1 - \mathbf{a}_1) + \mathbf{b}_5 + \mathbf{b}_4 + (\mathbf{b}_3 - \mathbf{a}_3) - \mathbf{a}_2 = \mathbf{0} \quad (3)$$

represents the second five-bar  $AFDCB$ .

It is clear that the last equation is not independent of the first two. Therefore, we say that a Stephenson-I mechanism consists of two *independent* sub-chains, viz. a four-bar and a five-bar.

If we multiply\* equation (1) by the constant complex number  $\mathbf{b}_3/\mathbf{a}_3$  and subtract the result from (2), we get

$$\mathbf{b}_1 - (\mathbf{b}_3/\mathbf{a}_3)\mathbf{a}_1 + \mathbf{b}_5 + \mathbf{b}_4 - (\mathbf{b}_3/\mathbf{a}_3)\mathbf{a}_2 + \mathbf{a}_0(1 - \mathbf{b}_3/\mathbf{a}_3) = \mathbf{0} \quad (4)$$

which is a five-bar of the form

$$\mathbf{c}_1 + \mathbf{b}_5 + \mathbf{b}_4 + \mathbf{c}_2 + \mathbf{c}_0 = \mathbf{0}. \quad (4a)$$

This represents the five-bar  $A'_0FDCB'_0$  (see Fig. 4).

Interchanging the sequence of the bars in the kinematic chain represented by equation (3), means a permutation in the sequence of the terms in this equation.

Therefore, the equation

$$(\mathbf{b}_1 - \mathbf{a}_1) + \mathbf{b}_5 + \mathbf{b}_4 - \mathbf{a}_2 + (\mathbf{b}_3 - \mathbf{a}_3) = \mathbf{0} \quad (5)$$

represents the five-bar  $AFDCB'$ .

Both five-bars represented by the equations (4) and (5) are independent of each other. If we subtract the left-hand side of (5) from (4), we obtain the vector equation of

\*Here the center of similitude coincides with turning-joint  $A$ .

the four-bar  $A'_0AB'B'_0$

$$\mathbf{a}_1(1 - \mathbf{b}_3/\mathbf{a}_3) + (\mathbf{a}_3 - \mathbf{b}_3) + \mathbf{a}_2(1 - \mathbf{b}_3/\mathbf{a}_3) + \mathbf{a}_0(1 - \mathbf{b}_3/\mathbf{a}_3) = \mathbf{0}. \quad (6)$$

The link  $A'_0B'_0$  of the five-bar (4) and of the four-bar (6), however, is only a *translating* link. It can be made a *fixed* link if we multiply the left-hand sides of (4), (5) and (6) by the complex number  $\mathbf{b}_1/(\mathbf{b}_1 - (\mathbf{b}_3/\mathbf{a}_3)\mathbf{a}_1)$ , for then the point  $A'_0$  of the link  $A'_0B'_0$  which is already only translating, transforms itself into the fixed center  $A_0$ . And so the cognate Stephenson-1 mechanism will be obtained. The center of similitude of the last multiplication coincides with turning joint  $F$ . Therefore, the transformed moving planes 1 and 5 remain attached to the initial ones.

The multiplied five-bar (4), for instance, will then contain the vector  $\mathbf{b}_1$  again, that is the bar  $A_0F$ , of the initial mechanism. Thus, the dyad  $A_0FE$  is a common dyad to both cognates.

The fact that no other cognates for the generation of the curve produced by point  $E$  exist, will be proved by indirect demonstration (*reductio ad absurdum*).

The generating point  $E$  is connected with the fixed center  $A_0$  through the dyad  $A_0FE$ . This dyad is represented by the vector sum  $(\mathbf{b}_1 + \mathbf{d}_5)$ . It is not possible to find a cognate by interchanging the terms in this vector sum. This is due to the fact that a four-bar linkage of which the sides move parallel to the respective planes 0, 2, 3 and 5, cannot be composed. Likewise, it is prohibited to interchange the moving planes 3 and 4. And lastly, the fixed link 0 may never be interchanged, since, generally, no other moving plane of the initial mechanism moves parallel to the fixed link.

We assume that in a supposed cognate, point  $E$  has to be connected with the fixed link through some dyad. Instead of going through all the arising possibilities we just take one dyad and prove that such an arbitrarily chosen combination cannot be effectuated. Suppose, then, that the dyad that connects point  $E$  of the cognate with the fixed link, is represented by the vector sum  $(\mathbf{e}_3 + \mathbf{e}_4)$ . Here,  $\mathbf{e}_3$  is a vector moving parallel to a fixed line in plane 3, and  $\mathbf{e}_4$  moves parallel to plane 4. Although the fixed center does not necessarily coincide with  $A_0$  or  $B_0$  we may nevertheless indicate a hexagon of which the triad  $B_0CDE$  takes a part.

So we may write

$$\mathbf{e}_0 + \mathbf{e}_3 + \mathbf{e}_4 + (\mathbf{b}_5 - \mathbf{d}_5) + \mathbf{b}_4 + \mathbf{b}_3 = \mathbf{0}$$

or by interchanging the terms through making use of linkage parallelograms, we get the four-bar

$$\mathbf{e}_0 + (\mathbf{e}_3 + \mathbf{b}_3) + (\mathbf{e}_4 + \mathbf{b}_4) + (\mathbf{b}_5 - \mathbf{d}_5) = \mathbf{0}.$$

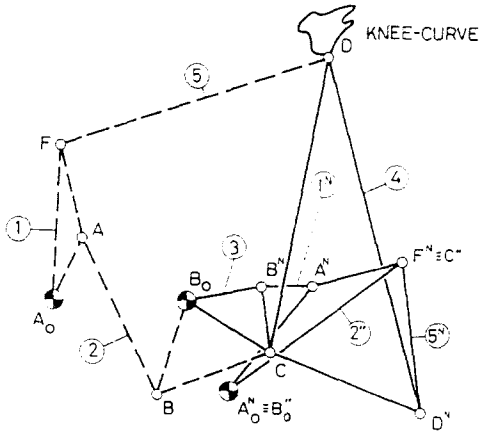
Since it is not possible to design a four-bar with links moving parallel to the planes 0, 3, 4 and 5, we encounter a contradiction. Therefore, the assumption made that the dyad mentioned may be represented by the vector sum  $(\mathbf{e}_3 + \mathbf{e}_4)$ , is not valid.

In this way going through all other combinations we find that, unless  $E$  coincides with  $D$ , there is only one dyad possible, viz. the dyad  $\mathbf{b}_1 + \mathbf{d}_5$ , the one used in the cognate already found.

In the case that  $E$  coincides with turning joint  $D$ , we may also consider this point a generating point of the moving plane 4. Since the latter takes a similar position in the Stephenson-1 mechanism, we may also interchange the planes 1 and 2 instead of the



planes 3 and 2 as was done for the preceding cognate. We thus obtain another cognate for the *knee-curve* as demonstrated in Fig. 6. So all in all there are three cognates generating the knee-curve.



**Figure 6.** Two, out of three, cognates for the knee-curve.

From the design shown in Fig. 4 we derive that

$$\diamond FA''B''C''D'' \sim \diamond FAB'CD; \quad \diamond FA_0B_0''C''D'' \sim \diamond FA_0B_0'CD.$$

and

$$\square A_0A''B''B_0'' \sim \square A_0'A'B'B_0'.$$

It also follows that

$$\triangle FAA'' \sim \triangle FA_0'A_0 \sim \triangle FDD''$$

and

$$\triangle C''B''B_0'' \sim \triangle B_0BC \sim \triangle A_0A'A_0'.$$

Similarly, Fig. 6 demonstrates that

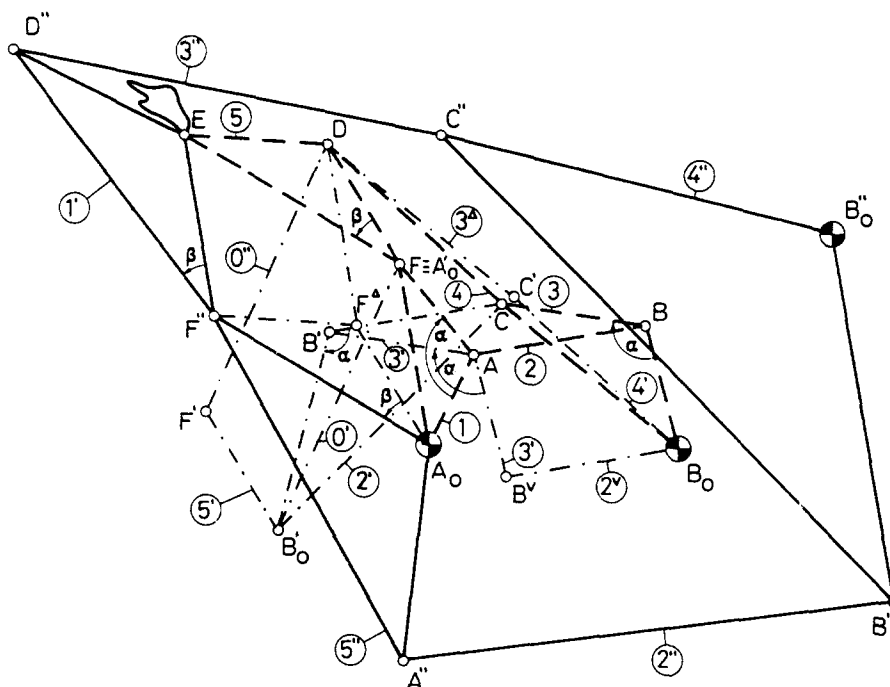
$$\triangle F^NA^NA_0^N \sim \triangle A_0AF.$$

