On the conservatism of upper bound tests for structured singular value analysis

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Abstract

Because of the well-known difficulties of exact real/mixed μ computation [19, 16, 3], efficiently computable upper bound tests are of great importance for both μ analysis and synthesis. However, another important issue is the introduced conservatism, and in this paper, we consider the worst case conservatism of these efficiently computable upper bound tests for real/mixed μ analysis. It shown that any upper bound test, $\overline{\mu}$, satisfying the condition $\mu(M) \leq \overline{\mu}(M) \leq C\ \dim(M)^{1-\epsilon}\ \mu(M)$, must itself be $\mathcal{NP}$-hard to compute. In other words, unless "$\mathcal{P} \neq \mathcal{NP}$" is false, for any efficiently computable upper bound test, $\overline{\mu}$, the worst case gap between the upper bound and the exact μ is not bounded by $\mathcal{O}(\dim(M)^{1-\epsilon})$. Therefore, unless "$\mathcal{P} \neq \mathcal{NP}$" is false, no matter which efficiently computable upper bound test we choose, there will be examples with arbitrarily large $\overline{\mu}/\mu$ ratios, i.e. with arbitrarily large conservatism.

1 Introduction

In this paper, we consider the conservatism of efficiently computable[1] upper bound tests for real/mixed μ analysis [18, 9, 10]. It is already known that exact real/mixed μ computation is $\mathcal{NP}$-hard [19, 16, 3], and therefore efficiently computable upper bound tests are of great practical importance for real/mixed μ analysis. The focus of this paper is the conservatism of such efficiently computable upper bound tests, and our main result is the following: Unless "$\mathcal{P} \neq \mathcal{NP}$" is false, for any efficiently computable upper bound test, $\overline{\mu}$, define

$$g_{\overline{\mu}}(n) := \sup_M \left\{ \frac{\overline{\mu}(M)}{\mu(M)} \ :\ \mu(M) \neq 0,\ \dim(M) = n \right\},$$

then

$$\sup_n \frac{g_{\overline{\mu}}(n)}{n^{1-\epsilon}} = \infty,$$

i.e., the worst case gap between the upper bound and the exact μ, is not bounded by $\mathcal{O}(\dim(M)^{1-\epsilon})$.

An alternative statement of our result is the following: No matter how large $C \in \mathbb{R}^+$, and how small $\epsilon \in \mathbb{R}^+$ are chosen, any upper bound test, $\overline{\mu}$, satisfying the condition

$\forall M$ with $\mu(M) \neq 0$,

$$\mu(M) \leq \overline{\mu}(M) \leq C\ \dim(M)^{1-\epsilon}\ \mu(M)$$

must itself be $\mathcal{NP}$-hard to compute.

Note that, standard upper bounds for real/mixed μ are efficiently computable [18, 9, 10]. Therefore, these results also show that, unless "$\mathcal{P} \neq \mathcal{NP}$" is false, the worst case gap between standard upper bounds and the exact μ, is not bounded by $\mathcal{O}(\dim(M)^{1-\epsilon})$. The conservatism of standard upper bound tests has also been discussed in [21, 9, 4, 8, 17], however our approach is to investigate the worst case conservatism using some tools from the theory of computation.

As a related result, we would like to mention that, in [15], it has been shown that the worst case gap between complex μ and its standard upper bound grows no faster than $\mathcal{O}(\dim(M))$, i.e.,

$$\sup_n \frac{g_{\overline{\mu}}(n)}{n} < \infty.$$

Based on these results, two natural questions are "What are the best achievable conservatism levels for efficiently computable upper bound tests for real, mixed, and complex μ ?", and "Do standard upper

[1] i.e. required computation time is bounded by a polynomial of the problem size.

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bound tests have any optimality property with respect to the introduced conservatism ?", but to best of the author's knowledge these are still open problems.

As another related result, we also would like to mention that, in [6], using the PCP Theorem, it has been show that there exists an $\epsilon > 0$ such that, approximating real μ with conservatism bounded by $(1+\epsilon)$ is $\mathcal{NP}$-hard. However, the results of the present paper are proved without using the PCP Theorem (See [1], and references therein), and hence are more accessible compared to [6].

