Geometrical treatment of the PPP-P case in coplanar motion : three infinitesimally and one finitely separated positions of a plane

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Geometrical Treatment of the PPP–P Case in Coplanar Motion (Three infinitesimally and one finitely separated positions of a plane)

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Abstract
The geometrical construction of the center- and of the circle-point curve is investigated for the case in which three out of four separated positions of a rigid body are infinitesimally near to each other.
It is proved that the first curve osculates the return-circle and the second the inflection circle of the three coinciding positions.
If one of the curves degenerates into the line at infinity and an orthogonal hyperbola, the other curve degenerates into the inflection- or the return-circle and into the perpendicular bisector to the line segment $\overline{P_{12}P_{34}}$.
Generally, if the opposite vertex quadrilateral $\Pi_{13}P_{32}\Pi_{24}P_{41}$ has an inscribed circle, touching $P_{32}P_{41}$ at Π_{24}, both curves have a double point, co-ordinated through the relation of Euler–Savary.

Zusammenfassung—Geometrische Behandlung des PPP-P Falles in Ebener Bewegung (Drei unendlich und eine endlich getrennte hage einer Ebene): Dr. tech. Wet. E. A. Dijksman. Die geometrische Konstruktion der Mittelpunkt- und der Kreispunktkurve ist untersucht für den Fall in welchem drei von vier getrennten Lagen eines festen Körpers unendlich nahe zueinander liegen. Es wird bewiesen, dass die erste Kurve den Rückkehr-Kreis berührt und die zweite Kurve den Wendekreis der drei zusammenfasslenden Lagen. Im Falle einer von diesen Kurven degeneriert in die unendlich ferne Gerade und in eine orthogonale Hyperbel, fällt die entsprechende Kurve auseinander in den Wendekreis (oder in den Rückkehr-Kreis) und in eine senkrecht-halbierende zum Abschnitt $\overline{P_{12}P_{34}}$.
Im allgemeinen, falls das Gegenpolviereck $\Pi_{13}P_{32}\Pi_{24}P_{41}$ einen eingeschriebenen Kreis besitzt, der $P_{32}P_{41}$ in Π_{24} berührt, haben beide Kurven einen Doppelpunkt, aneinander verknüpft durch die Euler–Savary'sche Bedingung.

Резюме—Геометрическое разсмотрение *PPP–P* случая в плоском движении: (три безконечно-близкого и одно конечно раздельное положение плоскости): Др. Е. А. Дийксман. Геометрическое построение кривой центров и кривой круговых точек изследовано для случая, в котором три из четырех раздельных положений твердаго тела находятся безконечно близко друг к другу.
Доказывается что первая кривая касается круга возврата, а вторая касается круга перегиба для трех совпадающих положений. В случае, когда одна из двух кривых вырождается в бесконечную прямую и в ортогональную гиперболу, вторая кривая распадается в круг перегиба (или в круг возврата) и перпендикуляр к отрезку $\overline{P_{12}P_{34}}$. Вообще в случае если противополюсньй четырехуголвник $\Pi_{13}P_{32}\Pi_{24}P_{41}$ имеет вписанную окружность, которую касается $P_{32}P_{41}$ в точке Π_{24}, обе кривой имеют двойную точку, определенную уравнением Эйлера–Савари.

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1. Introduction

In 1888 it was Ludwig Burmester [1] who presented a theory about four (and five) separated positions of a rigid body in coplanar motion. Soon he was followed by Rodenberg [2], Grübler [3] and Müller [4] who studied the kinematic properties of those positions where all of them are infinitesimally near to each other. In 1959 Volmer [5] investigated the case in which only the first two and the last two positions are infinitesimally near.

Tesar [6] introduced the symbols P–P and PP representing two finitely and two infinitesimally separated positions, respectively. In this way all possible combinations of four separated positions can be listed easily: P–P–P–P, PP–P–P, PP–PP, PPP–P and PPPP.

Of the combination PPP–P, not yet discussed hitherto, an *analytical* investigation has been made by Tesar and Eschenbach [7]. The intention of *this* paper, therefore, is to show the *geometric* properties of this combination, thus allowing the designer to use such a geometrical analysis on his kinematic problems.

2. The Case *PPP–P*

The case under consideration actually deals with only two separated positions A_1B_1 and A_4B_4. In the first position of the rigid body, however, the centers of curvature α and β, corresponding to the respective points of path A and B are given. Since the two osculation circles (A, α) and (B, β) are defined by three infinitesimally near position-points (A_1, A_2, A_3) and (B_1, B_2, B_3), respectively, we may introduce three positions A_1B_1, A_2B_2 and A_3B_3 instead of one.

The three infinitesimally near positions A_1B_1, A_2B_2 and A_3B_3 define an instantaneous pole P, a pole-tangent p and an inflection circle C_W. The pole $P=(P_{12}=P_{23}=P_{31})$ may be found by intersecting the straight lines $A_1\alpha$ and $B_1\beta$ (see Fig. 1).

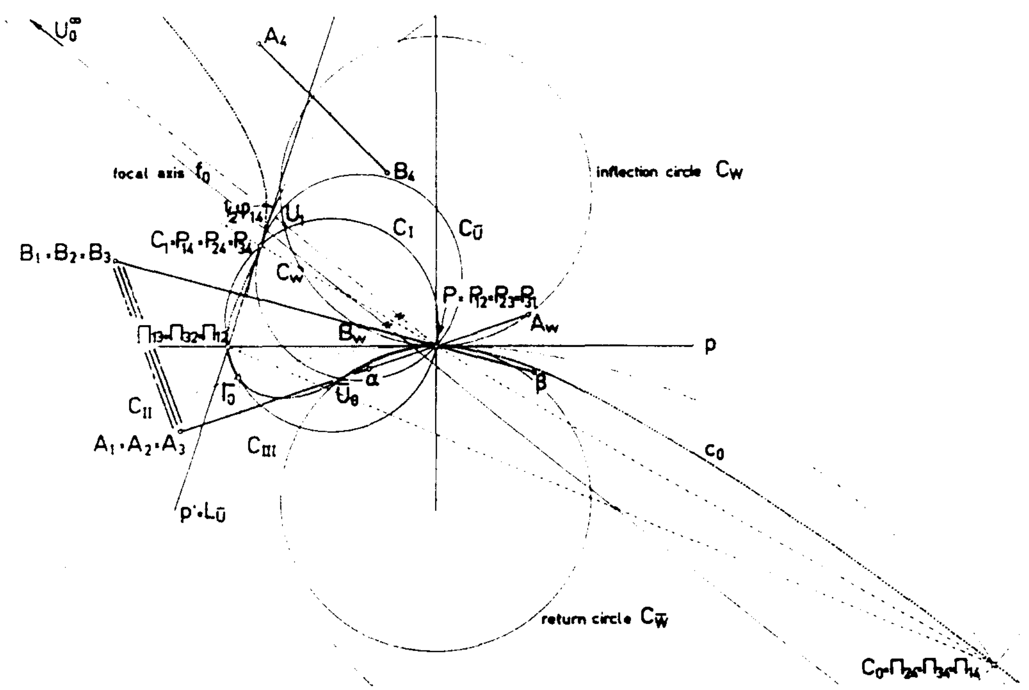

Figure 1.

