

Strong dynamic input-output decoupling: from linearity to nonlinearity

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STRONG DYNAMIC INPUT-OUTPUT DECOUPLING: FROM LINEARITY TO NONLINEARITY

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Abstract

We study the strong dynamic input-output decoupling problem (SDIODP) for nonlinear systems. It is shown that, given a generically satisfied assumption, the solvability of the SDIODP around an equilibrium point is equivalent to the solvability of the same problem for the linearization of the system around this equilibrium point. We introduce the Singh compensator, a dynamic state feedback of minimal order that solves the SDIODP. It is shown that, given the assumption mentioned above, the linearization of the Singh compensator around an equilibrium point is a Singh compensator for the linearization of the original nonlinear system around this equilibrium point.

1 Introduction

Since for output regulation tasks input-output decoupled systems are relatively easy to handle, the so called input-output decoupling problem has received a lot of attention in the literature. Today, this problem is quite well understood for nonlinear control systems, see e.g. [16], [1],[13] where a complete solution is described in terms of a dynamic compensator. An important feature of the decoupling compensator we consider in this paper, the so called Singh compensator, is that it is a decoupling compensator of minimal dimension, see [9], and therefore it is intuitively of minimal "complexity".

Control engineers are often led to handle a specific control problem in a concrete nonlinear system by linearizing the given model around an equilibrium point, and afterwards solve, if possible, the given control problem for the linearization. We show in the present paper that at least in a local sense such an approach may be successfully used, provided the system fulfills a generically satisfied regularity assumption. In particular it follows that the decoupling Singh compensator for the nonlinear system becomes, when linearized around the equilibrium point, a decoupling compensator for the linearization. Furthermore, we have, again under the same regularity assumption, a converse of the above result: for a decoupling Singh compensator for the linearization of a nonlinear system there exists a decoupling Singh compensator for

the nonlinear system having the formentioned compensator as its linearization. The practical consequence of this is that, at least in a sufficiently small neighborhood, the linear solution of the input-output decoupling problem for the linearization of a nonlinear system will act as a first order approximate solution of the decoupling problem for the original nonlinear system. In this sense, we consider this paper as a justification of engineering practice. Of course it remains to be studied for each specific system what an acceptable size of the region of applicability of this result will be.

With the hereforementioned relation between nonlinear and linearized input-output decouplability in mind, we conclude this paper with some remarks relating algebraic and geometric structure at infinity of the nonlinear system. These comments should be considered as a complement to the work in [11].

2 Preliminaries

Consider a square nonlinear system P , given by equations of the form

$$P \begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases} \quad (1)$$

with $x = \text{col}(x_1, \dots, x_n) \in \mathbb{R}^n$ local coordinates for the state space manifold \mathcal{X} , $u \in \mathbb{R}^m$ denoting the controls, and $y \in \mathbb{R}^m$ denoting the outputs. Furthermore, we assume all data to be analytic. Recall that a meromorphic function η is a function of the form $\eta = \pi/\theta$, where π and θ are analytic functions. Assume that the control functions $u(t)$ are n times continuously differentiable. Then define $u^{(0)} := u$, $u^{(i+1)} := (d/dt)u^{(i)}$. View $x, u, \dots, u^{(n-1)}$ as variables and let \mathcal{K} denote the field consisting of the set of rational functions of $(u, \dots, u^{(n-1)})$ with coefficients that are meromorphic in x . For the system (1) we define in a natural way (with $y^{(0)} := y$)

$$y^{(k+1)} = y^{(k+1)}(x, u, \dots, u^{(k)}) =$$

$$\frac{\partial y^{(k)}}{\partial x} [f(x) + g(x)u] + \sum_{i=0}^{k-1} \frac{\partial y^{(k)}}{\partial u^{(i)}} u^{(i+1)}$$

Note that $y, \dot{y}, \dots, y^{(n)}$ so defined have components in the field \mathcal{K} .