The rest of this paper is organized as follows: In Section 2, some approximation results about Legendre series expansions are presented, in Section 3, the main result of the paper, the conservatism theorem is proved, and finally in Section 4, some concluding remarks are made.

2 Legendre series expansion

In this section, we consider uniform approximations of $C([-1,1])$ functions by polynomials. By Weierstarss theorem, it is known that the set of polynomials is dense in $C([-1,1])$, and hence for any $f \in C([-1,1])$, and $\epsilon > 0$, there exists a $p \in \mathbb{Q}[t]$ such that

$$\sup_{t\in[-1,1]} |f(t) - p(t)| < \epsilon.$$

Note that, in the above formulation, there is no constraint on the degree of p. However if we fix the degree of p, then we obtain a completely different approximation problem.

Fact 2.1 ([13]): There exists a constant $L > 0$ such that, for any $f \in C([-1,1])$, and any positive integer d, there exists a $p_d \in \mathbb{Q}[t]$ of degree d such that

$$\sup_{t\in[-1,1]} |f(t) - p_d(t)| < L\, \omega_f\left(\frac{2}{d}\right),$$

where the function ω_f is called the modulus of continuity of f, and is equal to

$$\omega_f(\delta) = \sup\{|f(t_2) - f(t_1)| \ : \ t_1, t_2 \in [-1,1],\ |t_2 - t_1| \leq \delta\}.$$

Fact 2.2 ([13]): In the statement of Fact 2.1, the L value cannot be chosen arbitrarily small, i.e., there exists an $L_o > 0$ value such that the statement of Fact 2.1 is not true for $L = L_o$.

Namely, by a d^{th} order polynomial, we can guarantee $\mathcal{O}(\omega_f(\frac{2}{d}))$ uniform approximation, and in some sense this is the best that we can obtain (Fact 2.2).

In the proof of the conservatism theorem, we need a similar strong result on approximation of $C([-1,1])$ functions by fixed order polynomials. However, because of the techniques used in the proof, we need a constructive result, rather than an existence result. In the following, we discuss some approximation results about Legendre series expansions [13]. This will provide a constructive (infact a polynomial time computable) $\mathcal{O}(\log(d)\ \omega_f(\frac{4}{d}))$ approximation, and will be sufficient for the proof of the conservatism theorem.

Define $L_k \in \mathbb{Q}[t]$, $k \geq 0$ as,

$$L_0(t) = 1, \quad L_1(t) = t, \tag{1}$$

and

$$L_{k+1}(t) = \frac{1}{k+1}\left((2k+1)tL_k(t) - kL_{k-1}(t)\right). \tag{2}$$

The set of all L_k's are called Legendre polynomials, and satisfies the following inner product relation [13]:

$$\int_{-1}^{+1} L_j(t)L_k(t)dt = \frac{2}{k+j+1}\delta_{j,k}, \tag{3}$$

i.e. they form an orthogonal set in $C([-1,1])$. Furthermore, by Weierstrass theorem, and the recursion relation (2), the set $\{L_k \ : \ k \geq 0\}$ is dense, and hence forms an orthogonal basis in $C([-1,1])$.

For a given $f \in C([-1,1])$, the d^{th} order Legendre series expansion is defined as:

$$S_d(t) = \sum_{k=0}^{d} a_k L_k(t), \tag{4}$$

where

$$a_k = \frac{2k+1}{2}\int_{-1}^{+1} f(t)L_k(t)dt. \tag{5}$$

The Legendre series expansions have the following approximation properties [13]:

$$\lim_{d\to\infty} S_d(t) = f(t), \quad \text{for all } t \in [-1,1], \tag{6}$$

and

$$|f(t) - S_d(t)| \leq H\log(d)\omega_f\left(\frac{2}{d}\right), \quad t \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{7}$$

where H is a universal rational constant. The inequality (7) will play a key role in the proof of the conservatism theorem.