The pole-tangent p is to be found with the so-called construction of Bobillier. The inflection circle C_W finally, touches p at P and goes through point A_w which is defined by the relation $\overline{PA_1^2}=\overline{A_1\alpha}\,.\,\overline{A_1A_w}$, which is a way to express the relation of Euler–Savary.

Now we can say that the three infinitesimally near separated positions are represented by the instantaneous pole P, the pole tangent p and the inflection circle C_W.

The fourth position, represented by the position A_4B_4, is also defined by the rotation-center P_{14} and the rotation angle φ_{14}. Since the first three positions are coinciding with each other, it is clear that $P_{14}=P_{24}=P_{34}$ and $\varphi_{14}=\varphi_{24}=\varphi_{34}$.

In the theory of finitely separated positions, the halves of the rotation angles appear in the so-called pole-triangles. For instance the angle $\varphi_{14}/2=\sphericalangle P_{21}P_{14}P_{42}$ appears in the pole triangle $\Delta P_{21}P_{14}P_{42}$. In the case under consideration the poles P_{14} and P_{24} are infinitesimally close, which means that P_{24} approaches P_{14} on a line p', making an angle $\varphi_{14}/2$ with the side $P_{41}P_{12}$ of that pole triangle. Since also

$$\begin{aligned}\varphi_{14}/2&=\sphericalangle P_{21}P_{14}P_{42}=\sphericalangle P_{31}P_{14}P_{43}=\\ \varphi_{24}/2&=\sphericalangle P_{32}P_{24}P_{43}=\sphericalangle P_{12}P_{24}P_{41}=\\ \varphi_{34}/2&=\sphericalangle P_{13}P_{34}P_{41}=\sphericalangle P_{23}P_{34}P_{42}\end{aligned}$$

it can be said that the straight line $p'=P_{14}P_{42}=P_{24}P_{43}=P_{34}P_{41}=P_{14}P_{43}=P_{24}P_{41}=P_{34}P_{42}$.

In the pole-triangle $\Delta P_{12}P_{23}P_{31}$, the pole P_{23} approaches P_{12} along the *fixed polode*, since P_{23} is the next-near rotation center in the sequence P_{12}–P_{23}–P_{34}–P_{45}–........ (The polygon represented by these points approaches the fixed polode as long as the corresponding positions approach infinitesimally near to each other).

From this it follows that the pole-tangent $p=P_{12}P_{23}$. Since also $\sphericalangle P_{21}P_{13}P_{32}=\varphi_{13}/2$ approaches the value zero, it is to be seen that $p=P_{12}P_{23}=P_{13}P_{32}=P_{21}P_{13}$.

So p and p' are opposite sides in the three coinciding opposite-pole quadrilaterals $\square P_{12}P_{23}P_{34}P_{41}$, $\square P_{23}P_{31}P_{14}P_{42}$ and $\square P_{31}P_{12}P_{24}P_{43}$. According to the notations of by Burmester, the opposite sides $P_{12}P_{23}$ and $P_{34}P_{41}$ intersect at Π_{13} and so on. It thus follows that p and p' intersect at $\Pi_{13}=\Pi_{21}=\Pi_{32}$. Also according to Burmester the opposite sides $P_{12}P_{41}$ and $P_{23}P_{34}$ intersect at Π_{24}. Interchanging the positions 1, 2 and 3 it follows that $P_{12}P_{41}$, $P_{23}P_{34}$, $P_{23}P_{42}$, $P_{31}P_{14}$, $P_{31}P_{43}$ and $P_{12}P_{24}$ all intersect at the point $\Pi_{24}=\Pi_{34}=\Pi_{14}$. But since all these sides are coinciding, the location of Π_{24} on $P_{12}P_{41}$ has to be found by another method.

In the theory of four finitely separated positions a so-called pole curve is defined as the locus of those points at which two opposite sides from an opposite-pole quadrilateral subtend the same angle. It can be shown that such a curve is independent from the choice of the opposite-pole quadrilateral. It is known also that this curve is identical with the so-called center-point curve. If the four positions X_1, X_2, X_3 and X_4 of a point X of the rigid body are lying on a circle, the locus of points X_1 for which such is the case, is called the circle-point curve of position 1. The locus of the corresponding centers X_0 of these circles is called the center-point curve.

By kinematic inversion the center-point curve of position 1 becomes the circle-point curve and the circle-point curve of position 1 the center-point curve. So likewise the poles P_{12}, P^1_{23}, P_{13}, P^1_{24}, P^1_{34} and P_{14} of the four positions of the fixed plane with respect to the initial position 1 of the moving plane define also a pole-curve which is identical with the circle-point curve of position 1.

It can be shown that the center-point curve c_0 passes through the poles P_{12}, P_{23}, P_{31}, P_{14}, P_{42} and P_{43} and also through the points Π_{12}, Π_{23}, Π_{31}, Π_{14}, Π_{42} and Π_{43}. And in the same way that the circle-point curve c_1 passes through the poles P_{12}, P^1_{23}, P_{31}, P_{14}, P^1_{42} and P^1_{43} and also through the points Π^1_{12}, Π^1_{23}, Π^1_{31}, Π^1_{14}, Π^1_{42} and Π^1_{43}. (See Fig. 2). Finally it is known that both curves c_0 and c_1 are *circular cubics*. Now assuming, but also proved later on, that this final property is not affected by letting three positions approach infinitesimally near to each other, it is clear that in general any straight line intersects c_0 in

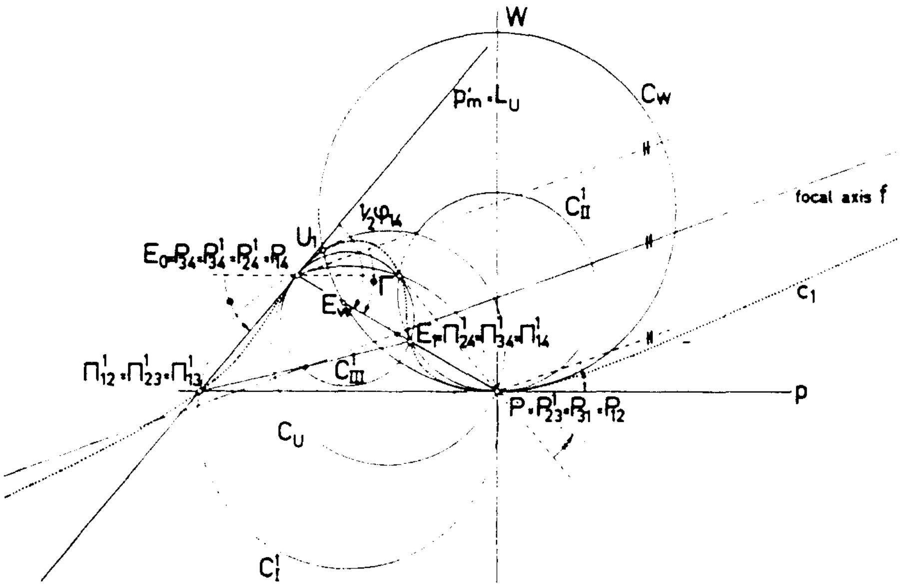

Figure 2.

