EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

RANA 94-23
November 1994

On existence of solutions in plane quasistationary Stokes flow driven by surface tension

by

G. Prokert

Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands
ISSN: 0926-4507

On existence of solutions in plane quasistationary Stokes flow driven by surface tension

G. Prokert

Technical University Eindhoven, Faculty of Mathematics and Computer Science, P.O. Box 513, 5600 MB Eindhoven, The Netherlands; e-mail prokert@win.tue.nl

Abstract: Recently, the free boundary problem of quasistationary Stokes flow of a mass of viscous liquid under the action of surface tension forces has been considered by R.W. HOPPER, L.K. ANTANOVSKII, and others. The solution of the Stokes equations is represented by analytic functions, and a time dependent conformal mapping onto the flow domain is applied for the transformation of the problem to the unit disk. Two coupled Hilbert problems have to be solved there which leads to a Fredholm boundary integral equation. The solution of this equation determines the time evolution of the conformal mapping.

The question of existence of a solution to this evolution problem for arbitrary (smooth) initial data has not yet been answered completely. In this paper, local existence in time is proved using a theorem of OVSIANNIKOV on Cauchy problems in an appropriate scale of Banach spaces. The necessary estimates are obtained in a way that is oriented at the a priori estimates for the solution given by ANTANOVSKII. In the case of small deviations from the stationary solution represented by a circle, these a priori estimates together with the local results are used to prove even global existence of the solution in time.

1 Introduction

For the theoretical and numerical treatment of two-dimensional moving boundary problems as arising e.g. in flow problems, the application of time-dependent conformal mappings (so-called "pseudo-Lagrangian coordinates") is a successful technique. In the last years it has been applied to the problem of the free quasistatic motion of a viscous mass of liquid (a drop or an infinite mass enclosing a bubble) under the influence of the surface tension forces [1]–[3], [7], [9], [16]. In this model inertial forces are neglected because viscous forces ("inner friction") and surface tension forces are dominating. In practice, it has been applied very successfully for the description of viscous sintering of glass ([19], see also the references therein).

Theoretically, results by ANTANOVSKII and the author show that the behaviour of any given solution of the problem match the physical expectations: under "reasonable" assumptions, small perturbations of the circular shape are smoothed out exponentially (see below), and even in higher dimensions and for arbitrary initial domains the boundary of the fluid mass approaches the circular (or spherical) shape in an appropriate sense for $t \to \infty$. (For this, of course, global existence of the solution in time has to be assumed, see [13].)

Existence of the solution, however, has apparently not yet been proved completely. The main aim of this paper is to provide such a proof locally in time and for arbitrary initial shapes that fulfill relatively strong smoothness conditions. It is based on the work of L.K. ANTANOVSKII. The use of analytic functions enables us to use a Cauchy-Kovalevskaya type theorem on existence of solutions of certain nonlinear initial value problems [10] [11]. In their abstract form, such theorems are formulated in the framework of scales of Banach spaces. Since 1970, they have been applied to instationary free boundary flow problems in various geometries, for various driving mechanisms, and for various governing equations. Without attempting to be complete, we mention potential flow [11] [12], two-phase flow in porous media [6], coupled flow of surface and ground water [17], and Hele-Shaw flow [8] [14] [15]. In this paper, the same technique is applied to Stokes flow driven by surface tension.

Moreover, an a priori estimate given in [3] is used to prove global existence of the solution in time for initial states that are close to the stationary (equilibrium) state.

For the sake of brevity, we will restrict our attention here to the problem of thermocapillary motion of a bubble as discussed in [3], but the proof can be taken over to the case of a (simply-connected) drop straightforwardly (cf. [2]).

2 Problem formulation and derivation of the evolution equation

This section is devoted to the complete description of the considered problem and its reformulation in terms of a nonlocal Cauchy problem for a time-dependent real function on the boundary of the unit circle. It follows [3], therefore details will be omitted.

The Problem The fluid region at time t is taken to be the outer domain $\Omega_t \subset \mathbb{R}^2$ bounded by a simple closed curve Γ_t where Ω_0 is given. The velocity and pressure field in the fluid domain fulfill the Stokes equations

$$-\mu\Delta v + \nabla p = 0 \tag{1}$$

and the incompressibility condition

$$\operatorname{div} v = 0 \tag{2}$$

where μ is the the constant dynamic viscosity of the liquid. Near infinity, velocity and pressure are assumed to approach constant values:

$$v \to v_\infty,\ p \to p_\infty \qquad \text{as } |x| \to \infty \tag{3}$$

where $x \in \mathbb{R}^2$ denotes the space variable. These constants are time-dependent and a priori unknown. Moreover, incompressibility of the bubble has to be demanded:

$$\int_{\mathbb{R}^2\setminus\Omega_t} dx = \text{const} = \pi a^2. \tag{4}$$

This is an equivalent formulation for the condition that no fluid is injected or extracted at infinity.

At the free surface, as usual, two boundary conditions occur. The kinematic one is an equivalent expression for the fact that the set of the liquid particles at the boundary does not change in time:

$$V_n = v \cdot n \qquad \text{at } \Gamma_t \tag{5}$$

where V_n denotes the normal velocity of Γ_t with respect to the inner normal n. The dynamic boundary condition is given by the action of the surface tension force:

$$-\mathcal{T}n = \frac{d}{ds}\left(\sigma \frac{dx}{ds}\right) \tag{6}$$

where $\mathcal{T} = \mu\left((\nabla v) + (\nabla v)^T\right) - p\mathcal{I}$ is the stress tensor, σ is the surface tension coefficient, and s is the arclength parameter along Γ_t. As usual, $\mathcal{I}$ denotes the identity tensor. (The pressure inside the bubble is assumed to be zero.)

In order to consider thermocapillary motion, the dependence of σ on the temperature T has to be taken into account. For our purpose it is sufficient to take the simplest case of linear dependence:

$$\sigma = \sigma_* + \gamma T. \tag{7}$$

The temperature field is given as the solution of the auxiliary elliptic boundary value problem

$$\begin{aligned} \Delta T &= 0 \quad \text{in } \Omega_t \\ \frac{\partial T}{\partial n} &= 0 \quad \text{at } \Gamma_t \end{aligned} \tag{8}$$

with the asymptotic condition

$$T - C_\infty \cdot x \qquad \text{as } |x| - \infty \tag{9}$$

where C_∞ can be interpreted as temperature gradient at infinity.

Complex representation The velocity and pressure field fulfilling (1) and (2) in Ω_t can be represented by

$$v = \begin{pmatrix} -\partial_2\psi \\ \partial_1\psi \end{pmatrix}, \qquad p = -\mu\Delta\phi$$

where ∂_1, ∂_2 denote the partial spatial derivatives and (ϕ, ψ) is a pair of *conjugated biharmonic functions*, i.e. it fulfills

$$\partial_1^2\phi - \partial_2^2\phi = 2\partial_1\partial_2\psi, \qquad \partial_1^2\psi - \partial_2^2\psi = -2\partial_1\partial_2\phi$$

which is equivalent to the existence of two analytic functions w_0, w_1 in Ω_t such that

$$\phi + i\psi = w_0 + \bar{z}w_1.$$

(Of course, $\mathbb{R}^2$ has to be identified with the complex plane in the usual way.)

Rewriting the conditions (3)–(6) in terms of the new variables one gets

$$\operatorname{Im}\left(\frac{\partial\bar{z}}{\partial s}\frac{dz}{dt}\right) = \frac{\partial\psi}{\partial s} \text{ at } \Gamma_t \tag{10}$$

from the kinematic boundary condition,

$$\left.\begin{aligned} \operatorname{Re}(w_0 + \bar{z}w_1) &= 0 \\ 2\operatorname{Im}\left(\frac{\partial\bar{z}}{\partial s}w_1\right) &= \frac{\partial\psi}{\partial s} + \frac{\sigma}{2\mu} \end{aligned}\right\} \text{ at } \Gamma_t \tag{11}$$

after integration along Γ_t from the dynamic boundary condition, and

$$w_0 = O(1),\ w_1 = -\frac{p_\infty}{4\mu}z + v_\infty + O\left(\frac{1}{z}\right) \text{ as } |z| \to \infty \tag{12}$$

from the asymptotic conditions (3) if an appropriate moving system of coordinates is chosen.

