Some examples of discontinuous bifurcations

Some Examples of Discontinuous Bifurcations

R.I. Leine H. Nijmeijer

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Some Examples of Discontinuous Bifurcations

R.I. Leine (TUE)

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A New Book...

Series:

Lecture Notes in

Applied Mechanics

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Dynamics and Bifurcations of Non-smooth Mechanical Systems

R. I. LEINE AND H. NIJMEIJER

2004

Nonsmooth Systems

- Nonsmooth continuous systems:
  discontinuous Jacobian existence and uniqueness
  - one-sided elastic supports
- Filippov systems: differential equations with a discontinuous right-hand side: existence, uniqueness not guaranteed (Filippov's solution concept)
  - friction with bounded normal contact force
  - switches
- Systems with discontinuities in the state:
  existence and uniqueness ? ⟷ solution concept?
  - (frictional) impact
  - impulsive systems

Nonsmooth Continuous Systems

Nonsmooth continuous system

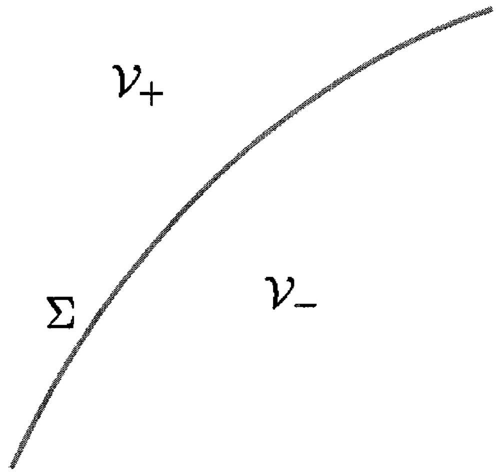
$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{f}_-(\boldsymbol{x},\mu), & \boldsymbol{x} \in \mathcal{V}_- \bigcup \Sigma \\ \boldsymbol{f}_+(\boldsymbol{x},\mu), & \boldsymbol{x} \in \mathcal{V}_+ \end{cases}$$

Switching boundary function $h(\boldsymbol{x},\mu)$

$$\begin{aligned} \mathcal{V}_- &= \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x},\mu) < 0\}, \\ \Sigma &= \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x},\mu) = 0\}, \\ \mathcal{V}_+ &= \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x},\mu) > 0\}. \end{aligned}$$

Continuity on Σ

$$\boldsymbol{f}_-(\boldsymbol{x},\mu) = \boldsymbol{f}_+(\boldsymbol{x},\mu) \quad \forall \boldsymbol{x} \in \Sigma.$$