either three real points or in only one real point (the other two points being of complex nature in the last case). Let us now consider the straight line $P_{12}P_{41}$. The three real points of intersection are P_{12}, P_{41} and Π_{24} in the case under consideration. Take point C_1 at P_{14}, it then follows that $C_3=C_2=C_1=P_{14}=C_4$. So C_0 which is the center of curvature regarding the first three positions, is a point of the center-point curve c_0. But C_0, C_1 and $P=P_{12}$ are lying at the same path normal $P_{12}P_{14}$ as is known from curvature theory. The exact location of C_0 on $P_{12}P_{14}$ may be found with the formula $\overline{PC_1^2}=\overline{C_1C_0}.\overline{C_1C_w}$, which represents the relation of Euler–Savary. It is clear that no more than three intersections of $P_{12}P_{14}$ with c_0 exist. Thus $C_0\equiv\Pi_{24}=\Pi_{34}=\Pi_{14}$. This establishes the exact location of Π_{24} on $P_{12}P_{14}$.

It is proved in the appendix of this paper that for finite separated positions the locus of points at which two opposite sides of the quadrilateral $\square\Pi_{31}P_{12}\Pi_{24}P_{43}$ subtend the same "looking-angle" is identical with the pole-curve c_0. Assuming this property holds if three out of four separated positions approach infinitesimally near to each other, it is clear that in the case under consideration $\square\Pi_{31}P_{12}\Pi_{24}P_{43}$ is more appropriate as a base for the construction of c_0 than the initial but folded opposite-pole quadrilateral $\square P_{12}P_{23}P_{34}P_{41}$.

The easiest way of drawing c_0 is using its property as *focal-curve*. The method is described by Burmester and noted by Beyer [8] in his book "The Kinematic Synthesis of Mechanisms". Thereby the focus Γ_0 of c_0 is defined as the intersection of the tangents at the isotropic points of the curve. But it is known that the focus Γ_0 can be found also at the common intersection point of the circles circumscribed about the triangles:

$$\left.\begin{array}{l}\Pi_{13}P_{23}P_{34},\,\Pi_{13}P_{21}P_{14},\,\Pi_{24}P_{12}P_{23},\,\Pi_{24}P_{14}P_{43},\\ \Pi_{21}P_{31}P_{14},\,\Pi_{21}P_{32}P_{24},\,\Pi_{34}P_{23}P_{31},\,\Pi_{34}P_{24}P_{41}\\ \Pi_{32}P_{12}P_{24},\,\Pi_{32}P_{13}P_{34},\,\Pi_{14}P_{31}P_{12}\text{ and }\Pi_{14}P_{34}P_{42}\end{array}\right\}\qquad\text{(A)}$$

In the case under consideration not more than three of these circles differ. They are: C_{I} circumscribing $\Delta\Pi_{13}P_{23}P_{34}$, C_{II} touching p at P and going through Π_{24}, and finally C_{III} touching p' at P_{34} and also going through Π_{24}.

The three circles intersect at Γ_0*.

* In the special case that A_4B_4 lies parallel to A_1B_1, it can be seen that the common intersection point of C_{I}, C_{II} and C_{III} coincides with the instantaneous pole P. Thus, if $P_{14}=P_{14}^{*}$, then $\Gamma_0=P$. And similar, if $P=P^{*}$ then $\Gamma_0=P_{14}$.

Another method of finding Γ_0 is using theorem 2, which is proved in the appendix, and which states that the point Γ_0 is *isogonally coordinated* to the asymptotic point of the pole curve c_0 with respect to any of the triangles mentioned under (A).

Using this theorem, it is essential to find the asymptotic point of c_0. This point, however, is the infinite point of the so-called *focal axis*, which line also plays an important part in the construction of the pole curve.

As is known, the focal axis coincides with the connecting line of the midpoints of the diagonals of the opposite-pole quadrilateral. As stated before, the case under consideration makes it more appropriate to use the diagonals of the quadrilateral $\Pi_{31}P_{12}\Pi_{24}P_{43}$ instead of the ones from the initial opposite-pole quadrilateral.

Thus the focal axis is the connecting-line of the mid-points of the line-segments $\overline{\Pi_{31}\Pi_{24}}$ and $\overline{P_{12}P_{34}}$. It is proved by Burmester that the points of the pole curve c_0 may be found by intersecting the members of a pencil of circles with the corresponding members of a pencil of rays. If the pencil of rays has the focal point Γ_0 as base-point, the correspondence between the two pencils is such that the ray through Γ_0 intersects the focal axis in the center of the corresponding circle. The latter then intersects the mentioned ray in two points of the curve. The radii of the corresponding circles will be found by finding the power axis (or the radical axis) of the pencil of circles. This can be done by drawing two members of the pencil, to be found by using two known points (for instance P_{12}, P_{14}, Π_{31} or Π_{24}) of the curve.

If the base-points of the pencil of circles are real, and not coinciding, the curve, produced by points lying on either side of the focal axis, consists of two branches, also lying on either side of the focal axis. The two branches do not meet, as long as two of the intersections of the focal axis with c_0 remain of complex nature, which is the case if the pencil of circles do not contain members with zero diameter. If, however, the base-points are of complex nature, two members exist with zero diameter, which causes the two produced parts of the curve to blend into each other at those places on the focal axis where the rays through Γ_0 touch the curve.*

In order to derive the equation of the curve c_0, a co-ordinate system x–0–y of reference will be chosen in such a way that the origin 0 coincides with P, the x-axis with the pole-tangent and the inflection pole $W(0, \delta)$ lies on the positive half of the y-axis (see Fig. 3).

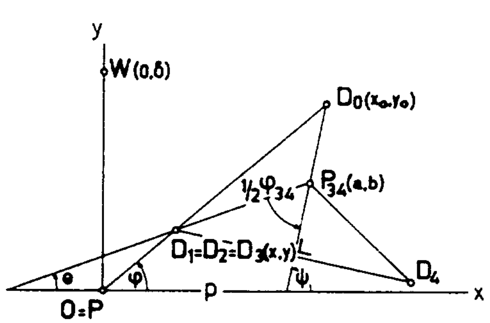

Figure 3.

