An inventory problem with constrained ordercapacity

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Introduction

In 1959 for the simplest stochastic inventory problem Scarf [1] proved the optimal orderstrategy being of the s-S type when the total expected discounted costs for a finite number of periods is used as criterion.

When one uses the total expected discounted costs for an infinite number of periods or the average costs, the optimal strategy is of the s-S type too. (See for instance Iglehart [2] and Tijms [3]).

In this report we consider the same inventory problem, only adding a capacity constraint for orders (or production).
We investigate the influence this constraint has on the structure of the optimal strategy.
Intuitively we expect the following:

If the inventory level $x < s$ order up to $\mathrm{Min}(x + R, S)$,

(R is the maximal orderquantity);

if the inventory level $x \geq s$ do not order.

We will call this type of strategy s-S strategies as well.
When there are no set-up costs the optimal strategy for the finite period total discounted costs case indeed is an s-S one (with $s = S$). But in the case of set-up costs this is not true in general.

For the average costs case we try to find conditions under which the optimal strategy is of the s-S type.
We do this in the following way:

Assume a certain strategy being the best one of all s-S strategies. The average costs for this strategy, $g_{s,S}$, are equal to $\frac{k_{s,S}(S)}{t_{s,S}(S)}$, where $k_{s,S}(\cdot)$ and $t_{s,S}(\cdot)$ are solutions of integral equations. $t_{s,S}(S)$ and $k_{s,S}(S)$ are the expected recurrence time and costs of state S(*

*) We assume here $S - R < s$ but we will come back to this later.

The value function of Howard is given by

$$v_{s,S}(u) = \begin{cases} K + k_{s,S}(u + R) - t_{s,S}(u + R)g_{s,S} & \text{if } u < S - R , \\ K & \text{if } S - R \le u < s , \\ k_{s,S}(u) - t_{s,S}(u)g_{s,S} & \text{if } s \le u , \end{cases}$$

where K are the set-up costs.
Differentiating $g_{s,S}$ with respect to s and S, which requires differentiating integral equations, gives some conditions for the function $J_{s,S}(.)$ given by $J_{s,S}(u) = k_{s,S}(u) - t_{s,S}(u)g_{s,S}$ and so for $v_{s,S}(.)$. In some cases these conditions suffice for the over all optimality of the best s-S strategy.

We found only two of such cases, namely:

a) The case K = 0 (s = S)

b) The case K ≥ 0 with linear holding- and shortage costs and a negative exponentially distributed demand.

So the results are rather poor but perhaps the very method used has some significance.

Description

We consider one inventory point. At the beginning of each period should be decided what quantity has to be ordered. Assume that the inventory is x at the beginning of a period and the ordered quantity is z ($0 \le z \le R$), then the ordering costs will be equal to $K\delta(z) + cz$, ($\delta(z) = 0$ if $z = 0$, $\delta(z) = 1$ if $z > 0$) and the expected inventory- and stockoutcosts in that period are:

$$L(x + z) = \begin{cases} \int_0^{x+z} h(x + z - \xi)\varphi(\xi)d\xi + \int_{x+z}^{\infty} p(\xi - x - z)\varphi(\xi)d\xi & \text{if } x + z \ge 0 , \\ \int_0^{\infty} p(\xi - x - z)\varphi(\xi)d\xi & \text{if } x + z < 0 , \end{cases}$$

where $\varphi(.)$ is the frequency function of the demand per period and h(.) and p(.) are the inventory- and stockoutcosts respectively, each depending on the inventory level at the end of the period.

Further assumptions:

L(.) is convex,
demand has to be backlogged.,
delivery of our own orders is without timelag.

I. Total expected discounted costs criterion

Assume that $C_n(x)$ is the expected discounted costs in the following n periods whilst the actual inventory level, before ordering, is x under the assumption that in future always the best decision will be made.

$$C_n(x) = \min_{x \le y \le x+R} \{K\delta(y - x) + (y - x)c + L(y) + \alpha \int_0^\infty C_{n-1}(y - \xi)\varphi(\xi)d\xi\} , \quad n = 1,2,3,\ldots$$

$C_0(x) \equiv 0$ by definition .