### 2.2 Special case where $\triangle A_0AF \sim \triangle B_0BC$

In this case  $A_0' \equiv F$  and the preceding design collapses. So another approach is adopted at the point where first the second multiplication could not be brought into effect. The design for this case may be carried out with the following instructions (see Fig. 7):

(a) Form the linkage parallelograms  $ABCB'$ ,  $ABB_0B_0'$ ,  $A_0FEF''$ ,  $A_0FDF^{\Delta}$  and  $B_0CDC'$ .

(b) Turn the four-bar  $A_0AB_0'B_0$  about  $A$  through the angle  $\alpha = \sphericalangle B^{\vee}AB' = \sphericalangle B_0BC = \sphericalangle A_0AF$  and multiply the size of the four-bar geometrically from  $A$  by the factor  $f_{\alpha} = B'A/B^{\vee}A = CB'/B_0B = FA'/A_0A$ .



**Figure 7.** Special case of the Stephenson-1 mechanism with  $\triangle A_0AF \sim \triangle B_0BC$ .

- (c) One obtains the four-bar  $\square FAB'B'_0 \sim \square A_0AB''B_0$ .
- (d) One forms the rigid triangle  $B'B'_0C \sim \triangle BCB_0 \sim \triangle AFA_0$ .
- (e) Next, one forms the linkage parallelogram  $B'_0FDF'$ .
- (f) Then, turn the five-bar  $\diamond A_0F\triangle DC'B_0$  about  $A_0$  through the angle  $\beta = \sphericalangle DFE = \sphericalangle F\triangle A_0F''$  and multiply the size of the five-bar geometrically from  $A_0$  by the factor  $f_\beta = EF/DF = F''A_0/F\triangle A_0$ .
- (g) One obtains the five-bar  $\diamond A_0F''D''C''B''_0 \sim \diamond A_0F\triangle DC'B_0$ .
- (h) Form the rigid triangle  $\triangle D''F''E \sim \triangle EFD$ .
- (i) Make the four-bar  $\square A_0B''_0B''A'' \sim \square F'DCB'_0$ .
- (j) Form the rigid triangles  $F''A_0A''$  and  $C''B''_0B''$ .

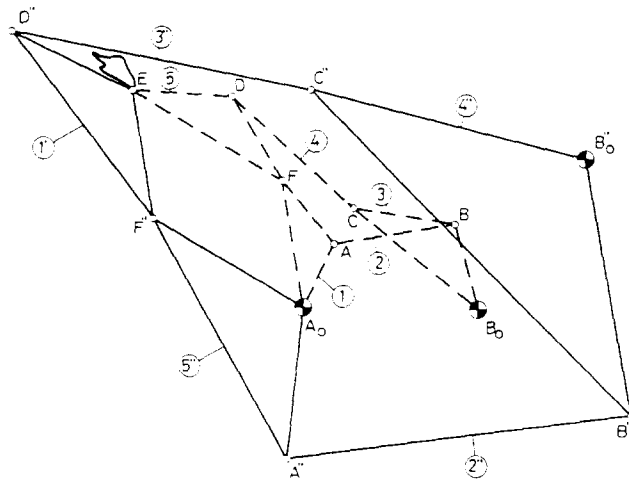
The cognate obtained consists of the five-bars  $\diamond A_0F''D''C''B''_0$ ,  $\diamond A''F''D''C''B''$ , the four-bar  $\square A_0B''_0B''A''$  and the rigid triangles  $\triangle A_0F''A''$ ,  $\triangle B''_0C''B''$  and  $\triangle D''F''E$  (see also Fig. 8).

In order to justify the correctness of the above result, it is sufficient to prove that the triangles to be indicated as rigid triangles are rigid indeed. In all cases arising, this may be done by proving that two sides of such a triangle are moving at identical angular velocity.

With regard to the geometric properties of the cognate obtained, one can prove that  $\triangle A_0F''A'' \sim \triangle AFA_0$ , and likewise that  $\triangle B''_0C''B'' \sim \triangle BCB_0$ . Thus

$$\triangle A_0F''A'' \sim \triangle B''_0C''B''.$$

So we see that for the initial mechanism as well as for the cognate one, the two triangles rotating about a fixed center are similar.

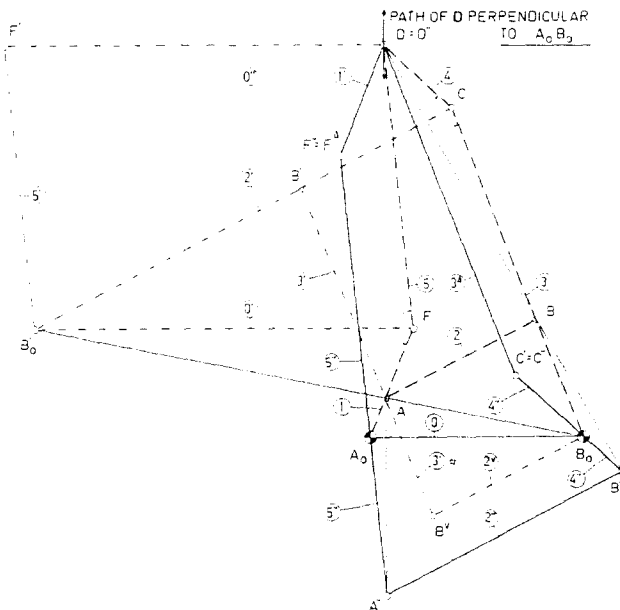


**Figure 8.** Special case of the Stephenson-1 mechanism with  $\triangle A_0AF \sim \triangle B_0BC$ .

We also find that

$$\triangle D''F''E \sim \triangle EFD \sim \triangle B''A_0B_0$$

2.2.1. *Application on a straight-line linkwork of Hart.* An exact straight-line linkwork of Hart[4], which is a special case of the mechanism of Kempe[5], which is also called the focal mechanism of Burmester[6], may be seen as a Stephenson-1 mechanism for which  $\triangle A_0AF \sim \triangle B_0BC$ . The four-bar  $A_0ABB_0$  in this linkwork (see Fig. 9) is chosen arbitrarily.



**Figure 9.** Two straight-line linkworks of Hart in cognation.

We took  $A_0A = 1$ ,  $AB = 4$ ,  $BB_0 = 3$  and  $A_0B_0 = 5$ . The remaining dimensions of the linkwork were calculated from formulae obtained by Wunderlich [7], and so we got  $A_0F = 25/9$ ,  $B_0C = 75/9$ ,  $FD = 20/3$  and  $CD = -20/9$ . The initial linkwork of Hart so consists of the five-bars  $A_0FDCB_0$ ,  $AFDCB$  and the four-bar  $A_0ABB_0$ . The knee-curve generated by point  $D$  is in this case an exact straight line perpendicular to the frame-line  $A_0B_0$ . Moreover, the angle  $\sphericalangle A_0FD \equiv \sphericalangle B_0CD$  as shown by Kempe.

If we apply the design procedure of the preceding section to this linkwork, we obtain the six-bar cognate consisting of the five-bars  $A_0F''D''C''B_0$ ,  $A''F''D''C''B''$  and the four-bar  $A_0A''B''B_0$ .