Now, if D_0 is a point of the center-point curve c_0, it is also a center of curvature to the point $D_1 = D_2 = D_3$ of the path and the corresponding circle of osculation passes through the position D_4 of the moving point D according to the definition of the center-point curve in the case under consideration. It then follows from the definition of P_{34}, that $P_{34}D_0$ is

* In the special case that the base-points are coinciding, only one member with zero diameter exists. The curve will then have a double-point at this place, making the cubic single-branched.

the perpendicular bisector of the line $\overline{D_3 D_4}$, while at the same time making an angle of $\varphi_{34}/2$ with the straight-line D_3P_{34}. If D_3P_{34} and $P_{34}D_0$ are making angles of θ and ψ with the x-axis, respectively, it can be seen that $\varphi_{34}/2=\psi-\theta$. Thus

$$\tan\varphi_{34}/2=\frac{\tan\psi-\tan\theta}{1+\tan\psi\,.\tan\theta}=\frac{\dfrac{y_0-b}{x_0-a}-\dfrac{b-y}{a-x}}{1+\dfrac{y_0-b}{x_0-a}\cdot\dfrac{b-y}{a-x}} \tag{1}$$

where the co-ordinates of P_{34} are a and b respectively.

The co-ordinate transformation from point D_1 of the path into the center of curvature D_0 (and *vice versa*), due to the quadratic relationship of Euler–Savary, will be represented by the relations:

$$\left.\begin{aligned} x_0&=-\frac{\delta xy}{x^2+y^2-\delta y}\\ y_0&=-\frac{\delta y^2}{x^2+y^2-\delta y}\end{aligned}\right\} \tag{2}$$

and

$$\left.\begin{aligned} x&=\frac{\delta x_0 y_0}{x_0^2+y_0^2+\delta y_0}\\ y&=\frac{\delta y_0^2}{x_0^2+y_0^2+\delta y_0}\end{aligned}\right\} \tag{3}$$

Substituting the expressions for x and y into (1), the equation of the center-point curve will be obtained:

$$\begin{aligned}&(y_0-b)\{a(x_0^2+y_0^2+\delta y_0)-\delta x_0y_0\}-(x_0-a)\{b(x_0^2+y_0^2+\delta y_0)-\delta y_0^2\}\\ &=\tan(\varphi_{34}/2)[(x_0-a)\{a(x_0^2+y_0^2+\delta y_0)-\delta x_0y_0\}+(y_0-b)\{b(x_0^2+y_0^2+\delta y_0)\\ &-\delta y_0^2\}]\;\ldots(c_0)\ldots\end{aligned} \tag{4}$$

This, indeed, is the equation of a *cubic* going through the two isotropic points* of the plane. It is therefore a *circular* curve of the *third* degree. Defining the value λ by

$$\lambda=\frac{x_0^2+y_0^2+\delta y_0}{\delta y_0}$$

and m_{34} by

$$m_{34}=\tan(\varphi_{34}/2)$$

(4) becomes

$$\begin{aligned}&m_{34}(x_0^2+y_0^2)+(b-am_{34})x_0-(a+bm_{34})y_0-\lambda\{(b+am_{34})x_0-(a-bm_{34})y_0\\ &-m_{34}(a^2+b^2)\}=0\end{aligned} \tag{6}$$

* Note: The isotropic points are the two common points of all circles of the plane.

Introducing the expressions

$$C_{\bar{U}} \equiv m_{34}(x_0^2+y_0^2)+(b-am_{34})x_0-(a+bm_{34})y_0=0 \tag{7}$$

$$L_{\bar{U}} \equiv (b+am_{34})x_0-(a-bm_{34})y_0-m_{34}(a^2+b^2)=0 \tag{8}$$

and

$$C_{\bar{W}} \equiv x_0^2+y_0^2+\delta y_0=0 \tag{9}$$

the equation of the center-point curve c_0 will be given by the parametric representation

$$\left.\begin{aligned} C_{\bar{W}}-\lambda\delta y_0&=0 \quad (10)\\ C_{\bar{U}}-\lambda L_{\bar{U}}&=0 \quad (11)\end{aligned}\right\} (c_0)$$

Equation (7) is the equation of a circle $C_{\bar{U}}$ passing through the origin P_{12} and touching p' at P_{34}.

Equation (8) represents p' and, finally, (9) is nothing else than the equation of the so-called *return circle* $C_{\bar{W}}$, which is the image of the inflection circle C_W reflected in the pole tangent. Thus family (10) represents a pencil of circles, of which all members are touching the pole tangent p at P_{12}. The family (11) also represents a pencil of circles; but the members of this family are touching p' at P_{34}.

The radius r_λ of any member of the family (11) has the value

$$r_\lambda=\tfrac{1}{2}(1-\lambda)\sqrt{(1+m_{34}^{-2})(a^2+b^2)} \tag{12}$$

The radius R_λ of the corresponding circle of the family (10) then has the value

$$R_\lambda=(1-\lambda)\delta/2. \tag{13}$$

Because of the last two equations it becomes possible to draw two corresponding circles of the pencils for any value of λ. The intersections of two corresponding circles are points of the center-point curve c_0, at the same time providing the designer with another construction of this curve.

If $\lambda=+1$ the two corresponding circles are circles with zero radii and located at P_{34} and P_{12}, respectively. If on the other hand $\lambda=0$, the corresponding circles become $C_{\bar{U}}$ and the return circle $C_{\bar{W}}$, which have intersection points in P_{12} and Ball's point $\bar{U}_0$ of the inverse motion.

Eliminating λ from the equations (10) and (11), another expression for c_0 is obtained:

$$C_{\bar{W}}\,.\,L_{\bar{U}}-C_{\bar{U}}\,.\,\delta y_0=0 \tag{14}$$

The intersections of the return circle with the curve c_0 can be found by combining (14) and (9). This leads to the two coincident intersections of $C_{\bar{W}}$ and the pole tangent ($y_0=0$), and also to the intersections $\bar{U}_0$ and P_{12} of $C_{\bar{W}}$ and $C_{\bar{U}}$. Thus there are three coincident intersections of c_0 with the return circle at P_{12}. Since P_{12} is not a double-point of c_0, the radius of curvature of c_0 at P_{12} is the same as the one of the return circle. So the return circle is coinciding with the osculation circle of the center-point curve at P_{12}. (The same result can be proved by direct calculation). It also follows from (14) that p and p' are tangents to the curve at P_{12} and P_{34} respectively.

The equation of the circle-point curve c_1 will be obtained by substituting the expressions (2) for x_0 and y_0 in (1). The same result will be achieved if in (4) the coordinates x_0 and y_0 are replaced by x and y, respectively, and if at the same time the values of δ and $-\delta$ and also m_{34} and $-m_{34}$ will be interchanged. Therefore the equation of the circle point curve c_1 becomes

$$C_W \, . \, L_U - C_U \, . \, \delta y = 0 \qquad (c_1) \tag{15}$$

where

$$C_U \equiv m_{34}(x^2+y^2)-(b+am_{34})x+(a-bm_{34})y=0$$

represents the equation of a circle touching a line p'_m at $P^1_{34}=P_{34}$ and going through $P_{12}=P$, and

$$L_U \equiv (b-am_{34})x-(a+bm_{34})y+m_{34}(a^2+b^2)=0$$

represents the equation of the line p'_m, which is the image of p' relative to $P_{12}P^1_{34}=P_{12}P_{34}$.