local Lipschitz condition gives existence and uniqueness of solutions

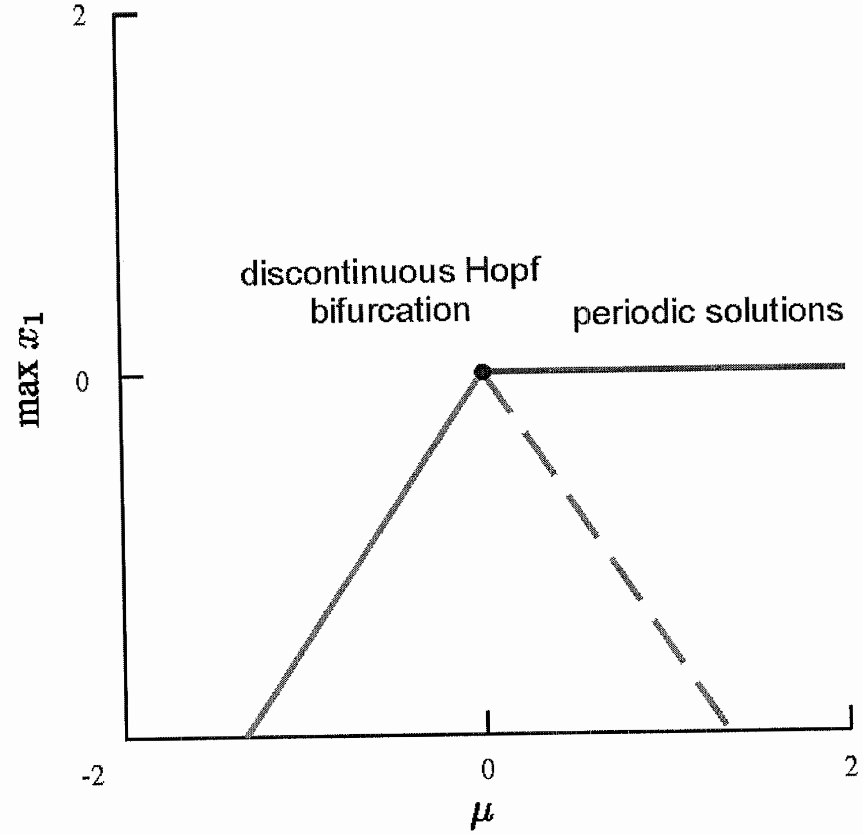
Example 1

system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - \tfrac{3}{2}|x_2 - \mu| - x_1$$

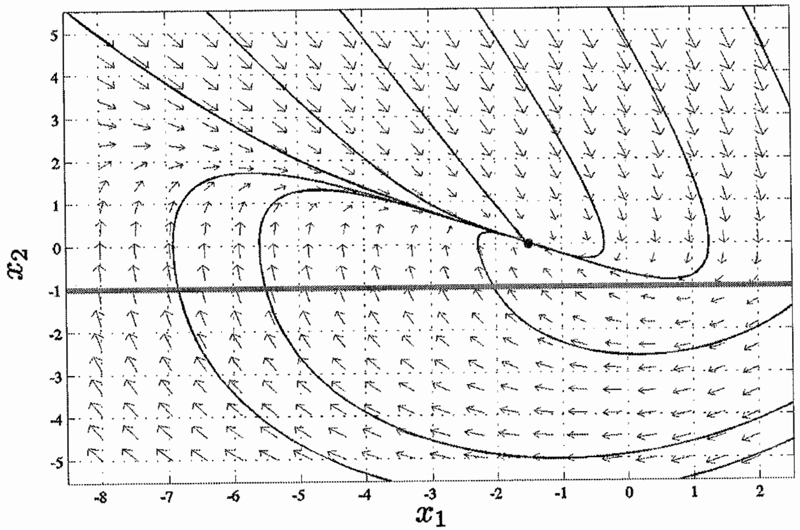
equilibrium

$$x_1 = -\frac{3}{2}|\mu|, \quad x_2 = 0$$

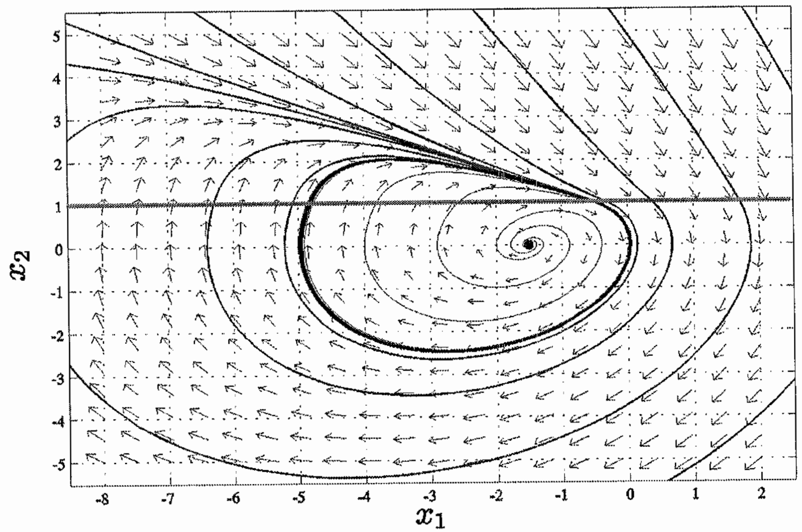


stable equilibrium — unstable equilibrium — stable periodic solution — unstable periodic solution

