On the differential equation yy" + 2xy' = 0

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by

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Abstract

The asymptotic behaviour for $x \to \infty$ of solutions $y(x)$ of the initial value problem $yy'' + 2xy' = 0$, $y(0) = \alpha > 0$, $y'(0) = \beta$, and also the asymptotic behaviour $\beta \to +\infty$ and for $\beta \to -\infty$ of $L := \lim_{x \to \infty} y(x)$ is investigated.

1. Introduction

In this paper we consider the following initial value problem

(1) $\qquad yy'' + 2xy' = 0 \qquad (x > 0)$,

(2) $\qquad y(0) = \alpha > 0$, $y'(0) = \beta$.

Equation (1) occurs in boundary layer theory (see [1], p. 22 - 23) and arises from the diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial C}{\partial z}\right) .$$

If $D = e^{-C}$, and if C only depends on $x := \frac{1}{2}z\,t^{-\frac{1}{2}}$, then $y := e^{-C}$ satisfies (1).

In boundary layer theory one is also interested in the boundary value problem (1), (3) with

(3) $\quad y(0) = \alpha > 0 \quad , \quad y(\infty) = L > 0 \; .$

A case of special interest is

(2') $\quad y(0) = 1 \quad , \quad y'(0) = \gamma \; ,$

since all solutions of (1), (2) are expressible in solutions of (1), (2') by means of the transformation (11) (see Section 2). We shall study the asymptotic behaviour of solutions $y(x,\gamma)$ of (1), (2') for $x \to \infty$. Moreover, we shall pay attention to the asymptotic behaviour of the limit $L(\gamma) := \lim_{x\to\infty} y(x,\gamma)$ for $\gamma \to \infty$ and $\gamma \to -\infty$, since physicists seem interested in large values of $|\gamma|$.

2. Results

In Section 3 the following fundamental results are proved. There exists a unique solution $y(\cdot,\alpha,\beta)$ of (1), (2), defined on $[0,\infty)$. This solution $y(x,\alpha,\beta)$ tends monotonically to a positive limit $L(\alpha,\beta)$ if $x \to \infty$. The transformation rule

$$(11) \qquad y(x,\alpha,\beta) = \alpha\, y(\alpha^{-\frac{1}{2}} x,\gamma) \quad , \quad L(\alpha,\beta) = \alpha\, L(\gamma) \quad , \quad \gamma = \alpha^{-\frac{1}{2}} \beta ,$$

enables us to consider only the solution $y(\cdot,\gamma)$ of (1), (2') and its limit $L(\gamma)$. This limit is a continuous increasing bijection $L : \mathbb{R} \to (0,\infty)$. Hence the boundary value problem (1), (3) has a unique solution.

In Section 4 the asymptotic behaviour of $y(x,\gamma)$ for $x \to \infty$ is determined.

$$(20) \qquad y(x,\gamma) = L - A\, \mathrm{erfc}\,(L^{-\frac{1}{2}} x) + \mathcal{O}(x^{-2} \exp[-2\,L^{-1} x^2]) \qquad (x \to \infty) ,$$

where

$$A = \tfrac{1}{2}\, \pi^{\frac{1}{2}}\, L^{\frac{1}{2}}\, \gamma \exp\left[2 \int_0^\infty s(L^{-1} - (y(s,\gamma))^{-1})\, ds \right] .$$

In Sections 5 and 6 bounds for the limit $L(\gamma)$ are determined.

$$(36) \qquad 2\, L_0 \log(1 + L_0\, \gamma^2) < L(\gamma) - L_0\, \gamma^2 < 47\, L_0 \log(1 + 3(16)^{-1} \gamma^2) + 47\, L_0 \qquad (\gamma > 1.8)$$

$$(56) \qquad 1 - \gamma^{-2} < L(\gamma) \exp[\tfrac{1}{2}\gamma^2 + \tfrac{1}{2}] < 1 + \gamma^{-2} \qquad (\gamma < -7) ,$$

where $L_0 = 0.3574...$ denotes the limit of the special solution y of (1) with initial values $y(0) = 0$, $y'(0) = 1$.

3. Preliminaries

We shall investigate existence and uniqueness properties, and prove the continuous dependence of the limit as functions on the initial values.

(4) Theorem. There exists a unique solution y of the initial value problem (1), (2) which is defined on $[0,\infty)$. This solution y is positive and monotone, and $y(x)$ tends to a positive limit L if $x \to \infty$.

