Spherical harmonics and combinatorics

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SPHERICAL HARMONICS AND COMBINATORICS

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Abstract. A quick introduction to spherical harmonics, the addition theorem and Gegenbauer polynomials, leading to the definitions and some theorems for spherical codes and designs. Analogously, the discrete sphere leads to Hahn polynomials and t-designs.

§1. Spherical harmonics

Let hom(k) denote the linear space of the homogeneous polynomials in d variables of degree k.

Lemma 1.1. $\dim \mathrm{hom}(k) = \binom{d+k-1}{d-1}$.

The Laplace operator

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \; : \; \mathrm{hom}(k) \to \mathrm{hom}(k-2)$$

is a map onto. Define the space of the harmonic polynomials of degree k by

$$\mathrm{harm}(k) := \ker \Delta \; .$$

Lemma 1.2. $\mathrm{hom}(k) \cong \mathrm{harm}(k) \oplus (\underline{x},\underline{x})\, \mathrm{hom}(k-2)$.

From now on we restrict our polynomials to the unit sphere in $\mathbb{R}^d$

$$\Omega := \{x \in \mathbb{R}^d \mid x_1^2 + \ldots + x_d^2 = 1\} \; ,$$

with standard measure ω. We define

Pol(k), the linear space of the polynomials in d variables, of degree $\leq$ k, restricted to Ω ,

Hom(k), the linear subspace of Pol(k) consisting of the homogeneous polynomials of degree k.

Harm(k), the linear subspace of Hom(k) consisting of the harmonic polynomials of degree k.

For functions f and g we use the inner product

$$\langle f,g\rangle := \frac{1}{\omega}\int_{\Omega} f(\xi)g(\xi)d\omega(\xi) .$$

Lemma 1.3. $\quad \mathrm{Pol}(k) \cong \mathrm{Hom}(k) \perp \mathrm{Hom}(k-1)$.

Indeed, since we work on the sphere we may put $1 = (\underline{x},\underline{x})$ for free.

Lemma 1.4. $\quad \mathrm{Hom}(k) \cong \mathrm{Harm}(k) \perp \mathrm{Hom}(k-2)$.

Theorem 1.5. $\quad \mathrm{Pol}(k) = \mathrm{Harm}(k) \perp \mathrm{Harm}(k-1) \perp \ldots \perp \mathrm{Harm}(1) \perp \mathrm{Harm}(0)$.

Every polynomial restricted to the sphere has a unique orthogonal decomposition into spherical harmonics.

Corollary 1.6. $\dim \mathrm{Pol}(k) = \binom{d+k-1}{d-1} + \binom{d+k-2}{d-1}$,

$$\dim \mathrm{Hom}(k) = \binom{d+k-1}{d-1} ,$$

$$\dim \mathrm{Harm}(k) = \binom{d+k-1}{d-1} - \binom{d+k-3}{d-1} .$$

With the right topology we also have

$$L^2(\Omega) \cong \sum_{i=0}^{\infty} \perp \operatorname{Harm}(i) .$$

§2. Zonal spherical harmonics

For any linear functional $\ell(f)$ defined on Harm(k) there exists a unique $\ell \in \operatorname{Harm}(k)$ such that

$$\ell(f) = \langle \ell, f\rangle , \quad f \in \operatorname{Harm}(k) .$$

Fix $\xi \in \Omega$ and define a linear functional on Harm(k) by

$$f \longmapsto f(\xi), \quad f \in \operatorname{Harm}(k) .$$

By the property above there exists a unique $Q_k(\xi,\cdot) \in \operatorname{Harm}(k)$ such that

$$\langle f, Q_k(\xi,\cdot)\rangle = f(\xi), \quad f \in \operatorname{Harm}(k) .$$

This $Q_k(\xi,\cdot)$ is called the k^{th} *zonal spherical harmonic* with pole ξ . It has the reproducing property, and it may be viewed as the projection onto Harm(k) of the Dirac function $\delta_\xi(\cdot)$ with pole ξ on Ω .

Theorem 2.1. $Q_k(\sigma\xi,\sigma\eta) = Q_k(\xi,\eta)$ for $\sigma \in O(d)$.

Proof. Let σ denote any orthogonal transformation of $\mathbb{R}^d$. Put $\zeta = \sigma\eta$ in

$$\int_\Omega f(\eta)Q_k)(\sigma\xi,\sigma\eta)d\omega(\eta) = \int_\Omega f(\sigma^{-1}\zeta)Q_k(\sigma\xi,\zeta)d\omega(\zeta) =$$
$$= f(\sigma^{-1}\sigma\xi) = f(\xi) .$$

This yields the result by uniqueness of Q_k .

Corollary 2.2. $Q_k(\xi,\cdot)$ is constant on parallels $\perp \xi$.