Define functions $G_n(.)$ by

$$G_n(y) = y.c + L(y) + \alpha \int_0^\infty C_{n-1}(y - \xi)\varphi(\xi)d\xi .$$

So

$$C_n(x) = \min_{x \le y \le x+R} \{G_n(y) + K\delta(y - x)\} - xc .$$

For the unconstrained case Scarf [1] showed that these functions $G_n(.)$ are K-convex and that therefore the optimal strategy is op the s-S type.*)

*) f(.) is K-convex iff $K + f(x + a) - f(x) - af'(x) \ge 0 \quad \forall x, \forall a > 0$.

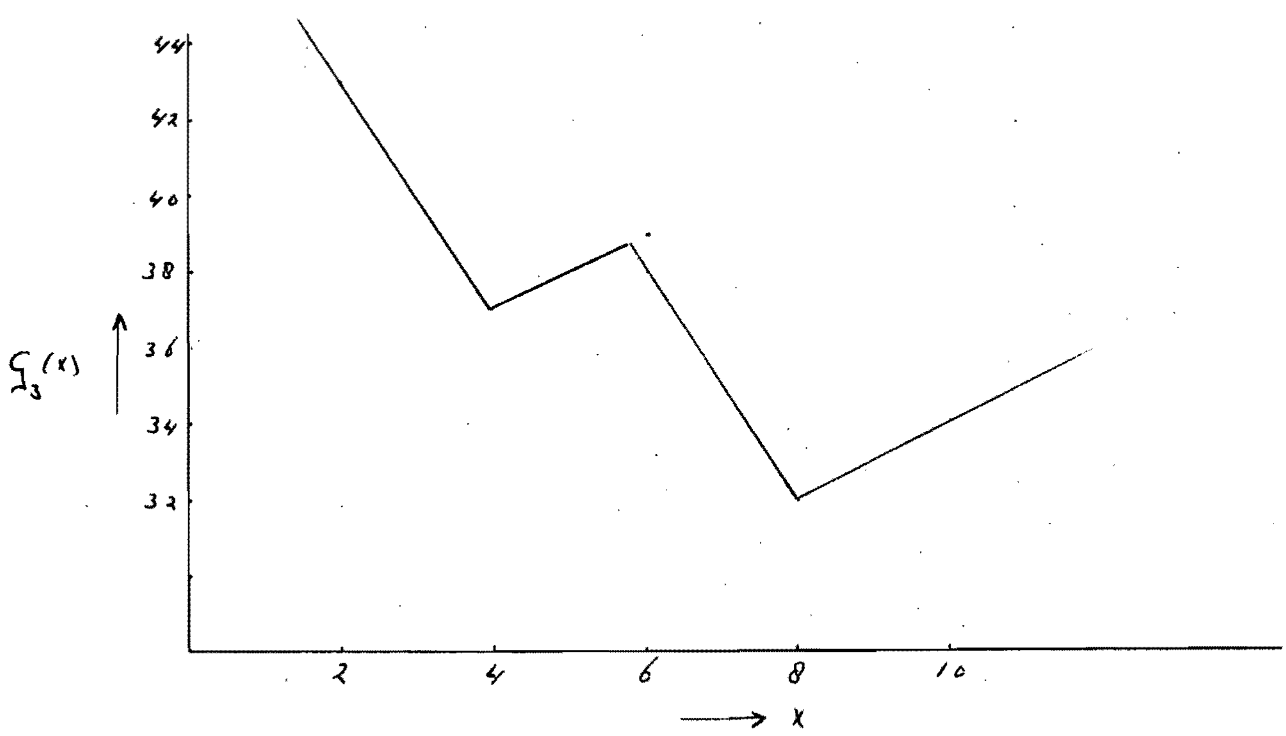
In our case it is easy to prove that if for n - 1 periods a strategy of the s-S type is optimal and if

$$K + G_{n-1}(x + a) - G_{n-1}(x) - a\,G'_{n-1}(x) \geq 0 \;, \forall x, \forall a, \; 0 \leq a \leq R$$

then also

$$K + G_{n}(x + a) - G_{n}(x) - a\,G'_{n}(x) \geq 0 \;, \forall x, \forall a, \; 0 \leq a \leq R \; .$$

But this does not imply the optimality of an s-S strategy for n periods. For instance if $p(x) = 3x$, $h(x) = x$, $c = 2$, $K = 9$, demand in each period is equal to 4 units, $R = 10$, $\alpha = 1$ then $G_3(x)$ is not unimodal (see fig. 1) and the optimal strategy for 3 periods certainly is not of the s-S type.



