Function theory in Banach algebras

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In order to explain what type of function theory we shall be occupied with, we start from a well-known fact. Let

$$f(\zeta) = \alpha_0 + \alpha_1 \zeta + \alpha_2 \zeta^2 + \dots$$

be a power series with a positive radius of convergence, R, say. Then if z is an $n \times n$ matrix whose eigenvalues all have absolute value $< R$, the series

$$\alpha_0 z^0 + \alpha_1 z + \alpha_2 z^2 + \dots$$

converges to an $n \times n$ matrix $F(z)$. This function $F(z)$ of course desires and deserves to be called analytic in the region under consideration. It cries for a suitable definition of analyticity of matrix-to-matrix functions, and for investigation of the possibilities of analytic continuation. The first thing one would try is to use expressions of the type

$$\beta_0 (z - z_0)^0 + \beta_1 (z - z_0) + \beta_2 (z - z_0)^2 + \dots,$$

where z_0 is any fixed matrix and the β's are complex numbers. However, such expressions are not very suitable if z and z_0 do not commute, and it is not our intention to exclude that case.

A useful definition has been given by G. Giorgi [3], and a more intricate one, also covering multi-valued functions, by M. Cipolla [1]. Both authors operated with the Jordan normal forms of the matrix variable in accordance with the principle that all functions to be dealt with have to satisfy the functional equation $F(aza^{-1}) = aF(z)a^{-1}$, if a is any non-singular $n \times n$

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* The author started working on this subject in 1953, stimulated by some questions of G. Szekeres. A preliminary program of a theory was drafted. In order to take care of all details a colloquium was devoted to the subject at the Mathematical Centre, Amsterdam, during the academic year 1955/56. In this colloquium Dr C. G. Lekkerkerker and Dr W. Peremans took an active part. Many ideas involved in the theory got their final form during this colloquium.

A preliminary report was given in a lecture at the Tata Institute of Fundamental Research, Bombay, 1956, but the text of this lecture has not been published. An elaborate account of the theory will be given in a forthcoming joint publication with W. Peremans.

matrix. If, for a moment, we restrict ourselves to diagonal matrices, Cipolla's definition admits functions like

$$F(\{\lambda_1, \ldots, \lambda_n\}) = \{f_1(\lambda_1), \ldots, f_n(\lambda_n)\},$$

where $f_1, \ldots, f_n$ are branches of one and the same analytic function f. (The diagonal matrix with diagonal elements $\mu_1, \ldots, \mu_n$ is denoted by $\{\mu_1, \ldots, \mu_n\}$.) In this general form the definition seems to be difficult, as analyticity at a point does not always mean that the function is defined in a neighbourhood. But if we add the restriction that the same branch should be taken for f_i and f_j if $\lambda_i = \lambda_j$, we get a very useful definition, to which we shall refer in the sequel as the modified Cipolla definition. It can be transformed into an even more suitable definition involving the Cauchy integral. (For the equivalence of several definitions, see R. F. Rinehart [6].) Apart from the fact that we get rid of the normal forms, the advantage is that the latter definition can be immediately generalized to arbitrary Banach algebras (see N. Dunford [2], and the exposition in E. Hille [4], also for further references).

A Banach algebra A is a ring with unit element e, which satisfies the following requirements. Multiplication by complex numbers (scalar multiplication) is defined, such that

$$\lambda(x+y) = \lambda x + \lambda y, \quad (\lambda+\mu)x = \lambda x + \mu x, \quad (\lambda\mu)x = \lambda(\mu x),$$
$$(\lambda x)(\mu y) = (\lambda\mu)(xy), \quad 1 \cdot x = x,$$

(here λ, μ denote arbitrary complex numbers, x, y denote arbitrary elements of A). To every $x \in A$ there is attached a non-negative number $||x||$, the norm of x. If we disregard the multiplication of elements of A, the algebra is a normed complex linear space. We require that it is a Banach space (i.e. $||x|| = 0$ if and only if $x = 0$; $||\lambda x|| = |\lambda| \cdot ||x||$; $||x+y|| \leq ||x|| + ||y||$, and the space should be complete with respect to the norm). Finally, normalization and multiplication are connected by the condition that $||xy|| \leq ||x|| \cdot ||y||$, and that $||e|| = 1$.