Proof. Take $\sigma \in O(d-1)$, $\sigma\xi = \xi$, then $Q_k(\xi,\eta) = Q_k(\xi,\sigma\eta)$.

Corollary 2.3. $Q_k(\xi,\eta)$ depends on (ξ,η) only.

Hence we may write

$$Q_k(\xi,\eta) = Q_k(z) \text{ with } z = (\xi,\eta) .$$

These $Q_k(z)$ are the Gegenbauer polynomials, cf. §3.

Addition

Theorem 2.4. $\quad Q_k(\xi,\eta) = \sum_{i=1}^{\mu_k} f_{k,i}(\xi) f_{k,i}(\eta)$,

where $f_{k,1},\ldots,f_{k,\mu_k}$ denotes an orthonormal basis of Harm(k).

Proof. Express $Q_k(\xi,\cdot)$ in the basis:

$$Q_k(\xi,\cdot) = \sum_{i=1}^{\mu_k} \langle Q_k(\xi,\cdot), f_{k,i}\rangle f_{k,i} = \sum_{i=1}^{\mu_k} f_{k,i}(\xi) f_{k,i}$$

and substitute η.

Corollary 2.5. $Q_k(1) = \dim \mathrm{Harm}(k)$.

Proof. $Q_k(1) = \sum_{i=1}^{\mu_k} f_{k,i}{}^2(\xi) =$

$$= \sum_{i=1}^{\mu_k} \frac{1}{\omega} \int_\Omega f_{k,i}^2(\xi)\, d\omega(\xi) = \sum_{i=1}^{\mu_k} 1 = \mu_k .$$

Corollary 2.6. $Q_k = H_k H_k{}^t$,

where $$Q_k = [Q_k(x,y)]_{\underline{x},\underline{y}\in X}$$

and $$H_k = [f_{k,i}(\underline{x})]_{\underline{x}\in X,\ i=1,\dots,\mu_k}$$

and X is a finite set of points on the sphere Ω .

Example for d = 2

Harm(k) has orthonormal basis $\sqrt{2}\cos k\Theta$, $\sqrt{2}\sin k\Theta$. The addition theorem reads

$$2\cos k\varphi(\xi)\cos k\varphi(\eta) + 2\sin k\varphi(\xi)\sin k\varphi(\eta) =$$

$$= 2\cos k(\varphi(\xi) - \varphi(\eta)) = 2\cos k\Theta ,$$

with $\cos\Theta = (\xi,\eta)$. Hence

$$Q_k(\cos\Theta) = 2\cos k\Theta .$$

§3. Gegenbauer polynomials

The Gegenbauer polynomials in one variable z, $Q_k(z)$, $k = 0,1,2,\dots, -1 \le z \le 1$, form a family of orthogonal polynomials with respect to the weight function $(1 - z^2)^{\frac{1}{2}(d-3)}$. Indeed, the reproducing property yields

$$\frac{1}{\omega}\int_\Omega Q_k(\xi,\eta)Q_\ell(\eta,\zeta)d\omega(\eta) = \delta_{k,\ell}Q_k(\xi,\zeta) .$$

Put $\xi = \zeta$ and $(\xi,\eta) = z$ then

$$\frac{\omega_{d-1}}{\omega_d} \int_{-1}^{1} Q_k(z)Q_\ell(z)\ (1 - z^2)^{\frac{1}{2}(d-3)} dz = \delta_{k,\ell} Q_k(1) .$$

The Gegenbauer polynomials satisfy the recurrence relation

$$\frac{k + 2}{d + 2k + 2} Q_{k+2}(z) = zQ_{k+1}(z) - \frac{d + k - 2}{d + 2k - 2} Q_k(z) .$$

The first few polynomials are

$$Q_0(z) = 1,\ Q_1(z) = dz,\ Q_2(z) = \frac{1}{2}(d + 2)(dz^2 - 1) ,$$

$$Q_3(z) = \frac{1}{6} d(d + 4)((d + 2)z^3 - 3z) ,$$

$$Q_4(z) = \frac{1}{24} d(d + 6)((d + 2)(d + 4)z^4 - 6(d + 2)z^2 + 3) .$$

Any polynomial $F(z)$ has a unique Gegenbauer expansion

$$F(z) = \sum_{k=0}^{\deg} f_k Q_k(z) .$$

Let X be any finite subset of Ω, and let A be the set of the inner products $\neq 1$ which occur among the elements of X.