Moreover

$$C_W \equiv x^2+y^2-\delta y=0$$

represents the equation of the inflection circle C_W. It may be clear that the curve c_1 is touching p'_m at P^1_{34} and has the inflection circle C_W as its osculation circle at point P_{12} of the curve. The intersection point U_1 of C_W and c_1, not coinciding with P_{12}, is identical with the intersection point U_1 of C_W and C_U. It is an inflection point of the path described by the moving point U of which the co-ordinated position U_4 lies on the inflection-tangent through U_1. (If position 4 comes infinitesimally near to position 3, such a point would become the so-called Ball's point of the instantaneous circling point curve). It is this point U_1 which certainly will be of some interest to the designer.

The construction of the circle-point curve c_1 based on its focal properties requires the location of the points Π^1_{31}, P_{12}, Π^1_{24} and P^1_{43}. Generally, the so-called *image poles* P^1_{23}, P^1_{34} and P^1_{24} are the images of the respective poles P_{23}, P_{34} and P_{24} relative to the sides $P_{21}P_{13}$, $P_{31}P_{14}$ and $P_{21}P_{14}$ respectively. In the case under consideration it follows that $P^1_{34}=P^1_{24}$ $=P_{34}=P_{24}=P_{14}$ and similarly $P^1_{23}=P_{23}=P_{31}=P_{12}$. Moreover, the sides $P_{12}P^1_{23}$ and $P^1_{34}P_{41}$ of the opposite-pole quadrilateral $P_{12}P^1_{23}P^1_{34}P_{41}$ are represented by p and p'_m, respectively. Thus p and p'_m intersect at Π^1_{13}. Now consider the straight-line $P_{12}P^1_{43}$. The three real points of intersection of $P_{12}P^1_{43}$ with c_1, are P_{12}, P^1_{43} and Π^1_{24}. Since $P^1_{43}=P_{43}$ is a point of c_0, we can take point E_0 at P^1_{43}. Thus E_1, coordinated to E_0, must lie at c_1. On the other hand this co-ordination is represented by the equation of Euler–Savary. So the relation $\overline{PE_1^2}=\overline{E_1E_0}\,.\,\overline{E_1E_w}$ holds and E_1 can be found on the path normal PE_0. Since c_1 is also of the third degree, no more than three intersections exist of $P_{12}P^1_{43}$ with c_1. Thus $\Pi^1_{24}=E_1$. Having established all vertices of the quadrilateral $\square\Pi^1_{31}P_{12}\Pi^1_{24}P^1_{43}$ the construction of c_1 can be done in a similar way as the construction of c_0 based on the quadrilateral $\Pi_{31}P_{12}\Pi_{24}P_{43}$.

2.1 *The special case with P_{34} on the pole tangent p.*

In this case Π_{31} coincides with P_{43} while Π_{24} coincides with P_{12}, since $C_1=P_{34}=P_{41}$ on p is then co-ordinated to point $C_0=P=P_{12}$. (See Fig. 4). Notwithstanding, it remains possible to establish the focal axis and the focal point Γ_0 in this case. The asymptotic

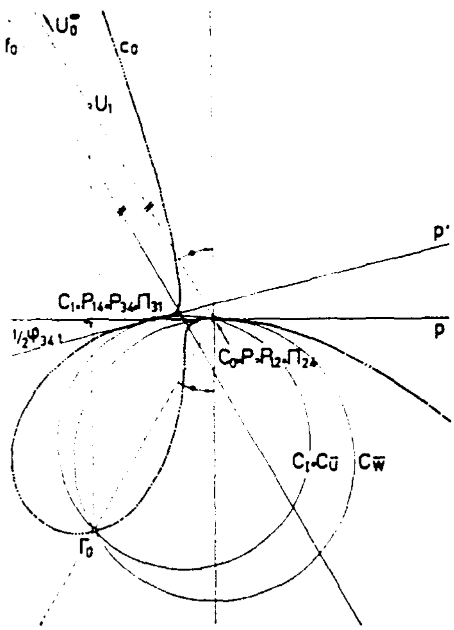

Figure 4.

direction of c_0, for instance, coincides with the path-normal PU_1, since $U_0 = U_0^\infty$. (See Fig. 5) And the point U_1 can be easily found as the intersection point of the circle C_U and the inflection circle C_W. Thus the focal axis of c_0, which passes through the mid-point of $\overline{PP}_{34}$,

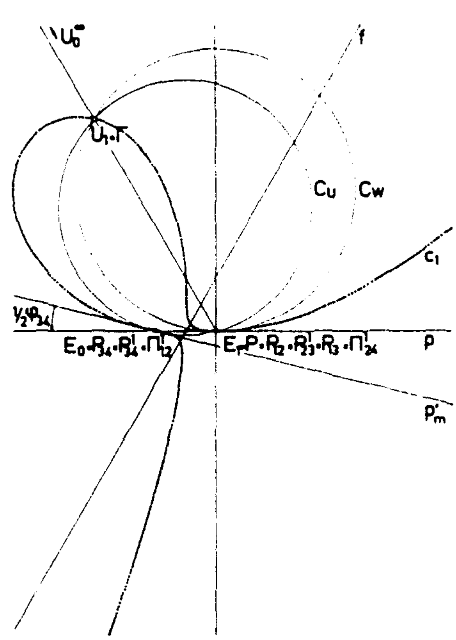

Figure 5.

is parallel to the path-normal PU_1. Theorem 2 of the appendix states that the focus Γ_0 is *isogonally* co-ordinated to the asymptotic point U_0^∞ of c_0 with respect to, for instance, the triangle $\Pi_{13}P_{23}P_{34}$. Moreover, it is known that the line at infinity is isogonally co-ordinated to a circle circumscribed about the fundamental triangle, used as the basic triangle in the isogonal transformation. Thus Γ_0 is lying on a circle circumscribed about $\Delta\Pi_{13}P_{23}P_{34}$.

Since c_0 is touching p' at $P_{34} = \Pi_{31}$, the line joining points P_{34} and Π_{31} of c_0 is represented by p'. Therefore the circle circumscribed about $\Delta\Pi_{13}P_{23}P_{34}$ coincides with the circle C_U, which touches p' at P_{34} and goes through $P_{23} = P$. Consequently, Γ_0 is lying on C_U in this case.

The isogonal transformation co-ordinates the points lying on a straight line through one of the vertices of the basic triangle into another line which is the image of the initial line relative to the bisector of the mentioned vertex of the triangle. In the case under consideration the bisector of the vertex P_{23} of $\Delta P_{23}P_{34}\Pi_{13}$ coincides with the pole tangent p. Since also Γ_0 is isogonally co-ordinated to U_0^∞, the focus Γ_0 lies on the image of $P_{23}U_0^\infty$ relative to the pole tangent. Thus Γ_0 coincides with the image of U_1 relative to the pole tangent p. The knowledge of the exact location of the focus Γ_0 and of the focal axis f_0 obtained in this way makes it possible to produce the points of the center-point curve c_0.