For future reference we quote some examples of Banach algebras.

1. The algebra A_0 of all complex numbers, with norm $||\alpha|| = |\alpha|$.

2. The algebra ${}^M A_n$ of all complex $n \times n$ matrices (the upper index stands for the word »matrix»), with the norm

$$||a|| = \max_i \Sigma_j |\alpha_{ij}|, \quad \text{if } a = (\alpha_{ij})_{i,j=1,\ldots,n}.$$

3. The direct product $A_0 \times A_0$ of A_0 with itself, consisting of all pairs (α,β), with the norm $||(\alpha,\beta)|| = \max(|\alpha|, |\beta|)$. It is isomorphic to the subalgebra of all diagonal matrices of ${}^M A_2$.

In a Banach algebra A, an element a is called regular if there is an element $b \in A$ with $ab = ba = e$. The set of all complex numbers such that $a - \lambda e$ is singular, is called the spectrum $\sigma(a)$ of a. So if $A = {}^M A_n$, $\sigma(a)$ is the set of eigenvalues of a.

It is well-known that $\sigma(a)$ is always a non-empty compact set in the complex plane. This is about all that can be said about the spectra, for every non-empty compact set is the spectrum of a suitable element in a suitable Banach algebra.

The spectrum $\sigma(a)$ depends continuously on a in the following relatively weak sense. If O_0 is any neighbourhood[1]) of $\sigma(a)$, then there is a neighbourhood O of a such that $\sigma(z) \subset O_0$ for all $z \in O$. It is a more difficult question whether the spectra are continuous if we adopt the Fréchet-Hausdorff topology in the set of all compact point-sets in the plane (see J. D. Newburgh [5]). The question is equivalent to the following one:

If $z_n \to z$ in A, and if none of the $\sigma(z_n)$ has any point inside the circle $|\zeta - \alpha| < \varepsilon$, then can it happen that $\alpha \in \sigma(z)$? This is definitely impossible if A is commutative, and also if $A = {}^M A_n$. In the general case the problem seems to be unsolved.

We can now phrase N. Dunford's definition of analyticity (see Hille [4], p. 122). Let a be an element of the Banach algebra A, and let O_0 be an open set covering $\sigma(a)$. Let $f(\zeta)$ be a locally analytic function on O_0 (i.e. on every component of O_0 it is an ordinary analytic function). Then we consider

$$F(z) = \frac{1}{2\pi i} \int (\lambda e - z)^{-1} f(\lambda)\, d\lambda, \tag{1}$$

where the integration is taken along a finite set of oriented closed curves in O_0 which are such that every point of $\sigma(a)$ is encircled, in total, just once (in the positive sense) by these curves. For simplicity we shall merely speak of an integration path encircling $\sigma(a)$, in order to indicate such a set of curves. The letter z stands for an element of A in a neighbourhood O of a, chosen such that for every $z \in O$ the spectrum $\sigma(z)$ is still encircled by the path. Then the value of $F(z)$, defined by (1), is independent of the choice of the path, as long as it lies in O_0 and encircles $\sigma(z)$. (In cases where $\sigma(z)$ is a finite set, it is merely the residue sum of $(\lambda e - z)^{-1} f(\lambda)$.)

Now a function F of an open set $S \subset A$ into A is called analytic at a point $a \in S$, if in some neighbourhood of a the function can be represented in the form (1), with a suitable f.