It follows from the design that  $\square A_0FDF''$  and  $\square B_0CDC''$  are linkage parallelograms. It may be proved that  $A''B''$  lies parallel to  $AB$  at any time. Further, one finds that the points  $D$ ,  $A$  and  $A''$  are always in line, and the same is true for the points  $D$ ,  $B$  and  $B''$ . We see that the *cognate* linkwork is also an exact straight-line linkwork of Hart, thus connected with the initial one (see Fig. 9).

In addition, we observe that the cognate linkwork may also be obtained by making the coupler  $AB$  of the initial linkwork the fixed link instead of  $A_0B_0$ , by simultaneously multiplying the entire linkwork by the factor  $CD/AF$  and, finally, by turning the linkwork thus obtained about some straight axis in the plane around by  $\pi$  radians. This proves that the turning joint  $D$  of the initial mechanism also moves along a straight line perpendicular to the coupler  $AB$  [7].

### 3. The 4 Cognates for a Stephenson-2 Mechanism with Respect to a Point E of Plane 2

#### 3.1. The two-fold generation of plane 2 and the 3/4-cognate for that plane

The initial Stephenson-2 mechanism consists of the five-bars  $A_0ABCC_0$ ,  $A_0FDCC_0$ , the four-bar  $\square ABDF$  and the rigid triangles  $\triangle A_0AF$ ,  $\triangle BCD$  and  $\triangle FDE$ . The centers of pivot on the frame in this mechanism are the points  $A_0$  and  $C_0$  (see Fig. 10). Design instructions for what is called the 3/4-cognate are as follows (see Fig. 11):

- (a) Form the parallelograms  $ABDB'$  and  $ABCB'$ .
- (b) Turn the five-bar  $\square A_0AB'CC_0$  about  $A$  through the angle  $\alpha = \sphericalangle B'AB' = \sphericalangle CBD$  and multiply the five-bar geometrically from  $A$  by the factor  $f_\alpha = B'A/B'A = DB/CB$ .
- (c) One obtains the five-bar  $\square A'_0AB'C'_0 \sim \square A_0AB'CC_0$  and the rigid triangles  $\triangle A_0A'_0A$  and  $\triangle B'C'D$ .
- (d) Next, turn the five-bar  $\square FA'_0C'_0D$ , together with the four-bar  $FAB'D$  about  $F$  through the angle  $\beta = \sphericalangle A'_0FA_0$  and multiply both geometrically from  $F$  by the factor  $f_\beta = FA_0/FA'_0$ .
- (e) One thus obtains the five-bar  $\square FA_0C''_0C''D'' \sim \square FA'_0C'_0C'D$  and the four-bar  $\square FA''B''D''$ .
- (f) Lastly, form the rigid triangles  $\triangle A_0A''F$ ,  $\triangle B''C''D''$ ,  $\triangle FD''D$  and  $\triangle FD''E$ .

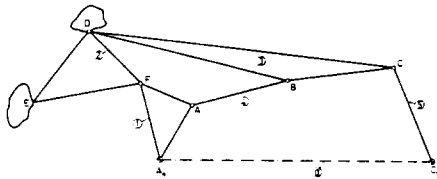
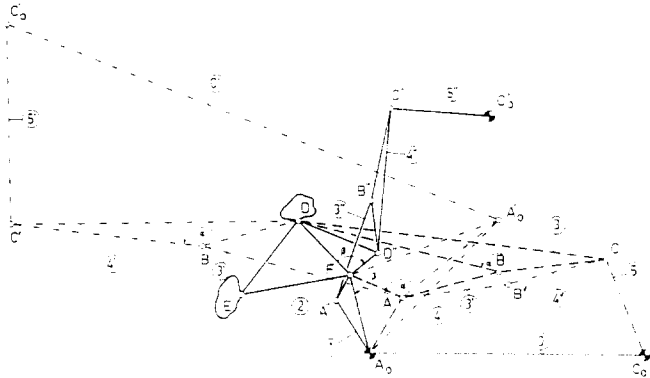


Figure 10. Initial Stephenson-2 mechanism.



**Figure 11.** The 3/4-cognate of a Stephenson-2 mechanism for plane 2.

One recognizes the cognate six-bar, consisting of the five-bars  $\diamond A_0A''B''C''C''_0$ ,  $\diamond A_0FD''C''C''_0$ , the four bar  $\square A''B''D''F$  and the rigid triangles  $\triangle A_0A''F$ ,  $\triangle B''C''D''$ ,  $\triangle FD''D$  and  $\triangle FD''E$ .

The cognate obtained has the same kinematic configuration as a Stephenson-2 mechanism (see Fig. 12). The angular velocities of the links in the cognate can at any time also be observed in the initial mechanism. With the exception of the angular velocities of the links 3 and 4, they are distributed in the same way among the links as is the case with the initial mechanism. One finds that only the angular velocities of the links 3 and 4 have been interchanged in the cognate. Therefore, the cognate is called a *3/4-cognate* for short.

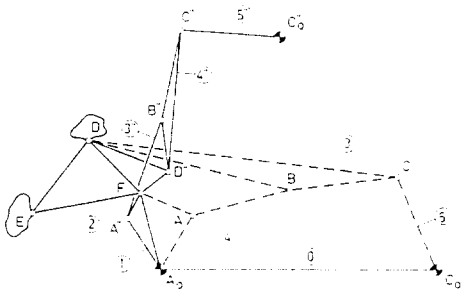
The other thing one observes is that the *entire* plane 2 of the initial mechanism is generated by the 3/4-cognate. Thus a two-fold generation of the moving plane 2 exists. A six-bar curve produced by any point  $E$  of this plane will then be generated by both cognates. The same holds for turning joint  $D$  of the initial mechanism (see Fig. 12). One notes also that the dyad  $A_0FE$  is always common to the initial and to the 3/4-cognate.

The geometric properties of the 3/4-cognate may be derived through the following observations (see Fig. 11):

$$\triangle AA_0A'_0 \sim \triangle B'DC' \sim \triangle BCD \sim \triangle B''D''C''.$$

From (d) we derive that

$$\diamond FA_0C''_0C''D'' \sim \diamond FA'_0C'_0C'D \text{ and } \square FA''B''D'' \sim \square FAB'D.$$



**Figure 12.** The 3/4-cognate of a Stephenson-2 mechanism for plane 2.



- (a) Form the parallelograms  $A_0FEF'$ ,  $A_0FDF''$  and  $AFDF'$ .  
 (b) Turn the five-bar  $A_0F''DCC_0$  about  $A_0$  through the angle  $\alpha' = \angle F'A_0F'' = \angle EFD$  and multiply the five-bar geometrically by the factor  $f_{\alpha'} = F'A_0/F''A_0 = EF/DF$ . Thus  $\square A_0F'D'C''C_0'' \sim \square A_0F''DCC_0$ .  
 (c) Form the rigid triangles  $\triangle A_0C_0''C_0$  and  $\triangle D'F'E$ .  
 (d) Make  $\square A''B''D'F' \sim \square ABDF$ .  
 (e) Form the rigid triangles  $\triangle C''B''D'$  and  $\triangle A''F'A_0$ .

The cognate obtained consists of the five-bars  $\square A_0A''B''C''C_0''$ ,  $\square A_0F'D'C''C_0''$ , the four-bar  $\square A''B''D'F'$  and the rigid triangles  $\triangle A_0A''F'$ ,  $\triangle B''C''D'$  and the coupler-triangle  $\triangle D'F'E$ . The alternative mechanism has the same kinematic configuration as the initial mechanism and is therefore a cognate Stephenson-2 mechanism. Here, only point  $E$  of the cognate generates the identical six-bar curve.

When one observes the distribution of angular velocities in the cognate obtained, one sees that only the angular velocities of the links 1 and 2 of the initial mechanism have interchanged. Therefore, the cognate is called a *1/2-cognate* for short (see Fig. 13).

The geometric properties derived from the figure are obtained in the following way:

From (b) it follows that

$$\triangle C_0''A_0C_0 \sim \triangle EFD \sim \triangle D'F'E.$$

Since  $\angle A''F'A_0 = \angle A_0FA$  and

$$\frac{A''F'}{F'A_0} = \frac{A''F'}{D'F'} \cdot \frac{D'F'}{EF} = \frac{FD}{FA} \cdot \frac{D'F'}{F'E} \cdot \frac{F'E}{EF} = \frac{FD}{FA} \cdot \frac{EF}{FD} \cdot \frac{F'E}{EF} = \frac{F'E}{FA} = \frac{A_0F}{FA}$$

we find that

$$\triangle A''F'A_0 \sim \triangle A_0FA.$$

Since  $\angle (D'C'', DC) = \alpha'$  and  $\angle (BD, B''D') = \alpha' + \pi - \angle A''F'A_0$  we have  $\angle B''D'C'' = 2\pi - (\angle BDC + \angle A_0FA)$ .