Example 1: Phase Portraits



$\mu = -1$



$\mu = 1$

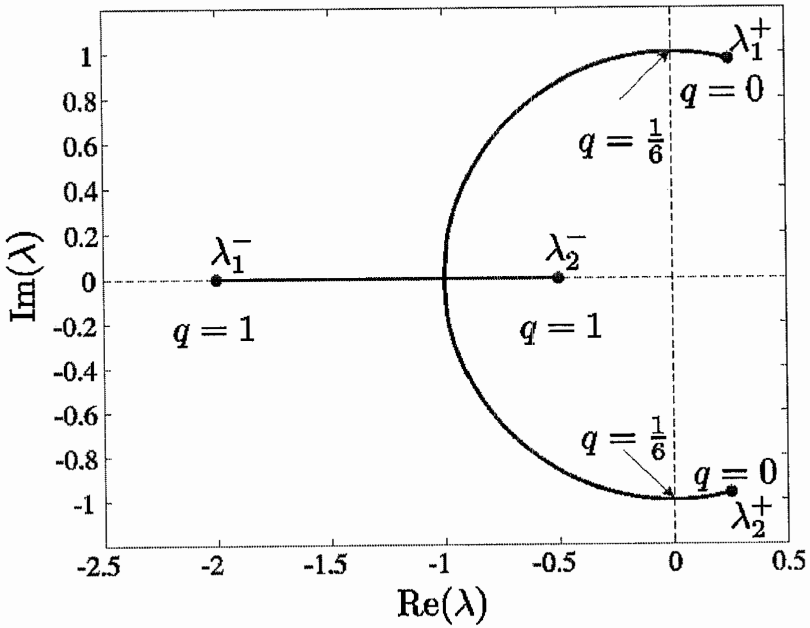
Example 1: Generalized Jacobian

Generalized Jacobian:

$$\boldsymbol{J}(\boldsymbol{0}) = \{\boldsymbol{J}_q, q \in [0,1]\}$$

with

$$\boldsymbol{J}_q = q\boldsymbol{J}_+ + (1-q)\boldsymbol{J}_- \text{ (convex)}$$



setvalued eigenvalues

Jacobians in each subspace:

$$\boldsymbol{J}_+ = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{bmatrix}, \quad \lambda_{1,2} = \tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{15}$$

$$\boldsymbol{J}_- = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \quad \lambda_1 = -\tfrac{1}{2},\ \lambda_2 = -2$$

..or using the set-valued sign function $\boldsymbol{J}(\boldsymbol{x},\mu) = \begin{bmatrix} 0 & 1 \\ -1 & -1 - \frac{3}{2}\,\mathrm{Sign}(x_2-\mu) \end{bmatrix}$

Example 2

Nonsmooth system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 + |x_1+\mu| - |x_1-\mu| - x_2 - |x_2+\mu| + |x_2-\mu|.\end{aligned}$$

Smooth system 1 (symmetric smoothing)