In the matrix case this definition coincides with the modified Cipolla definition apart from a minor detail. The two definitions would coincide if we would require that the function elements f, defined in the various components of O_0, all stem from branches of one and the same multi-valued

[1]) The notion neighbourhood of $\sigma(a)$ is taken in the sense of an open set covering $\sigma(a)$.

analytic function. However, it seems better to drop this condition as it is preferable to have a purely local definition of analyticity.

Against Dunford's definition one could raise the objection that it is a very complicated and seemingly not very natural one. It can be replaced, however, by an equivalent definition which is more transparent, although Dunford's is easier to handle in most applications.

Let $r(\zeta)$ be a rational function, i.e. the quotient of two polynomials $p(\zeta)$ and $q(\zeta)$, with complex coefficients. Then, if $z \in A$, we can define $r(z)$ as $p(z)\cdot\big(q(z)\big)^{-1}$, provided that $q(z)$ is a regular element of A (or, what amounts to the same, provided that no pole of $r(\zeta)$ belongs to $\sigma(z)$). We shall now explain analyticity in terms of uniform limits of sequences of rational functions. A function $F(z)$ defined in an open set O of A, with values in A, is called *strongly analytic* on O if there is a sequence $r_1, r_2, \ldots$ of rational functions such that $r_n(z)$ tends to $F(z)$ for all $z \in O$, with locally uniform convergence. $F(z)$ is called analytic at $a \in A$, if it is defined and strongly analytic in a neighbourhood of a. When showing the equivalence of this definition with Dunford's in one direction, we need the Runge-Montel theorem, which (in our present terminology) states that every locally analytic function on an open set O_0 of the complex plane is strongly analytic on O_0. In the other direction we proceed by proving that if $r_n(z)$ tends to a limit $F(z)$ in some neighbourhood of a, then $r_n(\zeta)$ tends to a limit function $f(\zeta)$ in some neighbourhood of $\sigma(a)$. F and f determine each other uniquely (as far as we have to do with the finite number of components of O_0 which contain parts of $\sigma(a)$).

The reason that we cannot use polynomials instead of rational functions lies in the fact that in our case the set O_0 can have components which are not simply connected, e.g. if A has elements whose spectrum consists of a closed curve.

Our new definition is quite natural: we of course want rational functions (with scalar coefficients) to be analytic, and we want uniform limits of analytic functions to be analytic. Furthermore, our definition immediately furnishes theorems about sums, products and quotients of analytic functions, and about analytic functions of an analytic function. Notice, however, that the concept is quite limited: not even a constant needs to be analytic (example: in $A_0 \times A_0$ the constant $(0,1)$ is not analytic at the point $(0,0)$, for if F is analytic, then the value of $F\big((0,0)\big)$ is scalar, i.e. a scalar multiple of the unit element $(1,1)$). The value of $F(a)$ always commutes with a, because all $r_n(a)$'s do. The spectra of a and $F(a)$ are related by Dunford's Spectral Mapping Theorem:

$$\sigma\big(F(a)\big) = f\big(\sigma(a)\big),$$

if F is analytic at a, and f is the corresponding function at $\sigma(a)$.

Naturally, $F(z)$ is called analytic in an open set $O \subset A$, if it is analytic at every point of O. However, it need not be strongly analytic on O (and a similar effect can be explained in terms of Dunford's definition). We give a counterexample in $A_0 \times A_0$. Consider the curve described by $(e^{i\varphi}, -e^{i\varphi})$, if φ runs through the interval $0 \leq \varphi \leq \pi$. In a certain neighbourhood O of this curve we can define $\log z$, such that $\log (e^{i\varphi}, -e^{i\varphi}) = (i\varphi, i\varphi + i\pi)$. This function can easily be shown to be analytic on O. If it were strongly analytic, $\log (1, -1)$ and $\log (-1, 1)$ (corresponding to $\varphi = 0$ and $\varphi = \pi$) would have the same spectrum, which is not the case.

In Dunford's terminology this example means that a one-valued function F in a region of A need not correspond to a one-valued function f in some region of A_0.