Moreover,

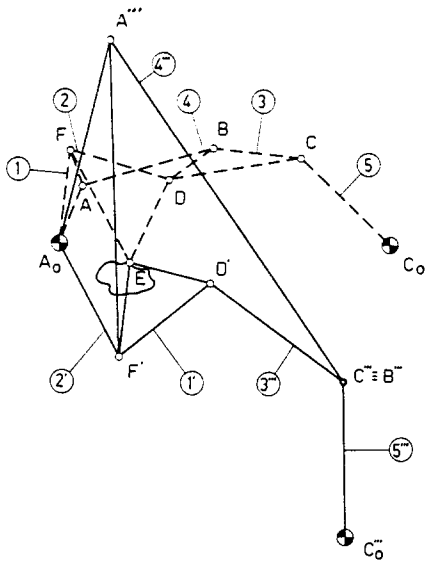
$$\frac{B''D'}{C''D'} = \frac{B''D'}{F'D'} \cdot \frac{F'D'}{A_0F} \cdot \frac{CD}{C''D'} \cdot \frac{A_0F}{CD} = \frac{DB}{CD} \cdot \frac{A_0F}{FA}.$$

Hence, if  $\triangle B^vC^vD^v \sim \triangle BCD$  and  $B^v \equiv A$  and  $D^v \equiv F$  we find that

$$\triangle B''D'C'' \sim \triangle A_0FC^v.$$

See also Fig. 13.

3.2.1. *Special case, where  $\triangle A_0AF \sim \triangle CBD$ .* In this case we find that  $A_0 \equiv C^v$  and so  $B'' \equiv C''$ . So point  $B''$  of the 1/2-cognate describes a circle about the fixed centre  $C_0''$ . One thus obtains a cognate of a special form, consisting of  $\square A_0F'D'C''C_0''$ ,  $\square A_0A''C''C_0''$ ,  $\square A''C''D'F'$  and the coupler triangle  $\triangle F'D'E$  (see Fig. 14). Such an alternative mechanism may be regarded as a special form of Stephenson-2, but also as a special mechanism of Watt's form. In addition, we know [8] that for a mechan-

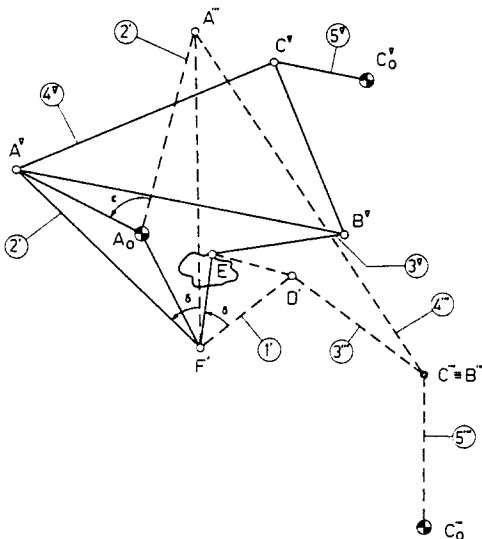


**Figure 14.** The 1/2-cognate of a Stephenson-2 mechanism in case  $\triangle A_0AF \sim \triangle CBD$ .

ism of Watt's form, the entire moving plane in which point  $E$  is fixed, may be generated by  $\infty^2$  different mechanisms of Watt's form. See for example the cognate obtained in Fig. 15, where point  $E$  is made a turning joint between the moving links  $F'E$  and  $B'E$  (This cognate consists of the four-bars  $\square A_0A^\nabla C^\nabla C_0^\nabla$ ,  $\square F'EB^\nabla A^\nabla$  and the rigid triangles  $\triangle A_0F'A^\nabla$  and  $\triangle A^\nabla B^\nabla C^\nabla$ ).

The cognate obtained is a special one, because  $\delta = \sphericalangle D'F'E$  and  $f_\delta = F'E/F'D'$ .

Should other values for  $\delta$  and  $f_\delta$  be chosen, one of the  $\infty^2$  different cognates should be obtained.



**Figure 15.** Special cognate of Watt's form by making  $E$  a turning joint of the mechanism.



3.3. The 1/2-3/4 cognate of a Stephenson-2 mechanism (with  $E$  a point of plane 2)

Starting with the same initial mechanism as in the preceding Sections 3.1 and 3.2, the design instructions for the cognate presented here are (see Fig. 16):

- (a) Form the linkage parallelograms  $A_0FEF'$ ,  $ABDB'$  and  $ABCB^v$  and also the rigid triangle  $\triangle AB^vB'$ .
- (b) Turn the four-bar  $\square AB'DF$  about  $A$  through the angle  $\alpha = \sphericalangle B^vAB' = \sphericalangle CBD$  and multiply the four-bar geometrically from  $A$  by the factor  $f_\alpha = B^vA/B'A = CB/DB$ .
- (c) One obtains the four-bar  $\square AB^vD^\Delta F^\Delta \sim \square AB'DF$ .
- (d) Form the rigid triangles  $A_0AF^\Delta$  and  $CB^vD^\Delta$ .
- (e) Form the linkage parallelogram  $A_0F^\Delta D^\Delta F'$  and the rigid triangle  $F'A_0F'$ .
- (f) Turn the five-bar  $\diamond A_0F'D^\Delta C C_0$  about  $A_0$  through the angle  $\gamma = \sphericalangle F'A_0F'$  and multiply the five-bar geometrically from  $A_0$  by the factor  $f_\gamma = F'A_0/F'A_0 = EF/C^bF$ .
- (g) One obtains the five-bar  $\diamond A_0F'D^s C^s C_0^s \sim \diamond A_0F'D^\Delta C C_0$ .
- (h) Form the rigid triangles  $\triangle A_0C_0C_0^s$  and  $\triangle D^sF'E$ .

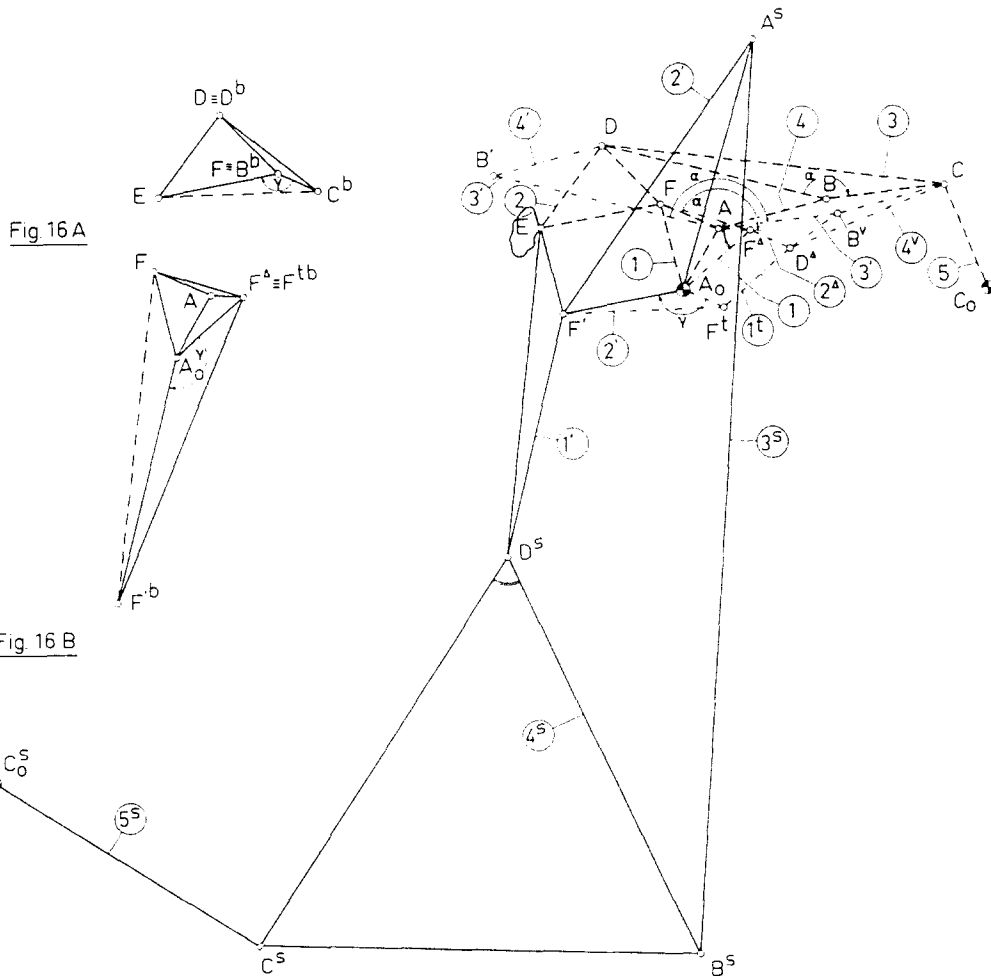


Figure 16. The 1/2-3/4 cognate of a Stephenson-2 mechanism.