Conformal mapping In order to transform our moving boundary problem to a fixed domain, one introduces now a time-dependent conformal mapping $z(\zeta, t)$ from the unit disk G onto the flow domain Ω_t. It can be shown that such a conformal mapping is of the form

$$z(\zeta, t) = \sum_{k=-1}^{\infty} z_k(t)\zeta^k$$

where $z_{-1} \in \mathbb{R}$, $z_{-1} > 0$ without loss of generality. In the following, all variables will be considered as functions of ζ but the same notation as before will be used. Obviously, the functions w_0 and w_1 are analytic in $G\backslash\{0\}$ for any t. Transforming the equations (10) and (11) to the new variable yields

$$\mathrm{Re}\,\frac{\partial z/\partial t}{\tau z'} + u = 0 \text{ at } \partial G \tag{13}$$

from the kinematic boundary condition,

$$\mathrm{Re}(w_0 + \bar{z}w_1) = 0 \text{ at } \partial G \tag{14}$$

and

$$2\,\mathrm{Re}\,\frac{w_1}{\tau z'} + u + A = 0 \text{ at } \partial G \tag{15}$$

from the dynamic boundary condition where $\tau = e^{i\theta} \in \partial G$, $\theta \in [0, 2\pi)$,

$$A(\tau, t) = \frac{\sigma_* + \gamma T(z(\tau, t))}{2\mu|z'(\tau, t)|},$$

$$u(\tau, t) = \frac{\partial\psi/\partial\theta}{|z'(\tau, t)|^2}. \tag{16}$$

From (12) we get now asymptotic conditions for $\zeta - 0$:

$$\begin{aligned}
\frac{\partial z/\partial t}{\zeta z'} &= -\frac{d}{dt}\log|z_{-1}| + O(\zeta) \\
w_0 &= O(1) \\
w_1 &= -\frac{p_\infty}{4\mu}(z_{-1}\zeta^{-1} + z_0) + v_\infty + O(\zeta).
\end{aligned} \tag{17}$$

Moreover, the conformal mapping introduced above enables one to solve the problem (8), (9) explicitly. For the values of T at the unit circle one gets

$$T(\tau, t) = \mathrm{Re}(\overline{C}_\infty(2z_{-1}(t)\tau^{-1} + z_0(t))).$$

By introducing the scaling factors a for length, $\dfrac{\sigma_*}{2\mu}$ for velocity, $\dfrac{2\mu a}{\sigma_*}$ for time, and $\dfrac{\sigma_*}{a}$ for pressure, all equations can be made dimensionless. One thus obtains

$$A(\tau, t) = \frac{1 + \mathrm{Re}(\bar{c}(2z_{-1}(t)\tau^{-1} + z_0(t)))}{|z'(\tau, t)|} \tag{18}$$

where, again, the same notation as before is used and the dimensionless constant $c = \dfrac{\gamma C_\infty a}{\sigma_*}$ is the so-called crispation number.

Schwarz integral and Hilbert transform The structure of the (real) equations (13)–(15) strongly suggests the application of a technique for the *reconstruction of an analytic function in a simply-connected domain from its real part at the boundary*, which is (under suitable smoothness presumptions) possible up to an imaginary constant. (This idea, of course, is also to be found in [7], [9], and [16], but there it is not exploited for deriving an explicit evolution equation that lends itself for general considerations concerning existence of solutions.)

On a fixed standard domain as the unit disk, this can be done by integral operators whose kernels are explicitly known. Namely, if f is an arbitrary Hölder continuous real function at ∂G, then the complex singular integral

$$\mathbf{S}[f](\zeta) = \frac{1}{2\pi}\int_0^{2\pi} S(\lambda,\zeta) f(\lambda)\, d\nu, \quad S(\lambda,\zeta) = \frac{\lambda+\zeta}{\lambda-\zeta}, \quad \lambda = e^{i\nu}$$

is a regular analytic function in G which is called the *Schwarz integral* of f. In the investigation of the behaviour of this integral at the boundary ∂G, the Plemelj formula yields for any $\tau \in \partial G$

$$\lim_{\zeta\in G \to \tau} \mathbf{S}[f](\zeta) = f(\tau) + i\mathbf{H}[f](\tau)$$

where

$$\mathbf{H}[f](\tau) = \frac{1}{2\pi}\int_0^{2\pi} H(\tau,\lambda) f(\lambda)\, d\nu, \quad H(\tau,\lambda) = iS(\tau,\lambda) = \cot\frac{\theta-\nu}{2},$$

and the integral is to be understood as Cauchy principal value. The operation $\mathbf{H}$ is called *Hilbert transform.* Moreover, $\mathbf{S}[f](0) \in \mathbb{R}$.

As a consequence, from (13) and the first equation in (17) one gets

$$\frac{\partial z}{\partial t}(\zeta,t) + \zeta z' \mathbf{S}[u](\zeta) = 0.$$

Using the above idea again, it is possible to replace this evolution equation by another one for the real function h defined at ∂G by

$$h(\tau,t) = \mathrm{Re}(\tau z(\tau,t) - 1).$$

It has the form

$$\frac{\partial h}{\partial t} = u + B(h,u)$$

where

$$B(h,u) = \left(h - \frac{\partial \mathbf{H}[h]}{\partial\theta}\right) u - \left(\frac{\partial h}{\partial\theta} + \mathbf{H}[h]\right)\mathbf{H}[u]. \tag{19}$$

It remains to determine u from h. Using the dynamic boundary conditions (14) and (15) as well as the asymptotic conditions (17) and an identity

$$S(\lambda,\zeta)S(\zeta,\tau) = S(\tau,\lambda)(S(\tau,\zeta) - S(\lambda,\zeta)) - 1 \tag{20}$$

one arrives after a certain effort of calculation at a boundary integral equation

$$u = K[h](u + A) \text{ at } \partial G \tag{21}$$

with

$$
\begin{aligned}
K[h](f) &= |z'|^{-2}L(Z,f) \\
L(Z,f) &= \frac{\partial}{\partial\theta}(\mathbf{H}[Zf] - Z\mathbf{H}[f]) \\
Z(\tau,t) &= \frac{1}{2}\operatorname{Re}\left(\tau z'(\tau,t)\overline{z(\tau,t)} - \frac{1}{2\pi}\int_{\partial G} H(\tau,\lambda)\overline{z(\tau,t)}\,dz(\lambda,t)\right).
\end{aligned}
\tag{22}
$$

It has to be remarked that solving (21) is equivalent to the determination of the normal velocity V_n at Γ_t if the problem (1)–(3), (6) is solved for *fixed* t. Under the condition of incompressibility of the bubble, existence and uniqueness of the solution of this problem can be shown (see [1]). Hence, it is justified to introduce the operator $U[h]$ as the solution of (21).

Note, finally, that the equations (19), (20), and the last equation in (22) are slightly different from the corresponding ones in [3]. This has no influence on the following considerations.

3 Short-time existence of the solution

Summarizing, we consider the following nonlocal Cauchy problem for a real function $h : \partial G \times J \longrightarrow \mathbb{R}$ where J is a time interval containing 0:

$$
\begin{aligned}
\frac{\partial h}{\partial t} &= F(h) = U[h] + B(h, U[h]) \\
h(\tau,0) &= h_*(\tau)
\end{aligned}
\tag{23}
$$

where B is defined by (19) and U is the solution operator of the integral equation (21) with K and A given by (22) and (18), respectively. h_* is the function corresponding to the initial domain Ω_0. Finally, the dependence of z on h at ∂G is given by

$$
z = \frac{1}{\tau}(1 + h + i\mathbf{H}[h]). \tag{24}
$$

For the investigation of this problem, still according to [2],[3], a *scale of Banach spaces* is used, i.e. a family of Banach spaces $(B_\rho, \|\cdot\|_\rho)$ depending on a real parameter ρ varying in a certain interval $\mathcal{J}$ with the following properties: For all $\rho, r \in \mathcal{J}$ with $r < \rho$ the embedding $B_\rho \hookrightarrow B_r$ holds, and the corresponding embedding operator $I_{\rho,r}$ is injective and fulfills $\|I_{\rho,r}f\|_r \leq \|f\|_\rho$ for all $f \in B_\rho$.

