

Aymptotics in normal order statistics

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ASYMPTOTICS IN NORMAL ORDER STATISTICS

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ABSTRACT

In order statistics certain integrals involving the standard normal distribution play an important role. The asymptotic behaviour with respect to a large parameter is studied.

1. INTRODUCTION

The expectation and variance of the maximum in a random sample of size n from the standard normal distribution involve some of the integrals

$$(1) \quad M_j(n) := \int_{-\infty}^{\infty} x^{j-1} \phi(x) (1 - \phi^{n-1}(x)) dx \quad (n \in \mathbb{N}, j \in \mathbb{N}) ;$$

where

$$(2) \quad \phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-s^2/2} ds .$$

Integration by parts gives

$$(3) \quad \mu_j(n) = j M_j(n) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^j e^{-x^2/2} dx ,$$

where

$$(4) \quad \mu_j(n) := \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^j e^{-x^2/2} \phi^{n-1}(x) dx \quad (n \in \mathbb{N}, j \in \mathbb{N}) .$$

The problem posed by two colleagues ^{*}) of the author is to determine the asymptotic behaviour for $n \rightarrow \infty$ of $\mu_j(n)$ for $j = 1, 2, 3, 4$. Moreover, they are interested especially in the asymptotic behaviour of

$$(\mu_3 - \mu_1 \mu_2)(\mu_2 - \mu_1^2)^{-\frac{1}{2}} \quad \text{and} \quad (\mu_4 - \mu_2^2)^{\frac{1}{2}} .$$

We remark that the differences

$$M_j(n+1) - M_j(n) = \int_{-\infty}^{\infty} x^{j-1} (1 - \phi(x)) \phi^n(x) dx$$

are just the integrals occurring in the coefficients of the asymptotic formulas in a previous paper [1] (where f is defined by $f(x) = \phi(x\sqrt{2})$).

2. RESULTS

Let the asymptotic series of $(1 - \phi(x))(\phi'(x))^{-1}$ be denoted by A , i.e.

$$A := \sum_{\ell=0}^{\infty} (-1)^\ell (2\ell-1)!! x^{-2\ell-1} \quad (x \rightarrow \infty), \quad ((-1)!! = 1) .$$

Formal differentiations of A are denoted by A' , A'' , etc. Let $x_1 = x_1(n)$ be defined by $\phi(x_1) = 1 - \frac{1}{n}$. Then $\mu_j(n)$ has an asymptotic expansion in powers of x_1^{-1} , i.e.

$$\mu_j(n) \approx \sum_{k=-j}^{\infty} C(j,k) x_1^{-k} \quad (n \rightarrow \infty),$$

which can be computed as follows: x is considered to be a function of a variable z and $\frac{dx}{dz} \approx \frac{1}{2}A$. Higher derivatives can be computed by means of the chain rule. For instance, $\frac{d^2x}{dz^2} \approx \frac{1}{2}AA'$. Then

$$\mu_j(n) \approx \sum_{\ell=0}^{\infty} \left(\frac{d^\ell x^j}{dz^\ell} \right)_1 \frac{(-2)^\ell \Gamma(\ell)(1)}{\ell!} \quad (n \rightarrow \infty),$$

where the subscript 1 means that the value at $x = x_1$ has to be taken. x_1 has the following asymptotic series:

^{*}) F.W. Steutel and D.A. Overdijk, Department of Mathematics, Eindhoven University of Technology, The Netherlands.

$$x_1 \approx z^{\frac{1}{2}} \left(1 + \sum_{k=1}^{\infty} q_k(\log z) z^{-k} \right) \quad (n \rightarrow \infty)$$

where $z = 2 \log \frac{n}{\sqrt{2\pi}}$ and the q_k 's are polynomials of degree k . A few q_k 's are

$$q_1(t) = -\frac{1}{2} t, \quad q_2(t) = -\frac{1}{8} t^2 + \frac{1}{2} t - 1,$$

$$q_3(t) = -\frac{1}{16} t^3 + \frac{1}{2} t^2 - \frac{3}{2} t + \frac{7}{2}.$$

The coefficients $C(j,k)$ have the property that $C(j,-j+s) = 0$ if s is odd and $C(j,-j) = 1$. A few more coefficients $C(j,k)$ are:

$$C(1,1) = -\Gamma'(1), \quad C(1,3) = \Gamma'(1) - \frac{1}{2} \Gamma''(1),$$

$$C(1,5) = 3\Gamma'(1) + 2\Gamma''(1) - \frac{1}{2} \Gamma'''(1);$$

$$C(2,0) = -2\Gamma'(1), \quad C(2,2) = 2\Gamma'(1), \quad C(2,4) = -6\Gamma'(1) + 2\Gamma''(1),$$

$$C(2,6) = 30\Gamma'(1) - 14\Gamma''(1) + \frac{8}{3} \Gamma'''(1);$$

$$C(3,-1) = -3\Gamma'(1), \quad C(3,1) = 3\Gamma'(1) + \frac{3}{2} \Gamma''(1),$$

$$C(3,3) = -9\Gamma'(1) + \frac{1}{2} \Gamma'''(1),$$

$$C(3,5) = 45\Gamma'(1) - \frac{21}{2} \Gamma''(1) - \frac{1}{2} \Gamma'''(1) + \frac{3}{8} \Gamma^{(4)}(1);$$

$$C(4,-2) = -4\Gamma'(1), \quad C(4,0) = 4\Gamma'(1) + 4\Gamma''(1),$$

$$C(4,2) = -12\Gamma'(1) - 4\Gamma''(1), \quad C(4,4) = 60\Gamma'(1) - \frac{8}{3} \Gamma'''(1).$$

A routine computation shows that

$$\frac{\mu_3 - \mu_1 \mu_2}{(\mu_2 - \mu_1)^{\frac{1}{2}}} \approx d_0 + d_2 x_1^{-2} + d_4 x_1^{-4} + \dots \quad (n \rightarrow \infty)$$

$$(\mu_4 - \mu_2)^{\frac{1}{2}} \approx e_0 + e_2 x_1^{-2} + e_4 x_1^{-4} + \dots \quad (n \rightarrow \infty)$$

where

$$d_0 = e_0 = 2(\Gamma''(1) - (\Gamma'(1))^2)^{\frac{1}{2}}$$

and

$$d_2 = e_2 = -d_0 .$$

3. PROOF OF THE RESULTS

We transform the integral in (4) by putting

$$(5) \quad \phi(x) = 1 - \frac{s}{n} .$$

Then

$$(6) \quad \frac{dx}{ds} = - \frac{\sqrt{2\pi}}{n} e^{x^2/2} ,$$

whence

$$(7) \quad \mu_j(n) = \int_0^n x^j (1 - \frac{s}{n})^{n-1} ds .$$

We observe that $x = x(s)$ is monotonically decreasing on $[0, \infty)$, that $x(s) \rightarrow \infty$ ($s \downarrow 0$), $x(\frac{1}{2}n) = 0$ and $x(s) \rightarrow -\infty$ ($s \uparrow n$).

Now we shall prove that

$$(8) \quad \mu_j(n) = \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) \int_0^{\log n} x^j e^{-s} ds + O(n^{-1} (\log n)^{j/2}) \quad (n \rightarrow \infty) .$$

Since

$$\begin{aligned} \left| \int_{\pi/2}^n x^j (1 - \frac{s}{n})^{n-1} ds \right| &= \frac{n}{\sqrt{2\pi}} \int_{-\infty}^0 |x|^j e^{-x^2/2} \phi^{n-1}(x) dx \leq \\ &\leq \frac{n2^{-n}}{\sqrt{2\pi}} \int_0^{\infty} x^j e^{-x^2/2} dx \end{aligned}$$

we can write

$$(9) \quad \mu_j(n) = \int_0^{\frac{1}{2}n} x^j (1 - \frac{s}{n})^{n-1} ds + O(n2^{-n}) \quad (n \rightarrow \infty) .$$

Using

$$(10) \quad \Phi(x) \approx 1 - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (2\ell-1)!!}{x^{2\ell-1}} \quad (x \rightarrow \infty)$$

we derive easily that at $s = \log n$

$$(11) \quad x \sim \sqrt{2 \log n} \quad (n \rightarrow \infty) .$$

Hence

$$(12) \quad \int_{\log n}^{n/2} x^j \left(1 - \frac{s}{n}\right)^{n-1} ds = O\left((\log n)^{j/2} \int_{\log n}^{n/2} \left(1 - \frac{s}{n}\right)^{n-1} ds\right) = \\ = O(n^{-1} (\log n)^{j/2}) \quad (n \rightarrow \infty) .$$