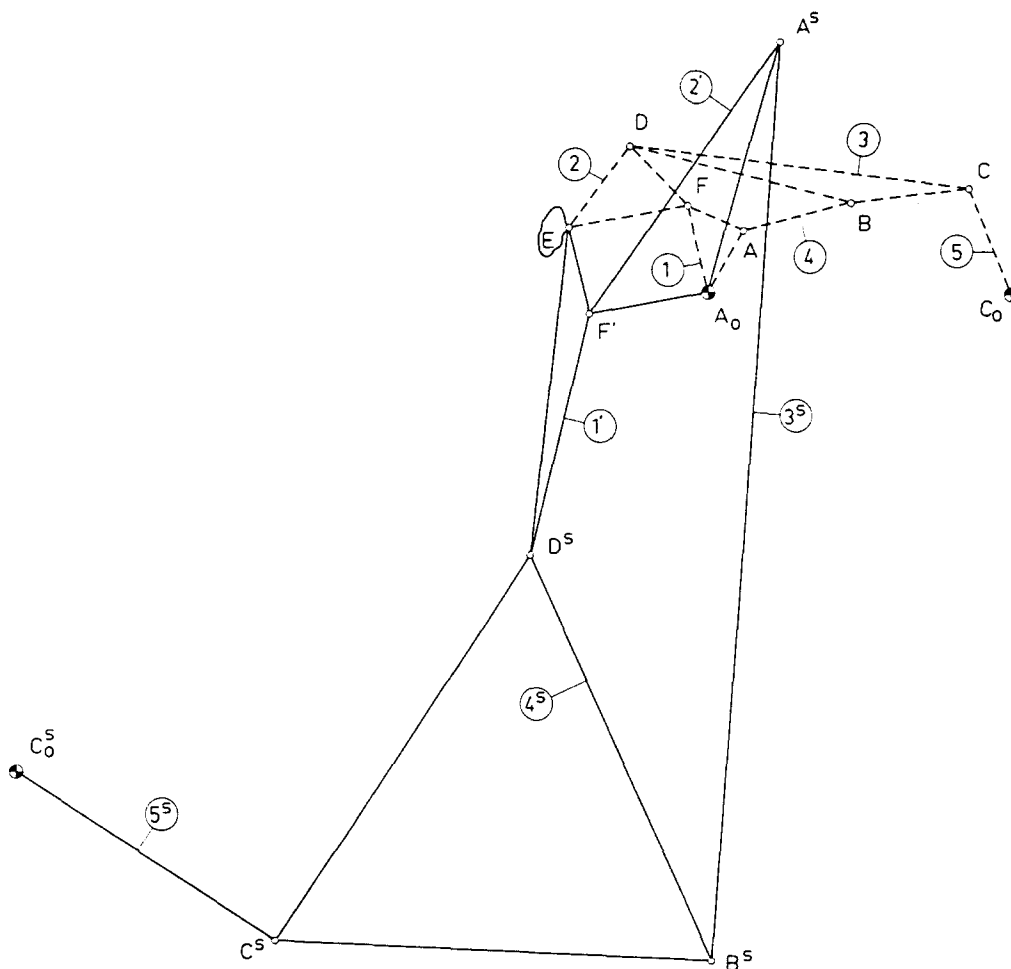
- (i) Make the four-bar  $\square F'D^sB^sA^s \sim \square FABD$ .  
 (j) Form the rigid triangles  $B^sC^sD^s$  and  $A_0A^sF'$ .

The alternative six-bar obtained consists of the five-bars  $\square A_0F'D^sC^sC_0^s$  and  $\square A_0A^sB^sC^sC_0^s$ , the four-bar  $\square A^sB^sD^sF'$  and the rigid triangles  $\triangle B^sC^sD^s$ ,  $\triangle A_0A^sF'$  and the coupler triangle  $\triangle D^sF'E$ .

The alternative six-bar has a kinematic configuration similar to the initial mechanism and may therefore be called a Stephenson-2 cognate. Here, only point  $E$  of the cognate generates the initial six-bar curve produced by the initial Stephenson-2 mechanism.

The angular velocities of the links 1 and 2 and also those of the links 3 and 4 are interchanged when one compares the distribution of angular velocities in the obtained cognate with the distribution in the initial six-bar. Therefore, the cognate is called a  $1/2-3/4$ -cognate (see Fig. 17).

The geometric properties of this cognate may be obtained from Fig. 16 in the following way.



**Figure 17.** The  $1/2-3/4$  cognate of a Stephenson-2 mechanism.

From

$$\begin{aligned} \frac{A^s F'}{F' A_0} &= \frac{A^s F'}{DF} \cdot \frac{DF}{F' A_0} = \frac{D^s F'}{D^{\Delta} F'} \cdot \frac{D^{\Delta} F'}{AF} \cdot \frac{DF}{F' A_0} = \frac{F' A_0}{F' A_0} \cdot \frac{F^{\Delta} A_0}{AF} \cdot \frac{DF}{F' A_0} \\ &= \frac{DF}{D^{\Delta} F^{\Delta}} \cdot \frac{F^{\Delta} A_0}{AF} = \frac{AF}{AF^{\Delta}} \cdot \frac{F^{\Delta} A_0}{AF} = \frac{F^{\Delta} A_0}{AF^{\Delta}}, \end{aligned}$$

and the fact that  $\sphericalangle A^s F' A_0 = \sphericalangle A_0 F^{\Delta} A$ .

We derive that

$$\triangle A^s F' A_0 \sim \triangle A_0 F^{\Delta} A.$$

From the same figure we also have

$$\triangle F' A_0 F^t \sim \triangle C_0^s A_0 C_0 \text{ with } \gamma = 2\pi - (\sphericalangle EFD + \sphericalangle DBC).$$

Since also

$$\frac{F' A_0}{A_0 F^t} = \frac{EF}{FD} \cdot \frac{FD}{F^{\Delta} D^{\Delta}} = \frac{EF}{FD} \cdot \frac{BD}{BC}$$

we find that

$$\triangle EFC^b \sim \triangle F' A_0 F^t$$

as long as

$$\triangle B^b C^b D^b \sim \triangle BCD \text{ with } D^b \equiv D \text{ and } B^b \equiv F.$$

Thus  $\triangle C_0^s A_0 C_0 \sim \triangle EFC^b$  (see Fig. 16A).

From Fig. 16 we see that

$$\triangle AF^{\Delta} F \sim \triangle BCD \sim \triangle B^v D^{\Delta} C$$

with step  $g$  of the design instructions we have

$$\frac{D^s C^s}{D^{\Delta} C} = \frac{EF}{FC^b}.$$

Moreover,

$$\frac{D^s B^s}{AB} = \frac{D^s F'}{D^{\Delta} F'} \cdot \frac{D^{\Delta} F'}{AF} = \frac{EF}{FC^b} \cdot \frac{F^{\Delta} A_0}{AF}.$$

Hence,

$$\frac{D^s C^s}{D^s B^s} = \frac{D^{\Delta} C}{AB} \cdot \frac{AF}{F^{\Delta} A_0} = \frac{CD}{BD} \cdot \frac{AF}{F^{\Delta} A_0} = \frac{F^{\Delta} F}{F A_0}.$$

On the other hand, one may observe that

$$\begin{aligned}\sphericalangle (D^s B^s, AB) &= \sphericalangle (F' D^s, FA) = -\gamma + \sphericalangle (F' D^\Delta, FA) \\ &= -\gamma + \sphericalangle (A_0 F^\Delta, FA)\end{aligned}$$

and

$$\sphericalangle (D^s C^s, AB) = -\gamma + \sphericalangle (D^\Delta C, CB^v) = -\gamma + \sphericalangle (F^\Delta F, FA).$$

Thus  $\sphericalangle B^s C^s D^s = \sphericalangle A_0 F F^\Delta$ .