Proof. According to well-known existence and uniqueness theorems (see for example Sections I.1 and I.2 of [2]), there is a positive member, say δ, such that there is a unique solution, say y, of (1), (2), defined on $[0,\delta)$. Let I be the maximal interval of tbe form $[0,a)$, (possibly a may be ∞) on which y is positive and satisfies (1), (2). Dividing both sides of (1) by yy', integrating over $[0,x]$ with $x \in I$ we get

$$(5) \qquad y'(x) = \beta \exp\left[-2 \int_0^x s(y(s))^{-1} ds\right] \qquad (x \in I) ,$$

a result also correct if $y' = 0$. It follows that

$$(6) \qquad 0 \le y'(x) \operatorname{sgn} \beta \le |\beta| \qquad (x \in I) .$$

Let $a > 0$, $a \in I$. Dividing both sides of (1) by xy integrating from a to x, $x \in I$, $x > a$, taking exponentials, we find

$$(7) \qquad y(x) = y(a) \exp\left[\tfrac{1}{2}a^{-1} y'(a) - \tfrac{1}{2}x^{-1} y'(x) - \tfrac{1}{2} \int_a^x s^{-2} y'(s) ds\right] .$$

From (6) and (7) we infer that

(8) $$y(x) \lesseqgtr y(a) \exp[\tfrac{1}{2}a^{-1} y'(a)] \qquad (a < x,\ x \in I)\ ,$$

where ≤ holds if $\beta \geq 0$ and > if $\beta < 0$. It follows from (5) and (8) that y is bounded, and increasing if $\beta > 0$, and that y is bounded away from zero and decreasing if $\beta < 0$, and that $I = [0,\infty)$. Clearly $\lim_{x\to\infty} y(x)$ exists and is positive.

We shall denote the solution of (1), (2) by $y(x,\alpha,\beta)$ and its limit by $L(\alpha,\beta)$. In the case of (1), (2') we write $y(x,\gamma)$ and $L(\gamma)$. Simply by inspection we can prove

(9) <u>Theorem</u>. For every $\lambda > 0$

(10) $$y(x,\lambda^2\alpha,\lambda\beta) = \lambda^2 y(\lambda^{-1}x,\alpha,\beta) \quad , \quad L(\lambda^2\alpha,\lambda\beta) = \lambda^2 L(\alpha,\beta)\ .$$

Theorem (9) enables us to consider, without loss of generality, only the initial value problem (1), (2'). We have

(11) $$y(x,\alpha,\beta) = \alpha\, y(\alpha^{-\frac{1}{2}}x,\gamma) \quad , \quad L(\alpha,\beta) = \alpha\, L(\gamma) \quad , \quad \gamma = \alpha^{-\frac{1}{2}}\beta\ .$$

(12) <u>Theorem</u>. $L(\cdot) : \mathbb{R} \to (0,\infty)$ is an increasing continuous bijection.

<u>Proof</u>. Let $\gamma_2 > \gamma_1$. Then there is a $\delta > 0$ such that $y(x,\gamma_2) > y(x,\gamma_1)$ on $[0,\delta)$. Suppose that there exists a positive x_0 such that $y(x_0,\gamma_2) = y(x_0,\gamma_1)$ and $y(x,\gamma_2) > y(x,\gamma_1)$ for $0 < x < x_0$. Clearly $y'(x_0,\gamma_2) < y'(x_0,\gamma_1)$. Dividing (1) by y and integrating from 0 to x we find

(13) $$y'(x,\gamma) - \gamma + 2x \log y(x,\gamma) = 2 \int_0^x s \log y(s,\gamma)\,ds\ .$$

Applying (13) twice with $\gamma = \gamma_2$ and $\gamma = \gamma_1$ and substracting both results we get for $x = x_0$

$$y'(x_0,\gamma_2) - y'(x_0,\gamma_1) - \gamma_2 + \gamma_1 = 2 \int_0^x s \log (y(s,\gamma_2)/(y(s,\gamma_1))ds \ ,$$

a contradiction since both sides have different signs. It follows that $y(x,\gamma_2) > y(x,\gamma_1)$ for all $x > 0$.

In the sequel of the proof we need the formula

$$\text{(14)} \qquad \log L = \tfrac{1}{2} \int_0^\infty (\gamma - y'(s))s^{-2} ds \ ,$$

where $y = y(\cdot,\gamma)$, $L = L(\gamma)$, and which is obtained by integration of $y'y^{-1} = -\frac{1}{2} x^{-1} y''$. The integral in (14) exists since $y''(s) = -2\gamma s + o(s)$ $(s \downarrow 0)$ and $\gamma - y'(s) = \gamma s^2 + o(s^2)$ $(s \downarrow 0)$. Applying (14) to $y_2 := y(\cdot,\gamma_2)$ and $y_1 := y(\cdot,\gamma_1)$ respectively, and substracting the results we get

$$\text{(15)} \qquad \log L_2 L_1^{-1} = \tfrac{1}{2} \int_0^\infty [\gamma_2 - \gamma_1 + y_1'(s) - y_2'(s)]s^{-2} ds \ ,$$

where $L_2 := L(\gamma_2)$ and $L_1 := L(\gamma_1)$.

