The mathematics of Paul Erdös I, II / ed. by R.L. Graham, J. Nestril

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Book Review

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The Mathematics of Paul Erdős I, II; ed. by R. L. GRAHAM and J. NEŠETŘIL. Berlin, etc.: Springer-Verlag, 1997. 399 p. +577 p., price DM 148,- + DM 148,- (hc) (Algorithms and Combinatorics; 13, 14).
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On September 20, 1996 Paul Erdős died at the age of 83. He was attending a conference in Warsaw, where he gave two talks, and on his way to yet another meeting. Many of his mathematical statements began with "*If I live ..*"; this was not to be but, again in his words, "*he left us*" in the manner he wanted to. The appearance of these two volumes in the following year might give the impression that already a survey of his work has been written. This is only partly true. The idea for these volumes was born at a small conference held in Prague in 1992 at the occasion of the awarding of an honorary doctorate to Erdős by the Univerzita Karlova (Charles University) (one of his fifteen honorary doctorates). It is a collection of about 70 sollicited papers illustrating various aspects of Erdős' work.

Since Erdős was certainly the most prolific mathematician of this century, one cannot expect a complete survey. Volume II ends with a list of 1414 papers by Erdős (up to March 1996; many more have appeared since then). If these papers had been "covered" by the present authors, each would have treated about 20 of them in his (or her) contribution, with an average length of 11 pages.

There are several papers that do survey a coherent piece of Erdős' work, pointing out the development of ideas, mentioning open problems and often including some new results. Together, these give a very interesting picture of the mathematics of Paul Erdős, justifying the title of the volumes. However, there are ten contributions (totalling 150 pages) not related to Erdős' work at all. Furthermore, we find a number of highly technical research papers in which some paper of Erdős is mentioned but which do not shed light on his work. For the authors it would have been better to publish these papers in a research journal; for the general mathematical reader, only a few of them will be of interest. If the editors had insisted on contributions related to the title of these two volumes, a more affordable homage would have resulted.

Erdős made very important contributions to several parts of mathematics. The papers are divided into sections in accordance with this division of interests. Those who know Erdős' work will not be surprised that *Number Theory* and *Combinatorics and Graph Theory* play a central rôle. Each of the seven sections has an introduction to the area and Erdős' influence on it written by the editors. Quite often they contain sentences beginning with "I proved that . . .". Apparently the reader is assumed to know the literature so well that he realizes who "I" is.

Both volumes start with several wonderful pictures of Erdős (from age 3 to 80). The first contribution (preceding Section 1) is by B. Bollobás. In 1958, at the age of 14, Bollobás, already a winner of several mathematical competitions in Hungary, attended a lecture by Erdős. Erdős recognized his abilities and subsequently had a strong influence on his development. He became a true disciple of Erdős and coauthor of 14 papers. Bollobás' contribution is an excellent survey of both the life of Erdős and his contributions to mathematics. Little anecdotes like the fact that the great I. Schur referred to Erdős as "der Zauberer von Budapest" are relevant. This happened after Erdős (at age 20) had solved one of Schur's problems. The mathematical part of this survey shows the origin of *probabilistic number theory, extremal graph theory, Ramsey theory, the calculus of partitions* and of course a lot about analysis and number theory including the famous elementary proof of the prime number theorem with A. Selberg in 1948. For this, Erdős received the Cole Prize of the American Mathematical Society and Selberg the Fields Medal. The first section "Early Days" starts with a paper by Erdős himself, called *Some of My Favorite Problems and Results.* They are all quite hard and the total monetary reward that he offers for solutions or counterexamples exceeds \$ 20000, including an offer of \$ 10000 for a proof that for some $\epsilon > 0$

$$p_{n+1} - p_n > (\log n)^{1+\epsilon}.$$

The four other papers in this section are short papers with recollections by authors who met Erdős during his first period abroad : 1934–1938 in Manchester. Three are by authors of the paper establishing the theory of dissections of rectangles into distinct squares (R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, The dissection of rectangles into squares, *Duke Math. J.* **7** (1940), 312–340). Smith suggests that perhaps Erdős saved Western civilization with this problem! Apparently Erdős had made a conjecture about this problem in a lecture and a member of the audience had told it to Stone, then a pupil at a boys' school. The four authors met each other as students at Cambridge, where they wrote the paper. The fact that Tutte, then a chemistry student, worked at Bletchley Park during World War II could be ascribed to the fact that the other three had made it known that he was such a good mathematician. Tutte's contribution to the work done there on the *enigma* explains the suggestion by Smith.