And so, finally, we have

$$\triangle B^s C^s D^s \sim \triangle A_0 F F^\Delta.$$

Since

$$\frac{EF'}{D^s F'} = \frac{FA_0}{D^\Delta F'} \cdot \frac{D^\Delta F'}{D^s F'} = \frac{FA_0}{F^\Delta A_0} \cdot \frac{F' A_0}{F' A_0}$$

and  $\sphericalangle EF' D^s = -\sphericalangle F^\Delta A_0 F + (2\pi - \gamma) = \sphericalangle FA_0 F'^b$ , we conclude that

$$\triangle EF' D^s \sim \triangle FA_0 F'^b.$$

Here, the point  $F'^b$  is defined by the relationship  $\triangle A_0 F^\Delta F'^b \sim \triangle A_0 F' F' \sim \triangle FC^b E$  (see Fig. 16B).

For the particular point  $E$  defined by the relation  $\triangle FDE \sim \triangle BDC$ , one finds that  $E \equiv C^b$ . Then  $\gamma = 0$  and  $f_\gamma = 1$ .

Consequently

$$\begin{aligned}C_0^s &= C_0, \\ C^s &= C, \\ D^s &= D^\Delta, \\ F' &= F'.\end{aligned}$$

Thus

$$\frac{A_0 F'}{CD^\Delta} = \frac{F^\Delta D^\Delta}{CD^\Delta} = \frac{F^\Delta D^\Delta}{BA} \cdot \frac{DB}{DC} = \frac{FD}{BA} \cdot \frac{DB}{DC} \cdot \frac{CB}{DB} = \frac{DF}{BA} \cdot \frac{CB}{CD} = \frac{A^s F'}{B^s D^\Delta} \cdot \frac{CB}{CD}.$$

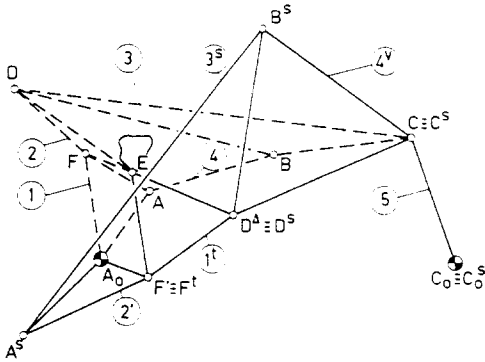
Moreover  $\triangle F' E D^\Delta \sim \triangle A_0 F F^\Delta \sim \triangle B^s C D^\Delta$ .

In this particular case, therefore, the ‘‘coupler-point’’  $E$  of the 1/2-3/4 cognate is defined in a way similar to the one in the initial six-bar (see Fig. 18).

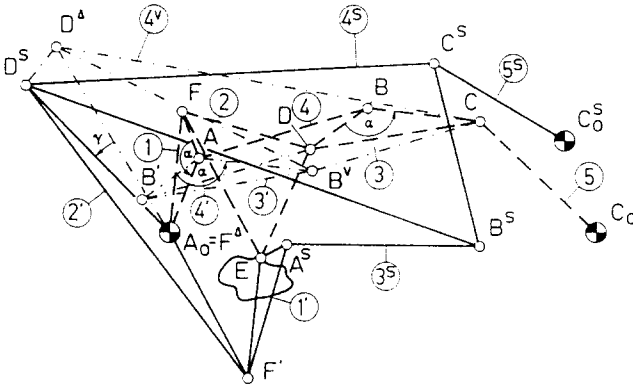
3.3.1. *Special case, where  $\triangle A_0 A F \sim \triangle C B D$ .* In this case we have  $F^\Delta \equiv A_0$ . Therefore, the method described in Section 3.3 collapses. Thus another method has to be developed, of which the design instructions are as follows (see Fig. 19):

(a) Form the linkage parallelograms  $A_0 F E F'$ ,  $A B D B'$  and  $A B C B^v$  and also the fixed triangle  $\triangle A B^v B'$ .

(b) Turn the four-bar  $\square A B' D F$  about  $A$  through the angle  $\alpha = \sphericalangle B^v A B' = \sphericalangle C B D = \sphericalangle A_0 A F$ , and multiply the four-bar by the factor  $f_\alpha = B^v A / B' A = C B / D B = A_0 A / F A$ .



**Figure 18.** The  $1/2-3/4$  cognate of a Stephenson-2 mechanism in case  $E$  is bound by:  $\triangle FDE \sim \triangle BDC$ .



**Figure 19.** Transformation of a Stephenson-2 mechanism, for which  $\triangle A_0AF \sim \triangle CBD$ , into one of  $\infty^2$  Watt's six-bar cognates.

- (c) One obtains the four-bar  $\square AB^vD^dA_0 \sim \square AB'DF$ .
- (d) Form the rigid triangle  $\triangle CB^vD^d$ .
- (e) Turn the four-bar  $\square A_0D^dCC_0$  about  $A_0$  through an *arbitrarily* chosen angle  $\gamma = \sphericalangle D^sA_0D^d$  and multiply the four-bar geometrically with an *arbitrarily* chosen factor  $f_\gamma = D^sA_0/D^dA_0$ .
- (f) One obtains the four-bar  $\square A_0D^sC^sC_0^s \sim \square A_0D^dCC_0$ .
- (g) Form the rigid triangle  $\triangle A_0D^sF'$ .
- (h) Make the four-bar  $\square A^sB^sD^sF' \sim \square AB'DF$ .
- (i) Finally, form the rigid triangles  $F'A^sE$  and  $B^sC^sD^s$ .

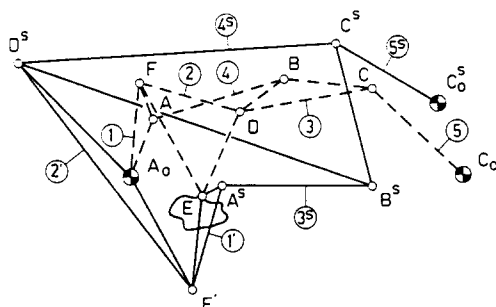
The cognate obtained consists of the four-bars  $\square A_0D^sC^sC_0^s$  and  $\square A^sB^sD^sF'$  and the rigid triangles  $\triangle F'A^sE$ ,  $\triangle B^sC^sD^s$  and  $\triangle A_0D^sF'$ .

Since  $\gamma$  and also  $f_\gamma$  may be chosen arbitrarily, one finds  $\infty^2$  cognates of Watt's form. This result is in agreement with the known\* fact [8] that a mechanism of Watt with point

\*This fact is only recently known, since the paper concerning the *six-bar cognates of Watt's type* has only just been accepted for presentation in the coming Fall 1970 at the A.S.M.E. Conference on Mechanisms, Columbus, Ohio, U.S.A. It should also be noted that a special case of these cognates has already been investigated by A. H. Soni [9]. He found cognates of Watt's type in case one of the four-bar sub-chains of the source mechanism was parallelogram-shaped.

$E$  arbitrarily chosen in the plane as shown in Fig. 20, has indeed  $\infty^2$  cognates of this type.

A special cognate of this type will be obtained by making the generating point  $E$  at the same time a turning joint of the mechanism. The resulting mechanism will then be the same as the one obtained in Fig. 15, since the mechanism obtained through this section, will have the same distribution of angular velocities over the links, and the points  $A_0$  and  $E$  will be identical points for the two resulting mechanisms. (The latter remark connects the result of the section under consideration with that of Section 3.2.1. Therefore, it is of little importance which design procedure one prefers: in the end one obtains the same mechanism.)



**Figure 20.** Transformation of a particular Stephenson-2 mechanism, for which  $\triangle A_0AF \sim \triangle CBD$ , into one of the  $\infty^2$  Watt's six-bar cognates.

### 3.4. General remarks about the 4 cognates of the Stephenson-2 mechanism with $E$ a point of plane 2

Including the initial six-bar, we now have 4 cognates of a Stephenson-2 mechanism. One may prove that the 3/4-cognate of the 1/2-cognate is identical to the 1/2-3/4 cognate. One may also prove that the 1/2-cognate of the 3/4-cognate turns into the 1/2-3/4 cognate. And, finally, that the 1/2-3/4 cognate of the 1/2-cognate becomes identical to the 3/4-cognate of the initial mechanism. It is clear that all transformations are reversible.