If $\gamma_1 > 0$ then by (5), $y_2' > \gamma_2\gamma_1^{-1} y_1'$ which implies directly $L_2 > 1 + \gamma_2\gamma_1^{-1}(L_1 - 1)$, and, by (15) and (14)

$$\log L_2 L_1^{-1} < (\gamma_2 - \gamma_1)\gamma_1^{-1} \log L_1 < (\gamma_2 - \gamma_1)\gamma_1^{-1} \log L_2 \ .$$

If $\gamma_1 = 0$ then by (5), $y_2' < \gamma_2 \exp[-x^2 L_2]$ which upon integration gives $L_2 < (1 + \frac{1}{2}\pi^{\frac{1}{2}}\gamma_2)^2$.

If $\gamma_2 = 0$ then by (5) $y_1' > \gamma_1 \exp[-x^2]$ which leads to $L_1 > 1 + \frac{1}{2}\pi^{\frac{1}{2}}\gamma_1$.

If $\gamma_2 < 0$ then by (5) $y_2' < \gamma_2\gamma_1^{-1} y_1'$ which implies

$$L_2 - L_1 < (1 - \gamma_2\gamma_1^{-1})(1 - L_1) < (1 - \gamma_2\gamma_1^{-1}) ,$$

and by (15) and (14) $\log L_2 L_1^{-1} > (1 - \gamma_1\gamma_2^{-1}) \log L_2$.

It follows that $L(\cdot)$ is continuous and strictly increasing on $\mathbb{R}$, and, moreover $L(\gamma) \to \infty$ if $\gamma \to \infty$, $L(\gamma) \downarrow 0$ if $\gamma \to -\infty$.

4. Asymptotic behaviour

First we investigate the asymptotic behaviour of solutions $y := y(\cdot,\gamma)$ with $\gamma \geq 0$. Using the inequality $y(x) \leq L := L(\gamma)$ in (5) we get

$$y'(x) \leq \gamma \exp[-x^2/L] ,$$

from which it follows by integration over $[x,\infty)$ that

$$y(x) \geq L - \tfrac{1}{2}\pi^{\frac{1}{2}} L^{\frac{1}{2}} \operatorname{erfc}(x L^{-\frac{1}{2}}) .$$

Hence

$$(16) \qquad y(x) = L + \mathcal{O}(x^{-1} \exp[-x^2/L]) \qquad (x \to \infty) .$$

It follows that

$$(17) \qquad Q := 2 \int_0^\infty s((y(s))^{-1} - L^{-1}) ds$$

exists and that

$$(18) \qquad 2 \int_x^\infty s((y(s))^{-1} - L^{-1}) ds = \mathcal{O}(x^{-1} \exp[-x^2/L]) \qquad (x \to \infty) .$$

Using (17) and (18) in (5) we get

$$(19) \qquad y'(x) = \gamma \exp[-Q - x^2/L] (1 + \mathcal{O}(x^{-1} \exp[-x^2/L])) \qquad (x \to \infty) ,$$

which upon integration from x to ∞ gives

$$(20) \qquad y(x) = L - \tfrac{1}{2}\pi^{\frac{1}{2}} L^{\frac{1}{2}} \gamma e^{-Q} \operatorname{erfc}(x L^{-\frac{1}{2}}) + \mathcal{O}(x^{-2} \exp[-2x^2/L]) \qquad (x \to \infty) .$$

Secondly we consider the case $\gamma < 0$. Then using $L < y(x) < 1$ in (5) we get $y'(x) \geq \gamma \exp[-x^2]$. By integration over $[x,\infty)$ it follows that

$$y(x) = L + \mathcal{O}(x^{-1} \exp[-x^2]) \qquad (x \to \infty) .$$

Repeating the same kind of arguments (after (16)) as in the case $\gamma \geq 0$ we arrive at the same Formula (20) except for the order term which is now $\mathcal{O}(x^{-2} \exp[-x^2(L^{-1} + 1)])$. Hence Formula (16) holds. Again by repeating the arguments after (16) we get (20).