The following question, which has some theoretical significance, is quite difficult. If O is a neighbourhood of $a \in A$, does there exist a fixed neighbourhood O_0 of $\sigma(a)$ such that every function F which is analytic in O corresponds to a function f around $\sigma(a)$ that can be continued analytically and one-valued throughout O_0? An equivalent question is whether there is a fixed neighbourhood O^* of a such that every function that is analytic in O, is automatically strongly analytic in O^*.

The answer to this question is affirmative in ${}^{M}A_n$, and more generally in every Banach algebra in which every spectrum is locally connected, but there do exist algebras for which the answer is negative.

In order to get a clearer insight into the relations in the large between F and f, and in order to be able to study multi-valued functions, we shall use the notion of A-manifolds, which constitute the direct generalization of the Riemann surface. For it is only on the Riemann surface that questions about ordinary multi-valued functions can adequately be studied. It is so natural to generalize this concept that it seems strange that the generalization to Banach algebras had not been undertaken before. The reason may be that already in the theory of one-valued functions there are some unpleasant facts. But this means that the manifolds are even more necessary for accurate description of what happens than they are already in ordinary function theory.

A first deviation from ordinary function theory is that if a function F is analytic but not constant, then the image of an open set under F need not be open. This is not so surprising, however, since the function f can be constant in one part of the plane and non-constant somewhere else. Example in $A_0 \times A_0$: The function $F\big((\alpha, \beta)\big) = (\alpha, 0)$ is analytic at any point where $\alpha \neq \beta$. But even if f is nowhere constant it need not be true: In ${}^{M}A_2$ the function $F(z) = z - z^2$ maps a neighbourhood O of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ into a neighbourhood of the zero matrix. If O is chosen so small that no $z \in O$ has a double eigenvalue, then the image $F(O)$ fails to be open, since it does not contain the matrices $\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$, however small t may be (every matrix z in O can be transformed into the diagonal form, and so the same thing holds for $F(z)$).

We have to be very careful with one-to-one analytic mappings. Even if an analytic function F provides a topological mapping of an open set onto an open set, then the inverse function need not be analytic. An example can already be given in $A_0 \times A_0$: Let k be a positive number such that the function $\zeta - \zeta^2$ is schlicht in the circle $|\zeta| < k$. Let O be the set of all points (α, β) of $A_0 \times A_0$ with $|\alpha| < k$, $|1 - \beta| < k$. Then $z - z^2$ gives a topological mapping of O onto a neighbourhood of the origin, but the inverse function is not analytic at the origin (if G is analytic at $(0,0)$, its value at $(0,0)$ should be scalar, and $G\big((0,0)\big) = (0,1)$).

It was already mentioned that power series expansions (with scalar coefficients) about a point a are useless in non-commutative algebras. But even in the commutative case it is not true that every analytic function F, analytic around $a \in A$, can be expanded as a power series in powers of $z - a$. It is true, however, if a is a scalar multiple of e.

Next we mention that if F is analytic around a, and if $z_n \to a$, $F(z_1) = F(z_2) = \ldots = 0$, then F need not vanish identically. An example is obtained by taking $a = 0$, $F(z) = z^2$ in ${}^{M}A_2$, with

$$z_n = \begin{pmatrix} 0 & n^{-1} \\ 0 & 0 \end{pmatrix}.$$

Finally we mention that in non-commutative algebras there is not much to be expected from integration along curves in A, neither from differentiation with respect to a variable point of A. Integrals of course do play a part in our theory, but they are always taken along curves in the spectral plane.