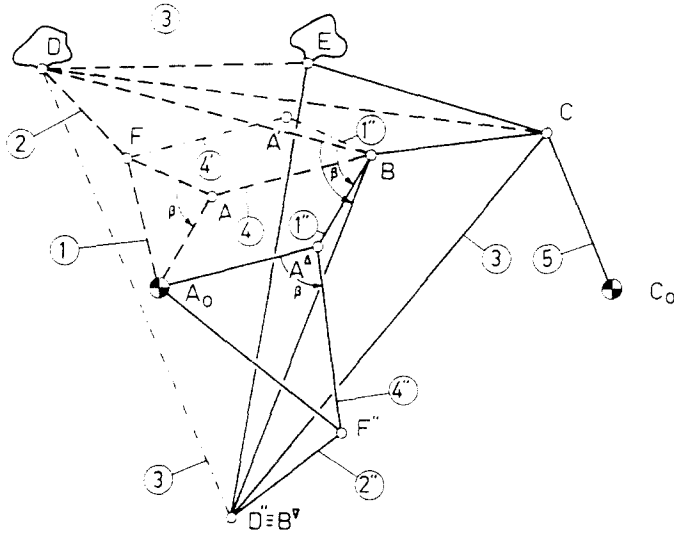
In a similar way as the one of Section 2.1 one may prove that no other cognates exist than the 4 already obtained. In order to shorten the length of the manuscript, all proofs of the facts just mentioned have been omitted and the reader is invited to compose them himself.

## 4. The Three-Fold Generation of Plane 3 of a Stephenson-2 Mechanism

### 4.1 The 1/2-cognate for plane 3

The initial mechanism is the same as the one of Section 3 and consists of the five-bars  $\square A_0ABCC_0$  and  $\square A_0FDCC_0$ , the four-bar  $\square ABDF$ , the rigid quadrangle  $BCDE$  and the rigid triangle  $\triangle A_0AF$ . The instructions for the design of the 1/2-cognate are successively (see Fig. 21):

- (a) Form the parallelograms  $A_0FDF^\nabla$  and  $AFDF^\wedge$ .
- (b) Turn the four-bar  $\square DF^\wedge AB$  about  $D$  through the angle  $\alpha = \sphericalangle F^\wedge DF^\nabla = \sphericalangle AFA_0$  and multiply the four-bar geometrically from  $D$  by the factor  $f_\alpha = F^\nabla D / F^\wedge D = A_0F / AF$ .



**Figure 21.** The three-fold generation of plane 3 of a Stephenson-2 mechanism (the 1/2 cognate).

(c) One obtains the four-bar  $\square DF^\nabla A^\nabla B^\nabla \sim \square DF^\wedge AB$ .

(d) Finally, one forms the rigid triangles  $F^\nabla F^\wedge D$ ,  $A^\nabla A_0 F^\nabla$ ,  $B^\nabla CD$  and  $CDE$ .

The 1/2- cognate obtained consists of the five-bars  $\diamond A_0 A^\nabla B^\nabla C C_0$  and  $\diamond A_0 F^\nabla D C C_0$ , the four-bar  $\square A^\nabla B^\nabla D F^\nabla$  and the rigid triangles  $\triangle A^\nabla A_0 F^\nabla$ ,  $\triangle B^\nabla CD$  and  $\triangle CDE$ .

One may observe that

$$\triangle B^\nabla BD \sim \triangle A_0 AF \sim \triangle F^\nabla F^\wedge D \sim \triangle A^\nabla A_0 F^\nabla.$$

In the special case where  $\triangle A_0 AF \sim \triangle CBD$ , one finds that  $B^\nabla \equiv C$  and the cognate has the similar relation  $\triangle A_0 A^\nabla F^\nabla \sim \triangle BCD$ .

One also notes that the cognate six-bar generates the entire moving plane 3 and also the curves produced by any point  $E$  of that plane.

#### 4.2. The 1/4-cognate for plane 3

Starting with the same initial mechanism as in the preceding section, one obtains the 1/4-cognate, using the instructions (see Fig. 22):

(a) Form the parallelograms  $A_0 A B A^\Delta$  and  $F A B A'$ .

(b) Turn  $\square BA'FD$  about  $B$  through  $\beta = \sphericalangle A'BA^\Delta = \sphericalangle FAA_0$  and multiply the four-bar by  $f_\beta = A^\Delta B/A'B = A_0 A/FA$ .

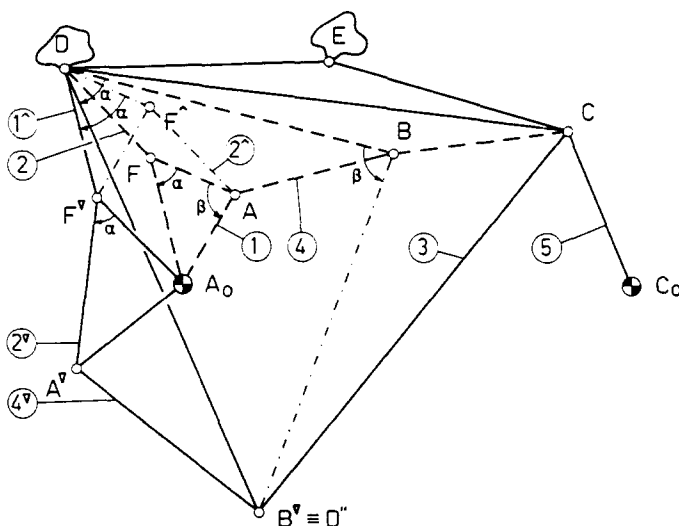
(c) One obtains the four-bar  $\square BA^\Delta F''D'' \sim \square BA'FD$ .

(d) Finally, one forms the rigid triangles  $A'BA^\Delta$ ,  $A_0 A^\Delta F''$ ,  $DBD''$  and  $BCD''$ .

The 1/4-cognate obtained consists of the five-bar  $\diamond A_0 A^\Delta B C C_0$  and  $\diamond A_0 F'' D'' C C_0$ , the four-bar  $\square A^\Delta B D'' F''$  and the rigid triangles  $A_0 A^\Delta F''$ ,  $CB D''$  and  $\triangle C D'' E$ .

One observes that  $\triangle F^\nabla A_0 A^\nabla \sim \triangle F A A_0 \sim A'BA^\Delta \sim \triangle A_0 A^\Delta F'' \sim \triangle DBD''$ . Comparing the 1/4-cognate with the 1/2-cognate, one finds that  $D'' \equiv B^\nabla$ .

In the special case where  $\triangle A_0 AF \sim \triangle CBD$ , one sees that  $D'' \equiv C \equiv B^\nabla$  and consequently  $\triangle A_0 A^\Delta F'' \sim \triangle DBC$ .



**Figure 22.** The three-fold generation of plane 3 of a Stephenson-2 mechanism (the 1/4 cognate).

By indirect demonstration (*reductio ad absurdum*) one may easily prove that no other cognates for an arbitrarily chosen point  $E$  of plane 3 exist. Thus, including the initial mechanism, there are three Stephenson-2 mechanisms generating the identical curve produced by any point  $E$  of the moving plane 3.

### 5. The Five-Fold Generation of the Curve Produced by Point $D$ of a Stephenson-2 Mechanism

The curve generated by point  $D$  of the mechanism is *algebraic of order 16* and has the *genus 7* [3]. Turning joint  $D$  of a Stephenson-2 mechanism may be regarded as a special point both of link 2 and of the moving plane 3. Therefore, the design instructions of the two Sections 3 and 4 may be applied.

In case  $E \equiv D$ , however, the 1/2-cognate of Section 3 turns into the 1/2-cognate of Section 4.