5. Bounds for the limit if $\gamma > 0$

By (10) we have

(21) $$L(1,\gamma) = \gamma^2 L(\gamma^{-2},1) .$$

We shall prove that there are positive constants C_1, C_2 such that for $\alpha > 0$ sufficiently small

(22) $$C_1 \alpha |\log \alpha| < L(\alpha,1) - L(0,1) < C_2 \alpha |\log \alpha| .$$

For simplicity we write y, L, y_0, L_0 instead of $y(\cdot;\alpha,1)$, $L(\alpha,1)$, $y(\cdot;0,1)$, $L(0,1)$ respectively. It is easily seen that

(23) $$y_0(x) = x - x^2 + \mathcal{O}(x^3) \qquad (x \downarrow 0) .$$

Clearly, Formula (5) with $\beta = 1$ holds for y_0. It follows by (5) that initially $y'(x) > y_0'(x)$, since initially $y(x) > y_0(x)$. By standard reasoning we can conclude that

(24) $$y'(x) > y_0'(x) \quad , \quad y(x) > y_0(x) + \alpha \qquad (x > 0) .$$

For later use we shall prove the following facts:

(25) $$y_0(x) > x - x^2 + \tfrac{1}{3}x^3 > x - x^2 \qquad (0 < x \le 1)$$

(26) $$y_0(x) < \tfrac{1}{2}(1 - e^{-2x}) \qquad (x > 0)$$

(27) $$L_0 = 0.35742210059...$$

<u>Proof of (25), (26) and (27)</u>. From $y_0'' > 0$ and $y_0(0) = 0$ it follows that $y_0(x) > x\,y_0'(x)$ $(x > 0)$. From (1) we deduce $y_0'' > -2$ which leads to $y_0 > x - x^2$. Using this last result in (5) we find $y_0' > (1 - x)^2$ $(0 < x \le 1)$ whence $y_0 > x - x^2 + \frac{1}{3}x^3$ $(0 < x \le 1)$. Clearly $L_0 > y_0(1) > \frac{1}{3}$. Using $y_0(x) < x$ $(x > 0)$ in (5) we get $y_0' < e^{-2x}$ and $y_0 < \frac{1}{2}(1 - e^{-2x})$. Hence $L_0 < \frac{1}{2}$. A numerical computation gives (27). We also need

$$\text{(28)} \qquad L < (1 + 2\sqrt{\alpha})^2 L_0 \; .$$

Let $0 < a \le \frac{1}{2}$. The line $y = a - a^2 + (1 - 2a)(x - a)$ is tangent to the curve $y = x - x^2$ in the point $(a, a - a^2)$. It follows by (25) that $y_2(x) := y(x; a^2, 1 - 2a)$ equals $y_0(x)$ for some $x = b < a$ since $y_2(a) < y_0(a)$. At $x = b$ we have $y_0(b) = y_2(b)$ and $y_0'(b) > y_2'(b)$. We show that $y_0(x) > y_2(x)$ for all $x > b$. Immediately to the right of $x = b$ we have $y_0(x) > y_2(x)$. Then in some interval $(b, b + \delta)$

$$y_0'(x) = y_0'(b)\exp\left[-2\int_b^x s(y_0(s))^{-1}ds\right] > y_0'(b)\exp\left[-2\int_0^x s(y_2(s))^{-1}ds\right]$$

$$= y_0'(b)(y_2'(b))^{-1}\,y_2'(x) > y_2'(x) \; .$$

By standard arguments it follows that $y_0(x) > y_2(x)$ $(x > b)$.

Letting $x \to \infty$ we find $L(a^2, 1 - 2a) < L_0$. By (10) we have $L(a^2, 1 - 2a) = (1 - 2a)^2 L(a^2(1 - 2a)^{-2}, 1)$. Substituting $a = \alpha^{\frac{1}{2}}(1 + 2\alpha^{\frac{1}{2}})^{-1}$ we get (28).

Now we proceed with the proof of (22).

The function K defined by (29) satisfies (30).

(29) $$K(x) := y'(x)/y_0'(x) \ .$$

(30) $$K'(x)/K(x) = 2x((y_0(x))^{-1} - (y(x))^{-1}) \ , \qquad K(0) = 1 \ .$$

Using (24) and the inequalities $y_0(x) < x$, $y_0'(x) < 1$ for $x > 0$, we see that

$$K'/K > 2\alpha(y_0 + \alpha)^{-1} y_0' \ ,$$

from which it follows by integration

$$K > (1 + y_0/\alpha)^{2\alpha} \ .$$

Using (29) and integrating we get

$$y > \alpha + \alpha(2\alpha + 1)^{-1}[(1 + y_0/\alpha)^{2\alpha+1} - 1] \ .$$

Letting $x \to \infty$ we find for all $\alpha > 0$

$$L > \alpha + \alpha(2\alpha + 1)^{-1}[(1 + L_0/\alpha)^{2\alpha+1} - 1] \ .$$

Using that $L_0 < \frac{1}{2}$ we derive

(31) $$L > L_0 + 2L_0 \alpha \log(1 + L_0/\alpha) \ .$$

Now we turn to the proof of the right hand inequality of (22).