From (9) and (12) it follows that

$$(13) \quad \mu_j(n) = \int_0^{\log n} x^j \left(1 - \frac{s}{n}\right)^{n-1} ds + O(n^{-1} (\log n)^{j/2}) \quad (n \rightarrow \infty) .$$

Furthermore

$$(14) \quad \int_0^{\log n} x^j \left(1 - \frac{s}{n}\right)^{n-1} ds = \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) \int_0^{\log n} x^j e^{-s} ds \quad (n \rightarrow \infty)$$

since

$$(15) \quad \left(1 - \frac{s}{n}\right)^{n-1} = e^{-s} \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) \quad (0 \leq s \leq \log n, n \rightarrow \infty) .$$

Clearly (13) and (14) imply (8).

On the interval $0 \leq s \leq \log n$, corresponding with large values of x , we can use (10) in order to solve x from (5) as a function of s .

Introduction of

$$(16) \quad z = 2 \log \frac{n}{\sqrt{2\pi}} - 2 \log s$$

transforms (5) into

$$(17) \quad \Phi(x) = 1 - \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z} .$$

Clearly $z \rightarrow \infty$ if $n \rightarrow \infty$ and $0 < s < \log n$.

Using (10) and taking logarithms we get

$$(18) \quad z \approx x^2 + \log(x^2) + \frac{2}{x^2} - \frac{5}{x^4} + \frac{74/3}{x^6} \dots \quad (x \rightarrow \infty)$$

By asymptotic iteration we find

$$(19) \quad x^2 \approx z - \log z + \sum_{k=1}^{\infty} z^{-k} p_k(\log z) \quad (z \rightarrow \infty)$$

where the p_k 's are polynomials of degree k . A few polynomials p_k are

$$(20) \quad \begin{aligned} p_1(t) &= t - 2 \\ p_2(t) &= \frac{1}{2} t^2 - 3t + 7 \\ p_3(t) &= \frac{1}{3} t^3 - \frac{3}{2} t^2 + 17t - \frac{107}{3} . \end{aligned}$$

The asymptotic expansions for x^j have the form

$$(21) \quad x^j \approx z^{j/2} \left(1 + \sum_{k=1}^{\infty} p_{jk}(\log z) z^{-k} \right) \quad (z \rightarrow \infty)$$

where the p_{jk} are polynomials of degree k .

The individual terms in the asymptotic expansions (21) have the following property: Let $f(z)$ be such a term occurring in the right side of (21).

Then $f(z)$ is of the form

$$f(z) = (\log^m z) z^{-k+\frac{1}{2}j} \quad \text{where } m \leq k .$$

Let $z_1 := 2 \log \frac{n}{\sqrt{2\pi}}$. Clearly $z_1 \geq 2$ if $n \geq 7$. Let $n \geq 7$. Then the power-series about $z = z_1$

$$(22) \quad f(z_1 + \varepsilon) = \sum_{\ell=0}^{\infty} \frac{f^{(\ell)}(z_1)}{\ell!} \varepsilon^{\ell}$$

is convergent for $|\varepsilon| < z_1$. Now it is easily seen that this powerseries has the property that for all $N \in \mathbb{N}$, $N \geq \frac{1}{2}j - k$

$$(23) \quad f(z_1 + \varepsilon) = \sum_{\ell=0}^N \frac{f^{(\ell)}(z_1)}{\ell!} \varepsilon^{\ell} + O((\log^m z_1) z_1^{-k+\frac{1}{2}j-N-1} \varepsilon^{N+1})$$

($-\frac{1}{2}z_1 < \varepsilon < \infty$, $z_1 \geq 2$) .

Then it follows that upon substitution $\varepsilon = -2 \log s$ in (22) we get an asymptotic expansion for $0 < s < \log n$, $n \rightarrow \infty$, i.e. for all $N \geq \frac{1}{2}j-k$

$$(24) \quad f(z_1 - 2 \log s) = \sum_{\ell=0}^N \frac{f^{(\ell)}(z_1)}{\ell!} (-2 \log s)^\ell + \\ + O((\log^m z_1) z_1^{-k+\frac{1}{2}j-N-1} (\log s)^{N+1}) \quad (0 < s < \log n, n \rightarrow \infty)$$

since $2 \log \log n < \frac{1}{2}z_1$ for n sufficiently large. The hidden constant in the 0-term is independent of n .