As a consolation, we mention a few theorems of classical function theory which do have a generalization to Banach algebras.

a. The maximum modulus principle (only in its weak form, for a function having constant norm in a region need not be constant. Example in $A_0 \times A_0$: $F\big((\zeta_1, \zeta_2)\big) = (\zeta_1, 3)$ is analytic in the region $|\zeta_1| < 1$, $|\zeta_2 - 2| < 1$, and in this region it has the constant norm 3).

b. Liouville's theorem that a bounded entire function is a constant; it can even be shown that this constant is scalar.

c. The theorem that analytic continuation is possible in at most one way.

d. The theorem that a function f, analytic at a, with $f'(a) \neq 0$, provides a conformal mapping of a neighbourhood of a. The generalization is as follows: If F is analytic at $a \in A$, if f is the corresponding function in a neighbourhood of $\sigma(a)$, if, furthermore, $f(\alpha) \neq f(\beta)$ whenever $\alpha \in \sigma(a)$, $\beta \in \sigma(a)$, $\alpha \neq \beta$, and if finally $f'(\alpha) \neq 0$ $(\alpha \in \sigma(a))$, then F provides a one-to-one mapping of some neighbourhood of a onto a neighbourhood of $F(a)$, and this mapping is analytic in both directions.

The way from the algebra A with its transitive notion of analyticity, to the A-manifold is a well-known one in modern mathematics. An A-manifold M is a Hausdorff topological space, covered by a collection of open sets Ω_i (i runs through some index set S, which need not be finite or countable). Every Ω_i is mapped by a topological mapping φ_i onto an open set O_i of A. These φ_i's are called uniformizing parameters, or parameters for short. The crucial condition is that if a point $P \in M$ belongs to any intersection $\Omega_i \cap \Omega_j$, then there is an analytic function F around $\varphi_i(P)$ such that $\varphi_j(Q) = F(\varphi_i(Q))$ for all Q in some neighbourhood of P. Notice that an A_0-manifold is a Riemann surface, or rather a collection of Riemann surfaces, since we did not require connectivity.

On behalf of this condition we can call a function of M to A analytic if it is, at each point of M, an analytic function of one of the parameters defined at that point.

It should be remarked that the A-manifolds are generally not so homogeneous as in the case of A_0. The reason is that the value $F(a)$ of an analytic function at a point $a \in A$ cannot be any arbitrary element of A. For example: if a has a finite spectrum, then $F(a)$ has a finite spectrum; if a is scalar, then $F(a)$ is scalar, etc. And notions about points of A which are invariant under bi-analytic mapping, can directly be carried over to notions about points on an A-manifold. For example, on an ${}^{M}A_n$-manifold we can speak about scalar points, diagonal points, triangular points, symmetrical points.

We shall give some examples of A-manifolds.

1. A itself is an A-manifold, if we take just one Ω $(=A)$, and if for the φ we take the identity.

2. From $A_0 \times A_0$ we make a manifold M_1 as in the previous example. However, we can make a second manifold M_2 by taking a different set of parameters. We take two Ω's. Ω_1 consists of all non-scalar elements of $A_0 \times A_0$, i.e. the set of all (α, β) with $\alpha \neq \beta$. For φ_1 we take the identity. Ω_2 covers the set of scalar points:

$$\Omega_2 = \{(\alpha, \beta) \mid |\alpha - \beta| < 1\}.$$

We define φ_2 by

$$\varphi_2((\alpha, \beta)) = (\alpha, \beta+2).$$

The effect is that the value of a parameter is nowhere scalar, so that M_2 does not contain scalar points. This gives a peculiar difference between

M_1 and M_2. The function $F\big((\alpha, \beta)\big) = (0, 1)$ is analytic on M_2, but in M_1 it is not analytic at a scalar point. The mapping $(\alpha, \beta) \to (\alpha, \beta+1)$ of M_2 onto M_1 is analytic, but the inverse mapping is not.