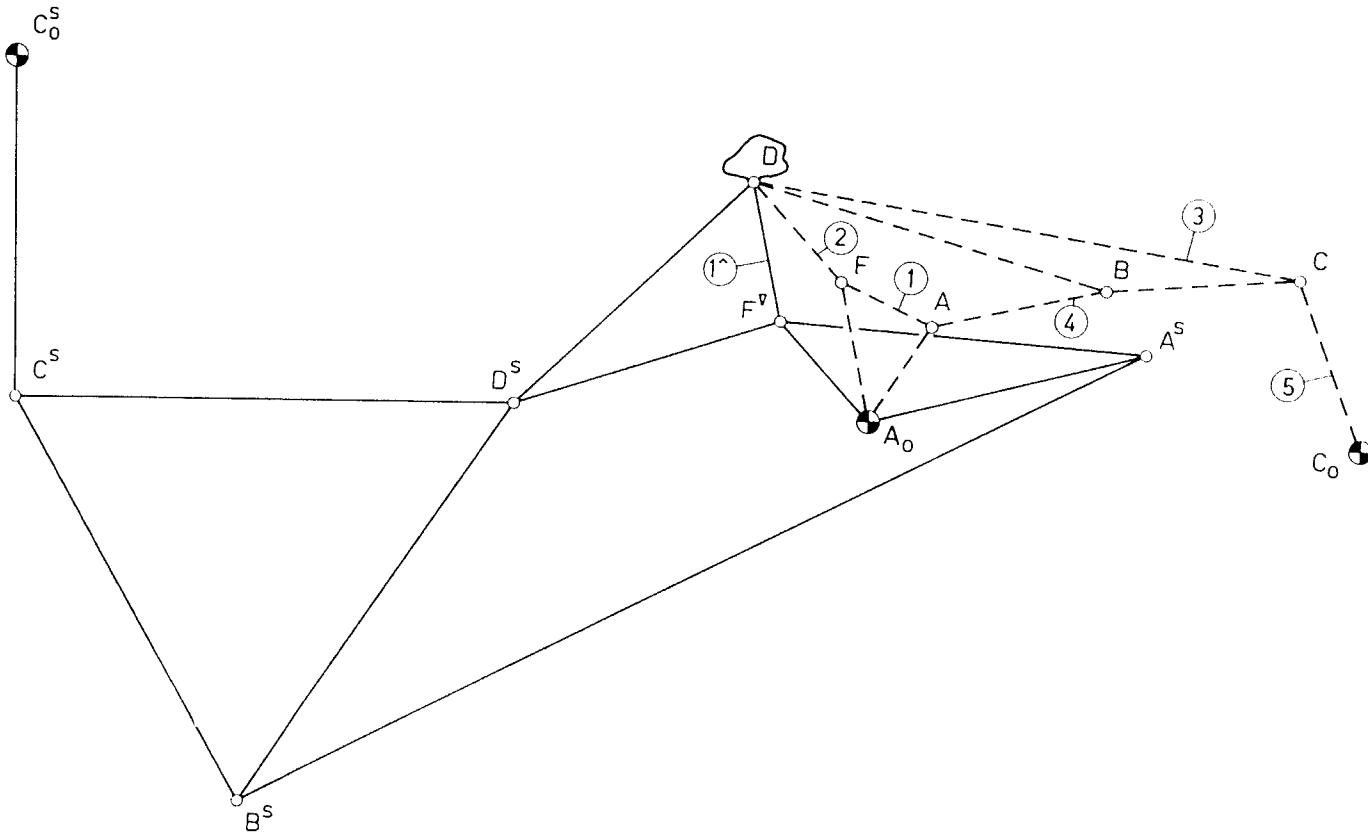
Therefore, we have the five cognates:

- (a) the initial six-bar with  $D$  a turning joint between the moving planes 2 and 3 (see Fig. 10),
- (b) the 1/2-cognate with  $D$  being a similar turning joint (see Fig. 21),
- (c) the 3/4-cognate with  $D$  merely a specific point of plane 2 (see Fig. 12),
- (d) the 1/2-3/4-cognate, also with  $D$  a specific point of a similarly situated plane in the kinematic configuration (see Fig. 23), and
- (e) the 1/4-cognate, where  $D$  is merely a specific point of plane 3 (see Fig. 22).

There are no other cognates obtainable. With two cognates, point  $D$  turns out to be a turning joint. In the remaining three cognates, point  $D$  is merely a specific coupler-point attached to some moving link.

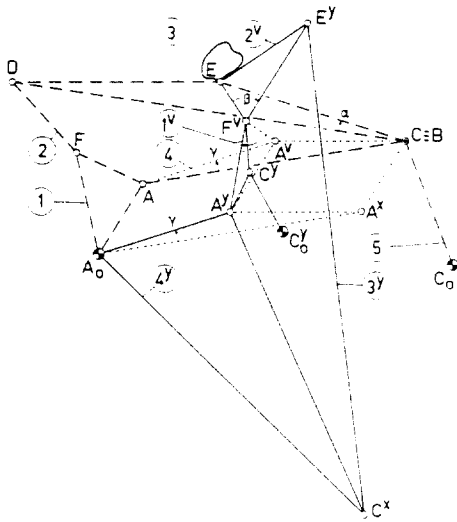
The six-bars listed under  $a$ ,  $b$ ,  $c$  and  $d$  are cognates discussed in Section 3. The cognates listed under  $a$ ,  $b$  and  $e$  are of the kind treated in Section 4.



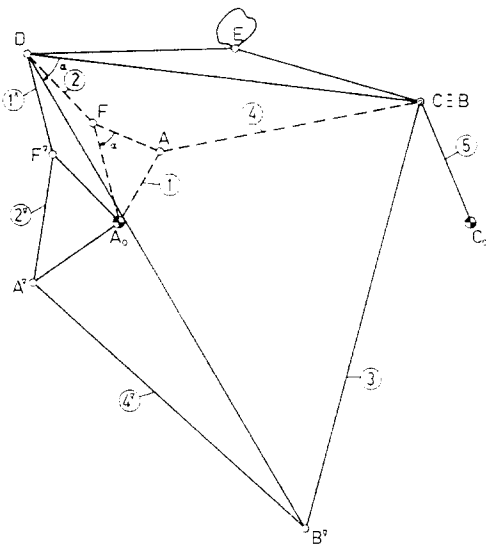


**Figure 23.** The 1/2-3/4 cognate of a Stephenson-2 mechanism with respect to the curve generated by turning joint *D*.





**Figure 26.** 1/4–2/3 cognate in case  $B \equiv C$ . Five-fold generation of the curve.



**Figure 27.** The 1/2-cognate for plane 3, if  $B \equiv C$ . (The five-fold generation of the curve produced by point  $E$  of that plane).

initial mechanism is looked upon as a special configuration of the Watt mechanism[8]. But besides that, we also find two other cognates of Watt's type of the initial mechanism. These are the 2/3-cognate (see Fig. 25) and the 1/4–2/3 cognate (see Fig. 26). In designing the Watt's 2/3-cognate of the 1/4-cognate, one obtains the 1/4–2/3 cognate again. And finally, the 1/4-cognate taken from the Watt's 2/3-cognate also produces the 1/4–2/3 cognate.

The design procedure for the 2/3-cognate will here be given briefly (see Fig. 25):

- (a) Frame the parallelograms  $FDCD^\Delta$  and  $FDED^*$ .
- (b) Make four-bar  $\square FA^*B^*D^* \sim \square FABD^\Delta$ .

(c) Form the rigid and mutually similar triangles  $F A A^*$ ,  $F D^{\Delta} D^*$  and  $D^* E B^*$  (all similar to  $\Delta D C E$ ).

(d) Make the four-bar  $\square A_0 A^* C^* C_0^* \sim \square A_0 A C C_0$ .

(e) And finally, form the rigid triangles  $A_0 A^* F$ ,  $A_0 C_0 C_0^*$  and  $A^* B^* C^*$ .

The mechanism obtained is indicated in Fig. 25 by thick solid lines.

The design instructions for the 1/4–2/3 cognate are briefly (see Fig. 26):

(a) Make the four-bar  $\square C E F^{\vee} A^{\vee} \sim \square C D F A$ .

(b) Frame the rigid triangle  $\Delta A A^{\vee} C$ , the parallelogram  $A_0 A A^{\vee} A^{\vee}$  and the rigid triangle  $\Delta F^{\vee} A^{\vee} A^{\vee}$ .

(c) Make the four-bar  $\square F^{\vee} A^{\vee} C^{\vee} E^{\vee} \sim \square F A C D$ .

(d) Frame the rigid triangles  $A_0 A^{\vee} C^{\vee}$  and  $\Delta F^{\vee} E^{\vee} E$ , the linkage parallelogram  $A_0 A C A^{\vee}$  and the rigid triangle  $\Delta A_0 A^{\vee} A^{\vee}$ .

(e) Make the four-bar  $\square A_0 A^{\vee} C^{\vee} C_0^{\vee} \sim \square A_0 A^{\vee} C C_0$ .

(f) Frame the rigid triangles  $A^{\vee} C^{\vee} F^{\vee}$  and  $A_0 C_0 C_0^{\vee}$ .

The six-bar obtained is shown by Fig. 26 by thick solid lines and has one degree of freedom in movement like the initial mechanism.

There are no other cognates to be obtained. The initial six-bar, and its 1/4-cognate only, have coincident turning joints  $B$  and  $C$ . This is not the case with the three remaining cognates. On these grounds, no other cognates are to be found in this particular case.

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