Further, for every $\ell \in \mathbb{N}$,

$$(25) \quad \int_{\log n}^{\infty} \log^\ell s e^{-s} ds = O(n^{-1} (\log \log n)^\ell) \quad (n \rightarrow \infty).$$

Therefore we can proceed as follows: In the asymptotic expansion (21) of x^j we substitute $z = z_1 - 2 \log s$ and we expand formally into a power-series about z_1 . After multiplication with e^{-s} and integration over $(0, \infty)$ we get an asymptotic expansion for $\mu_j(n)$.

Denoting the asymptotic expansion (21) of x^j by X_j and writing $X_j^{(k)}$ for its formal derivatives we have proved that

$$(26) \quad \mu_j(n) \approx \sum_{k=0}^n X_j^{(k)}(z_1) (-2)^k \Gamma^{(k)}(1) (k!)^{-1} \quad (n \rightarrow \infty),$$

where we have used that

$$(27) \quad \int_0^{\infty} e^{-s} \log^k s ds = \Gamma^{(k)}(1).$$

If we carry out the above program then, for instance, we find

$$(28) \quad \mu_1 = z^{1/2} - \frac{1}{2} z^{-1/2} \log z + \gamma z^{-1/2} - \frac{1}{8} z^{-3/2} \log^2 z + \\ + \frac{1}{2} (1+\gamma) z^{-3/2} \log z - (1-\gamma - \frac{1}{2} \gamma^2 - \frac{1}{12} \pi^2) z^{-3/2} + \\ + O(z^{-5/2} \log^3 z) \quad (n \rightarrow \infty)$$

where $z = z_1$. We have used that $\Gamma'(1) = \gamma$ (Euler's constant) and $\Gamma''(1) = \gamma^2 + \frac{1}{6} \pi^2$.

Of course we can also find asymptotic results for μ_2 , μ_3 and μ_4 . We will not do so since there is a more convenient way to obtain asymptotic expansions for $\mu_j(n)$. We shall show that $\mu_j(n)$ has an asymptotic power-series expansion in powers of x_1^{-1} , where x_1 is the value of x at $z = z_1 := 2 \log \frac{n}{\sqrt{2\pi}}$, i.e. there are sequences $(C(j,k))_{k=-j}^{\infty}$ of real numbers such that

$$(29) \quad \mu_j(n) \approx \sum_{k=-j}^{\infty} C(j,k) x_1^{-k} \quad (n \rightarrow \infty) .$$

Considering x as a function of z defined by (17) we can write the integral in (8) as

$$(30) \quad \int_0^{\log n} x^j(z_1 - 2 \log s) e^{-s} ds .$$

We shall prove that we can find the asymptotic expansion of (30) by term-wise integration of the formal powerseries expansion of $x^j(z_1 - 2 \log s)$ about z_1 . We shall give the details of the proof for the case $j = 1$. The other cases $j > 1$ can be treated analogously. So for the moment being we suppose $j = 1$. Obviously we are done with the problem if we have proved that

$$(31) \quad \forall_{k \in \mathbb{N}} \left(\frac{d^k x}{dz^k} \right)_1 \text{ has an asymptotic powerseries in } x_1^{-1} .$$

$$(32) \quad \forall_{k \in \mathbb{N}} \left(\frac{d^{k+1} x}{dz^{k+1}} \right)_1 = o \left(\left(\frac{d^k x}{dz^k} \right)_1 \right) \quad (n \rightarrow \infty)$$

$$(33) \quad \forall_{N \in \mathbb{N}} \exists_{A > 0} x(z_1 - 2 \log s) = x_1 + \sum_{k=1}^N \left(\frac{d^k x}{dz^k} \right)_1 \frac{(-2 \log s)^k}{k!} + \\ + o \left(\left(\frac{d^{N+1} x}{dz^{N+1}} \right)_1 (\log s)^{N+1} \right) \quad (0 < s < \log n, n > A)$$

where the subscript 1 in $()_1$ means the value at z_1 .