3. The complex plane can be extended to the complex sphere, to the effect that the fractional linear mappings $\zeta \to (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$ $(\alpha\delta - \beta\gamma \neq 0)$ can be extended to analytic mappings of the sphere into itself. The similar thing can be done with a Banach algebra A, by considering formal quotients $(\alpha z + \beta)/(\gamma z + \delta)$ and defining the obvious identifications. We thus get a space that can be parametrized in an obvious way, and the resulting A-manifold will be called the A-sphere. We can embed A into the A-sphere by $z \to (1 \cdot z + 0)/(0 \cdot z + 1)$, and the points of the A-sphere which are not covered in this embedding will be called improper points.

The A-sphere can be used for defining meromorphic functions on an A-manifold M, as analytic mappings of M into the A-sphere, and this provides us with the means for the description of polar singularities of a function. These, however, do not always have the isolated nature like in A_0. For the improper points on the A-sphere are usually not isolated. The set of improper points can even have inner points, viz. if A contains elements whose spectra have inner points.

Another surprise is, that the ${}^M A_n$-sphere is not compact (in the Bolzano-Weierstrass sense) if $n > 1$, the set of matrices $\begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ $(k = 1, 2, 3, \ldots)$ having no limit point on the sphere.

4. The following example is not very exciting, since it gives an A-manifold which is just a piece of A itself. It is mentioned here as a preparation to the next example.

Let O_0 be an open set in the complex plane, and let $M(O_0)$ be the set of all $a \in A$ such that $\sigma(a) \subset O_0$. Then $M(O_0)$ is, with the identity as the only parameter, again an A-manifold. This $M(O_0)$ need not be connected. For example, take $A = {}^M A_2$, and let O_0 consist of two disjoint regions O_{01} and O_{02}. Now $M(O_0)$ consists of three pieces: (i) the set of all matrices having two eigenvalues in O_{01}, (ii) the set of all matrices having two eigenvalues in O_{02}, (iii) the set of all matrices having one eigenvalue in O_{01} and one in O_{02}. Notice that if we replace ${}^M A_2$ by $A_0 \times A_0$, then the manifold mentioned under (iii) splits into two components.

Returning to the case of a general O_0 and a general A, we notice that every function f that is analytic on O_0 generates a function F that is analytic on $M(O_0)$, and that we always have $\sigma\big(F(a)\big) = f\big(\sigma(a)\big)$.

5. If R is an arbitrary A_0-manifold (i.e. a collection of Riemann surfaces), we can obtain an A-manifold $M(R)$ in such a way that the relation between O_0 and $M(O_0)$ is a special case. It can be done as follows. On R we choose an open set Ω_0 which is such that it can be mapped by an analytic 1-1-

mapping φ onto a set $\varphi(\Omega_0) = O_0$ in A_0. We next construct $M(O_0) = O \subset A$. If we consider all possible Ω_0's, the corresponding O's can be fitted together to an A-manifold[1]).

If f is a function, analytic throughout R, then on $M(R)$ we can construct the corresponding analytic function F, in analogy with the case discussed in example 4. Moreover, to every point P of $M(R)$ we can attach a point set of R, which we can call the spectrum $\sigma(P)$, and which has the property that, for all f, we have $\sigma\big(F(P)\big) = f\big(\sigma(P)\big)$ (notice that $F(P) \in A$, whence $\sigma\big(F(P)\big)$ is a spectrum in the ordinary sense).

Just now we discussed a mapping f of R into A_0, from which we constructed a mapping F of $M(R)$ into A. This can be generalized as follows: Let R and R^* be A_0-manifolds, and let f be an analytic mapping of R into R^*. Then f induces[2]) an analytic mapping F of $M(R)$ into $M(R^*)$, and we have the generalized spectral mapping theorem

$$\sigma\big(F(P)\big) = f\big(\sigma(P)\big).$$

Notice that P is a point of $M(R)$, $F(P)$ a point of $M(R^*)$, $\sigma\big(F(P)\big)$ a point set on R^*, $\sigma(P)$ a point set on R, and $f\big(\sigma(P)\big)$ again a point set on R^*.