For our purpose, we will use a scale of spaces B_ρ consisting of (complex-valued) functions f on ∂G having a Fourier series

$$
f(\tau) = \sum_{k\in\mathbb{Z}} f_k \tau^k
$$

for which the expression

$$
\|f\|_\rho = \sum_{k\in\mathbb{Z}} |f_k| e^{|k|\rho} \tag{25}
$$

is finite. It is to be seen immediately that these spaces form a scale of Banach spaces with $\mathcal{J} = \mathbb{R}$ where the embedding operators are the identity. From (25) follows directly that

$$
\left\|\overline{f}\right\|_\rho = \|f\|_\rho, \qquad \|\operatorname{Re} f\|_\rho \leq \|f\|_\rho
$$

etc. Furthermore, taking into account that

$$\mathbf{H}[f](\tau) = \sum_{k\in\mathbb{Z}} \operatorname{sgn}(k) f_k \tau^k$$

one finds that

$$\|\mathbf{H}[f]\|_\rho \leq \|f\|_\rho \,. \tag{26}$$

The special choice of the scale ensures compactness of the embedding $B_\rho \hookrightarrow B_r$ for all $r < \rho$. In order to prove this, we approximate the embedding operator $I_{\rho,r}$ by a sequence of finite-rank operators $I_n \in L(B_\rho, B_r)$ defined by truncation of the Fourier series:

$$I_n = \sum_{|k|\leq n} f_k \tau^k.$$

For the difference, one gets

$$\|(I_{\rho,r} - I_n)f\|_r = \sum_{|k|>n} |f_k| e^{|k|r} = \sum_{|k|>n} |f_k| e^{|k|\rho} e^{-|k|(\rho-r)} \leq \|f\|_\rho \, e^{-n(\rho-r)},$$

hence $I_n \to I_{\rho,r}$ in $L(B_\rho, B_r)$ and therefore $I_{\rho,r}$ is compact.

For what follows it will be highly important that

$$\frac{\partial^n}{\partial\rho^n} \|f\|_\rho = \left\| \frac{\partial^n f}{\partial\theta^n} \right\|_\rho \qquad (n \in \mathbb{N}) \tag{27}$$

for all $f \in B_\rho$ for which these expressions exist. Consequently, the function $\rho \mapsto \|f\|_\rho$ is monotone and convex in its domain of definition for any fixed f, and a "triangle inequality" holds for all derivatives of the norms:

$$\frac{\partial^n}{\partial\rho^n} \|f+g\|_\rho \leq \frac{\partial^n}{\partial\rho^n} \|f\|_\rho + \frac{\partial^n}{\partial\rho^n} \|g\|_\rho \,.$$

Additionally, in the estimates below the obvious inequality

$$\frac{\partial}{\partial\rho} \|f\|_\rho \leq \frac{\partial^2}{\partial\rho^2} \|f\|_\rho$$

will play a role.

Finally, two important remarks have to be made concerning the spaces B_ρ with $\rho \geq 0$. It is easily checked that they are Banach algebras, i.e. $\|fg\|_\rho \leq \|f\|_\rho \|g\|_\rho$. In combination with (27) this leads to the estimate

$$\frac{\partial}{\partial\rho} \|fg\|_\rho \leq \frac{\partial}{\partial\rho} \|f\|_\rho \|g\|_\rho + \|f\|_\rho \frac{\partial}{\partial\rho} \|g\|_\rho \,. \tag{28}$$

Moreover, the Weierstrass criterion ensures the continuity of all $f \in B_\rho$, $\rho \geq 0$. If even $f \in B_\rho$ for a $\rho > 0$ then f admits an analytic continuation into a the annulus $\mathcal{A}_\rho = \{\zeta \mid e^{-\rho} < |\zeta| < e^\rho\}$ around ∂G, and therefore it is infinitely differentiable. On the other hand, it is clear from the uniqueness of analytic continuation and the absolute convergence that the restriction of a function which is analytic in $\mathcal{A}_\rho$ to ∂G belongs to B_ρ.

In order to give the existence proof now, we introduce a modified family of Banach spaces $(\tilde{B}_\rho, \frac{\partial}{\partial\rho}\|\cdot\|_\rho)$ where $\tilde{B}_\rho$ consists of all equivalence classes of functions $f \in B_\rho$ for which also

$\frac{\partial f}{\partial \theta} \in B_\rho$ and which differ only by a constant. It is immediately seen that these spaces form a scale in the above sense. A function $h(\cdot, t)$ corresponding to a certain domain Ω_t will be considered as an element of $\tilde{B}_\rho$ without loss of information, because the conservation of area yields

$$h_0(t) = \left(1 + 4\sum_{k=2}^{\infty}(k-1)|h_k(t)|^2\right)^{\frac{1}{2}} - 1, \tag{29}$$

where h_k are the Fourier coefficients of h, in other words, by means of the area conservation the original function h may be reconstructed from an element of $\tilde{B}_\rho$ in an unique way. Therefore, in the following we will use the notation h for elements of $\tilde{B}_\rho$ as well, and the Fourier coefficient h_0 will be considered as a function on $\tilde{B}_\rho$.

The crucial step in the existence proof will be an inequality which ensures that F is a *quasidifferential operator* in a scale of spaces $\tilde{B}_\rho$ in the sense of OVSIANNIKOV [11]. As a preparation for this, we introduce the notation

$$\mathcal{U}_\rho(h_*, r) = \left\{h \in \tilde{B}_\rho \;:\; \frac{\partial}{\partial \rho}\|h - h_*\|_\rho < r\right\}$$

for all ρ for which $h_* \in \tilde{B}_\rho$.

Lemma 1 *Assume $h_* \in \tilde{B}_{\rho_*}$ for a certain $\rho_* > 0$ and let Ω_0 be a $C^{1,\alpha}$-domain with $\alpha > 0$. Then there are constants $\hat{\rho} \in (0, \rho_*)$, $r > 0$ and $C > 0$ such that for all $\rho \in (0, \hat{\rho})$ and all $h_1, h_2 \in \mathcal{U}_\rho(h_*, r)$ the inequality*

$$\begin{aligned}\frac{\partial}{\partial \rho}\|F(h_1) - F(h_2)\|_\rho \;\leq\; & C\left(\frac{\partial^2}{\partial \rho^2}\|h_1 - h_2\|_\rho + \right. \\ & + \left. \left(\frac{\partial^2}{\partial \rho^2}\|h_1\|_\rho + \frac{\partial^2}{\partial \rho^2}\|h_2\|_\rho\right)\frac{\partial}{\partial \rho}\|h_1 - h_2\|_\rho\right)\end{aligned} \tag{30}$$

holds.

Proof: The statement of the lemma will be proved by a sequence of inequalities which are obtained from working with the Fourier coefficients in a manner similar to [3]. Furthermore, perturbation arguments are used to ensure the boundedness of certain expressions. The smoothing property of the operator $K[h](\cdot)$ is used to apply a compactness argument and to ensure the uniformity of some estimates with respect to ρ.