Let $u := y - y_0$. Then u satisfies

(32) $$y u'' = -2x u' - y_0'' u \quad ; \quad u(0) = \alpha \quad , \quad u'(0) = 0 \ .$$

We want an upper bound for $u(\frac{1}{4})$.

From (24) we know that $u' > 0$, and also $y > \alpha + y_0$. Moreover, we have $-y_0'' \le 2$, and, using (30), $y_0(x) > x(1 - x + \frac{1}{3}x^2) > \frac{3}{4}x$ on $[0,\frac{1}{4}]$. Hence

$$u'' \le 2(\alpha + \tfrac{3}{4}x)^{-1}u \qquad (0 \le x \le \tfrac{1}{4}) \;. \tag{33}$$

We introduce a function v as follows: $v(0) = \alpha$, $v'(0) = 0$, v satisfies (32) with equality. Then $v(x) \ge u(x)$ on $[0,\frac{1}{4}]$. This follows by standard arguments using the differential (in-)equalities for $\varphi := u'/u$, $\psi := v'/v$. Since v is convex we have $v(s) \le 4s\,v_1 + \alpha(1 - 4s)$ $(0 \le s \le \frac{1}{4})$ where $v_1 := v(\frac{1}{4})$. Hence $v''(s) \le 2(\alpha + \frac{3}{4}s)^{-1}(4s\,v_1 + \alpha(1 - 4s))$. Integrating twice we find

$$v_1 - \alpha \le \tfrac{1}{3}(v_1 - \alpha) - \tfrac{256}{9}\alpha(v_1 - \alpha - \tfrac{3}{16})[(\tfrac{1}{4} + \tfrac{4}{3}\alpha)\log(1 + 3(16\alpha)^{-1}) - \tfrac{1}{4}] \;,$$

from which an upper bound for v_1 arises, thus also for $u(\frac{1}{4})$. We get

$$u(\tfrac{1}{4}) < \alpha + \alpha\log(1 + 3(16\alpha)^{-1}) \;. \tag{34}$$

Finally we shall show that $u(\infty)/u(\frac{1}{4}) < C$, where C is independent of α for $\alpha \in (0,\frac{1}{3})$. Defining $\varphi := u'/u$ we get from (32):

$$\varphi' = 2x(y_0y)^{-1}y_0' - 2xy^{-1}\varphi - \varphi^2 =: F(x,\varphi) \quad , \quad \varphi(0) = 0 \;.$$

Let $\psi(x) := (y_0' + (2xy_0')^{\frac{1}{2}})(y_0)^{-1}$. Then $\psi'(x) - F(x,\psi(x)) = (2x)^{\frac{1}{2}}(y_0')^{3/2}(y_0)^{-2} + (2xy_0')^{\frac{1}{2}}(y_0)^{-1}(x(2(y)^{-1} - (y_0)^{-1}) + (2xy_0)^{-1})$. Clearly $\psi' \_ F(x,\psi) > 0$ for $0 < x < 2^{-\frac{1}{2}}$. Further $\psi' - F(x,\psi) > 0$ for $x \ge 2^{-\frac{1}{2}}$ provided that $2(y)^{-1} - (y_0)^{-1} > 0$ on $[2^{-\frac{1}{2}},\infty)$. This condition is fullfilled, since by (25) and (28), we have

$$2y_0(2^{-\frac{1}{2}}) > (7 - 3\sqrt{2})(3\sqrt{2})^{-1} > (1 + 2\sqrt{\alpha})^2\,\tfrac{1}{2} > (1 + 2\sqrt{\alpha})^2 L_0 > L > y(x) \;,$$

and the second inequality holds if $0 < \alpha \le \frac{1}{3}$.