In the following a key tool will be to find an appropriate polynomial $F(z)$, to express in two ways the quantity

$$\sum_{\underline{x},\underline{y}\in X} F((\underline{x},\underline{y})) = |X|\ F(1) + \sum_{\alpha\in A} \text{frqu}\ \ F(\alpha)\ ,$$

$$\sum_{\underline{x},\underline{y}\in X} F((\underline{x},\underline{y})) = f_0\ |X|^2 + \sum_{k=1}^{\deg} f_k \sum_{\underline{x},\underline{y}\in X} Q_k(\underline{x},\underline{y})\ ,$$

and to observe that the addition theorem implies that

$$\sum_{\underline{x},\underline{y}\in X} Q_k(\underline{x},\underline{y}) \geq 0\ .$$

§4. Spherical codes

Let $A \subset [-1,1[$.

A *spherical* A-*code* is a finite subset X of the unit sphere Ω such that

$$(\underline{x},\underline{y}) \in A \quad \text{for all} \quad \underline{x} \neq \underline{y} \in X\ .$$

<u>Theorem 4.1.</u> If $|A| = s$ then for any A-code X:

$$|X| \leq \binom{d+s-1}{d-1} + \binom{d+s-2}{d-1}\ .$$

<u>Example.</u> $n \leq \frac{1}{2}\,d(d+3)$ for $s = 2$.

<u>Proof</u>. For each $y \in X$ define

$$F_{\underline{y}}(\underline{\xi}) := \prod_{\alpha\in A} \frac{(\underline{y},\underline{\xi}) - \alpha}{1-\alpha}\ , \quad \underline{\xi} \in \Omega\ .$$

These are $|X|$ polynomials of degree $\leq s$, independent since

$$F_{\underline{y}}(\underline{x}) = \delta_{y,x} \quad \text{for} \quad \underline{x},\underline{y} \in X\ .$$

Hence $|X| \leq \dim \mathrm{Pol}(s)$.

Sometimes we can do better. Call $F(z)$ *compatible* with the set A if

$$F(\alpha) \le 0 \quad \text{for all} \quad \alpha \in A .$$

The following theorem is a direct consequence of the remarks of §3.

Theorem 4.2. Let $F(z)$ be compatible with A, and let its Gegenbauer coefficients satisfy $f_0 > 0$ and $f_k \ge 0$. Then the cardinality of any A-code X satisfies

$$|X| \le F(1) / f_0 .$$

Example 1. $A = \{\alpha, -\alpha\}$. Take

$$F(z) = \frac{z^2 - \alpha^2}{1 - \alpha^2} = \frac{1 - d\alpha^2}{d(1 - \alpha^2)} + \frac{2}{d(d+2)(1-\alpha^2)} \frac{1}{2} (d+2)(dz^2 - 1) .$$

Hence $|X| \le \dfrac{d(1-\alpha^2)}{1 - d\alpha^2}$ for $\alpha^2 < \dfrac{1}{d}$.

For $\alpha = \frac{1}{3}$ this yields existing examples:

d	3	4	5	6	7	8
$\lvert X\rvert$	4	6	10	16	28	28

Example 2. $A = \{0, \frac{1}{2}, -\frac{1}{2}\}$ yields the root systems E_d

d	5	6	7	8	9
$\lvert X\rvert$	21	36	63	120	120

Example 3. The kissing number τ_d is the maximum number of nonoverlapping unit spheres that can touch a given unit sphere in $\mathbb{R}^d$. Application of the theorem to $A = \{-1 \le \alpha \le \frac{1}{2}\}$ makes it possible to determine τ_d in certain cases. For instance $\tau_8 = 240$ (Odlyzko–Sloane). Indeed,

$$F(z) = (z+1)(z+\tfrac{1}{2})^2 z^2 (z-\tfrac{1}{2}) =$$

$$= \frac{3}{320} Q_0 + \sum_{k=1}^{6} \text{pos.}\ Q_k(z)$$

yields $\tau_8 \le 240$, whereas an example with equality is known.

§5. Spherical designs

A finite subset X of the unit sphere Ω is a *spherical design* of *strength t* if

$$\frac{1}{|X|} \sum_{\underline{x} \in X} f(x) = \frac{1}{\omega} \int_{\Omega} f(\xi) d\omega(\xi) \quad \text{for all } f \in \mathrm{Pol}(t) .$$

Equivalently, if for $k = 1,2,\ldots,t$ the k^{th} moments of X equal the k^{th} moments of Ω. Equivalently, if for $k = 1,2,\ldots,t$,

$$\sum_{\underline{x} \in X} h(\underline{x}) = 0 \quad \text{for all } h \in \mathrm{Harm}(k) .$$

Equivalently, if for $k = 1,2,\ldots,t$ the characteristic matrices H_k have zero column sums.

By use of the techniques referred to above the following theorems are obtained.

Theorem 5.1. Let X be an A-code, $|A| = s$, and a $t = 2e$ - design. Then

$\dim \mathrm{Pol}(e) \leq |X| \leq \dim \mathrm{Pol}(s)$, hence $t \leq 2s$.

Moreover, if equality once then all.

Example. $d = 2,\ t = 4,\ |X| = 5;\ d = 6,\ t = 4,\ |X| = 27$.