For $k = 1, \dots, n$, introduce the Jacobian matrices (cf. [12]):

$$J_k(x, u, \dots, u^{(k-1)}) = \begin{pmatrix} \frac{\partial \dot{y}}{\partial u} & 0 & 0 & \dots & 0 \\ \frac{\partial y^{(2)}}{\partial u} & \frac{\partial y^{(2)}}{\partial u} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y^{(k)}}{\partial u} & \frac{\partial y^{(k)}}{\partial u} & \frac{\partial y^{(k)}}{\partial u^{(2)}} & \dots & \frac{\partial y^{(k)}}{\partial u^{(k-1)}} \end{pmatrix}$$

Then the rank $\rho^*(P)$ of P is defined by (see [3])

$$\rho^*(P) = \text{rank}_{\mathcal{K}} J_n - \text{rank}_{\mathcal{K}} J_{n-1} \quad (2)$$

Note that we always have $\rho^*(P) \leq m$. P is said to be of *full rank* if $\rho^*(P) = m$.

We next introduce Singh's algorithm for the system P . This algorithm has been introduced in [16] for calculation of a left inverse of a nonlinear system. The version of Singh's algorithm presented here is taken from [3].

Algorithm 2.1 Singh's algorithm

Step 0 Let $\tilde{y}_0^{(0)}$ be void and define $\hat{y}_0^{(0)} := y$.

Step k+1 For $r, s \in \mathbb{N}$, let $\mathcal{I}_{rs} := \{r, r+1, \dots, s\}$. Suppose that in Steps 1 through k , $\tilde{y}_1, \dots, \tilde{y}_k^{(k)}, \hat{y}_k^{(k)}$ have been defined so that

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u \\ &\vdots \\ \tilde{y}_k^{(k)} &= \tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k-1}, j \in \mathcal{I}_{ik}\}) \\ &\quad + \tilde{b}_k(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k-1}, j \in \mathcal{I}_{ik-1}\})u \\ \hat{y}_k^{(k)} &= \hat{y}_k^{(k)}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \in \mathcal{I}_{1k}, j \in \mathcal{I}_{ik}\}) \end{aligned}$$

Suppose also that the matrix $\tilde{B}_k := [\tilde{b}_1^T, \dots, \tilde{b}_k^T]^T$ has full rank equal to ρ_k , where the rank is taken with respect to the field of rational functions of $\{\tilde{y}_i^{(j)} \mid 1 \leq i \leq k-1, i \leq j \leq k-1\}$ with coefficients in the field of meromorphic functions of x . Then calculate

$$\hat{y}_k^{(k+1)} = \frac{\partial \hat{y}_k^{(k)}}{\partial x} [f(x) + g(x)u] + \sum_{i=1}^k \sum_{j=i}^k \frac{\partial \hat{y}_k^{(k)}}{\partial \tilde{y}_i^{(j)}} \tilde{y}_i^{(j+1)}$$

and write this as

$$\begin{aligned} \hat{y}_k^{(k+1)} &= a_{k+1}(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k}, j \in \mathcal{I}_{ik+1}\}) \\ &\quad + b_{k+1}(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k}, j \in \mathcal{I}_{ik}\})u \end{aligned}$$

Define $B_{k+1} := [\tilde{B}_k^T, b_{k+1}^T]^T$, and $\rho_{k+1} := \text{rank } B_{k+1}$ where the rank is taken with respect to the field of rational functions of $\{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k}, j \in \mathcal{I}_{ik}\}$ with coefficients in the field of meromorphic functions of x . Permute, if necessary, the components of $\hat{y}_k^{(k+1)}$ so that

the first ρ_{k+1} rows of B_{k+1} are linearly independent. Decompose $\hat{y}_k^{(k+1)}$ as $\hat{y}_k^{(k+1)} = (\tilde{y}_{k+1}^{(k+1)T} \hat{y}_{k+1}^{(k+1)T})^T$, where $\tilde{y}_{k+1}^{(k+1)}$ consists of the first $(\rho_{k+1} - \rho_k)$ rows. Since the last rows of B_{k+1} are linearly dependent on the first ρ_{k+1} rows, we can write

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u \\ &\vdots \\ \hat{y}_{k+1}^{(k+1)} &= \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k}, j \in \mathcal{I}_{ik+1}\}) \\ &\quad + \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k}, j \in \mathcal{I}_{ik}\})u \\ \hat{y}_{k+1}^{(k+1)} &= \hat{y}_{k+1}^{(k+1)}(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k+1}, j \in \mathcal{I}_{ik+1}\}) \end{aligned}$$

Finally, set $\tilde{B}_{k+1} := [\tilde{B}_k^T, \tilde{b}_{k+1}^T]^T$. ■

We associate a notion of regularity with Singh's algorithm in the following way (see [2] for a related, but somewhat different notion of regularity).