Further, $F(x,0) > 0$ $(x > 0)$, and $\varphi > 0$ $(x > 0)$. It follows (see [3] p. 177) that $0 < \varphi < \psi$. Therefore,

$$\log(u(\infty)/u(\tfrac{1}{4})) < \int_{\frac{1}{4}}^{\infty} \psi(x)dx = \log(L_0/y_0(\tfrac{1}{4})) +$$

$$+ \int_{\frac{1}{4}}^{\infty} (2xy_0')^{\frac{1}{2}}(y_0)^{-1}dx = \log(L_0/y_0(\tfrac{1}{4})) + 2(2y_0'(\tfrac{1}{4}))^{\frac{1}{2}} -$$

$$- 2^{-\frac{1}{2}} \int_{\frac{1}{4}}^{\infty} x^{-3/2}(y_0')^{\frac{1}{2}}dx < \log(L_0/y_0(\tfrac{1}{4})) + 2(2y_0'(\tfrac{1}{4}))^{\frac{1}{2}} .$$

Using $y_0(\frac{1}{4}) > 37/192$ (from (25)) and $y_0'(\frac{1}{4}) < e^{-\frac{1}{2}}$ (see proof of (26)) we find

(35) $$u(\infty)/u(\tfrac{1}{4}) < 47\, L_0 .$$

Combining (21), (31), (34) and (35) we can summarize the results of this section by

(36) $$2\, L_0 \log(1 + L_0\, \gamma^2) < L(\gamma) - L_0\, \gamma^2 < 47\, L_0 \log(1 + 3(16)^{-1}\gamma^2) + 47\, L_0$$

$$(\gamma > 1.8) .$$

<u>Remark</u>. Numerical experiments suggest that

$$L - L_0\, \gamma^2 \sim C \log \gamma \qquad (\gamma \to \infty) .$$

6. Bounds for the limit if $\gamma < 0$.

Integrating (1) in the form $y'' = -2xy^{-1}y'$ over $[1,x]$ we get

$$(37) \qquad y(x) = 1 + \gamma x + 2\int_0^x (s^2 - xs)(y(s))^{-1}y'(s)ds\ .$$

Writing $s^2 - xs = (s - \frac{1}{2}x)^2 - \frac{1}{4}x^2$ and taking exponentials we arrive at

$$(38) \qquad y(x) = \exp[2x^{-2} + 2\gamma x^{-1} - 2x^{-2}y(x) + I(x)]\ ,$$

where

$$(39) \qquad I(x) := 4x^{-2}\int_0^x (s - \frac{1}{2}x)^2(y(s))^{-1}y'(s)ds\ .$$

Since $I(x) < 0$ we have

$$(40) \qquad y(x) < \exp[2x^{-2} + 2\gamma x^{-1}] \qquad (x > 0)\ .$$

Obviously, we have the inequality $L < y(2|\gamma|^{-1}) < \exp[-\frac{1}{2}\gamma^2]$, but we can do much better. First we will show that

$$(41) \qquad L \geq y(2|\gamma|^{-1})(1 - \frac{1}{4}\gamma^2\exp[-8\sqrt{2}\,|\gamma|^{-3}\exp[\frac{1}{2}\gamma^2]])\ .$$

This inequality states that $y(2|\gamma|^{-1})$ is a very good approximation of L for large $|\gamma|$.

Secondly, we will obtain sharp inequalities for $y(2|\gamma|^{-1})$ which, by (41), will result in sharp inequalities for L.

Utilizing (5) with $x = \alpha := 2|\gamma|^{-1} - 2\sqrt{2}|\gamma|^{-2}$ and $x = \beta := 2|\gamma|^{-1}$ respectively, and (40) we have

$$y'(\beta) > \gamma \exp[-(\beta^2 - \alpha^2)(y(\alpha))^{-1}] >$$

$$> \gamma \exp[-(\beta^2 - \alpha^2) \exp -2\alpha^{-2} - 2\gamma\alpha^{-1}] >$$

$$> \gamma \exp[-8\sqrt{2}|\gamma|^{-3} \exp[\tfrac{1}{2}\gamma^2]] .$$

Using this in (8) with $a = 2|\gamma|^{-1}$ we get (41).

In order to obtain good estimates for $y(2|\gamma|^{-1})$ we have, by (38), to find good estimates for $I(2|\gamma|^{-1})$. Therefore we put

(42) $$u(t) := \gamma\, y(|\gamma|^{-1} t)(y'(|\gamma|^{-1} t))^{-1} .$$

Then, with $\varepsilon := 2|\gamma|^{-2}$ we get

(43) $$\dot{u}(t) = -1 + \varepsilon t\, u(t) \exp\left[\int_0^t (u(s))^{-1} ds\right] , \qquad u(0) = 1 ,$$

and

(44) $$I(2|\gamma|^{-1}) = \int_0^2 (t-1)^2 (u(t))^{-1} dt .$$

The solution u of (42) has the following global behaviour. It starts with the value 1, decreases till $t = t_0$, where it attains a minimum $u_0 := u(t_0) > 0$ and thereafter it increases very rapidly. Since $f(t) := u(t) \exp[{}_0\!\int^t (u(s))^{-1} ds]$ is increasing and $f(t_0) = (\varepsilon\, t_0)^{-1}$ we have $\dot{u}(t) \le -1 + t/t_0$ $(0 \le t \le t_0)$, from which it follows that $u(t) \le 1 - t + \tfrac{1}{2}t^2/t_0$, whence $0 < u_0 < 1 - \tfrac{1}{2}t_0$. Hence $t_0 < 2$.