At first, a suitable $\hat{\rho}$ has to be determined. Let z_* denote the initial conformal map corresponding to h_*. By the smoothness presumption on Ω_0 and KELLOGG's theorem, z_*' may be extended continuously to ∂G, and it is not vanishing there, i.e.

$$|z_*'|^2 > \gamma > 0 \qquad \text{at } \partial G \tag{31}$$

due to a compactness argument. An easy calculation analogously to the derivation of inequality (37) below shows that $z_*' \in B_{\rho_*}$ and hence $|z_*'|^2 = z_*'\overline{z_*'} \in B_{\rho_*}$ from the Banach algebra property. This means there is a function w that is analytic in $\mathcal{A}_{\rho_*}$ whose restriction to ∂G is $|z_*'|^2$. From (31) and continuity and compactness arguments follows that we can choose a $\hat{\rho} \in (0, \rho_*)$ such that $\operatorname{Re} w > 0$ in $\mathcal{A}_{\hat{\rho}}$. In this smaller annulus, the functions $w^{\frac{1}{2}}$ and $w^{-\frac{1}{2}}$ are analytic. (Here and in the following, we preserve single-valuedness by choosing the branch of the square root which maps positive real numbers to positive real numbers.) Restriction of these functions to ∂G yields $|z_*'|, |z_*'|^{-1} \in B_{\hat{\rho}}$.

Let $\rho \in (0, \hat{\rho})$ be arbitrary, $r > 0$ small. (The upper bounds that are to be imposed on r will become clear from the arguments used within the proof.) Let $h, h_1, h_2 \in \mathcal{U}_\rho(h_*, r)$ be arbitrary, having the Fourier coefficients $h_k, h_k^{(1)}, h_k^{(2)}$, respectively.

It is clear that the functions A, z, and Z have to be considered now as functions of h with values in B_ρ. Throughout the proof, the index 1 or 2 will indicate the values of them at h_1 and h_2, respectively. If no index is used, the value of these functions at h is meant. Furthermore, all occuring constants will be denoted by C if their actual value is of no interest. Without explicit statement in every single case, all inequalities are to be understood in the sense that they hold *with the same constant(s)* C for all $h, h_1, h_2 \in \mathcal{U}_\rho(h_*, r)$ and for all $\rho \in (0, \hat{\rho})$.

It is immediately clear that

$$\frac{\partial}{\partial\rho}\|h\|_\rho \leq \frac{\partial}{\partial\rho}\|h_*\|_\rho + r \leq C. \tag{32}$$

From this and (29) one obtains (cf. [3])

$$|h_0| \leq \left(\frac{\partial}{\partial\rho}\|h\|_\rho\right)^2 \leq C. \tag{33}$$

With the notation

$$\eta_j = 4\sum_{k=2}^{\infty}(k-1)|h_k^{(j)}|^2 > 0, \qquad j = 1, 2$$

one can write

$$|h_0^{(1)} - h_0^{(2)}| = |\sqrt{1+\eta_1} - \sqrt{1+\eta_2}| = \left|\frac{\eta_1 - \eta_2}{\sqrt{1+\eta_1} + \sqrt{1+\eta_2}}\right|$$

and thus

$$\begin{aligned}
|h_0^{(1)} - h_0^{(2)}| &\leq 2\sum_{k=2}^{\infty}(k-1)\left||h_k^{(1)}|^2 - |h_k^{(2)}|^2\right| \\
&\leq 2\sum_{k=2}^{\infty}(k-1)(|h_k^{(1)}| + |h_k^{(2)}|)|h_k^{(1)} - h_k^{(2)}| \\
&\leq \frac{\partial}{\partial\rho}(\|h_1\|_\rho + \|h_2\|_\rho)\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho \leq C\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho .
\end{aligned} \tag{34}$$

The Fourier coefficients of Z_j are (cf. [3])

$$Z_k^{(j)} = -h_0^{(j)}h_k^{(j)} - h_k^{(j)} + 2\sum_{m=2}^{\infty}(m-1)\overline{h_m^{(j)}}h_{m+k}^{(j)}.$$

Therefore ($n = 1, 2$),

$$\begin{aligned}
&\frac{\partial^n}{\partial\rho^n}\|Z_1 - Z_2\|_\rho \leq \\
&\leq 4\sum_{k=1}^{\infty}\sum_{m=2}^{\infty}k^n(m-1)|\overline{(h_m^{(1)} - h_m^{(2)})}h_{m+k}^{(1)} - \overline{h_m^{(2)}}(h_{m+k}^{(1)} - h_{m+k}^{(2)})|e^{k\rho}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\partial^n}{\partial\rho^n}\|h_1-h_2\|_\rho + 2\sum_{k=1}^{\infty} k^n |(h_0^{(2)}-h_0^{(1)})h_k^{(2)} + h_0^{(1)}(h_k^{(2)}-h_k^{(1)})| e^{k\rho} \\
&\leq \quad 4\sum_{k=1}^{\infty}\sum_{m=2}^{\infty}(k+m)^n m(|h_m^{(1)}-h_m^{(2)}||h_{k+m}^{(1)}| \\
&\quad\quad +|h_m^{(2)}||h_{m+k}^{(1)}-h_{m+k}^{(2)}|)e^{(k+m)\rho}e^{-m\rho} \\
&\quad\quad +\frac{\partial^n}{\partial\rho^n}\|h_1-h_2\|_\rho + |h_0^{(1)}-h_0^{(2)}|\frac{\partial^n}{\partial\rho^n}\|h_2\|_\rho + |h_0^{(1)}|\frac{\partial^n}{\partial\rho^n}\|h_1-h_2\|_\rho \\
&\leq \quad \frac{\partial}{\partial r}\|h_1-h_2\|_r\Big|_{r=-\rho}\frac{\partial^n}{\partial\rho^n}\|h_2\|_\rho + \frac{\partial}{\partial r}\|h_1\|_r\Big|_{r=-\rho}\frac{\partial^n}{\partial\rho^n}\|h_1-h_2\|_\rho \\
&\quad\quad +\frac{\partial^n}{\partial\rho^n}\|h_1-h_2\|_\rho + |h_0^{(1)}-h_0^{(2)}|\frac{\partial^n}{\partial\rho^n}\|h_2\|_\rho + |h_0^{(1)}|\frac{\partial^n}{\partial\rho^n}\|h_1-h_2\|_\rho \\
&\leq \quad C\left(\frac{\partial^n}{\partial\rho^n}\|h_1-h_2\|_\rho + \frac{\partial^n}{\partial\rho^n}\|h_1\|_\rho\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\right) \qquad (35)
\end{aligned}
$$

where (33) and (34) have been used. For $n=1$, this may be estimated further by

$$
C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho
$$

using (32) again.

Replacing h_1 by h and h_2 by 0 in the estimate (35) yields

$$
\frac{\partial^n}{\partial\rho^n}\|Z\|_\rho \leq C\frac{\partial^n}{\partial\rho^n}\|h\|_\rho \qquad (36)
$$

which for $n=1$ reduces to $\frac{\partial}{\partial\rho}\|Z\|_\rho \leq C$.

Because of (24) and

$$
z' = -\frac{i}{\tau}\frac{\partial z}{\partial\theta}
$$

~~one finds~~

$$
\begin{aligned}
z' &= -\frac{i}{\tau^2}\frac{\partial}{\partial\theta}(h+i\mathbf{H}[h]) - \frac{1}{\tau^2}(1+h+i\mathbf{H}[h]) \\
\frac{\partial}{\partial\theta}z' &= -\frac{i}{\tau^2}\frac{\partial^2}{\partial\theta^2}(h+i\mathbf{H}[h]) - \frac{3}{\tau^2}\frac{\partial}{\partial\theta}(h+i\mathbf{H}[h]) + \frac{2i}{\tau^2}(1+h+i\mathbf{H}[h]).
\end{aligned}
$$

Taking the norms $\|\cdot\|_\rho$ of these expressions and applying the properties introduced above, one gets for $n=0,1$

$$
\frac{\partial^n}{\partial\rho^n}\|z'\|_\rho \leq C\left(\frac{\partial^{n+1}}{\partial\rho^{n+1}}\|h\|_\rho + 1\right). \qquad (37)
$$

In an analogous way,

$$
\frac{\partial^n}{\partial\rho^n}\|z_1'-z_2'\|_\rho \leq C\frac{\partial^{n+1}}{\partial\rho^{n+1}}\|h_1-h_2\|_\rho \qquad (38)
$$

may be obtained. An immediate consequence is

$$
\begin{aligned}
\left\||z_1'|^2-|z_2'|^2\right\|_\rho &= \left\|\overline{z_1'}z_1' - \overline{z_2'}z_2'\right\|_\rho \\
&\leq (\|z_1'\|_\rho + \|z_2'\|_\rho)\|z_1'-z_2'\|_\rho \leq C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho. \qquad (39)
\end{aligned}
$$