For instance if the inventory level is -4 it is optimal to order up to 4 instead of up to 6.
In this case the demand per period is deterministic but by taking a demand distribution with mean 4 and small variance this $G_3(.)$ may be approximated arbitrarily well.

If $K = 0$ then $G_n(.)$ is convex for any n.
The convexity of $G_n(.)$ implies the optimality of an s-S strategy with $s = S$.
The proof is obvious and will be omitted therefore.

II. Average costs criterion

Here we will use a continuous state version of the dynamic programming method of Howard which we proved for this case in [4].

1) Assumptions and definitions.

An orderstrategy is defined by a measurable function f(.), where f(u) is the quantity to order if the inventory level is u at the decisionmoment. We only consider strategies of the following form:

$$f(u) = R \quad \text{if } u < m_f ,$$
$$u + f(u) \le M_f \quad \text{if } u \le M_f \tag{1}$$
for some pair (m_f, M_f) with $m_f < M_f$.

Define the space $E_\infty(-\infty,b]$ as follows:

$\rho(.) \in E_\infty(-\infty,b] \iff \rho(.)$ is a measurable function on $(-\infty,b]$ and

$\dfrac{\rho(u)}{e^{-u}}$ is bounded a.e.

Define an operator $T_f^\star$ on $E_\infty(-\infty,M_f]$ by

$$(T_f^\star\rho)(u) = \int_{-\infty}^{M_f} \varphi(u + f(u) - v)\rho(v)dv , \tag{2}$$

where $\varphi(.)$ is the frequency function of the demand per period.

Assumptions about $\varphi(.)$ and L(.):

$$\varphi(\xi) > 0 \quad \text{if } \xi > 0 ;$$
$$\varphi(\xi) = 0 \quad \text{if } \xi < 0 ; \tag{3}$$
$\varphi(.)$ is continuous and differentiable on $[0,\infty)$;

$$\int_0^\infty \varphi(\xi)e^\xi d\xi < e^R \quad ; \tag{4}$$

$$\frac{L(u)}{e^{-u}} \text{ is bounded .} \tag{5}$$

We define $r_f(u)$ as the expected costs in a period which we begin with an inventory level u and where we apply orderstrategy f(.).
Then (5) implies the existence of $T_f^* r_f$.
The average costs under strategy f(.) are (see [4])

$$g_f = \lim_{n\to\infty} (T_f^{*n} r_f)(u) \quad . \tag{6}$$

2) We suppose that among the s-S strategies with R > S - s is a best one*) and we want to derive some conditions for the valuefunction of Howard. But for doing so we first have to say something about this "value-function".
Define an operator $P_{s,S}$ on $E_\infty(-\infty,S]$ by

$$(P_{s,S}\rho)(u) = \int_s^u \varphi(u - v)\rho(v)dv + \int_{-\infty}^{S-R} \varphi(u - v)\rho(v + R)dv \quad . \tag{7}$$

If $\lambda(.) \in E_\infty(-\infty,S]$, the equation $\rho - P_{s,S}\rho = \lambda$ has a unique solution, given by

$$\sum_0^\infty (P^n_{s,S}\lambda) \quad , \quad \text{(see [4])} \ .$$

Define $k_{s,S}(.)$ as the unique solution of

$$\rho(u) - (P_{s,S}\rho)(u) = L(u) + K \int_{-\infty}^{s} \varphi(u - v)dv \tag{8}$$

*) R = S - s can be handled as a limitcase of R > S - s.
If we include R = S - s it is obvious that among the s-S strategies is a best one.
But we will not work out here the transition to the limit.

and $t_{s,S}(\cdot)$ as the unique solution of

$$\rho(u) - (P_{s,S}\rho)(u) = 1 \ . \tag{9}$$

$k_{s,S}(u)$ are the expected costs between now, beginning with an inventory level u, after ordering, and the moment S is reached for the first time*).
$t_{s,S}(u)$ is the corresponding expected time.