Definition 2.2 Consider the nonlinear system P and let a point $x_0 \in \mathcal{X}$ be given. We call $(x, y) = (x_0, 0)$ a strongly regular point for P if for each application of Singh's algorithm to P we have

$$\text{rank}_{\mathbb{R}} \tilde{B}_k(x_0, 0) = \rho_k \quad (k = 1, \dots, n) \quad (3)$$

If $(x, y) = (x_0, 0)$ is a strongly regular point for P , the rank of P can be calculated by evaluating J_n and J_{n-1} at a single point (cf. [6]):

$$\rho^*(P) = \text{rank}_{\mathbb{R}} J_n(x_0, 0) - \text{rank}_{\mathbb{R}} J_{n-1}(x_0, 0) \quad (4)$$

3 The nonlinear SDIOP and linearization

In this section a strong version of the dynamic input-output decoupling problem is studied. For a nonlinear system P , define the relative degrees r_i ($i = 1, \dots, m$) as the smallest $k \in \mathbb{N}$ for which

$$\frac{\partial y_i^{(k)}}{\partial u} \neq 0 \quad (5)$$

If all relative degrees are finite, define the decoupling matrix $A(x)$ with entries $a_{ij}(x) = (\partial y_i^{(r_i)} / \partial u_j)(x)$ ($i, j \in \mathcal{I}_{1,m}$). A system P is said to be *input-output decoupled* if each of its inputs influences one and only one of its outputs. The system is said to be *strongly input-output decoupled* (see [15]) if all relative degrees are finite, its decoupling matrix is an invertible diagonal matrix, and

$$\frac{\partial y_i^{(k)}}{\partial u_j} = 0 \quad (i = 1, \dots, m; j \neq i; k \geq r_i + 1) \quad (6)$$

Definition 3.3 Consider a nonlinear system P and let $x_0 \in \mathcal{X}$ be given. Then the strong dynamic input-output decoupling problem (SDIODP) is said to be solvable around x_0 if there exist an integer ν , a dynamic state feedback Q on \mathbb{R}^ν of the form

$$Q \begin{cases} \dot{z} = \alpha(x, z) + \beta(x, z)v \\ u = \gamma(x, z) + \delta(x, z)v \end{cases} \quad (7)$$

with $z \in \mathbb{R}^\nu$, $v \in \mathbb{R}^m$ denoting the new controls and $\alpha, \beta, \gamma, \delta$ analytic, a neighborhood $U \subset \mathcal{X}$ of x_0 and an open subset \mathcal{Z} of \mathbb{R}^ν such that the system $P \circ Q$ restricted to $U \times \mathcal{Z}$ is strongly input-output decoupled. ■

In [3] it was shown that if $(x_0, 0)$ is a strongly regular point for P , the SDIODP solvable around x_0 if and only if $\rho^*(P) = m$ (see also [1], [13]).

We now present a special sort of dynamic state feedback that solves SDIODP around strongly regular points for P . This dynamic state feedback is obtained via Singh's algorithm and we call it a Singh compensator. The Singh compensator is obtained as follows (see [7] and also [16]). Consider the nonlinear system P and let $(x_0, 0)$ be a strongly regular point for P . Furthermore assume that $\rho^*(P) = m$. Apply Singh's algorithm to P . This yields at the n -th step:

$$\begin{aligned} \dot{\tilde{Y}}_n &= \tilde{A}_n(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1n-1}, j \in \mathcal{I}_{in}\}) \\ &+ \tilde{B}_n(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1n-1}, j \in \mathcal{I}_{in-1}\})u \end{aligned} \quad (8)$$

where $\tilde{Y}_n = (\tilde{y}_1^T \dots \tilde{y}_n^{(n-1)T})^T$ and \tilde{A}_n is an m -vector with entries \tilde{a}_k . Moreover there exist a neighborhood $U \subset \mathcal{X}$ of x_0 , and a neighborhood Y_0 of the point $(\tilde{y}_{i_0}^{(j)} \mid i \in \mathcal{I}_{1n-1}, j \in \mathcal{I}_{in}) = 0$ such that \tilde{B}_n is invertible on $U \times Y_0$. Then on $U \times Y_0$ (8) yields in particular:

$$u = \tilde{B}_n^{-1}[\dot{\tilde{Y}}_n - \tilde{A}_n] \quad (9)$$

For $i = 1, \dots, m$, let γ_i be the lowest time-derivative and δ_i be the highest time-derivative of y_i appearing in (9). Then we can rewrite (9) as

$$\begin{aligned} u &= \phi_1(x, \{y_i^{(j)} \mid i \in \mathcal{I}_{1m}, j \in \mathcal{I}_{\gamma_i, \delta_i-1}\}) \\ &+ \sum_{i=1}^m \phi_{2i}(x, \{y_i^{(j)} \mid i \in \mathcal{I}_{1m}, j \in \mathcal{I}_{\gamma_i, \delta_i-1}\})y_i^{(\delta_i)} \end{aligned}$$

for certain vector-valued functions ϕ_1, ϕ_{2i} ($i = 1, \dots, m$). Let z_i ($i = 1, \dots, m$) be a vector of dimension $\delta_i - \gamma_i$ and consider the system:

$$\begin{cases} \dot{z}_i = A_i z_i + B_i v_i \quad (i = 1, \dots, m) \\ u = \phi_1(x, z_1, \dots, z_m) + \\ \quad \sum_{i=1}^m \phi_{2i}(x, z_1, \dots, z_m)v_i \end{cases} \quad (10)$$

with (A_i, B_i) in Brunovsky canonical form. Then (10) is called a *Singh compensator* for P around x_0 . Inspection of Singh's algorithm and the construction of a Singh compensator (10) gives that for the compensated system we have that $y_i, \dots, y_i^{(\delta_i-1)}$ are independent of the new controls and that $y_i^{(\delta_i)} = v_i$ ($i = 1, \dots, m$). Thus the decoupling matrix of the compensated system is given by $A(x) = I_m$, and (6) holds. Hence any Singh compensator (10) around a strongly regular point x_0 solves the SDIODP around x_0 . In [9] it was shown that in fact a Singh compensator is a dynamic state feedback of minimal order solving the SDIODP.

Let $x_0 \in \mathcal{X}$ be an equilibrium point for P , i.e., $f(x_0) = 0$. Assume (without loss of generality) that $h(x_0) = 0$. Let the (Jacobian) linearization LP of P around x_0 be given by

$$LP \begin{cases} \dot{\xi} = F\xi + Gu \\ \eta = H\xi \end{cases} \quad (11)$$

In the sequel we make the following assumption:

Assumption 3.4 $(x, y) = (x_0, 0)$ is a strongly regular point for P . ■

We investigate the connection between the solvability of the SDIODP for P and LP . A first result is:

Proposition 3.5 Consider a square nonlinear system P . Let $x_0 \in \mathcal{X}$ an equilibrium point for P that satisfies Assumption 3.4. Then $\rho^*(LP) = \rho^*(P)$.

Proof By (4) we have that

$$\rho^* = \text{rank}_{\mathbb{R}} J_n(x_0, 0) - \text{rank}_{\mathbb{R}} J_{n-1}(x_0, 0) \quad (12)$$

Analogously to Section 2, define Jacobian matrices J_k^ℓ ($k = 1, \dots, n$) for the linearized plant LP . It is easily checked that

$$J_k^\ell = \begin{pmatrix} HG & 0 & 0 & \dots & 0 \\ HFG & HG & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ HF^{k-1}G & HF^{k-2}G & \dots & \dots & HG \end{pmatrix} \quad (13)$$

We show that $J_k(x_0, 0) = J_k^\ell$ ($k = 1, \dots, n$), i.e., for $k = 1, \dots, n$, $\ell = 0, \dots, k-1$ we have

$$\frac{\partial y^{(k)}}{\partial u^{(\ell)}}(x_0, 0) = HF^{k-\ell-1}G \quad (14)$$