Theorem 5.2. Let X be an antipodal $(2e + 1)$-design with s inner products $\neq \pm 1$. Then

$2 \dim \mathrm{Hom}(e) \leq |X| \leq 2 \dim \mathrm{Hom}(s)$, hence $e \leq s$.

Moreover, if equality once then all.

Example. $d = 3,\ t = 5,\ |X| = 12$ (the icosahedron).

Theorem 5.3. Let X be an A-code and a t-design. Let $F(z)$ have Gegenbauer coefficients satisfying $f_0 > 0$ and $f_k \leq 0$ for $k > t$, and let $F(1) > 0$, $F(\alpha) \geq 0$ for $\alpha \in A$. Then

$|X| \geq F(1) / f_0$.

§6. The discrete sphere

Given $v \geq 2k > 0$, the discrete sphere is defined to be the set of all k-subsets (blocks) of a v-set:

$$\Omega := \left\{ \underline{x} \in \mathbb{R}^v \mid x_1^2 + \ldots + x_v^2 = k, \quad x_i \in \{0,1\} \right\}.$$

For the discrete sphere we define:

Hom(t), the linear space of the homogeneous polynomials of degree ≤ 1 in each of the v variables, of total degree t, restricted to Ω.

Harm(t), the linear subspace of Hom(t) consisting of the polynomials vanishing under

$$\Delta := \frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_v} .$$

For the inner product

$$\langle f,g\rangle := \sum_{x\in\Omega} f(x)g(x)$$

we have the following decomposition.

<u>Theorem 6.1</u>. $\mathrm{Hom}(t) = \mathrm{Harm}(t) \perp \mathrm{Harm}(t-1) \perp \ldots \perp \mathrm{Harm}(0)$.

<u>Corollary 6.2</u>. $|\Omega| = \binom{v}{k}$, $\dim \mathrm{Hom}(t) = \binom{v}{t}$,

$$\dim \mathrm{Harm}(t) = \binom{v}{t} - \binom{v}{t-1} .$$

The following addition theorem holds for any orthonormal basis $f_{t,1},\ldots,f_{t,\mu_t}$ of Harm(t).

<u>Theorem</u> $$\sum_{i=1}^{\mu_t} f_{t,i}(\xi)f_{t,i}(\eta) = Q_t\Big((\xi,\eta)\Big) .$$

Here $Q_t(z)$ denote the Hahn polynomials, a family of polynomials in the discrete variable z, orthogonal with respect to the weight function $w(z)$:

$$z \in \{0,1,\ldots,k\} , \quad w(z) = \binom{k}{z}\binom{v-k}{z} .$$

Any polynomial $F(z)$ has a unique Hahn expansion. As in §3 a key tool will be to find appropirate $F(z)$ and to express in two ways

$$\sum_{\underline{x},\underline{y}\in X} F((x,y))$$

for a subset X of Ω.

§7. t-designs

A t-design $t-(v,k,\lambda)$ is a collection X of k-subsets (blocks) of a v-set such that each t-subset is in a constant number λ of blocks.

Examples. $2-(q^2+q+1, q+1, 1)$, the lines of $PG(2,\mathbb{F}_q)$.

$2-(35,3,1)$, the lines of $PG(3,\mathbb{F}_2)$.

$5-(24,8,1)$, the Steiner system,

the weight 8 vectors in the (24,12) Golay code.

The above definition is equivalent to the one of §5 applied to the discrete sphere. Indeed, in both definitions we require

$$\sum_{x\in X^\sigma} f(\underline{x}), \quad f \in \mathrm{Hom}(t)\ ,$$

to be constant with respect to the elements σ of a group. In §5 this is the orthogonal group $O(d)$. In the present § this is the symmetric group $\mathrm{Sym}(v)$.

Example. The 5-design property of the Steiner system is expressed in terms of the set X of blocks by

$$1 = \sum_{\underline{x}\in X} x_1 x_2 x_3 x_4 x_5 = \sum_{\underline{x}\in X} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} x_{\sigma(5)}$$

for any permutation σ of the 24 varibles x_i.

Theorem 7.1. A set of blocks X forms a t-design whenever

$$\sum_{\underline{x}\in X} h(\underline{x}) = 0, \quad \forall h \in \sum_{i=1}^{t} \mathrm{Harm}(i)\ .$$

The method of §6 leads to the following generalization of Fisher's inequality.

Theorem 7.2. $|X| \geq \binom{v}{e}$ for any 2e-design X. In the case of equality the block-intersections of X are uniquely determined.

Proof. Apply the method of §3 and 6 to

$$F(z) = (Q_e(z) + Q_{e-1}(z) + \ldots + Q_0(z))^2\ .$$

Example. The 253 blocks of 4 - (23,7,1).

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