We need the relations

$$(45) \qquad t(\dot{u} + 1)^{-1} = \varepsilon^{-1} - \int_0^t s(u(s))^{-1} ds ,$$

$$(46) \qquad \int_0^t s^2 (u(s))^{-1} ds = \varepsilon^{-1} + (u + t) \left(\int_0^t s(u(s))^{-1} ds - \varepsilon^{-1} \right) ,$$

the correctness of both is easily checked by differentiation. Substituting $t_0 = t_0$ in (42), (45) and (46) we obtain

$$(47) \qquad \int_0^{t_0} (u(s))^{-1} ds = -\log(\varepsilon\, t_0\, u_0) ,$$

$$(48) \qquad \int_0^{t_0} s(u(s))^{-1} ds = \varepsilon^{-1} - t_0 ,$$

$$(49) \qquad \int_0^{t_0} s^2 (u(s))^{-1} ds = \varepsilon^{-1} - t_0^2 - u_0\, t_0 .$$

We will now prove that $t_0 > 1$ for $\varepsilon > 0$ sufficiently small. For suppose that $t_0 \le 1$. Then on the one hand we have

$$t_0^2 + u_0\, t_0 - t_0 = \int_0^{t_0} (s - s^2)(u(s))^{-1} ds \ge \int_0^{t_0} (s - s^2) ds =$$

$$= \frac{1}{2} t_0^2 - \frac{1}{3} t_0^3 ,$$

which implies $u_0 > 1 - \frac{1}{2} t_0 - \frac{1}{3} t_0^2 \ge \frac{1}{6}$. On the other hand we have

$$\varepsilon^{-1} - 1 \le \int_0^{t_0} s(u(s))^{-1} ds \le \int_0^{t_0} (u(s))^{-1} ds = -\log(\varepsilon\, t_0\, u_0) < -\log(\varepsilon\, u_0) ,$$

implying that $u_0 < \varepsilon^{-1} \exp[1 - \varepsilon^{-1}]$ which contradicts $u_0 \ge \frac{1}{6}$ for $0 < \varepsilon < 0.23$.

In the sequel we need the inequalities

(50) $$(\varepsilon\, t_o)^{-1} + \log(\varepsilon\, t_0) - 1 < \log(u_0^{-1}) < \varepsilon^{-1} + \log(\varepsilon\, t_0) + 1 - t_0 \;.$$

Using (47), (48) and the inequality $u(s) \geq 1 - s$ $(0 \leq s \leq 1)$ we can write

$$\varepsilon^{-1} - t_0 = \int_0^{t_0} s(u(s))^{-1}ds < t_0 \int_0^{t_0} (u(s))^{-1}ds = -t_0 \log(\varepsilon\, t_0\, u_0) \;;$$

$$\varepsilon^{-1} - t_0 > \int_0^1 s(u(s))^{-1}ds + \int_1^{t_0} (u(s))^{-1}ds =$$

$$= \int_0^1 (s - 1)(u(s))^{-1}ds + \int_0^{t_0} (u(s))^{-1}ds > -1 + \log(\varepsilon\, t_0\, u_0)^{-1} ,$$

from which (50) follows.

The next stage in our treatment is the proof of

(51) $$t_0 - 1 < \tfrac{1}{2}\varepsilon + 4\varepsilon^2|\log\varepsilon| \qquad (0 < \varepsilon \leq 0.05) \;.$$

Since $\ddot{u} = t^{-1}(\dot{u} + 1) + u^{-1}(\dot{u} + 1)^2 > 0$ on $[0,t_0]$ we have

(52) $$u(t) \leq t_0 + u_0 - t \qquad (0 \leq t \leq t_0) \;.$$

Therefore

(53) $$\int_1^{t_0} (s^2 - s)(u(s))^{-1}ds \geq \int_1^{t_0} (s^2 - s)(t_0 + u_0 - s)^{-1}ds =$$

$$= (t_0 - 1)[(u_0 + t_0)\log(u_0^{-1}(u_0 + t_0 - 1)) + \frac{1}{2} - \frac{3}{2}t_0 - u_0] >$$

$$> (t_0 - 1)[t_0 \log(u_0^{-1}(t_0 - 1)) + \frac{1}{2} - \frac{3}{2}t_0 - u_0] \;.$$

Using (48), (49), (50) and the inequality $u(t) \geq 1 - t$ on $[0,1]$ we find

$$t_0(1 - t_0 - u_0) = \int_0^1 (s^2 - s)(u(s))^{-1}ds + \int_1^{t_0} (s^2 - s)(u(s))^{-1}ds >$$

$$> -\frac{1}{2} + \int_1^{t_0} (s^2 - s)(u(s))^{-1}ds ,$$

which with (53) leads to

$$(t_0 - 1)[\varepsilon^{-1} + t_0 \log(\varepsilon\, t_0(t_0 - 1)) - \frac{3}{2}t_0 + \frac{1}{2} + u_0(t_0 - 1)^{-1}] < \frac{1}{2}$$

from which (51) follows.