First it is shown that for $k = 0, 1, 2, \dots$

$$\frac{\partial y^{(k)}}{\partial x}(x_0, 0) = HF^k \quad (15)$$

For $k = 0, 1$ (15) is immediate. Assume that (15) holds for $k = 1, \dots, \ell - 1$. Then

$$\begin{aligned} \frac{\partial y^{(\ell)}}{\partial x}(x_0, 0) &= \frac{\partial}{\partial x} \left(\frac{\partial y^{(\ell-1)}}{\partial x} [f(x) + g(x)u] + \right. \\ &\quad \left. \sum_{r=0}^{\ell-2} \frac{\partial y^{(\ell-1)}}{\partial u^{(r)}} u^{(r+1)} \right)_{(x_0, 0)} = \\ &\quad \frac{\partial y^{(\ell-1)}}{\partial x}(x_0, 0) \frac{\partial f}{\partial x}(x_0) = HF^{\ell-1}F = HF^\ell \end{aligned}$$

Hence (15) holds for $k = 1, 2, \dots$. Next we show that for $k = 1, \dots, n$:

$$\frac{\partial y^{(k)}}{\partial u}(x_0, 0) = HF^{k-1}G \quad (16)$$

For $k = 1$ (16) is trivially satisfied. Assume that (16) holds for $k = 0, \dots, \ell - 1$. Then, using (15), we have:

$$\begin{aligned} \frac{\partial y^{(\ell)}}{\partial u}(x_0, 0) &= \frac{\partial}{\partial u} \left(\frac{\partial y^{(\ell-1)}}{\partial x} [f(x) + g(x)u] + \right. \\ &\quad \left. \sum_{r=0}^{\ell-2} \frac{\partial y^{(\ell-1)}}{\partial u^{(r)}} u^{(r+1)} \right)_{(x_0, 0)} = \\ &\quad \frac{\partial y^{(\ell-1)}}{\partial x}(x_0, 0)g(x_0) = HF^{\ell-1}G \end{aligned} \quad (17)$$

Let $k \in \{1, 2, \dots, n\}$, $\ell \in \{1, \dots, k-1\}$. Then:

$$\begin{aligned} \frac{\partial y^{(k)}}{\partial u^{(\ell)}}(x_0, 0) &= \frac{\partial}{\partial u^{(\ell)}} \left(\frac{\partial y^{(k-1)}}{\partial x} [f(x) + g(x)u] + \right. \\ &\quad \left. \sum_{r=0}^{k-2} \frac{\partial y^{(k-1)}}{\partial u^{(r)}} u^{(r+1)} \right)_{(x_0, 0)} = \frac{\partial y^{(k-1)}}{\partial u^{(\ell-1)}}(x_0, 0) = \\ &\quad \frac{\partial y^{(k-\ell)}}{\partial u}(x_0, 0) = HF^{k-\ell-1}G \end{aligned} \quad (18)$$

(17) and (18) establish that (14) holds for $k = 1, 2, \dots, n$, $\ell = 0, \dots, k-1$ and hence for $k = 1, \dots, n$ we have $J_k(x_0, 0, \dots, 0) = J_k^\ell$. Together with (12) this establishes that $\rho^*(LP) = \rho^*(P)$. ■

As an immediate consequence of Proposition 3.5 we have:

Theorem 3.6 Consider a square nonlinear plant P . Let $x_0 \in \mathcal{X}$ be an equilibrium point that satisfies Assumption 3.4 for P . Denote the linearization of P around x_0 by LP . Then the SDIODF is solvable for P around x_0 if and only if it is solvable for LP . ■

Remark 3.7 The result of Theorem 3.6 can be found in [10]. It generalizes a result of [4], where a similar result was obtained for the strong input-output decoupling problem via static state feedback. In this case, the Singh compensator reduces to a regular static state feedback that renders Δ^* , the maximal locally controlled invariant distribution in $\text{Ker } dh$, invariant. The result is then established by showing that $\Delta^*(x_0) = \mathcal{V}^*$ (or rather: $\Delta^*(x_0)$ can be identified

with \mathcal{V}^*), where \mathcal{V}^* is the maximal controlled invariant subspace in $\text{Ker } H$ for LP (cf. [4]). It seems likely that also the result of Theorem 3.6 may be proved in a similar way by employing the same techniques as in Section 4 of [7]. Also the approximation result of [5] is connected with the result of Theorem 3.6. ■