Note that, it is not guaranteed that S_d converges uniformly to f on the interval $[-1, 1]$. As a final remark, note that if P_d is the d^{th} order Legendre series expansion of

$$F(x) = \begin{cases} f(-1) & \text{if } -1 \le x \le -\frac{1}{2} \\ f(2x) & \text{if } -\frac{1}{2} \le x \le +\frac{1}{2} \\ f(1) & \text{if } +\frac{1}{2} \le x \le +1 \end{cases},$$

then

$$\sup_{t\in[-1,1]} |f(t) - P_d(t/2)| \le H \ \log(d) \ \omega_f \left(\frac{4}{d}\right),$$

therefore by using this Legendre series based method, we obtain a simple construction for an $\mathcal{O}(\log(d) \ \omega_f \left(\frac{4}{d}\right))$ approximation.

The above approximation results will be sufficient for the proof of the conservatism theorem. For further approximation results which are not discussed here, we refer the reader to [13] and references therein.

3 The conservatism theorem

In this section, we prove the results stated in Section 1. The proof is based on the $\mathcal{NP}$-hardness of the maxcut problem: It is shown that, if there exists an efficiently computable upper bound test $\overline{\mu}$ satisfying the conservatism conditions stated below, then for any given graph G, one can construct in polynomial time a rational matrix M_G, such that the decision version of the maxcut problem is equivalent to checking whether "$\overline{\mu}(M_G) < 1$" holds or not. The construction of M_G involves several steps, and the polynomial approximation results of the previous section will be used in this construction.

After the above discussion to motivate the main idea of the proof, we now state and prove the conservatism theorem:

Theorem 3.1 (The conservatism theorem): In the context of real/mixed μ analysis, unless "$\mathcal{P} \neq \mathcal{NP}$" is false, for any efficiently computable upper bound test, $\overline{\mu}$, with

$$g_{\overline{\mu}}(n) := \sup_M \left\{ \frac{\overline{\mu}(M)}{\mu(M)} \ : \ \mu(M) \neq 0, \ \dim(M) = n \right\},$$

then

$$\sup_n \frac{g_{\overline{\mu}}(n)}{n^{1-\epsilon}} = \infty,$$

i.e., the worst case gap between the upper bound and the exact μ, is not bounded by $\mathcal{O}(\dim(M)^{1-\epsilon})$.

In other words, no matter how large $C \in \mathbb{R}^+$, and how small $\epsilon \in \mathbb{R}^+$ are chosen, if an upper bound test, $\overline{\mu}$, satisfies, $\mu(M) \le \overline{\mu}(M) \le C \ \dim(M)^{1-\epsilon} \ \mu(M)$, for all M with $\mu(M) \neq 0$, then, $\overline{\mu}$ itself must be $\mathcal{NP}$-hard to compute.

Proof: Let $C > 2$, and $1 > \epsilon > 0$, be fixed rational numbers. We will first show that, if an upper bound test, $\overline{\mu}$, satisfies

$\forall M$ with $\mu(M) \neq 0$,

$$\mu(M) \le \overline{\mu}(M) \le C \ \dim(M)^{1-\epsilon} \ \mu(M), \qquad (8)$$

then, $\overline{\mu}$ itself must be $\mathcal{NP}$-hard to compute. Note that, this proves the second statement, as well as the first one.

Let $G = (N, E)$ be a (undirected) graph with n nodes, and e branches $(n, e > 1)$, and let m_G denote the maxcut of G. Furthermore, let k be an integer ≥ 2. We have either $m_G \ge k$, or $m_G \le k-1$, but the problem of deciding which one holds is $\mathcal{NP}$-hard [11].

In the rest of the proof, we outline a polynomial time[2] construction of a rational matrix M_G. Each step presented below is a polynomial time operation, and at the end we explain how checking whether "$\overline{\mu}(M_G) < 1$" holds or not, can be used to solve the above decision version of the maxcut problem. By the $\mathcal{NP}$-hardness of the maxcut problem, we will complete the proof of the above statement (8), and hence the first and second parts of the conservatism theorem.

Now, we start the construction of M_G: As in [6], define $A_G \in \mathrm{Mat}(n, \mathbb{Z})$ as

$$(A_G)_{i,j} = \begin{cases} -1 & \text{if } i \neq j, \text{ and } \{i,j\} \in E \\ 0 & \text{if } i \neq j, \text{ and } \{i,j\} \notin E \\ \sum_{\{i,k\}\in E} 1 & \text{if } i = j \end{cases},$$

then,

$$\sup_{X\in[r_1,r_2]^n} X^T A_G X = m_G \ |r_1 - r_2|^2. \qquad (9)$$

[2]In this Section, polynomial time means polynomial time with respect to the number of nodes of G.