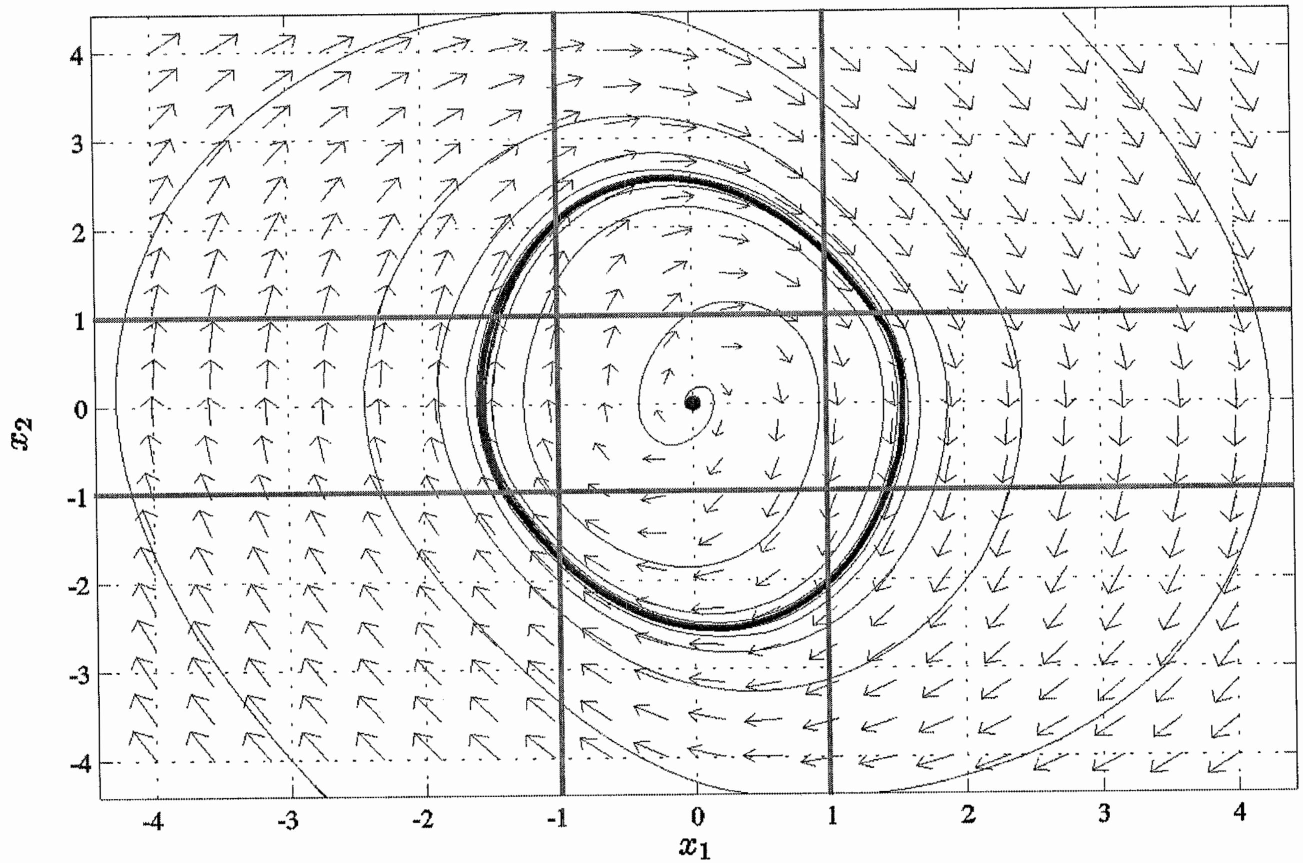
$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 + \frac{2}{\pi}\arctan(\varepsilon(x_1+\mu))(x_1+\mu) - \frac{2}{\pi}\arctan(\varepsilon(x_1-\mu))(x_1-\mu)\\ &\quad - x_2 - \frac{2}{\pi}\arctan(\varepsilon(x_2+\mu))(x_2+\mu) + \frac{2}{\pi}\arctan(\varepsilon(x_2-\mu))(x_2-\mu)\end{aligned}$$

Smooth system 2 (asymmetric smoothing)

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 + \frac{2}{\pi}\arctan(\varepsilon(x_1+\mu))(x_1+\mu) - \frac{2}{\pi}\arctan(\varepsilon(x_1-\mu))(x_1-\mu)\\ &\quad - x_2 - \frac{2}{\pi}\arctan(\varepsilon(x_2+\mu))(x_2+\mu) + \frac{2}{\pi}\arctan(\varepsilon(x_2-\mu))(x_2-\mu) + \frac{1}{\varepsilon}.\end{aligned}$$