Writing $g_{s,S}$, $r_{s,S}$ and $T^*_{s,S}$ instead of g_f, r_f and T^*_f we have, (see [4])

$$g_{s,S} = \lim_{n\to\infty} (T^{*n}_{s\ S} r_{s\ S})(u) = \frac{k_{s,S}(S)}{t_{s,S}(S)} \ . \tag{10}$$

$J_{s,S}(\cdot)$, defined by $J_{s,S}(u) = k_{s,S}(u) - g_{s,S}t_{s,S}(u)$, (11)

is the solution of

$$\rho(u) - (P_{s,S}\rho)(u) = L(u) + K \int_{-\infty}^{s} \varphi(u - v)dv - g_{s,S}\cdot 1 \tag{12}$$

Let $v_{s,S}(\cdot)$ be a function, defined by:

$$v_{s,S}(u) = K + J_{s,S}(u + R) \quad \text{if } u < S - R \ , \tag{13a}$$

$$v_{s,S}(u) = K \quad \text{if } S - R \le u < s \ , \tag{13b}$$

$$v_{s,S}(u) = J_{s,S}(u) \quad \text{if } s \le u \ . \tag{13c}$$

It is obvious that $v_{s,S}(\cdot)$, the value function, is a solution of

$$\rho(u) - (T^*_{s,S}\rho)(u) = r_{s,S}(u) - g_{s,S} \ . \tag{14}$$

In [4] we proved the following optimality criterion for the strategy (s , S): Iff

$$v_{s,S}(u) - \int_{-\infty}^{M} \varphi(u + a - x)v_{s,S}(x)dx \le K\delta(a) + L(u + a) - g_{s,S}$$

$$\forall u,\ m \le u \le M \text{ and } \forall a,\ 0 \le a \le R \ , \tag{15}$$

*) In the average costs case one can take c = 0 without loss of generality.

then this (s,S) strategy is the best one of all the strategies with $u + f(u) \le M$, $\forall u \le M$ and $f(u) = R$, $\forall u < m$, $(m < s \le S < M)$.
We can translate this optimality condition into a condition on $J_{s,S}(.)$ for

$$\int_{-\infty}^{M} \varphi(u + a - x) v_{s,S}(x) dx = J_{s,S}(u + a) \ . \tag{16}$$

Together with (13a,b,c) and (15) this yields the following theorem:

Theorem: Iff $J_{s,S}(.)$ satisfies the following conditions

1) if $u < S - R$: a) $J_{s,S}(u + R) \le J_{s,S}(u + a)$, $\forall a$, $0 < a \le R$, (17a)

b) $K + J_{s,S}(u + R) \le J_{s,S}(u)$; (17b)

2) if $S - R \le u < s$: a) $J_{s,S}(S) \le J_{s,S}(u + a)$, $\forall a$, $0 < a \le R$, (17c)

b) $K + J_{s,S}(S) \le J_{s,S}(u)$; (17d)

3) if $u \ge s$: $J_{s,S}(u) \le K + J_{s,S}(u + a)$, $\forall a$, $0 < a \le R$; (17e)

then this (s,S) strategy is the best one of all the strategies f(.) for which there is a pair (m_f, M_f), $m_f < s \le S < M_f$ such that $u + f(u) \le M_f$, $\forall u \le M_f$ and $f(u) = R$, $\forall u < m_f$.

3) Now we can derive some conditions for $J_{s,S}(.)$ using the assumption that the strategy (s,S) is the best one of all the (s,S) strategies.
Define

$$k^{0}_{p,q}(x) = \frac{\partial k_{p,q}(x)}{\partial q} \ , \tag{18a}$$

$$k^{\nabla}_{p,q}(q) = \frac{\partial k_{p,q}(q)}{\partial q} \ , \tag{18b}$$

$$k^{*}_{p,q}(x) = \frac{\partial k_{p,q}(x)}{\partial p} \ , \tag{18c}$$

and the same for $t_{p,q}(x)$. (19a,b,c)

The existence of the derivatives follows from the sequel when we need it.