In the same way it is to be seen that the focus Γ of the circle-point curve c_1 coincides with U_1, which is the image of Γ_0 relative to p. It appears also that the focal axis f is the image of f_0 relative to p and furthermore, that p'_m and p', making an angle of φ_{34}, also are each other's image. Therefore both curves c_1 and c_0, co-ordinated through the relation of Euler–Savary, are the image of each other relative to the pole tangent p.

If the fourth position comes infinitesimally near to the third one and in such a way that this happens to remain in accordance with the continuous movement of the rigid body, this is only possible if P_{34} approaches $P_{12} = P_{23}$ on the pole tangent p.

Up to the very last moment of coincidence, however, the inflection circle serves as the osculation circle of curve c_1 at $P_{12} = P_{23}$ and $\Gamma = U_1$. Since it is known that the instantaneous circling-point curve, which can be defined as the limit curve of c_1, generally, has another osculation circle at the instantaneous pole P and also has a Ball's point U which does not coincide with the instantaneous focus Γ, it is clear that at the moment of coincidence an abrupt change appears in the curvature-radius of the curve at P. The same can be said about the location of "Ball's point" $U_1 \to U$ and of the focal point of the curve.

The abrupt change in the curvature-radius is due to the fact that at the moment of coincidence the curve c_1 obtains a double point at P and the curve at this point suddenly takes on another form of continuity.

2.2 *The degenerated case where P_{34} lies on the inflection circle and $\Pi_{13} \to \Pi_{13}^\infty$.*

In this special case the co-ordination of the point of path $C_1 = C_2 = C_3 = P_{34} = C_4$ with the center point C_0 is such that $C_0 \to C_0^\infty$. Since $\Pi_{24} = C_0$, we have the case in which both points Π_{13}^∞ and Π_{24}^∞ lie at different points on the line at infinity. Since the circular cubic c_0, generally, intersects this line only at the asymptotic point of c_0 and at the two isotropic points I and J, another point of intersection means the line at infinity becomes part of the cubic. Therefore the cubic degenerates into the infinity line and into a remaining part which must be a conic (see Fig. 6). Since the conic has at least one asymptote, the conic is a hyperbola. The opposite pole quadrilateral $P_{12}P_{23}P_{34}P_{41}$ may be regarded as a degenerate parallelogram since $\Pi_{13} \to \Pi_{13}^\infty$ and $\Pi_{24} \to \Pi_{24}^\infty$. In addition, it is known that if the opposite pole quadrilateral becomes a parallelogram, the center-point curve will be an *orthogonal* hyperbola in the finite part of the plane. In this special case, the mid-point of $\overline{P_{12}P_{34}}$ becomes the intersection point of the two existing asymptotes of the hyperbola. These are the bisectors of the two lines which pass through this intersection point and lie parallel to the pole tangent and $P_{12}P_{34}$. Since also the tangents at P_{12} and P_{34} are known, it is easy to find a construction for this hyperbola—for instance, by drawing a pencil of rays through P_{12} and another one through P_{34} and finding the second intersection of each ray with the hyperbola. Each ray then intersects the asymptotes in points lying at equal distances from the respective intersection points of the ray with the hyperbola.

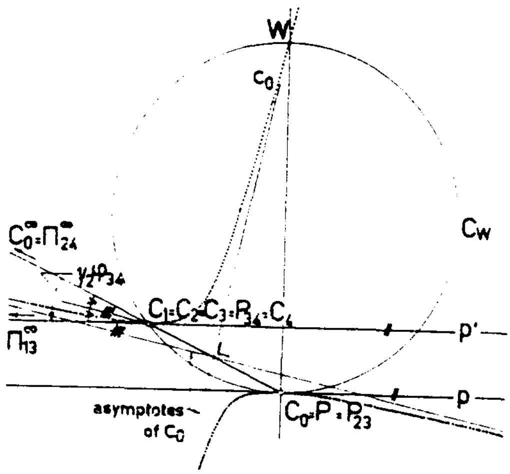

Figure 6.

Next, it will be shown that the corresponding curve c_1 is degenerated into a circle and into a straight line. Since $\Pi_{13} \to \Pi_{13}^{\infty}$ the tangent p' to c_0 is parallel to the pole tangent p. It then follows that the tangent p'_m to c_1 is also a tangent to the inflection circle C_W. (See Fig. 7). Therefore $\Delta P_{12}P_{34}^1\Pi_{13}^1$ is an isosceles triangle. Moreover, since $E_0 = P_{34} = P_{34}^1 = P_{41} = E_w$ and $\overline{PE_1^2} = \overline{E_1E_0} \cdot \overline{E_1E_w}$, we derive that $E_1 = \Pi_{24}^1$ coincides with the mid-point of $\overline{P_{12}P_{14}}$. Thus $\Pi_{13}^1\Pi_{24}^1 = f$, the focal-axis of c_1.

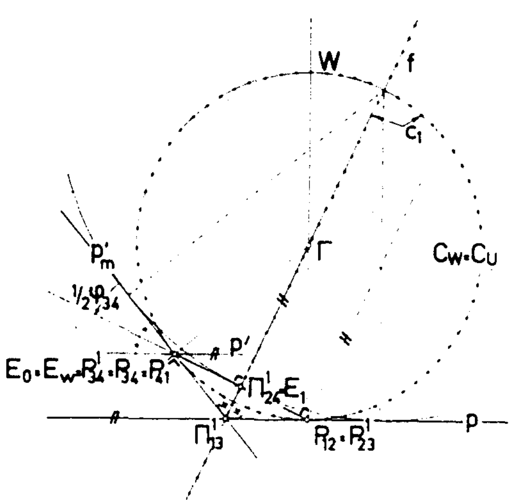

Figure 7.

The focal-point Γ is isogonally co-ordinated to the infinite point of f with respect to $\Delta\Pi_{13}^1P_{12}P_{41}$. Therefore Γ coincides with the center of the inflection circle in this case. Since also f passes through this point, the focal-point lies on the focal axis. The construction of c_1, based on its focal properties, then yields the inflection circle C_W and the focal axis f as two parts of the circular cubic c_1. It may now be clear that the inflection circle C_W is co-ordinated to the infinite part of c_0, while the focal axis f is co-ordinated to both parts of the orthogonal hyperbola.

The curve c_1 has two double-points, coinciding with the centers of the two *inscribed* circles of $\square\Pi_{13}^1P_{32}^1\Pi_{24}^1P_{41}$. (Each circle touches $P_{32}^1P_{41}$ at Π_{24}^1, otherwise neither of the two bisectors at the vertex Π_{24}^1 of the quadrilateral should pass through the center of such an inscribed circle).

Since the inflection circle coincides with a branch of c_1, each point of C_W may be regarded as a Ball's point U_1. So there is an infinite number of such points in this case.