Next we show that

$$(54) \qquad \int_{t_0}^{2} (t - 1)^2(u(t))^{-1}dt \leq 3\varepsilon^2 \qquad (0 < \varepsilon \leq 0.05) .$$

Since $\dddot{u} = 3(\dot{u} + 1)^2(tu)^{-1} + (\dot{u} + 1)^2(\dot{u} + 2)u^{-2} > 0$ we have $\ddot{u}(t) > \ddot{u}(t_0) = t_0^{-1} + u_0^{-1} > u_0^{-1}$ for $t > t_0$ whence $u(t) > u_0 + \frac{1}{2}u_0^{-1}(t^2 - t_0^2) \quad (t > t_0)$.
Now, using this latter inequality and (50), (51) we have

$$(55) \qquad \int_{t_0}^{2} (s - 1)^2(u(s))^{-1}ds = \int_{t_0}^{2} [(t_0 - 1)^2 + 2(t_0 - 1)(s - t_0) +$$

$$+ (s - t_0)^2](u(s))^{-1}ds \leq$$

$$\leq (t_0 - 1)^2 2^{-\frac{1}{2}}\pi + (2(t_0 - 1)u_0 + u_0^2)\log(1 + \frac{1}{2}u_0^{-2}) < 3\varepsilon^2 \quad (0 < \varepsilon \leq 0.05).$$

Finally, we are able to estimate $\int_0^{t_0} (1 - s)^2(u(s))^{-1}ds$ as follows, using

(48), (51) and (55),

$$\int_0^{t_0} (1 - s)^2 (u(s))^{-1} ds < \int_0^1 (1 - s)^2 (1 - s)^{-1} ds +$$

$$+ (t_0 - 1)^2 \int_0^{t_0} s(u(s))^{-1} ds \le$$

$$\le \frac{1}{2} + (\frac{1}{4}\varepsilon^2 + 4\varepsilon^2 |\log \varepsilon| + 16\varepsilon^4 \log^2 \varepsilon)(\frac{1}{\varepsilon} - t_0) \le \frac{1}{2} + \frac{1}{3}\varepsilon \qquad (0 < \varepsilon \le 0.05) ,$$

and, using (52) and (51) and (50)

$$\int_0^{t_0} (1 - s)^2 (u(s))^{-1} ds > \int_0^1 (1 - s)^2 (u_0 + t_0 - s)^{-1} ds =$$

$$= \frac{1}{2} - (t_0 - 1) + [(u_0 + 2t_0 - 2)u_0 + (t_0 - 1)^2] \log(u_0 + t_0)(u_0 + t_0 - 1)^{-1}) - u_0 >$$

$$> \frac{1}{2} - (t_0 - 1) - u_0 \quad \frac{1}{2} - \frac{1}{2}\varepsilon - 4\varepsilon^2 |\log \varepsilon| - \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}} \qquad (0 < \varepsilon \le 0.05) .$$

Combining the last two inequalities with (38), (41), (44), (54) and (55) we can obtain

(56) $$1 - \gamma^{-2} \quad L(\gamma) \exp[\tfrac{1}{2}\gamma^2 + \tfrac{1}{2}] < 1 + \gamma^{-2} \qquad (\gamma < -7) .$$

Remark. Numerical experiments suggest that

$$L(\gamma) \exp[\tfrac{1}{2}\gamma^2 + \tfrac{1}{2}] - 1 \sim C\gamma^{-2} \qquad (\gamma \to -\infty) .$$

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References.

[1] Jongen, P.H.M.; De warmte-overdracht van een gas met vibratierelaksatie naar de schokbuisachterwand.
Thesis 1971, Eindhoven University of Technology, Dep. of Physics, Eindhoven, The Netherlands.

[2] Hale, J.K.; Ordinary differential equations.
(2nd ed.) 1980, Robert E. Krieger publ. cy., New York.

[3] Bruijn, N.G. de; Asymptotic Methods in Analysis.
1981, Dover Publications, Inc. New York.