The pair (s,S) has to meet the following conditions

i) $$\frac{\partial g_{s,S}}{\partial S} = \frac{k^{\nabla}_{s,S}(S)t_{s,S}(S) - t^{\nabla}_{s,S}(S)k_{s,S}(S)}{\{t_{s,S}(S)\}^2} = 0\ , \tag{20}$$

ii) $$\frac{\partial g_{s,S}}{\partial s} = \frac{k^{*}_{s,S}(S)t_{s,S}(S) - t^{*}_{s,S}(S)k_{s,S}(S)}{\{t_{s,S}(S)\}^2} = 0\ , \tag{21}$$

or

$$k^{\nabla}_{s,S}(S) - g_{s,S}t^{\nabla}_{s,S}(S) = 0\ , \tag{20a}$$

and

$$k^{*}_{s,S}(S) - g_{s,S}t^{*}_{s,S}(S) = 0\ . \tag{21a}$$

4. First we shall prove that the condition (20) implies

$$J'_{s,S}(S) = 0$$
$$\left(J'_{s,S}(x) = \frac{dJ_{s,S}(x)}{dx}\right) \tag{22}$$

$$k_{s,S+\Delta}(x) - k_{s,S}(x) = \int_{s}^{x} \varphi(x - z)\{k_{s,S+\Delta}(z) - k_{s,S}(z)\}dz + $$
$$+ \int_{-\infty}^{S} \varphi(x + R - y)\{k_{s,S+\Delta}(y) - k_{s,S}(y)\}dy + $$
$$+ \int_{S}^{S+\Delta} \varphi(x + R - y)k_{s,S+\Delta}(y)dy\ .$$

So

$$k^0_{s,S}(x) = \int_s^x \varphi(x - z)k^0_{s,S}(z)dz + \int_{-\infty}^S \varphi(x + R - y)k^0_{s,S}(y)dy +$$

$$+ \varphi(R)k_{s,S}(S) \ .$$

And in the same way

$$t^0_{s,S}(x) = \int_s^x \varphi(x - z)t^0_{s,S}(z)dz +$$

$$+ \int_{-\infty}^S \varphi(x + R - y)t^0_{s,S}(y)dy + \varphi(R)t_{s,S}(S) \ .$$

Therefore

$$k^0_{s,S}(x) - t^0_{s,S}(x)\cdot g_{s,S} = P_{s,S}(k^0_{s,S} - g_{s,S}t^0_{s,S})(x)$$

and by the uniqueness of the solution of $\rho = P_{s,S}\rho + \lambda$ then follows

$$k^0_{s,S}(x) - g_{s,S}t^0_{s,S}(x) \equiv 0 \ . \qquad (23)$$

Further

$$k_{s,S+\Delta}(S + \Delta) - k_{s,S}(S) = L(S + \Delta) - L(S) +$$

$$+ K \int_{-\infty}^s \{\varphi(S + \Delta - z) - \varphi(S - z)\}dz +$$

$$+ \int_s^{S+\Delta} \varphi(S + \Delta - z)\{k_{s,S+\Delta}(z) - k_{s,S}(z)\}dz +$$

$$+ \int_s^{S+\Delta} k_{s,S}(z)\{\varphi(S + \Delta - z) - \varphi(S - z)\}dz +$$

$$+ \int_{S}^{S+\Delta} \varphi(S - z)k_{s,S}(z)dz +$$

$$+ \int_{-\infty}^{S+\Delta} \varphi(S + \Delta + R - y)\{k_{s,S+\Delta}(y) - k_{s,S}(y)\}dy +$$

$$+ \int_{-\infty}^{S+\Delta} \{\varphi(S + \Delta + R - y) - \varphi(S + R - y)\}k_{s,S}(y)dy +$$