We now investigate the connection between Singh compensators for P and LP . In what follows we use the following lemma, which can easily be verified:

Lemma 3.8 Consider the equation

$$\begin{aligned} \tilde{Y} &= \tilde{A}(Z) + \tilde{B}(Z)U \\ \hat{Y} &= \hat{A}(Z) + \hat{B}(Z)U \end{aligned} \quad (19)$$

where $\tilde{A}(Z_0) = 0$, $\hat{A}(Z_0) = 0$, $\tilde{B}(Z)$ has full row rank on a neighborhood of Z_0 , and each of the rows of $\hat{B}(Z)$ is linearly dependent on the rows of $\tilde{B}(Z)$. Moreover, consider the linearization of (19) around $(Z, U, \tilde{Y}, \hat{Y}) = (Z_0, 0, 0, 0)$:

$$\begin{aligned} \tilde{Y}^\ell &= \tilde{A}^\ell Z^\ell + \tilde{B}^\ell U^\ell \\ \hat{Y}^\ell &= \hat{A}^\ell Z^\ell + \hat{B}^\ell U^\ell \end{aligned} \quad (20)$$

Let $\tilde{B}^+(Z)$ be a right inverse of $\tilde{B}(Z)$ on a neighborhood of Z_0 where $\tilde{B}(Z)$ has full row rank. Then (19) and (20) yield:

$$\begin{aligned} U &= \tilde{B}^+(Z)[\tilde{Y} - \tilde{A}(Z)] \\ \hat{Y} &= \hat{A}(Z) + \hat{B}(Z)\tilde{B}^+(Z)[\tilde{Y} - \tilde{A}(Z)] \end{aligned} \quad (21)$$

$$\begin{aligned} U^\ell &= \tilde{B}^{\ell+} Z^\ell [\tilde{Y}^\ell - \tilde{A}^\ell Z^\ell] \\ \hat{Y}^\ell &= \hat{A}^\ell Z^\ell + \hat{B}^\ell \tilde{B}^{\ell+} [\tilde{Y}^\ell - \tilde{A}^\ell Z^\ell] \end{aligned} \quad (22)$$

where (22) can be obtained by linearizing (21) around $(Z, U, \tilde{Y}, \hat{Y}) = (Z_0, 0, 0, 0)$. ■

Lemma 3.9 Consider a square nonlinear plant P of full rank. Let x_0 be a strongly regular equilibrium point satisfying Assumption 3.4. Let Q be a Singh compensator for P around x_0 . Then $(x, z) = (x_0, 0)$ is an equilibrium point for $P \circ Q$.

Proof See [6]. ■

Proposition 3.10 Consider a square nonlinear system P of full rank. Let $x_0 \in \mathcal{X}$ be an equilibrium point satisfying Assumption 3.4. Apply Singh's algorithm to P , yielding a reordering $\tilde{y}_1, \dots, \tilde{y}_n$ of the outputs, such that

$$\begin{aligned} \tilde{y}_k^{(k)} &= \tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k-1}, j \in \mathcal{I}_{ik}\}) + \\ &\quad \tilde{b}_k(x, \{\tilde{y}_i^{(j)} \mid i \in \mathcal{I}_{1k-1}, j \in \mathcal{I}_{ik-1}\})u \\ \hat{y}_k^{(k)} &= \hat{y}_k^{(k)}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq k, i \leq j \leq k\}) \end{aligned} \quad (23)$$

Then there is an application of Singh's algorithm to LP that results in the same reordering of the outputs and where the result of Singh's algorithm applied to LP can alternatively be obtained by linearizing the result of Singh's algorithm applied to P around $(x, u, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq n, i \leq j \leq n\}) = (x_0, 0, 0, \dots, 0)$.