It is possible to generate in polynomial time, rational numbers δ, q_1, q_2 satisfying

$$0 < \delta < \frac{1}{3}\left(\sqrt{\frac{n}{k}} - \sqrt{\frac{n}{k-1}}\right),$$

$$\sqrt{\frac{n}{k}} + \delta \leq q_1 \leq q_2 - \delta \leq \sqrt{\frac{n}{k-1}}.$$

Let H be the universal rational constant defined in (7) of Section 2, and $c_\epsilon = \frac{4}{\epsilon}$. Note that, this choice for c_ϵ will guarantee the condition

$$\forall x \geq 1, \quad \log(x) \leq c_\epsilon x^{\frac{\epsilon}{4}}.$$

It is possible to generate in polynomial time, an integer d, and a rational number K, such that,

$$d \geq \left(16 q_2 H C c_\epsilon \delta^{-1}\right)^{\frac{1}{4\epsilon}} + (2n)^{\frac{1}{2\epsilon}}, \tag{10}$$

and

$$C d^{1-\frac{\epsilon}{2}} < K < 2C d^{1-\frac{\epsilon}{2}}.$$

Define $\Phi_0 : \mathbb{R} \to \mathbb{R}$ as

$$\Phi_0(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ q_1 x & \text{if } 0 \leq x \leq 1 \\ q_1 + m(x-1) & \text{if } x \geq 1 \end{cases},$$

where $m = (q_2 - \delta - q_1)/(K-1)$. Furthermore, define $\Phi_1, \Phi : \mathbb{R} \to \mathbb{R}$ as

$$\Phi_1(x) = \Phi_0(x) + \delta, \quad \Phi(x) = \frac{1}{2}(\Phi_1(x) + \Phi_0(x)).$$

Let $S_d(x)$ be the Legrendre series expansion of $\phi(x) := \Phi(2Kx)$ of degree d [13]. Then,

$$|S_d(x) - \phi(x)| \leq H \ \log(d) \ \omega_\phi\left(\frac{2}{d}\right), \qquad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

where ω_ϕ is the modulus of continuity [13] of $\phi(x)$. Since $\omega_\phi(\frac{2}{d}) \leq 4Kq_2 d^{-1}$, it follows that

$$\frac{d}{\log d} \geq \frac{8Kq_2}{\delta} H \tag{11}$$

implies

$$|S_d(x) - \phi(x)| \leq \delta/2, \qquad x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

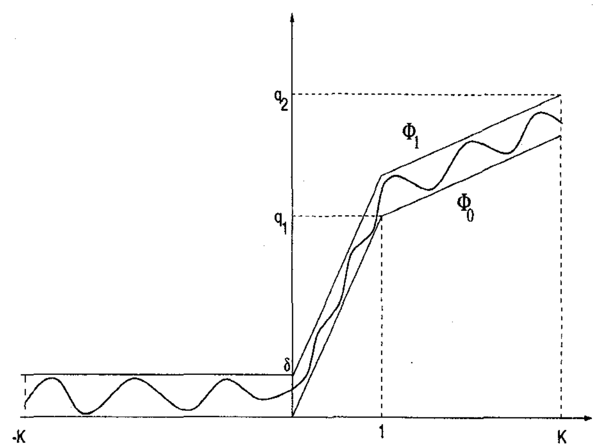

Figure 1: $\mathcal{S}_d(x)$ lies between $\Phi_1(x)$, and $\Phi_0(x)$

Note that, by our choice of d and K, the condition (11) holds, and we get

$$\Phi_0(x) \leq \mathcal{S}_d(x) \leq \Phi_1(x), \quad x \in [-K, K].$$

where $\mathcal{S}_d(x) := S_d(x/2K)$.