$$+ \int_{S}^{S+\Delta} \varphi(S + R - y)k_{s,S}(y)dy$$

shows the existence of $k^{\nabla}_{s,S}(S)$

$$k^{\nabla}_{s,S}(S) = L'(S) + K \int_{-\infty}^{s} \varphi'(S - z)dz + \int_{s}^{S} \varphi(S - z)k^{0}_{s,S}(z)dz +$$

$$+ \int_{s}^{S} \varphi'(S - z)k_{s,S}(z)dz + \varphi(0)k_{s,S}(S) + \varphi(R)k_{s,S}(S) +$$

$$+ \int_{-\infty}^{S} \varphi(S + R - y)k^{0}_{s,S}(y)dy + \int_{-\infty}^{S} \varphi'(S + R - y)k_{s,S}(y)dy .$$

In the same way

$$t^{\nabla}_{s,S}(S) = \int_{s}^{S} \varphi(S - z)t^{0}_{s,S}(z)dz + \int_{s}^{S} \varphi'(S - z)t_{s,S}(z)dz +$$

$$+ \varphi(0)t_{s,S}(S) + \int_{-\infty}^{S} \varphi(S + R - y)t^{0}_{s,S}(y)dy +$$

$$+ \int_{-\infty}^{S} \varphi'(S + R - y)t_{s,S}(y)dy + \varphi(R)t_{s,S}(S) .$$

Using (23) we get

$$k^{\nabla}_{s,S}(S) - g_{s,S}\cdot t^{\nabla}_{s,S}(S) = L'(S) + K\int_{-\infty}^{s}\varphi'(S - z)dz +$$

$$+ \int_{s}^{S}\varphi'(S - z)J_{s,S}(z)dz + \int_{-\infty}^{S}\varphi'(S + R - y)J_{s,S}(y)dy \ . \tag{24}$$

With aid of (24) it is easy to show that

$$J'_{s,S}(S) = k^{\nabla}_{s,S}(S) - g_{s,S}t^{\nabla}_{s,S}(S)$$

and from (20a) then follows

$$J'_{s,S}(S) = 0 \ .$$

5) Now we consider the condition (21a).
By the same way as in point 4) we find

$$k^{*}_{s,S}(x) = \varphi(x - s)\{K - k_{s,S}(s)\} + \int_{s}^{x}\varphi(x - z)k^{*}_{s,S}(z)dz +$$

$$+ \int_{-\infty}^{S}\varphi(x + R - y)k^{*}_{s,S}(y)dy \tag{25}$$

and

$$t^{*}_{s,S}(x) = \varphi(x - s)(-t_{s,S}(s)) + \int_{s}^{x}\varphi(x - z)t^{*}_{s,S}(z)dz +$$

$$+ \int_{-\infty}^{S}\varphi(x + R - y)t^{*}_{s,S}(y)dy \ . \tag{26}$$

The uniqueness of the solution of $\rho - P_{s,S}\rho = \lambda$ implies

$$\frac{k^*_{s,S}(x)}{t^*_{s,S}(x)} = \frac{k_{s,S}(s) - K}{t_{s,S}(s)} \tag{27}$$

and together with (21a) this yields

$$\frac{k_{s,S}(s) - K}{t_{s,S}(s)} = g_{s,S} \quad \text{or} \quad J_{s,S}(s) = K \;. \tag{28}$$

6) We can use (22) and (28) in order to obtain an easy equation for $J'_{s,S}(x)$:

$$J'_{s,S}(x) = L'(x) + K \int_{-\infty}^{s} \varphi'(x - z)dz + \int_{s}^{x} \varphi'(x - z)J_{s,S}(z)dz +$$

$$+ \varphi(0)J_{s,S}(x) + \int_{-\infty}^{S} \varphi'(x + R - y)J_{s,S}(y)dy \;.$$

$$\int_{s}^{x} \varphi'(x - z)J_{s,S}(z)dz = -\varphi(0)J_{s,S}(x) + \varphi(x - s)J_{s,S}(s) +$$

$$+ \int_{s}^{x} \varphi(x - z)J'_{s,S}(z)dz$$

$$\int_{-\infty}^{S} \varphi'(x + R - y)J_{s,S}(y)dy = \int_{-\infty}^{S} \varphi(x + R - y)J'_{s,S}(y)dy \;.$$

Hence

$$J'_{s,S}(x) = L'(x) + \int_{s}^{x} \varphi(x - z)J'_{s,S}(z)dz + \int_{-\infty}^{S} \varphi(x + R - y)J'_{s,S}(y)dy \;. \tag{29}$$

This equation is analytically usable as we shall see by the nest two points.