Proof We first consider $k = 1$. We have:

$$\dot{y} = \frac{\partial h}{\partial x}(x)[f(x) + g(x)u] =: a_1(x) + b_1(x)u \quad (24)$$

where $\text{rank}_{\mathcal{X}} b_1(x) = \rho_1$. By Assumption 3.4 there exists a neighborhood of x_0 such that $\text{rank}_{\mathbb{R}} b_1(x) = \rho_1$ on this neighborhood. After a possible permutation of the outputs (24) yields:

$$\begin{pmatrix} \dot{\hat{y}}_1 \\ \dot{\tilde{y}}_1 \end{pmatrix} = \begin{pmatrix} \tilde{a}_1(x) + \tilde{b}_1(x)u \\ \hat{a}_1(x) + \hat{b}_1(x)u \end{pmatrix} \quad (25)$$

where \tilde{b}_1 has full row rank ρ_1 on a neighborhood of x_0 . Let $\tilde{b}_1^+(x)$ be a right inverse of $\tilde{b}_1(x)$ on this neighborhood. Then (25) yields

$$\dot{\hat{y}}_1 = \tilde{a}_1(x) + \tilde{b}_1(x)u \quad (26)$$

$$\dot{\tilde{y}}_1 = \hat{a}_1(x) + \hat{b}_1(x)\tilde{b}_1^+(x)(\dot{\hat{y}}_1 - \tilde{a}_1(x))$$

Now consider LP . We have:

$$\dot{\eta} = HF\xi + HGu =: a_1^\ell \xi + b_1^\ell u \quad (27)$$

where

$$a_1^\ell = HF = \frac{\partial}{\partial x} \left(\frac{\partial h}{\partial x} f \right) (x_0) = \frac{\partial a_1}{\partial x}(x_0) \quad (28)$$

$$b_1^\ell = HG = \frac{\partial h}{\partial x}(x_0)g(x_0) = b_1(x_0) \quad (29)$$

Employing the same permutation of the outputs as for P , (27) yields

$$\begin{pmatrix} \dot{\hat{\eta}}_1 \\ \dot{\tilde{\eta}}_1 \end{pmatrix} = \begin{pmatrix} \tilde{a}_1^\ell \xi + \tilde{b}_1^\ell u \\ \hat{a}_1^\ell \xi + \hat{b}_1^\ell u \end{pmatrix} \quad (30)$$

where, by Assumption 3.4, \tilde{b}_1^ℓ has full row rank ρ_1 . Note that by (28) and (29) $\tilde{a}_1^\ell = (\partial \tilde{a}_1 / \partial x)(x_0)$, $\hat{a}_1^\ell = (\partial \hat{a}_1 / \partial x)(x_0)$, $\tilde{b}_1^\ell = \tilde{b}_1(x_0)$, $\hat{b}_1^\ell = \hat{b}_1(x_0)$. Let $\tilde{b}_1^{\ell+}$ be a right inverse of \tilde{b}_1^ℓ . Then (30) yields

$$\dot{\hat{\eta}}_1 = \tilde{a}_1^\ell \xi + \tilde{b}_1^\ell u \quad (31)$$

$$\dot{\tilde{\eta}}_1 = \hat{a}_1^\ell \xi + \hat{b}_1^\ell \tilde{b}_1^{\ell+}(\dot{\hat{\eta}}_1 - \tilde{a}_1^\ell \xi)$$

Then by Lemma 3.8, (31) can be obtained by linearizing (26) around $(x, u, \dot{\hat{y}}_1) = (x_0, 0, 0)$ and hence our claim holds for $k = 1$. Now consider the case $k = 2$. We have:

$$\dot{\hat{y}}_1^{(2)} = \frac{\partial \dot{\hat{y}}_1}{\partial x}[f(x) + g(x)u] + \frac{\partial \dot{\hat{y}}_1}{\partial \dot{\hat{y}}_1} \dot{\hat{y}}_1^{(2)} =: \quad (32)$$

$$a_2(x, \dot{\hat{y}}_1, \dot{\tilde{y}}_1^{(2)}) + b_2(x, \dot{\hat{y}}_1)u$$

and

$$\dot{\hat{\eta}}_1^{(2)} = \frac{\partial \dot{\hat{\eta}}_1}{\partial \xi}[F\xi + Gu] + \frac{\partial \dot{\hat{\eta}}_1}{\partial \dot{\hat{\eta}}_1} \dot{\hat{\eta}}_1^{(2)} =: \quad (33)$$

$$a_{20}^\ell \xi + a_{22}^\ell \tilde{\eta}_1^{(2)} + b_2^\ell u$$

We find from (26),(31),(32),(33):

$$a_{20}^\ell = \frac{\partial \dot{\hat{\eta}}_1}{\partial \xi} F = (\hat{a}_1^\ell - \tilde{a}_1^\ell)F =$$