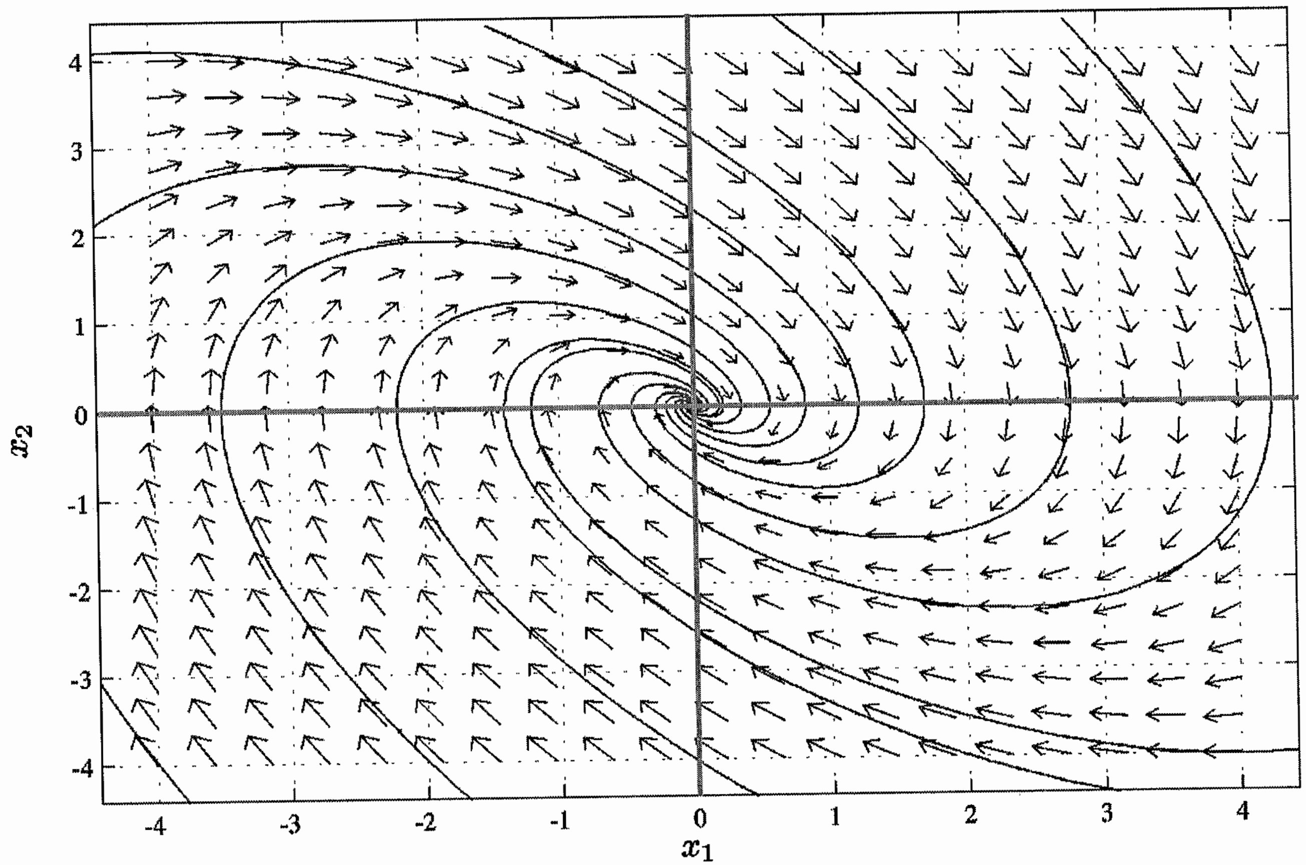
Example 2: Phase Portrait μ = -1

$\mu = -1$



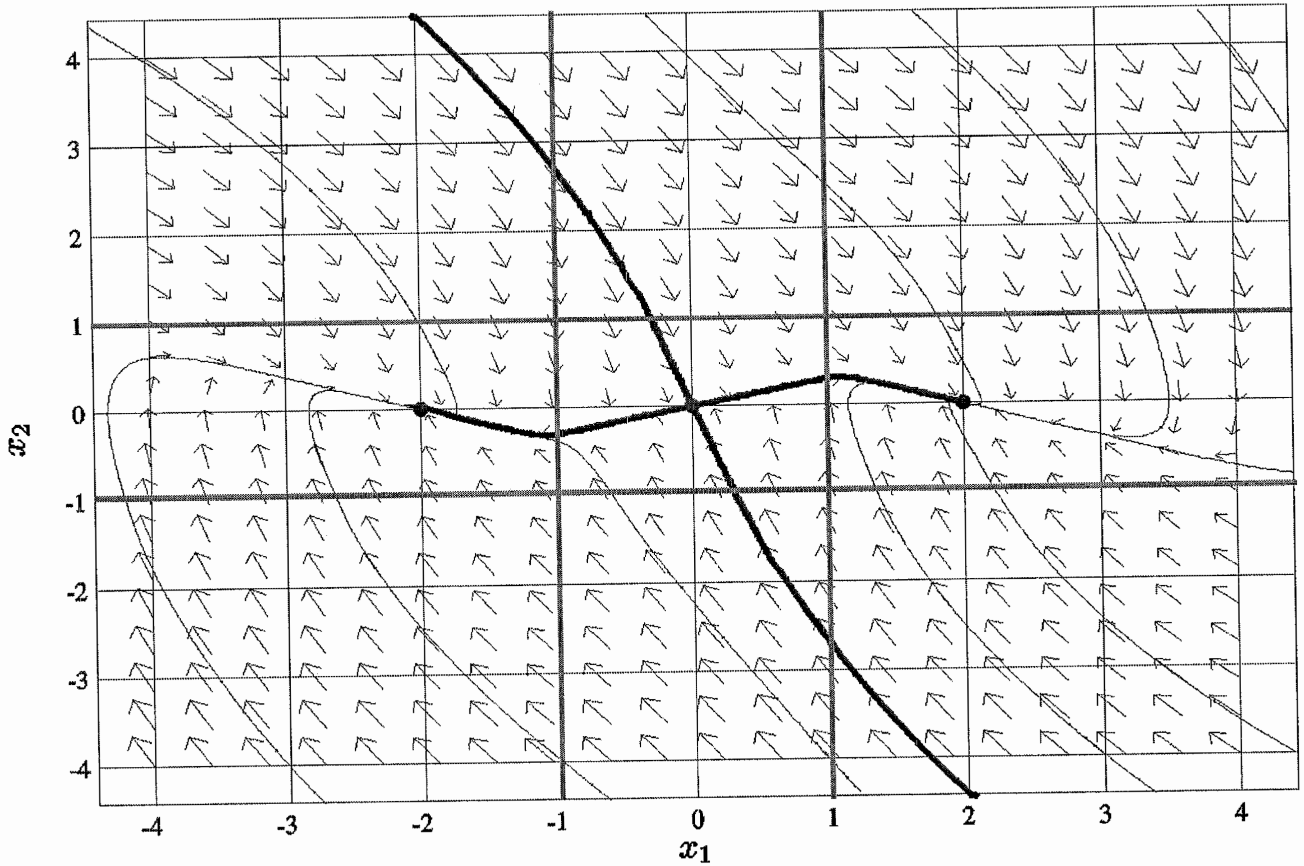
Example 2: Phase Portrait $\mu = 0$

$\mu = 0$

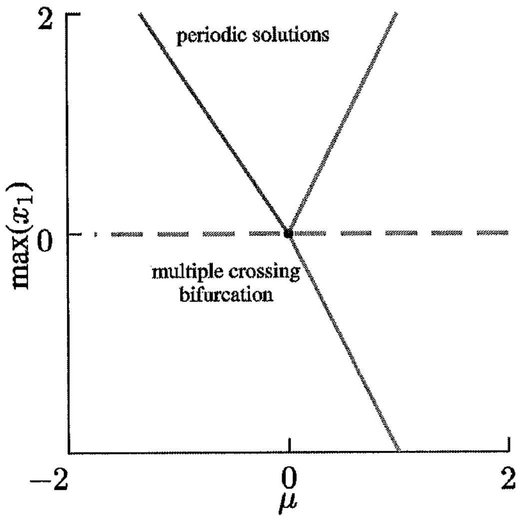


Example 2: Phase Portrait $\mu = 1$

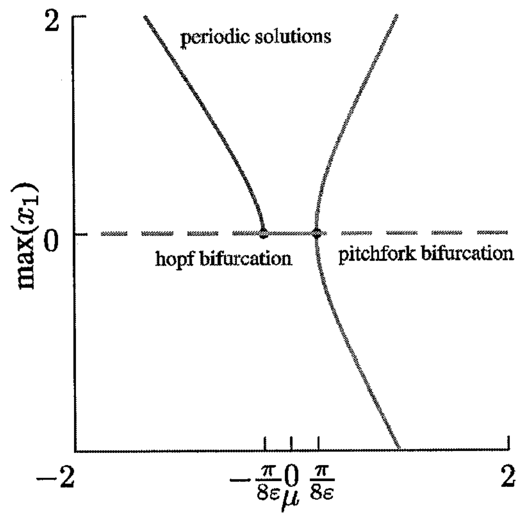
$\mu = 1$



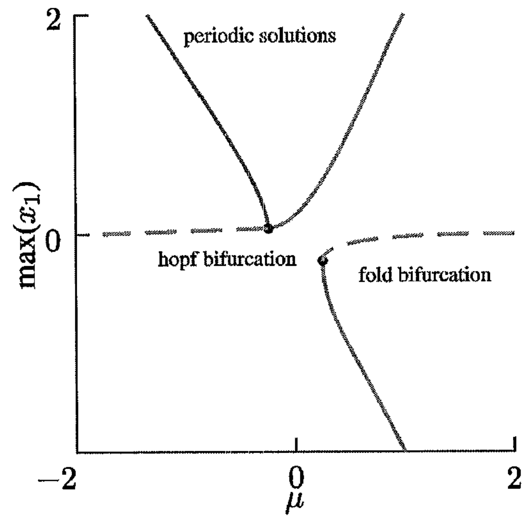
Example 2: Bifurcation Diagrams



Nonsmooth system



Smooth system 1



Smooth system 2

$$\left.\begin{array}{c}\text{unstable focus}\\ \text{periodic solution}\end{array}\right\} \xrightarrow{\text{multiple crossing bifurcation}} \left\{\begin{array}{l}\text{stable node}\\ \text{saddle}\\ \text{stable node}\end{array}\right.$$

Example 2: Bifurcation Structure

Nonsmooth system