This construction includes the construction of meromorphic functions on $M(R)$, for if R^* is the complex sphere then it can be shown that $M(R^*)$ is the A-sphere.

We mention another simple example of the relation of R to $M(R)$. Let $A = A_0 \times A_0$, and let R consist of two planes. Then $M(R)$ can be shown to consist of four pieces. Two of them are of the type M_1, and two are of the type M_2 (with the notation used in the second example of an A-manifold).

As A-manifolds of the type $M(R)$ are so easily accessible, we of course propose the question whether a general A-manifold M can be embedded into an $M(R)$ (for sufficiently small pieces of M this is trivial). Unfortunately, the answer is negative; there is a counterexample, already working in $A_0 \times A_0$, by W. Peremans. The fact that this M cannot be embedded is proved by showing that there are no non-constant meromorphic functions on M. In a way these non-embeddable M's behave like non-Hausdorffian

[1]) It would take some time to describe how this fitting together actually works; there is, however, only one reasonable way to do it. The proof that $M(R)$ is an A-manifold is straightforward, apart from the proof that $M(R)$ has the Hausdorff property. A much simpler proof for this detail would be possible if we were allowed to assume spectral continuity in the strong sense.

[2]) A peculiar restriction has to be made here, which comes into effect if A contains elements whose spectra contain inner points. We have to require that for $P \in M(R)$, the set $f(\sigma(P))$ still has a neighbourhood that can be mapped conformally onto an open set of the complex plane. This is not the case, e.g., if R^* is the sphere and if $f(\sigma(P))$ covers the entire sphere.

Riemann surfaces. The following theorem is typical: Let M be an A-manifold. Let $F_1, \ldots, F_k$ be a finite collection of functions, analytic throughout M, and assume that at each point P one of the F_j ($1 \leq j \leq k$) can be used as a parameter. Then there is an A_0-manifold R, with analytic functions $\varphi_1, \ldots, \varphi_k$, which give rise to analytic functions $\Phi_1, \ldots, \Phi_k$ on $M(R)$, and there is an analytic mapping χ of M into $M(R)$, such that $F_j = \Phi_j \chi$ for all j. The mapping is locally bi-analytic. Two points P and Q of M are mapped onto the same point of $M(R)$ if and only if neighbourhoods of P and Q admit a bi-analytic mapping ψ (with $\psi(P) = Q$) such that, for each j, $F_j = F_j \psi$ around P.

The theorem remains true if M is non-Hausdorffian, and in particular it is true if M is a non-Hausdorffian Riemann surface.

Fortunately, our M's are $M(R)$'s in most applications, because the complete analytic continuation of a given function element in A can always be described in terms of an $M(R)$, and in fact it always suffices to take for R an A_0-manifold with only a finite number of components.

Our original concept of analyticity can now be generalized by taking two analytic functions z and w on an $M(R)$, and considering w as a multi-valued function of z. However, owing to some of the peculiarities already described in this lecture, it is not always wise to consider the A-manifold as a covering manifold of the z-»plane», like we do in ordinary function theory, for by that procedure the topological structure becomes obscure[1]).

We finally mention that the general case of Cipolla's definition can be obtained from the generalized analytic function, and it appears that our present definition also gives cases not covered by Cipolla. It should be noted that Cipolla defined »values of an analytic function» like $w = z^{\frac{1}{2}}$, without giving any definition of analyticity.

With these superficial remarks and examples I hope to have succeeded in giving you the impression that this type of function theory is somewhat entangled, but also that it would be much more so if we would not use A-manifolds, both for local and for global purposes.

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[1]) A very »regular» example is the following one. Let, in M_2 (see the second example of an A-manifold) z and w be defined by $z((\alpha, \beta)) = (\alpha, \beta)$, $w((\alpha, \beta)) = (\alpha, \beta + 1)$. Now w is a continuous and one-valued function of z. It is, however, not analytic throughout $A_0 \times A_0$ in the original sense.

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