A series expansion for the square root gives

$$|z'| - |z'_*| = \sqrt{|z'|^2} - \sqrt{|z'_*|^2} = |z'_*| \sum_{n=1}^{\infty} \frac{a_n}{|z'_*|^{2n}} \left(|z'|^2 - |z'_*|^2\right)^n$$

where all the coefficients a_n fulfill $|a_n| < 1$. Hence, using (39),

$$\begin{aligned} \left\| |z'| - |z'_*| \right\|_\rho &\leq \left\| |z'_*| \right\|_\rho \sum_{n=1}^{\infty} \left(\left\| |z'_*|^{-1} \right\|_\rho^2 \left\| |z'|^2 - |z'_*|^2 \right\|_\rho \right)^n \\ &\leq \frac{C \frac{\partial}{\partial \rho} \|h - h_*\|_\rho}{1 - C \frac{\partial}{\partial \rho} \|h - h_*\|_\rho} \leq C \frac{\partial}{\partial \rho} \|h - h_*\|_\rho \end{aligned} \tag{40}$$

if r is small. This yields, moreover, $\left\| |z'| \right\|_\rho \leq C$, and by repeating the above argument we get

$$\left\| |z'_1| - |z'_2| \right\|_\rho \leq C \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho \tag{41}$$

Furthermore, using (40),

$$\begin{aligned} \left\| |z'|^{-1} - |z'_*|^{-1} \right\|_\rho &\leq \frac{\left\| |z'_*|^{-1} \right\|_\rho^2 \left\| |z'| - |z'_*| \right\|_\rho}{1 - \left\| |z'_*|^{-1} \right\|_\rho \left\| |z'| - |z'_*| \right\|_\rho} \\ &\leq C \left\| |z'| - |z'_*| \right\|_\rho \leq C \frac{\partial}{\partial \rho} \|h - h_*\|_\rho \end{aligned} \tag{42}$$

for sufficiently small r, hence

$$\left\| |z'|^{-1} \right\|_\rho \leq C. \tag{43}$$

With the use of (43), one obtains analogously to (40)

$$\left\| |z'_1|^{-1} - |z'_2|^{-1} \right\|_\rho \leq C \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho \tag{44}$$

and from this and (43)

$$\begin{aligned} \left\| |z'_1|^{-2} - |z'_2|^{-2} \right\|_\rho &\leq \left(\left\| |z'_1|^{-1} \right\|_\rho + \left\| |z'_2|^{-1} \right\|_\rho \right) \left\| |z'_1|^{-1} - |z'_2|^{-1} \right\|_\rho \\ &\leq C \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho . \end{aligned} \tag{45}$$

As a next step, some derivatives with respect to ρ have to be estimated. We find

$$\begin{aligned} \frac{\partial}{\partial \rho} \left\| |z'|^{-1} \right\|_\rho &= \left\| \frac{\partial}{\partial \theta} |z'|^{-1} \right\|_\rho \leq \left\| |z'|^{-2} \right\|_\rho \left\| \frac{\partial}{\partial \theta} |z'| \right\|_\rho \leq C \left\| \frac{\partial}{\partial \theta} \sqrt{z' \overline{z'}} \right\|_\rho \\ &\leq C \left\| |z'|^{-1} \right\|_\rho \left\| \frac{\partial z'}{\partial \theta} \right\|_\rho \|z'\|_\rho \leq C \left(\frac{\partial^2}{\partial \theta^2} \|h\|_\rho + 1 \right) \end{aligned} \tag{46}$$

from (43) and (37),

$$\frac{\partial}{\partial\rho}\left\||z_1'| - |z_2'|\right\|_\rho =$$

$$= \left\||z_1'|^{-1}\frac{\partial z_1'}{\partial\theta}z_1' - |z_2'|^{-1}\frac{\partial z_2'}{\partial\theta}z_2'\right\|_\rho \leq \left\||z_1'|^{-1}\right\|_\rho \left\|\frac{\partial z_1'}{\partial\theta}\right\|_\rho \|z_1' - z_2'\|_\rho$$

$$+ \left\||z_1'|^{-1}\right\|_\rho \left\|\frac{\partial}{\partial\theta}(z_1' - z_2')\right\|_\rho \|z_2'\|_\rho + \left\||z_1'|^{-1} - |z_2'|^{-1}\right\|_\rho \left\|\frac{\partial z_2'}{\partial\theta}\right\|_\rho \|z_2'\|_\rho$$

$$\leq C\left(\frac{\partial^2}{\partial\rho^2}\|h_1 - h_2\|_\rho + \left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho + \frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho\right) \tag{47}$$

where (37), (38), (43), and (44) have been used. Moreover, using (37), (45), and (47),

$$\frac{\partial}{\partial\rho}\left\||z_1'|^{-1} - |z_2'|^{-1}\right\|_\rho \leq \left\||z_1'|^{-2}\frac{\partial}{\partial\theta}|z_1'|\right\|_\rho - \left\||z_1'|^{-2}\frac{\partial}{\partial\theta}|z_1'|\right\|_\rho$$

$$\leq \left\||z_1'|^{-2}\right\|_\rho \frac{\partial}{\partial\rho}\left\||z_1'| - |z_2'|\right\|_\rho + \frac{\partial}{\partial\rho}\left\||z_2'|\right\|_\rho \left\||z_1'|^{-2} - |z_2'|^{-2}\right\|_\rho$$

$$\leq C\left(\frac{\partial^2}{\partial\rho^2}\|h_1 - h_2\|_\rho\right.$$

$$\left. + \left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho + \frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho\right). \tag{48}$$

In a similar way,

$$\frac{\partial}{\partial\rho}\left\||z_1'|^{-2} - |z_2'|^{-2}\right\|_\rho \leq \left(\frac{\partial}{\partial\rho}\left\||z_1'|^{-1}\right\|_\rho + \frac{\partial}{\partial\rho}\left\||z_2'|^{-1}\right\|_\rho\right)\left\||z_1'|^{-1} - |z_2'|^{-1}\right\|_\rho$$

$$+ \left(\left\||z_1'|^{-1}\right\|_\rho + \left\||z_2'|^{-1}\right\|_\rho\right)\frac{\partial}{\partial\rho}\left\||z_1'|^{-1} - |z_2'|^{-1}\right\|_\rho$$

$$\leq C\left(\frac{\partial^2}{\partial\rho^2}\|h_1 - h_2\|_\rho\right.$$

$$\left. + \left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho + \frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho\right) \tag{49}$$

applying (43), (46), (44), and (48).

The estimates concerning the operator L have already been given in [3]. We repeat them here only for the sake of completeness. For the Fourier coefficients g_k of $L(Z, f)$ one easily calculates

$$g_k = k\sum_{m\in\mathbb{Z}}(\mathrm{sgn}(k) - \mathrm{sgn}(m))Z_{k-m}f_m, \qquad g_{-k} = \overline{g_k}$$

and therefore $(n = 0, 1)$

$$\frac{\partial^n}{\partial\rho^n}\|L(Z, f)\|_\rho = 2\sum_{k=1}^{\infty}k^n|g_k| \leq$$

$$\leq 2\sum_{k=1}^{\infty}k^{n+1}\left(2\sum_{m=1}^{\infty}|f_m||Z_{k+m}| + |f_0||Z_k|\right)e^{k\rho}$$

$$
\begin{aligned}
&\leq \; 4\sum_{m=1}^{\infty}|f_m|e^{-m\rho}\sum_{k=1}^{\infty}k^{n+1}|Z_{k+m}|e^{(k+m)\rho}+|f_0|\frac{\partial^{n+1}}{\partial\rho^{n+1}}\|Z\|_\rho \\
&\leq \; \|f\|_0\frac{\partial^{n+1}}{\partial\rho^{n+1}}\|Z\|_\rho\leq C\|f\|_0\frac{\partial^{n+1}}{\partial\rho^{n+1}}\|h\|_\rho
\end{aligned}
\tag{50}
$$

because of $\rho > 0$ and (36).