2.3 *The case where P_{34} lies on the return circle and $\Pi_{13}^1 \to \Pi_{13}^{1\infty}$.*

In this case $E_0 = P_{34}$ is co-ordinated to $E_1^\infty = \Pi_{24}^1$. Thus $\Pi_{13}^1 \to \Pi_{13}^{1\infty}$ and $\Pi_{24}^1 \to \Pi_{24}^{1\infty}$. The cubic c_1 degenerates into the line at infinity and into an orthogonal hyperbola. The hyperbola osculates the inflection circle at P_{12} and intersects the same circle at Ball's point U_1, which is to be found in a direction parallel to the focal axis f_0 of c_0. The center-point curve c_0 degenerates into the return circle and into the focal axis $f_0 = \Pi_{13}\Pi_{24}$ passing through the center Γ_0 of the return circle.

2.4 *The special case where $\square \Pi_{13}^1 P_{32}^1 \Pi_{24}^1 P_{41}$ has an inscribed circle.*

It must be stated beforehand that an opposite-vertex quadrilateral has an inscribed circle if and only if the bisectors at the four vertices all pass through *one* point, which is the center of the inscribed circle. Since the sides $P_{32}^1\Pi_{24}^1$ and $\Pi_{24}^1 P_{41}$ lie along the same straight-line $P_{23}^1 P_{41}$, the inscribed circle has to touch this line at Π_{24}^1. (See Fig. 8a).

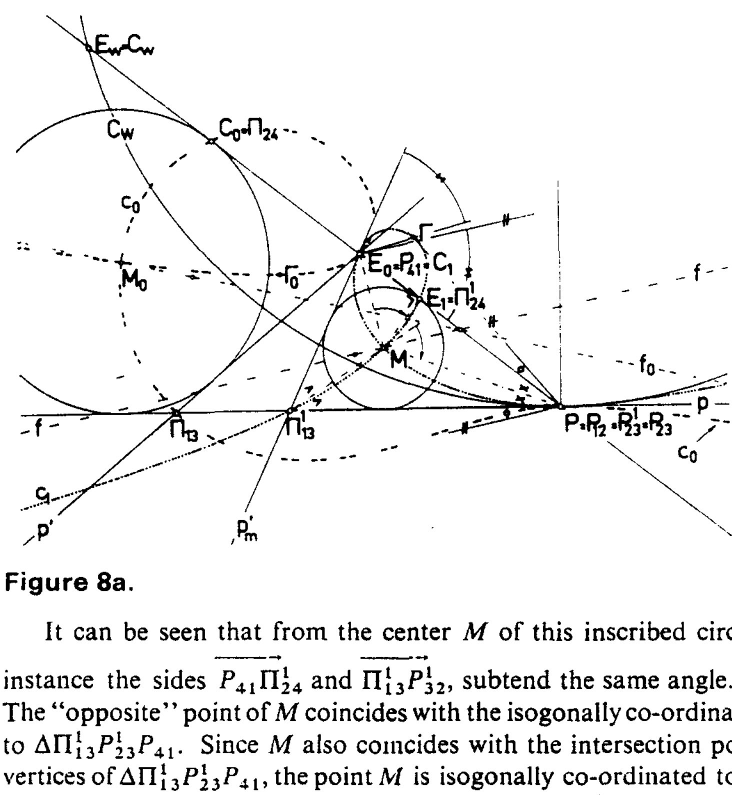

Figure 8a.

It can be seen that from the center M of this inscribed circle two opposite sides, for instance the sides $\overrightarrow{P_{41}\Pi_{24}^1}$ and $\overrightarrow{\Pi_{13}^1 P_{32}^1}$, subtend the same angle. Thus M is a point of c_1. The "opposite" point of M coincides with the isogonally co-ordinated point of M with respect to $\Delta\Pi_{13}^1 P_{23}^1 P_{41}$. Since M also coincides with the intersection point of the bisectors of the vertices of $\Delta\Pi_{13}^1 P_{23}^1 P_{41}$, the point M is isogonally co-ordinated to itself. Thus the opposite points M of the opposite-vertex quadrilateral $P_{41}MP_{32}^1M$ are coincident. This quadrilateral can be used as a base for the construction of c_1, as in fact any opposite-vertex quadrilateral can be used. Therefore c_1 is the locus of those points at which the opposite sides $\overrightarrow{P_{41}M}$ and $\overrightarrow{MP_{32}^1}$ subtend the same angle. (It then follows that the focal axis f goes through M and also the midpoint of $\overline{P_{32}^1 P_{41}}$.) Each point of c_1 can be found by intersecting a member of a pencil of circles (base points P_{41} and M) with a co-ordinated member of another pencil of circles (base points M and P_{32}^1). Any pair of co-ordinated members intersect at M and at another point of c_1. Each time only one point of c_1 is produced. Thus c_1 is *unicursal*. If the two intersection points should coincide at M, the two co-ordinated members touch at M and the common tangent is also a tangent to c_1. The co-ordination of any two co-ordinated

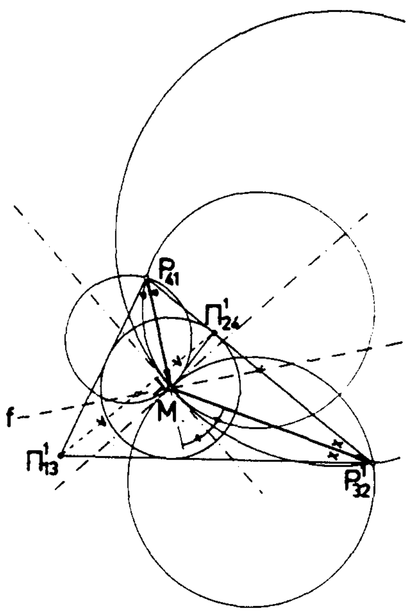

Figure 8b.

members is such that $\overrightarrow{P_{41}M}$ and $\overrightarrow{MP^1_{32}}$ subtend the same angle at the co-ordinated centers of these members. Therefore the common tangent coincides with one of the two bisectors at the vertex M of $\Delta MP^1_{32}P_{41}$. The two bisectors are perpendicular to each other. Thus the touching of two co-ordinated members occurs twice. Thus the curve c_1 has a *double-point* at M with perpendicular tangents to the curve at M.

The curve c_0 can be obtained from c_1 through the quadratic relationship of Euler–Savary. Generally, such a transformation transforms a double point into another one. Therefore, it becomes clear that $\square\Pi_{13}P_{32}\Pi_{24}P_{41}$ also has an inscribed circle, touching $P_{32}P_{41}$ at Π_{24}. Thus c_0 has a double-point M_0 with perpendicular tangents to the curve at M_0.