$$\left.\begin{array}{c}\text{unstable focus}\\ \text{periodic solution}\end{array}\right\} \xrightarrow{\text{multiple crossing bifurcation}} \left\{\begin{array}{l}\text{stable node}\\ \text{saddle}\\ \text{stable node}\end{array}\right.$$

Smooth system 1 (symmetric smoothing)

$$\left.\begin{array}{c}\text{unstable focus}\\ \text{periodic solution}\end{array}\right\} \xrightarrow[\text{bifurcation}]{\text{Hopf}} \text{stable focus} \xrightarrow[\text{transition}]{\text{focus–node}} \text{stable node} \xrightarrow[\text{bifurcation}]{\text{pitchfork}} \left\{\begin{array}{l}\text{stable node}\\ \text{saddle}\\ \text{stable node}\end{array}\right.$$

Smooth system 2 (asymmetric smoothing)

$$\left.\begin{array}{c}\text{unstable focus}\\ \text{periodic solution}\end{array}\right\} \xrightarrow[\text{bifurcation}]{\text{Hopf}} \text{stable focus} \xrightarrow[\text{transition}]{\text{focus–node}} \text{stable node} \qquad\qquad \text{stable node}$$

$$\emptyset \xrightarrow[\text{bifurcation}]{\text{fold}} \left\{\begin{array}{l}\text{saddle}\\ \text{stable node}\end{array}\right.$$

Example 2: Generalized Jacobian

Generalized Jacobian

$$\boldsymbol{J}(\boldsymbol{x},\mu) = \begin{bmatrix} 0 & 1 \\ -1 + \mathrm{Sign}(x_1+\mu) - \mathrm{Sign}(x_1-\mu) & -1 - \mathrm{Sign}(x_2+\mu) + \mathrm{Sign}(x_2-\mu) \end{bmatrix}$$

at the bifurcation point

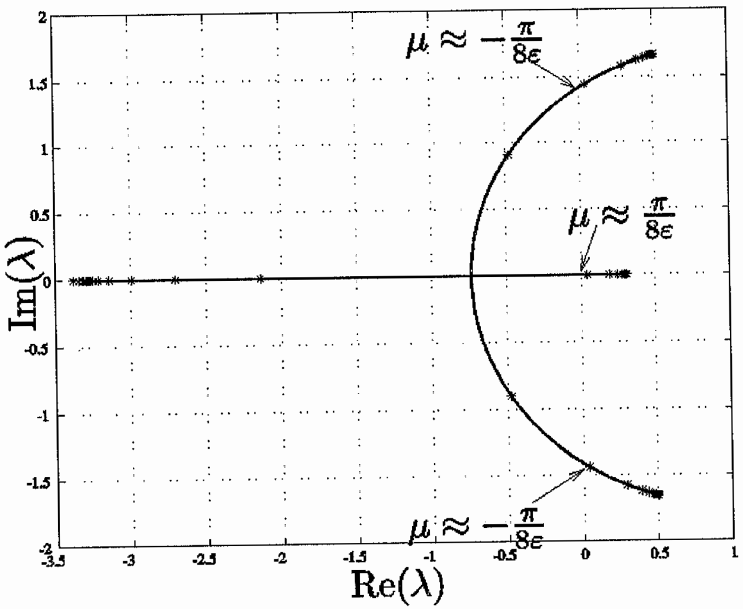
$$\boldsymbol{J}(\boldsymbol{0},0) = \{\boldsymbol{J}_q,\ q_i \in [0,1],\ i=1,\dots,4\},\ \text{with } \boldsymbol{J}_q = \begin{bmatrix} 0 & 1 \\ -1-2q_1+2q_2 & -1-2q_3+2q_4 \end{bmatrix}$$