Now we are able to investigate the crucial question of dependence of the operator K on h. Applying the linearity of L in the first argument, (35), (36), (45), and (43), we find

$$
\begin{aligned}
&\|(K[h_1]-K[h_2])(f)\|_\rho \leq \\
\leq \;& \left\||z_1'|^{-2}\right\|_\rho\|L(Z_1-Z_2,f)\|_\rho+\left\||z_1'|^{-2}-|z'^2|^{-2}\right\|_\rho\|L(Z_2,f)\|_\rho \\
\leq \;& C\frac{\partial}{\partial\rho}\|Z_1-Z_2\|_\rho\|f\|_0+C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\frac{\partial}{\partial\rho}\|Z_2\|_\rho\|f\|_0 \\
\leq \;& C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\|f\|_0.
\end{aligned}
\tag{51}
$$

For the derivative one obtains

$$
\begin{aligned}
&\frac{\partial}{\partial\rho}\|(K[h_1]-K[h_2])(f)\|_\rho \leq \\
\leq \;& \frac{\partial}{\partial\rho}\left\||z_1'|^{-2}L(Z_1-Z_2,f)\right\|_\rho+\frac{\partial}{\partial\rho}\left\|(|z_1'|^{-2}-|z_2'|^{-2})L(Z_2,f)\right\|_\rho \\
\leq \;& \frac{\partial}{\partial\rho}\left\||z_1'|^{-2}\right\|_\rho\|L(Z_1-Z_2,f)\|_\rho+\left\||z_1'|^{-2}\right\|_\rho\frac{\partial}{\partial\rho}\|L(Z_1-Z_2,f)\|_\rho \\
&+\frac{\partial}{\partial\rho}\left\||z_1'|^{-2}-|z_2'|^{-2}\right\|_\rho\|L(Z_2,f)\|_\rho\left\||z_1'|^{-2}-|z_2'|^{-2}\right\|_\rho\frac{\partial}{\partial\rho}\|L(Z_2,f)\|_\rho \\
\leq \;& C\left(\frac{\partial^2}{\partial\rho^2}\|h_1-h_2\|_\rho\right. \\
&\left.+\left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho+\frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\right)\|f\|_0
\end{aligned}
\tag{52}
$$

where (43), (46), (50), (36), (35), and (45) have been used.

By help of (24) we may rewrite (18) as

$$
A(\tau,t)=\frac{1+2\,\mathrm{Re}(\bar{\tau}((h_0(t)+1)\tau^{-1}+h_1(t)))}{|z'(\tau,t)|}
$$

and using (43) and (44) it is straightforward to prove

$$
\|A\|_0\leq C \tag{53}
$$

and

$$
\|A_1-A_2\|_0\leq C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\,. \tag{54}
$$

For estimates concerning the solution operator U of (21) it is important to remark that

$$
\begin{aligned}
\|K[h](f)\|_\rho \;&\leq\; \left\||z'|^{-2}\right\|_\rho\|L(Z,f)\|_\rho\leq C\frac{\partial}{\partial\rho}\|Z\|_\rho\|f\|_0 \\
&\leq\; C\frac{\partial}{\partial\rho}\|h\|_\rho\|f\|_0\leq C\|f\|_0
\end{aligned}
\tag{55}
$$

for all $f \in B_0$ because of (43), (50), and (36), therefore $K[h]$ is continuous from B_0 in B_ρ. Together with the compactness of the imbedding $B_\rho \hookrightarrow B_0$ this ensures compactness of $K[h]$ in B_0. Hence, the Fredholm alternative holds for the operator $I - K[h]$ in this space. According to the above remark, the integral equation (21), which may be written as

$$(I - K[h])(u) = K[h](A),$$

has a unique solution. Therefore $I - K[h]$ is a homeomorphism of B_0. This means, in particular, $(I - K[h_*])^{-1} \in L(B_0, B_0)$.

In the following, if $\|\cdot\|_0$ is applied to an operator instead of a function on ∂G, it will denote the usual norm in $L(B_0, B_0)$. Note that from (51) follows

$$\|K[h_1] - K[h_2]\|_0 \leq C\frac{\partial}{\partial \rho}\|h_1 - h_2\|_\rho .$$

Thus, applying a standard perturbation result concerning the inverse of regular linear operators,

$$\begin{aligned}\left\|(I - K[h])^{-1} - (I - K[h_*])^{-1}\right\|_0 &\leq \frac{\|(I - K[h_*])^{-1}\|_0^2 \, \|K[h] - K[h_*]\|_0}{1 - \|K[h] - K[h_*]\|_0 \, \|(I - K[h_*])^{-1}\|_0} \\ &\leq Cr\end{aligned}$$

and therefore $\left\|(I - K[h])^{-1}\right\|_0 \leq C$ (with C independent of h) if r is chosen small enough. Consequently,

$$\left\|(I - K[h_1])^{-1} - (I - K[h_2])^{-1}\right\|_0 \leq C\,\|K[h_1] - K[h_2]\|_0 \leq C\frac{\partial}{\partial \rho}\|h_1 - h_2\|_\rho .$$

Moreover,

$$\begin{aligned}\frac{\partial}{\partial \rho}\|K[h](f)\|_\rho &\leq \frac{\partial}{\partial \rho}\left\||z'|^{-2}\right\|_\rho \|L(Z,f)\|_\rho + \left\||z'|^{-1}\right\|_\rho^2 \frac{\partial}{\partial \rho}\|L(Z,f)\|_\rho \\ &\leq C\left(\frac{\partial^2}{\partial \rho^2}\|h\|_\rho + 1\right)\|f\|_0 \end{aligned} \tag{56}$$

where (46), (43), (50), and (36) have been used.

After these preparations, the necessary estimates for U can be given. Indeed,

$$U[h] = (I - K[h])^{-1}K[h](A) = K[h](I - K[h])^{-1}(A)$$

and therefore

$$\|U[h]\|_\rho \leq \left\|(I - K[h])^{-1}(A)\right\|_0 \leq C\,\|A\|_0 \leq C, \tag{57}$$

$$\frac{\partial}{\partial \rho}\|U[h]\|_\rho \leq C\left(\frac{\partial^2}{\partial \rho^2}\|h\|_\rho + 1\right)\left\|(I - K[h])^{-1}(A)\right\|_0 \leq C\left(\frac{\partial^2}{\partial \rho^2}\|h\|_\rho + 1\right), \tag{58}$$

$$\begin{aligned}\|U[h_1] - U[h_2]\|_\rho &\leq \left\|K[h_1](I - K[h_1])^{-1}(A_1 - A_2)\right\|_\rho \\ &\quad + \left\|K[h_1]((I - K[h_1])^{-1} - (I - K[h_2])^{-1})(A_2)\right\|_\rho\end{aligned}$$

$$
\begin{aligned}
&+ \left\|(K[h_1]-K[h_2])(I-K[h_2])^{-1}(A_2)\right\|_\rho \\
\leq\; & C\left(\left\|(I-K[h_1])^{-1}(A_1-A_2)\right\|_0\right. \\
&+ \left\|((I-K[h_1])^{-1}-(I-K[h_2])^{-1})(A_2)\right\|_0 \\
&\left.+ \left\|(I-K[h_2])^{-1}(A_2)\right\|_0 \frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\right) \\
\leq\; & C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho ,
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\rho}\|U[h_1]-U[h_2]\|_\rho \leq\; & \frac{\partial}{\partial\rho}\left\|K[h_1](I-K[h_1])^{-1}(A_1-A_2)\right\|_\rho \\
&+\frac{\partial}{\partial\rho}\left\|K[h_1]((I-K[h_1])^{-1}-(I-K[h_2])^{-1})(A_2)\right\|_\rho \\
&+\frac{\partial}{\partial\rho}\left\|(K[h_1]-K[h_2])(I-K[h_2])^{-1}(A_2)\right\|_\rho \\
\leq\; & C\left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho+1\right)\left(\left\|(I-K[h_1])^{-1}(A_1-A_2)\right\|_0\right. \\
&\left.+\left\|((I-K[h_1])^{-1}-(I-K[h_2])^{-1})(A_2)\right\|_0\right) \\
&+C\left(\frac{\partial^2}{\partial\rho^2}\|h_1-h_2\|_\rho+\left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho+\frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\cdot\right. \\
&\left.\cdot\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\right)\left\|(I-K[h_2])^{-1}(A_2)\right\|_0 \\
\leq\; & C\left(\frac{\partial^2}{\partial\rho^2}\|h_1-h_2\|_\rho\right. \\
&\left.+\left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho+\frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\right)
\end{aligned}
\tag{60}
$$

using the above estimates concerning K, $(I-K[h])^{-1}$, and A.