Erdős' first results were on number theory. At age 17 he gave a simple proof of Chebyshev's theorem that there is a prime between n and $2n$. In his second year at the university he extended these results to arithmetic progressions, thus essentially completing his Ph. D. thesis. He had to wait two more years before he could submit the thesis, graduating in Budapest at the age of 21. From very early on, Erdős was interested in density theorems. A typical example (1935) is the statement that if $a_1, a_2, \dots$ is an increasing sequence of integers with $1 \neq a_1$ and $a_i \not| a_j$ for $i \neq j$, then

$$\sum \frac{1}{a_i \log a_i} < \infty.$$

A paper by R. Ahlswede and L. H. Khatchatrian mentions several such results and then continues as a research paper on some new problems in this area.
A. Sárkőzy and V. T. Sós are among the great friends of and super coauthors with Erdős. Sárkőzy is the record holder with 57 joint papers. Vera Sós met Erdős in her first year at the university and has 33 joint papers. Of course she also witnessed the great collaboration of Erdős and her husband Paul Turán. These two authors give a long and quite readable survey of regularity properties and value distribution of additive representation functions. The paper quotes 16 papers by Erdős on this subject.

More than thirty years ago, Erdős and Heilbronn conjectured that if $\{a_1, a_2, \dots, a_k\}$ is a set of k integers mod p (p a prime), then the set of sums $a_i + a_j$ ($i \neq j$) contains at least $\min\{p, 2k-3\}$ elements. This was proved in 1994 : J. A. Dias da Silva and Y. O. Hamidoune, Cyclic spaces for Grassmann derivatives and additive theory, *Bull. London Math. Soc.* **26** (1994), 140–146. The paper by M. B. Nathanson is an elementary exposition of this proof.

In the paper "On Divisibility Properties of Sequences of Integers" Sárkőzy returns to his first paper with Erdős (together with Szemerédi)(1966) with a survey and a related problem.

In 1975 Erdős and Selfridge proved that the product of two or more consecutive integers is never a cube or a higher power. T. N. Shorey and R. Tijdeman sketch the proof and then show how Erdős' methods can be extended to obtain results for arithmetic progressions.

In 1938, anticipating the coming events in Europe, Erdős moved to the USA. He spent the year 1938–1939 at the Institute for Advanced Study in Princeton. He has stated that this was his most successful year. The papers with M. Kac and with A. Wintner mark the beginning of probabilistic number theory. This is the topic of the section *Randomness and Applications.*

In a long paper J. Beck discusses positional games like tic-tac-toe and describes ideas initiated by Erdős to handle such games. About half of the paper is on new results. Computationally these games are hopeless. The author mentions that a 1977 program by O. Patashnik found the first first-player winning strategy for 3-dimensional $4 \times 4 \times 4$ tic-tac-toe after 1500 hours of computing time.
M. Karoński and A. Ruciński survey eight joint papers of Erdős and A. Rényi written between 1959 and 1968 that mark the origin of the theory of random

graphs. Like Szekeres and Turán, Rényi was one of the great Hungarian mathematicians and lifelong friends of Erdős that met as young boys. How much more wonderful theory would they have developed if Rényi had not died so young?

The probablistic method in combinatorics developed from "something that Erdős was good at" to a "general tool" mainly through the very successful book "*Probabilistic Methods in Combinatorics*" by him and Joel Spencer (coauthor of 22 publications). In the paper *The Erdős Existence Argument* Spencer reexamines some of the critical early papers on this method. We mention a typical example concerning Ramsey numbers. One instance of Ramsey's theorem states that there is a number $R(k,k)$ such that if $n > R(k,k)$, any coloring of the edges of a complete graph K_n on n vertices with the colors red and blue must contain a complete subgraph K_k with all edges red or one with all edges blue. We call a coloring with no such subgraphs a (k,k)-coloring. There are $2^{\binom{n}{2}}$ ways of coloring K_n. Given a specific subgraph K_k, there are $2^{1+\binom{n}{2}-\binom{k}{2}}$ ways of coloring K_n such that the given K_k is monochromatic. There are $\binom{n}{k}$ ways of choosing the subgraph K_k. It follows that a (k,k)-coloring exists if

$$2^{\binom{n}{2}} > \binom{n}{k} 2^{1+\binom{n}{2}-\binom{k}{2}}.$$

Using simple estimates this leads to $R(k,k) > 2^{k/2}$. This is a typical example of the method: by counting, one shows that something exists even though one cannot provide an example.

A long survey article by J. Kahn summarizes progress on several old hypergraph problems of Erdős (93 references).

In 1976 Erdős and E. G. Straus wrote a paper with the title *How Abelian is a Finite group?*. In a paper with the same title, L. Pyber solves one of the problems considered there. He proves that every group of order n contains an abelian subgroup of order at least $2^{\epsilon\sqrt{\log n}}$ for some $\epsilon > 0$.

The second volume starts with a section on *Combinatorics and Graph Theory*. The first book on graph theory, *Theorie der endlichen und unendlichen Graphen* was written by D. König in 1935. On the last page we find a proof of the infinite version of Menger's Theorem, due to the then 22 years old P. Erdős whom König refers to as a colleague! Later on, Erdős contributed completely new areas to graph theory, such as random graphs and extremal graph theory. The latter was born in 1946 when the Erdős - Stone Theorem appeared : If n is sufficiently large, then every graph of order n and size at least $(1-1/r+\epsilon)\binom{n}{2}$ contains $K_{r+1}(t)$ (the complete $(r+1)$-partite graph with t vertices in each class). Actually, in this journal, we should consider the origin of extremal graph theory to be Problem 28 in *Wiskundige Opgaven* **10** (1907). The problem by W. Mantel states that a graph on n vertices with more than $\frac{1}{4}n^2$ edges contains a triangle.