$$\left(\frac{\partial a_1}{\partial x}(x_0) - \frac{\partial \tilde{a}_1}{\partial x}(x_0) \right) \frac{\partial f}{\partial x}(x_0) = \quad (34)$$

$$\frac{\partial}{\partial x} \left[\left(\frac{\partial \hat{a}_1}{\partial x} - \frac{\partial \tilde{a}_1}{\partial x} \right) f \right] (x_0) =$$

$$\frac{\partial}{\partial x} \left(\frac{\partial \dot{\hat{y}}_1}{\partial x} f \right) (x_0) = \frac{\partial a_2}{\partial x}(x_0)$$

$$a_{22}^\ell = \frac{\partial \dot{\hat{\eta}}_1}{\partial \dot{\hat{\eta}}_1} = \tilde{b}_1^\ell \tilde{b}_1^{\ell+} = \tilde{b}_1(x_0) \tilde{b}_1^+(x_0) = \frac{\partial \dot{\hat{y}}_1}{\partial \dot{\hat{y}}_1}(x_0) =$$

$$\frac{\partial \dot{\hat{y}}_1^{(2)}}{\partial \dot{\hat{y}}_1^{(2)}}(x_0) = \frac{\partial a_{22}}{\partial \dot{\hat{y}}_1^{(2)}}(x_0) \quad (35)$$

$$b_2^\ell = \frac{\partial \dot{\hat{\eta}}_1}{\partial \xi} G = (\hat{a}_1^\ell - \tilde{a}_1^\ell)g(x_0) = \quad (36)$$

$$\left(\frac{\partial \dot{\hat{y}}_1}{\partial x} g \right) (x_0) = b_2(x_0, 0)$$

From (34),(35),(36) it follows that $\dot{\hat{\eta}}_1^{(2)}$ can be obtained by linearizing $\dot{\hat{y}}_1^{(2)}$ around $(x, u, \dot{\hat{y}}_1, \dot{\hat{y}}_1^{(2)}) = (x_0, 0, 0, 0)$. Then, using Lemma 3.8, it can be shown that our claim hold for $k = 2$. Proceeding as above, it can be shown that the claim also holds for $k = 3, 4, \dots, n$. ■

Remark 3.11 For nonlinear systems, an *algebraic* structure at infinity ([11]) and a *geometric* structure at infinity ([14]) can be defined. For linear systems both structures at infinity coincide ([11]). A consequence of Proposition 3.10 is that the algebraic structure at infinity of P coincides with that of LP . This could lead to the conjecture that for a nonlinear system P satisfying Assumption 3.4 the algebraic and geometric structures at infinity coincide. However, this is not the case, as follows from the example $\dot{x}_1 = u_1$, $\dot{x}_2 = x_3 + x_4 u_1$, $\dot{x}_3 = u_2$, $\dot{x}_4 = u_3$, $\dot{x}_5 = x_6$, $\dot{x}_7 = u_3$, $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_5$. A straightforward calculation shows that this system has 3 geometric zeros at infinity of orders 2,2,1 respectively and it has 3 algebraic zeros at infinity of orders 3,2,1 respectively. ■

From Proposition 3.10 we may draw the following conclusion:

Theorem 3.12 Consider a square nonlinear system P of full rank. Let $x_0 \in \mathcal{X}$ be an equilibrium point satisfying Assumption 3.4 and let LP denote the linearization of P around x_0 . Then:

- (i) The linearization of a Singh compensator for P is a Singh compensator for LP .
- (ii) Conversely, every Singh compensator for LP is a first order approximation of a Singh compensator for P around x_0 . ■

Theorem 3.12 has the following practical implication. In engineering practice one often studies a specific control problem by addressing the problem on the linearization around a given working point. Of course it then remains questionable if the control that was designed to solve the control problem for the linearization of the system is a approximate (first order) approximation of a control that solves the control problem for the original nonlinear system. In [4] an example for the strong input-output decoupling problem via static state feedback was given where indeed this was not the case. However, from Theorem 3.12 it follows that any Singh compensator for LP (which is a dynamic state feedback that solves the SDIODP for LP) is a first order approximation of a dynamic state feedback that solves the SDIODP for P .

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