For arbitrary $u, \tilde{h} \in B_\rho$, the estimate

$$
\begin{aligned}
&\frac{\partial}{\partial\rho}\left\|B(\tilde{h},u)\right\|_\rho \leq \\
\leq\; & \frac{\partial}{\partial\rho}\left\|\frac{\partial}{\partial\theta}\mathbf{H}[\tilde{h}]u\right\|_\rho+\frac{\partial}{\partial\rho}\left\|\tilde{h}u\right\|_\rho+\frac{\partial}{\partial\rho}\left\|\frac{\partial\tilde{h}}{\partial\theta}\mathbf{H}[u]\right\|_\rho+\frac{\partial}{\partial\rho}\left\|\mathbf{H}[\tilde{h}]\mathbf{H}[u]\right\|_\rho \\
\leq\; & 2\left(\frac{\partial^2}{\partial\rho^2}\left\|\tilde{h}\right\|_\rho\|u\|_\rho+\frac{\partial}{\partial\rho}\left\|\tilde{h}\right\|_\rho\frac{\partial}{\partial\rho}\|u\|_\rho\right. \\
&\left.+\frac{\partial}{\partial\rho}\left\|\tilde{h}\right\|_\rho\|u\|_\rho+\left\|\tilde{h}\right\|_\rho\frac{\partial}{\partial\rho}\|u\|_\rho\right)
\end{aligned}
\tag{61}
$$

holds. Using the bilinearity of B, we find

$$
\frac{\partial}{\partial\rho}\|B(h_1,u_1)-B(h_2,u_2)\|_\rho \leq \frac{\partial}{\partial\rho}\|B(h_1-h_2,u_1)\|_\rho+\frac{\partial}{\partial\rho}\|B(h_2,u_1-u_2)\|_\rho
\tag{62}
$$

for arbitrary $u_1, u_2 \in B_\rho$. The lemma follows now from the subsequent application of the inequalities (57)–(62). (Note that

$$\|h\|_\rho \leq C, \qquad \|h_1 - h_2\|_\rho \leq C\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho$$

as a consequence of (33), (34)). ∎

Additionally, (57), (58), and (61) yield

$$\frac{\partial}{\partial\rho}\|F(h_*)\|_\rho \leq C\left(\frac{\partial^2}{\partial\rho^2}\|h_*\|_\rho + 1\right) \tag{63}$$

for all $\rho \in (0, \hat{\rho})$. The inequalities (30) and(63) may be written as

$$\begin{aligned}
\frac{\partial}{\partial\rho}\|F(h_1) - F(h_2)\|_\rho &\leq C\frac{\partial}{\partial\rho}\left[\left(\frac{\partial}{\partial\rho}\|h_1\|_\rho + \frac{\partial}{\partial\rho}\|h_2\|_\rho + 1\right)\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho\right] \\
\frac{\partial}{\partial\rho}\|F(h_*)\|_\rho &\leq C\frac{\partial}{\partial\rho}\left[\frac{\partial}{\partial\rho}\|h_*\|_\rho + \rho\right].
\end{aligned}$$

The expressions in square brackets are positive convex functions of the real parameter ρ, hence for arbitrary $\rho, \rho' \in (0, \hat{\rho})$ with $\rho' < \rho$:

$$\begin{aligned}
\frac{\partial}{\partial r}\|F(h_1) - F(h_2)\|_r\Big|_{r=\rho'} &\leq C\frac{\left(\frac{\partial}{\partial\rho}\|h_1\|_\rho + \frac{\partial}{\partial\rho}\|h_2\|_\rho + 1\right)\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho}{\rho - \rho'} \\
&\leq \frac{C}{\rho - \rho'}\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho,
\end{aligned} \tag{64}$$

$$\frac{\partial}{\partial\rho}\|F(h_*)\|_\rho \leq C\frac{\frac{\partial}{\partial r}\|h_*\|_r\Big|_{r=\hat{\rho}} + \hat{\rho}}{\hat{\rho} - \rho} \leq \frac{C}{\hat{\rho} - \rho} \tag{65}$$

because of the usual estimate of the derivative by a difference quotient. According to a result of NISHIDA [10], the inequalities (64) (holding uniformly for all $h_1, h_2 \in \mathcal{U}_\rho(h_*, r)$) and (65) ensure the existence of a solution to (23) locally in time.

Proposition 1 *For arbitrary $h_* \in \tilde{B}_{\rho_*}$ with $\rho_* > 0$ there is a $\beta > 0$ and a $\hat{\rho} > 0$ such that (23) has a unique solution $h(t)$ in the time interval $\left(-\frac{\hat{\rho}}{\beta}, \frac{\hat{\rho}}{\beta}\right)$ with $h(t) \in \tilde{B}_{\hat{\rho}-\beta|t|}$. The number β is completely determined by r and C in (64), (65).*

For the proof, see e.g. [18]. The results for the problem "backward in time", i.e. for $t < 0$, are an immediate consequence of the autonomous character of (23): The function $h_-(t) = h(-t)$ is described by the initial value problem

$$\frac{\partial h_-}{\partial t} = -F(h_-), \qquad h_-(0) = h_*$$

for which existence, uniqueness, and smoothness properties of the solution for $t > 0$ can be obtained as in the original problem without any changes.

4 Global existence of the solution near equilibrium

It has to be pointed out that, up to now, positivity of the surface tension coefficient σ has not yet been demanded. Physically, of course, the condition $\sigma > 0$ makes sense and turns out to be essential for results on existence of the solution for all $t > 0$. Note that the condition $\sigma > 0$ at Γ_t is equivalent to the time-independent one $|c| < \frac{1}{2}$ where c is the crispation number (see [3]).

It is easily concluded from energy considerations that, in this case, the only stationary solution of the considered problem is given by a circular bubble. In the dimensionless coordinate system chosen above, this corresponds to a resting circle of radius 1 centered at the origin, i.e. the equilibrium state is described by $h = 0$.

In [3], the following a priori estimate concerning solutions with initial states near $h = 0$ has been proved:

Proposition 2 *Let $|c| < \frac{1}{2}$, $\alpha \in (0, 1 - 2|c|)$, $\rho_* > 0$ be given. There is a $q_1 > 0$ such that for all solutions h of (23) with $h_* \in \mathcal{U}_{\rho_*}(0, q_1)$ and for all $t > 0$ for which h exists in $[0,t]$ the estimate*

$$\left\| \frac{\partial h}{\partial \theta}(t) \right\|_{\rho_* + \alpha t} \le \left\| \frac{\partial h}{\partial \theta}(0) \right\|_{\rho_*} \tag{66}$$

holds.

An immediate consequence of (66) is

$$\left\| \frac{\partial h}{\partial \theta}(t) \right\|_{\rho_*} \le e^{-\alpha t} \left\| \frac{\partial h}{\partial \theta}(0) \right\|_{\rho_*} \tag{67}$$

and hence (see (33)) $\|h(t)\|_{\rho_*} \le Ce^{-\alpha t}$, i.e. small perturbations of the equilibrium are "smoothed out" exponentially in the considered norm.