Now, define,

$$\eta_G := \inf\left\{\eta \ : \ \sup_{Y\in[-\eta,+\eta]^n} \sum_{i,j=1}^{n} (A_G)_{i,j}\mathcal{S}_d(Y_i)\mathcal{S}_d(Y_j) = n\right\}. \tag{12}$$

Then by a very simple geometric argument and (9), it follows that

$$m_G \geq k \Rightarrow (0 < \eta_G \leq 1),$$

and

$$m_G \leq k-1 \Rightarrow (K \leq \eta_G < \infty).$$

Note that

$$1 - \frac{1}{n}\sum_{i,j=1}^{n} (A_G)_{i,j}\mathcal{S}_d(Y_i)\mathcal{S}_d(Y_j) =$$

$$1 + M_{12}\Delta_Y(1 - M_{22}\Delta_Y)^{-1}M_{21}, \tag{13}$$

where $\Delta_Y = \text{diag}(Y_1 I_{2d}, \cdots, Y_n I_{2d})$. This is because, $\mathcal{S}_d(Y)$ is an LFT with $\Delta = YI_d$, and $\text{diag}(\mathcal{S}_d(Y_1)\cdots\mathcal{S}_d(Y_n))$ is an LFT with $\Delta = \text{diag}(Y_1 I_d, \cdots, Y_n I_d)$.

Furthermore, the LFT on the right hand side of (13) is always well posed, therefore, $\det(I - M_{22}\Delta_Y) \neq 0$, and

$$\begin{bmatrix} I - M_{22}\Delta_Y & -M_{21} \\ M_{12}\Delta_Y & 1 \end{bmatrix},$$

is singular iff $1 + M_{12}\Delta_Y(1 - M_{22}\Delta_Y)^{-1}M_{21} = 1 - \frac{1}{n}\sum_{i,j=1}^{n}(A_G)_{i,j}\mathcal{S}_d(Y_i)\mathcal{S}_d(Y_j) = 0$. Similarly, since $1 \neq 0$,

$$\begin{bmatrix} I - M_{22}\Delta_Y & -M_{21} \\ M_{12}\Delta_Y & 1 \end{bmatrix},$$

is singular iff $I - (M_{22} - M_{12}M_{21})\Delta_Y$ is singular. Therefore,

$$\mu(M_G) = \frac{1}{\eta_G},$$

where $M_G = M_{22} - M_{12}M_{21}$. Note that, $\dim(M_G) = 2nd$.

Now, let $\overline{\mu}$ be an upper bound test satisfying

$\forall M$ with $\mu(M) \neq 0$,

$$\mu(M) \leq \overline{\mu}(M) \leq C \dim(M)^{1-\epsilon}\ \mu(M).$$

If we use this upper bound test for the above $\mu(M_G)$ computation, we get

$$\eta_G^{-1} \leq \overline{\eta}_G^{-1} \leq C\ (2nd)^{1-\epsilon}\ \eta_G^{-1},$$

where $\overline{\eta}_G := 1/\overline{\mu}(M_G)$. By our choice of d and K, we have

$$C(2nd)^{1-\epsilon} \leq Cd^{1-\frac{\epsilon}{2}} < K,$$

and hence,

$$\eta_G \geq \overline{\eta}_G > \frac{1}{K}\eta_G.$$

Therefore,

$$m_G \geq k \Rightarrow (0 < \eta_G \leq 1) \Rightarrow (0 < \overline{\eta}_G \leq 1),$$

and

$$m_G \leq k - 1 \Rightarrow (K \leq \eta_G < \infty) \Rightarrow (1 < \overline{\eta}_G < \infty).$$

Hence,

$$m_G \leq k - 1 \Longleftrightarrow \overline{\mu}(M_G) < 1.$$

Since the whole construction from G to M_G is polynomial time and the maxcut problem is $\mathcal{NP}$-hard, the computation of $\overline{\mu}$ must be $\mathcal{NP}$-hard too. This completes the proof. □

4 Conclusion

In this paper, we investigated the conservatism of efficiently computable upper bound tests for real/mixed μ analysis. It was shown that any upper bound test, $\overline{\mu}$, satisfying the condition, $\mu(M) \leq \overline{\mu}(M) \leq C\ \dim(M)^{1-\epsilon}\ \mu(M)$, must itself be $\mathcal{NP}$-hard to compute. In other words, unless "$\mathcal{P} \neq \mathcal{NP}$" is false, for any efficiently computable upper bound test, $\overline{\mu}$, the worst case gap between the upper bound and the exact μ, is not bounded by $\mathcal{O}(\dim(M)^{1-\epsilon})$.

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