From (17) it follows that

$$(34) \quad \frac{dx}{dz} = \frac{1}{2}a(x) ,$$

where

$$(35) \quad a(x) = \frac{1 - \Phi(x)}{\Phi'(x)} .$$

From (10) we see that

$$(36) \quad a(x) \approx \sum_{\ell=0}^{\infty} (-1)^{\ell} (2\ell-1)!! x^{-2\ell-1} \quad (x \rightarrow \infty) .$$

From (35) we derive that

$$(37) \quad \frac{da}{dx} = xa - 1 .$$

Clearly (36) and (37) imply that all derivatives $d^k a/dx^k$ have asymptotic powerseries in x^{-1} which can be obtained by formal differentiation of the asymptotic series in (36). By means of the chain rule we can compute from (34) all derivatives dx^k/dz^k ; clearly, dx^k/dz^k is a sum of products involving $a(x)$ and its derivatives $d^{\ell} a/dx^{\ell}$, $\ell = 1, 2, \dots, k-1$. It follows that $d^k x/dz^k$ has an asymptotic expansion in powers of x^{-1} for $x \rightarrow \infty$.

Using that

$$\frac{d^{\ell} a}{dx^{\ell}} \sim \frac{(-1)^{\ell} \ell!}{x^{\ell+1}} \quad (x \rightarrow \infty)$$

we easily derive that for $k \geq 2$

$$(38) \quad \frac{d^k x}{dz^k} \sim (-1)^{k+1} (2k-3)!! 2^{-k} x^{-2k+1} \quad (x \rightarrow \infty) .$$

Thus we have proved (31) and (32).

Let $N \in \mathbb{N}$. Let $h \in \mathbb{R}$. Then there is a number $\theta \in (0, 1)$ such that

$$(39) \quad x(z_1 + h) = x_1 + \frac{h}{1!} \left(\frac{dx}{dz} \right)_1 + \dots + \frac{h^N}{N!} \left(\frac{d^N x}{dz^N} \right)_1 + R_N$$

where

$$(40) \quad R_N = \frac{h^{N+1}}{(N+1)!} \frac{d^{N+1} x}{dz^{N+1}} (z_1 + \theta h) .$$

Restricting ourselves to $h > -\frac{1}{2}z_1$, we only have to prove that there is a number $A > 0$ and a number $K > 0$ such that for all $z > A$

$$(41) \quad \left| \frac{d^{N+1}x}{dz^{N+1}}(\eta) \right| < K \left| \frac{d^{N+1}x}{dz^{N+1}}(z) \right| \quad (\eta > \frac{1}{2}z) .$$

On account of (38) and the fact that $x(z) \sim z^{\frac{1}{2}}$ ($z \rightarrow \infty$), (41) is obviously true. Hence (33) holds, since $z_1 - 2 \log s > \frac{1}{2}z_1$ for $0 < s < \log n$ and n sufficiently large.

Now using (33) in (30) for $j = 1$ we get an asymptotic series in powers of x_1^{-1} for $\mu_1(n)$.

Analogously we can find asymptotic series for $\mu_j(n)$, $j = 2, 3, \dots$. Only small adaptations are necessary; for instance in (33) we have to change $\forall N \in \mathbb{N}$ into $\forall N \in \mathbb{N}, N \geq \frac{1}{2}j$.

REMARK. Especially for $(\mu_3 - \mu_1\mu_2)(\mu_2 - \mu_1^2)^{-\frac{1}{2}}$ and $(\mu_4 - \mu^2)^{\frac{1}{2}}$ the computations are most easily done if one postpones the replacement of $d^k x^j / dz^k$ by its asymptotic series as long as possible. For instance, in this way we get

$$\mu_4 - \mu_2^2 = (8B - 4A^2)x^2(x')^2 + (24C - 8AB)(x^2x'x'' + x(x')^3) + \mathcal{O}(x^4)$$

where

$$A = -2\Gamma'(1), \quad B = 2\Gamma''(1) \quad \text{and} \quad C = -\frac{4}{3}\Gamma'''(1),$$

and x' , x'' denote first and second derivatives with respect to z . Now using the asymptotic series

$$x' = \frac{1}{2x} - \frac{1}{2x^3} + \dots, \quad x'' = -\frac{1}{4x^3} + \frac{1}{x^5} + \dots$$

we see that

$$x^2x'x'' + x(x')^3 = \mathcal{O}(x^{-4}) \quad \text{and} \quad x^2(x')^2 = \frac{1}{4} - \frac{1}{2x^2} + \dots$$

We get

$$\mu_4 - \mu_2^2 = d_0^2 - 2d_0^2x^{-2} + \mathcal{O}(x^{-4}),$$

from which we can easily derive the result in section 2.

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