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Appendix

Theorem 1 *The pole curve based on an opposite vertex quadrilateral with alternating P and Π vertices, is identical to the pole curve based on an opposite pole quadrilateral.*

Proof. The pole curve p is defined as the locus of points at which two opposite sides of, for instance, $\square P_{12}P_{23}P_{34}P_{41}$ subtend equal angles. On the other hand the pole curve πp will be defined as the locus of points at which two opposite sides of $\square \Pi_{31}P_{12}\Pi_{24}P_{43}$ subtend equal angles. (See Fig. 9).

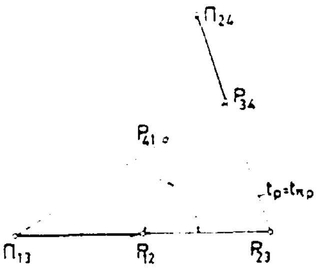

Figure 9.

The vertices Π_{31}, P_{12}, Π_{24} and P_{43} of $\square \Pi_{31}P_{12}\Pi_{24}P_{43}$ lie on p as well as on πp. At the poles P_{14} and P_{23} the two opposite sides of $\square \Pi_{31}P_{12}\Pi_{24}P_{43}$ do subtend equal angles. Therefore P_{14} and P_{23} are points of πp. They are also points of p, so they belong to both curves.

As is known, both curves πp and p are *circular* cubics. So the two isotropic points I and J are also common points of πp and p. Moreover, as P_{12} is a vertex of both quadrilaterals, therefore both $\overline{\Pi_{24}P_{34}}$ and $\overline{P_{12}\Pi_{31}}$ and also $\overline{P_{41}P_{34}}$ and $\overline{P_{12}P_{23}}$ subtend equal angles at P_{12}. This yields a construction for the tangent $t_{\pi p}$ at P_{12} to πp and also a construction for the tangent t_p at P_{12} to p. As can be seen from Fig. 9 $t_{\pi p}=t_p$. In the same way it can be proved that the tangents at P_{34} to πp and to p coincide. Therefore the points P_{12} and P_{34} have to be counted twice if we try to establish the number of common points of πp and p. It is now proved already that both curves at least have 10 points in common. Since both curves are cubics and they, generally, have no more than $3.3=9$ points in common, both curves must be identical. Thus $\pi p=p$.

It is already known that any of the three opposite-pole quadrilaterals $P_{12}P_{23}P_{34}P_{41}$, $P_{23}P_{31}P_{14}P_{42}$ and $P_{31}P_{12}P_{24}P_{43}$ may be used for obtaining the pole curve p.

Thus the same is true for the opposite vertex quadrilaterals $\Pi_{31}P_{12}\Pi_{24}P_{43}$, $\Pi_{23}P_{31}\Pi_{14}P_{42}$ and $\Pi_{31}P_{12}\Pi_{24}P_{43}$. They all produce the same pole curve p.

From this proof it may be clear, that also the quadrilaterals $P_{12}P^1_{23}P^1_{34}P_{41}$, $P^1_{23}P_{31}P_{14}P^1_{42}$, $P_{31}P_{12}P^1_{24}P^1_{43}$, $\Pi^1_{31}P_{12}\Pi^1_{24}P^1_{43}$, $\Pi^1_{23}P_{31}\Pi^1_{14}P^1_{42}$ and $\Pi^1_{31}P_{12}\Pi^1_{24}P^1_{43}$ produce the same pole curve p^1, which is identical to the circle-point curve.

Theorem 2 *The focus of the pole-curve p is isogonally co-ordinated to the asymptotic point of p with respect to any of the following triangles*

$$\Pi_{13}P_{23}P_{34},\quad \Pi_{13}P_{21}P_{14},\quad \Pi_{24}P_{12}P_{23},\quad \Pi_{24}P_{14}P_{43},$$

$$\Pi_{21}P_{31}P_{14},\quad \Pi_{21}P_{32}P_{24},\quad \Pi_{34}P_{23}P_{31},\quad \Pi_{34}P_{24}P_{41}$$

$$\Pi_{32}P_{12}P_{24},\quad \Pi_{32}P_{13}P_{34},\quad \Pi_{14}P_{31}P_{12} \text{ and } \Pi_{14}P_{34}P_{42}.$$

Proof. (See Fig. 10). Four separated positions determine six poles. However, if, regarding these positions, nothing more is given than the location of the poles P_{12}, P_{23}, P_{34} and P_{41}, the pole P_{13} may be chosen *anywhere* on the pole curve p, determined by $\square P_{12}P_{23}P_{34}P_{41}$. By doing this, the location of the opposite

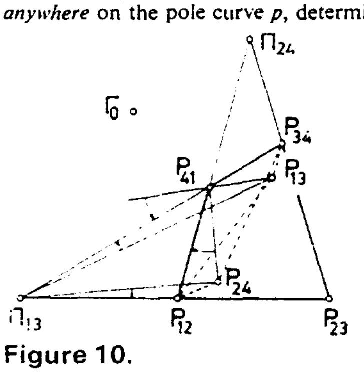

Figure 10.

pole P_{24} becomes fixed on p at the same time. Since Π_{13} lies on p, $\sphericalangle P_{12}\Pi_{13}P_{24} = \sphericalangle P_{13}\Pi_{13}P_{34} = \sphericalangle P_{13}\Pi_{13}P_{14}$. Thus the opposite poles P_{13} and P_{24} lie on isogonally co-ordinated straight lines going through the vertex Π_{13} of the basic triangle $\Pi_{13}P_{12}P_{14}$. Also the opposite sides $\overline{P_{13}P_{34}}$ and $\overline{P_{12}P_{24}}$ of $\square P_{12}P_{24}P_{43}P_{31}$ subtend equal angles at P_{14}.

Therefore $\sphericalangle P_{31}P_{14}P_{43} = \sphericalangle P_{21}P_{14}P_{42}$.

Thus the opposite poles P_{13} and P_{24} also lie on isogonally co-ordinated straight lines going through the vertex P_{14} of the basic triangle $\Pi_{13}P_{12}P_{14}$. It follows then that P_{13} and P_{24} are isogonally co-ordinated points with respect to $\Delta\Pi_{13}P_{12}P_{14}$. Since P_{13} has been taken arbitrarily on the pole curve, it follows immediately that the isogonally co-ordinated point of *any* point of the pole curve p lies on p.

On the other hand, the focus Γ_0 of p may be defined as the common intersection point of the circles circumscribed about the triangles $\Pi_{13}P_{12}P_{14}$, $\Pi_{13}P_{23}P_{34}$, $\Pi_{24}P_{32}P_{12}$ and $\Pi_{24}P_{34}P_{14}$. Moreover, the circle circumscribed about the basic triangle $\Pi_{13}P_{12}P_{14}$ is isogonally co-ordinated to the line at infinity. Since Γ_0 is a point of p, it is isogonally co-ordinated to another point of p, which also is a point of the infinity line. Therefore the focus Γ_0 is isogonally co-ordinated to the asymptotic point of p with respect to $\Delta\Pi_{13}P_{12}P_{14}$.

Finally, it should be clear that the proof holds also for any other triangle mentioned in the Theorem. Thus the theorem is proved.