The first paper in this section is a survey of the collaboration of Erdős with R. J. Faudree, C. C. Rousseau, and R. H. Schelp during a period of 20 years, which resulted in more than 40 papers. The remaining eleven papers in this section are actually all research papers, mainly on graph theory.

The following section contains more extremal graph theory and also Ramsey Theory. The paper *Paul Erdős' Influence on Extremal Graph Theory* by M. Simonovits is the longest of the book, with 45 pages (and 181 references, including 49 papers by Erdős). When Erdős first returned to Hungary after World War II, he happened to visit a class in a kindergarten. There he met a five year old boy, who nearly 50 years later (by then coauthor of 22 papers with Erdős) wrote this very nice survey.

The next authors are the editors, R. L. Graham and J. Nešetřil. Their purpose is to demonstrate the influence of Erdős on what is now known as Ramsey Theory (98 references). When this reviewer worked at Bell Laboratories in 1966, there was a call from P. Bateman in Urbana. Erdős was visiting them and he was becoming restless. Could Graham and I please come and amuse him for a week? We did and shortly afterwards my Erdős number was 1. Graham finally ended up in the top ten of Erdős' collaborators with over 25 joint papers. Several of these were on *Euclidean Ramsey Theory.* Graham later handled Erdős' financial affairs and he and his wife Fan Chung (also coauthor of many papers) even added a separate "Erdős room" to their house. One of the theorems from the famous Erdős-Szekeres paper (1935) is worth recalling here. It states that a sequence of n^2+1 distinct integers contains a subsequence with $n+1$ elements that is increasing or decreasing. The reader who does not know the proof should try to find it. It is a rewarding experience (elegance required !).

The two surveys fit in perfectly with the title of the book. They take up about two thirds of this section which ends with a number of short research papers.

The section called *Geometry* surprisingly starts with a paper by J. Aczél and L. Losonczi on extension of functional equations. The first extension theorem for the Cauchy functional equation was given in a paper by Aczél and Erdős in 1965.

In 1948, Erdős spent two months in Amsterdam. Here he worked with N. G. de Bruijn and with J. F. Koksma. The long sequence of joint papers with Rényi also starts there. The best known result of the collaboration with De Bruijn is the De Bruijn - Erdős theorem stating that in a linear space with more than one line, the number of lines is at least equal to the number of points, equality implying that any two lines meet. (Anecdote : at the seminar on the occasion of the 50th anniversary of De Bruijn's professorship, he admitted that he had never observed the misprint "combinatioral" in the title of that 1948 paper. Had anybody?) De Bruijn was an obvious author for a contribution to one of these volumes. In *Remarks on Penrose Tilings* he treats some of his results that until now had only been mentioned in lectures on the subject.

Most of the papers in this section are also research papers. One of them, "A Remark on Transversal Numbers" by J. Pach is related to Erdős' contribution to the book by König mentioned above. This was his first encounter with "infinity", the title of the last section.

Erdős and Rényi showed the paradoxical result that there is a unique countably infinite random graph. This graph and its automorphism group are described in a very nice survey by P. J. Cameron. One of the remarkable constructions is the following. Take as vertices the set of primes congruent to 1 mod 4. Join p and q by an edge if $(\frac{p}{q}) = 1$, (by quadratic reciprocity this is a proper definition). This graph is almost surely isomorphic to the graph obtained by selecting edges randomly with probability $\frac{1}{2}$ from the 2-element subsets of a countably infinite vertex set. Another remarkable fact is that the random graph has a very large automorphism group, i. e. it is highly symmetric!

In more than 50 joint papers (including a famous more than 100 page paper with R. Rado as third author) A. Hajnal and Erdős developed "partition calculus" and "transfinite combinatorics". In a 40 page survey, Hajnal describes some of the highlights of their cooperation. As a separate list of references he includes the 107 papers that Erdős wrote on set theory.

Erdős did not consider his infinite version of Menger's theorem the right extension and he made a conjecture as to what this right extension should be. In an expository paper R. Aharoni describes the development around this conjecture. The section has two more research papers and a short survey by P. Komjáth on Erdős' research on the border lines between set theory and geometry and analysis.

One of the things that Erdős was a master at was collaborating with other mathematicians. The last paper (by J. W. Grossman) analyses this collaboration and lists all mathematicians with Erdős number 1. Together, the authors of the contributions to these two volumes wrote between 300 and 400 papers with Erdős. They have indeed given a nice overview of the mathematics of this great mathematician who will certainly not be forgotten.