By help of this a priori estimate it is possible to globalize the existence result of the previous section in the case of such small perturbations:

Proposition 3 *Let $|c| < \frac{1}{2}$, $\rho_* > 0$ be given. Then there is a $q > 0$ such that all solutions of (23) for which $h_* \in \mathcal{U}_{\rho_*}(0, q)$ exist for all $t > 0$.*

Proof: The idea of the proof is to show that for a certain $q \in (0, q_1)$ there is a $T > 0$ such that all solutions of (23) with $h_* \in \mathcal{U}_{\rho_*}(0, q)$ exist on the interval $[0, T]$. The estimate (67) with arbitrary $\alpha \in (0, 1 - 2|c|)$ ensures then $h(T) \in \mathcal{U}_{\rho_*}(0, q)$, and a simple induction argument will finally prove the existence of h on the interval $[0, nT]$ for all $n \in \mathbb{N}$.

In other words, it is sufficient to find a uniform lower bound for the length of the existence intervals of the solutions of (23) with $h_* \in \mathcal{U}_{\rho_*}(0, q)$. This can be done by proving that lemma 1 holds with the same constants r and C for all $h_* \in \mathcal{U}_{\rho_*}(0, q)$ if $q > 0$ is chosen small enough.

Repeating the arguments for the proofs of (40) and (42) with z_* in place of z and ζ^{-1} in place of z_* we find

$$\big\| |z_*'| - 1 \big\|_{\rho_*} \le Cq, \qquad \left\| |z_*'|^{-1} - 1 \right\|_{\rho_*} \le Cq. \tag{68}$$

Note that in this case we can set $\hat{\rho} = \rho_*$ because no smoothness is lost when taking the square root of $|\zeta^{-1}|^2 = 1$ or its reciprocal. A reexamination of the proof of lemma 1 in this situation shows that for all h_* that fulfill

$$\max \left\{ \left\| \frac{\partial h_*}{\partial \theta} \right\|_{\rho_*}, \big\| |z_*'| \big\|_{\rho_*}, \left\| |z_*'|^{-1} \right\|_{\rho_*}, \left\| (I - K[h_*])^{-1} \right\|_0 \right\} \le M \tag{69}$$

the inequality (30) holds with C and r only depending on M. It it easily seen that $K[0]$ is the zero operator, and an estimate analogously to (51) with h_* in place of h_1 and 0 in place of h_2 shows that $\|K[h_*]\|_0 \leq Cq$, hence $\|(I - K[h_*])^{-1}\|_0 \leq 2$ for sufficiently small q.

From this and (68) follows now that if q is chosen sufficiently small, then (69) holds for all $h_* \in \mathcal{U}_{\rho_*}(0, q)$ with a certain fixed M. This completes the proof. ■

5 Conclusion

Based on the work of L.K. ANTANOVSKII, the problem of quasistationary thermocapillary motion of a bubble in viscous liquid is reformulated using complex representations and a time-dependent conformal mapping. One arrives at a nonlinear nonlocal Cauchy problem for a real function on the boundary of the unit disk. This technique is also applicable to more general distributions of the surface tension coefficient along the bubble boundary as well as to the motion of a free liquid drop driven by surface tension.

When identifying the character of the evolution problem one has to distinguish two cases which lead to qualitatively different results: If the initial domain is near the circle, the problem can be seen as a perturbed first-order linear parabolic (pseudodifferential) equation where the nonlinear nonlocal perturbation term is of the same order as the linear term, but remaining small. The non-perturbed linear problem can be shown to be well-posed if and only if the surface tension coefficient is strictly positive. The suitable decomposition of the operator F is given in [3]. Actually, it is the basis for deriving the a priori estimate which finally ensures global solvability and exponential decay to the state of rest. For an abstract setting describing a class of problems of this type see [5]. The corresponding problem for Hele-Shaw flow reveals very similar properties [4].

For the case of general initial domains, a comparable identification of the structure of the evolution problem is apparently not known. So the application of the Cauchy-Kovalevskaya theorem is natural because it enables us to obtain local solvability for analytic data without knowledge about special properties of the Cauchy problem. Therefore there is no difference in the treatment of the problem forward and backward in time and no restriction on the sign of the surface tension coefficient.

Finally, it has to be remarked that the analysis given here obviously cannot provide answers to the questions on the occurence of irregular behaviour like cusp formation or change of connectivity due to their local character in time and the presumptions on initial smoothness.

Acknowledgements

The author is indebted to L.K. Antanovskii for an interesting correspondence and to J. de Graaf for a lot of fruitful discussions.

References

[1] Antanovskii, L.K. 1991 Bianalytic stress-stream function in plane quasi-steady problems of capillary fluid mechanics, *Sibirsk. Mat. Zh.* **33**, No. 1, 3–15

[2] Antanovskii, L.K. 1991 Analyticity of a free boundary in plane quasi-steady flow of a liquid form subject to variable surface tension, in: *Proc. of Conference: The Navier-Stokes Equations II: Theory and Numerical Methods*, Oberwolfach 1991, LNM **1530**, **p.** 1–16, Springer 1993

[3] Antanovskii, L.K. 1992 Creeping thermocapillary motion of a two-dimensional deformable bubble: existence theorem and numerical simulation, *Eur. J. Mech., B/Fluids* **11** No. 6, 741–758

[4] Constantin, P., Pugh, M. 1993 Global solution for small data to the Hele-Shaw problem, *Nonlinearity* **6**, 393–415

[5] Duchon, J., Robert, R. 1985 Quasi-differential perturbation of a smoothing semigroup in a scale of Banach spaces, *C. R. Acad. Sci. Paris*, t. **301** Serie I, No. 11, 561–564 (French)

[6] Duchon, J., Robert, R. 1986 Estimates of integral operators of Cauchy type in Ovsiannikov scales and application, *Ann. Inst. Fourier*, **36**, No. 1, 83–95 (French)

[7] Graaf, J. de 1992 Mathematical addenda to Hopper's model of plane Stokes flow driven by capillarity on a free surface, in: *Proc. of Conference: Geometrical and quantum aspects of integrable systems, Scheveningen 1992*, LNP 424, Springer 1993, 167–185

[8] Hohlov, Yu.E., Reissig, M. On classical solvability for the Hele-Shaw moving boundary problems with kinetic undercooling regularization, *Euro. Journ. Appl. Math.* (to appear)

[9] Hopper, R.W. 1990 Plane Stokes flow driven by capillarity on a free surface, *J. Fluid Mech.* **213** 349–375

[10] Nishida, T. 1977 A note on a theorem of Nirenberg, *J. Diff. Geom.* **12** 629–633

[11] Ovsiannikov, L.V. 1971 A nonlinear Cauchy problem in a scale of Banach spaces, *Dokl. Akad. Nauk SSSR* **200** No. 4, 789–792

[12] Ovsiannikov, L.V. 1974 To the shallow water theory foundation, *Arch. Mech. Stos.* **26**, No. 3, 407–422

[13] Prokert, G. 1993 On a quasistatic model for the motion of a viscous capillary liquid drop, RANA 93-16, Eindhoven University of Technology

[14] Reissig, M., Wolfersdorf L.v. 1992 A simplified proof for a moving boundary problem for Hele-Shaw flows in the plane, *Arkiv för matematik*, **31**, 101–116

[15] Reissig, M. 1994 About the existence and uniqueness of analytic solutions for a moving boundary problem for Hele-Shaw flows in the plane, *Nonlinear Analysis* (to appear)

[16] Richardson, S. 1992 Two-dimensional slow viscous flow with time-dependent free boundaries driven by surface tension, *Euro. Journ. Appl. Math.* **3** 193–207

[17] Shelukhin, V.V. 1991 The justification of the conjugate conditions for the Euler's and Darcy's equations, in: *Proc. of Conference: Free boundary problems in continuum mechanics, Novosibirsk 1991*, Internat. Ser. Numer. Math. 106, Birkhäuser 1992, 293–300

[18] Tutschke, W. 1989 Solution of initial value problems in classes of generalized analytic functions, Springer

[19] Vorst, G.A.L.v.d. 1994 Modelling and Numerical Simulation of Viscous sintering, PhD Thesis, Eindhoven University of Technology