7) We consider the following special case

$$h(x) = x.h, \quad p(x) = x.p \quad \text{and} \quad \varphi(\xi) = \frac{\lambda^n \xi^{n-1} e^{-\lambda\xi}}{(n-1)!} \quad \text{if } \xi \geq 0 .$$

By differentiating (29) n times we find the following condition on $J'_{s,S}$

$$(D + \lambda)^n J'_{s,S} = \lambda^n (h + J'_{s,S}) \qquad \text{on } [0,\infty) \cap [s,\infty) ,$$

$$(D + \lambda)^n J'_{s,S} = \lambda^n h \qquad \text{on } [0,\infty) \cap [S - R, s) ,$$

$$(D + \lambda)^n J'_{s,S} = \lambda^n (h + J'_{s,S}(+ R)) \qquad \text{on } [0,\infty) \cap (-\infty, S - R) ,$$

where D is the differential operator and $J'_{s,S}(+ R)$ is defined by

$$J'_{s,S}(+ R)(x) = J'_{s,S}(x + R) .$$

For $x < 0$ we have the same conditions with h replaced by $-\rho$.

It is not easy to solve these equations. There are a lot of difficulties with boundary conditions.

If $n = 1$ we have the condition $DJ'_{s,S} = \lambda h$ on $[0,\infty) \cap [s,\infty)$.
Differentiating (29) one time gives

$$J''_{s,S}(x) = L''(x) + J'_{s,S}(s)\varphi(x - s) + \int_s^x \varphi(x - z) J''_{s,S}(z)dz +$$

$$+ \int_{-\infty}^{S} \varphi(x + R - y) J''_{s,S}(y)dy .$$

Now we assume that $s \geq 0$.
Then for $x \geq s$

$$J''_{s,S}(x) \geq 0$$

and for $x < s$

$$J''_{s,S}(x) = L''(x) + \int_s^S \varphi(x + R - y)\lambda h \, dy + \int_{-\infty}^{s} \varphi(x + R - y) J''_{s,S}(y)dy .$$

This implies $J''_{s,S}(x) \geq 0$ on $(-\infty,s)$ too and $J_{s,S}$ meets the conditions (17a,b,c,d,e).

That means: If the holding- and stockoutcosts are linear and the demand is negative exponentially distributed and if among the (s,S) strategies there is a best one with $R > S - s$ and $s \geq 0$ then this strategy is optimal over-all.

A very poor result but, considering the finite period case, we hardly could expect a better one.

8) The case without set-up costs ($K = 0$).

Now $s = S$ for the best one of the s-S strategies.

$$J'_{S,S} = L'(x) + \int_S^x \varphi(x - z)J'_{S,S}(z)dz + \int_{-\infty}^S \varphi(x + R - y)J'_{S,S}(y)dy \tag{30}$$

and

$$J''_{S,S}(x) = L''(x) + J'_{S,S}(S)\{\varphi(x - S) - \varphi(x + R - S)\} + $$

$$+ \int_S^x \varphi(x - z)J''_{S,S}(z)dz + \int_{-\infty}^S \varphi(x + R - y)J''_{S,S}(y)dy\ . \tag{31}$$

Thus for $x < S$

$$J''_{S,S}(x) = L''(x) + \int_{-\infty}^S \varphi(x + R - y)J''_{S,S}(y)dy \tag{32}$$

which implies $J''_{S,S}(x) \geq 0$ if $x < S$. Using this in (31) yields $J''_{S,S}(x) \geq 0$ everywhere.

It is easy to show then that this (S,S) strategy is optimal over-all.

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