On the equilibrium branch we have the sub-Jacobian

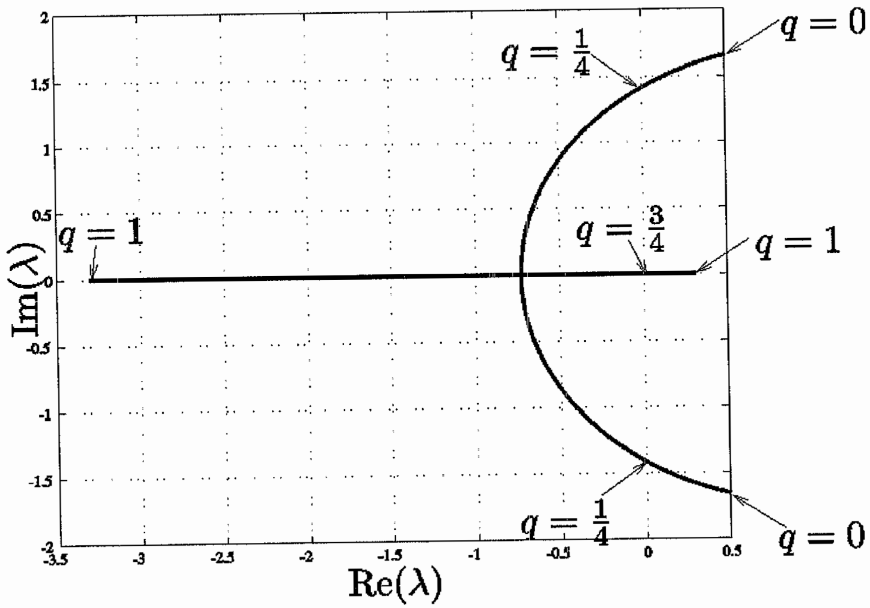
$$\boldsymbol{J}^{\mathrm{tr}} = \{(\boldsymbol{J}_+^{\mathrm{tr}} - \boldsymbol{J}_-^{\mathrm{tr}})q + \boldsymbol{J}_-^{\mathrm{tr}},\ q \in [0,1]\},$$

$$\boldsymbol{J}_-^{\mathrm{tr}} = \boldsymbol{J}(\boldsymbol{0},\mu<0) = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix},\quad \boldsymbol{J}_+^{\mathrm{tr}} = \boldsymbol{J}(\boldsymbol{0},\mu>0) = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

Example 2: Set-valued Eigenvalues



path of eigenvalues of
Smooth system 1



Set-valued eigenvalues of $\boldsymbol{J}(0,0)^{\mathrm{tr}}$

$$\boldsymbol{J}^{\mathrm{tr}} = \{(\boldsymbol{J}_{+}^{\mathrm{tr}} - \boldsymbol{J}_{-}^{\mathrm{tr}})q + \boldsymbol{J}_{-}^{\mathrm{tr}},\ q \in [0,1]\}$$

Example 3

Nonsmooth system

$$\begin{aligned}\dot{x}_1 &= x_1 + 2|x_1| + x_2,\\ \dot{x}_2 &= x_1 + 2|x_1| + \tfrac{1}{2}x_2 + \mu\end{aligned}$$

$\mu < 0$

$$\begin{aligned}&\text{equilibrium 1:} \quad x_1 = -\tfrac{2}{3}\mu, \quad x_2 = 2\mu,\\ &\text{equilibrium 2:} \quad x_1 = 2\mu, \quad x_2 = 2\mu\end{aligned}$$

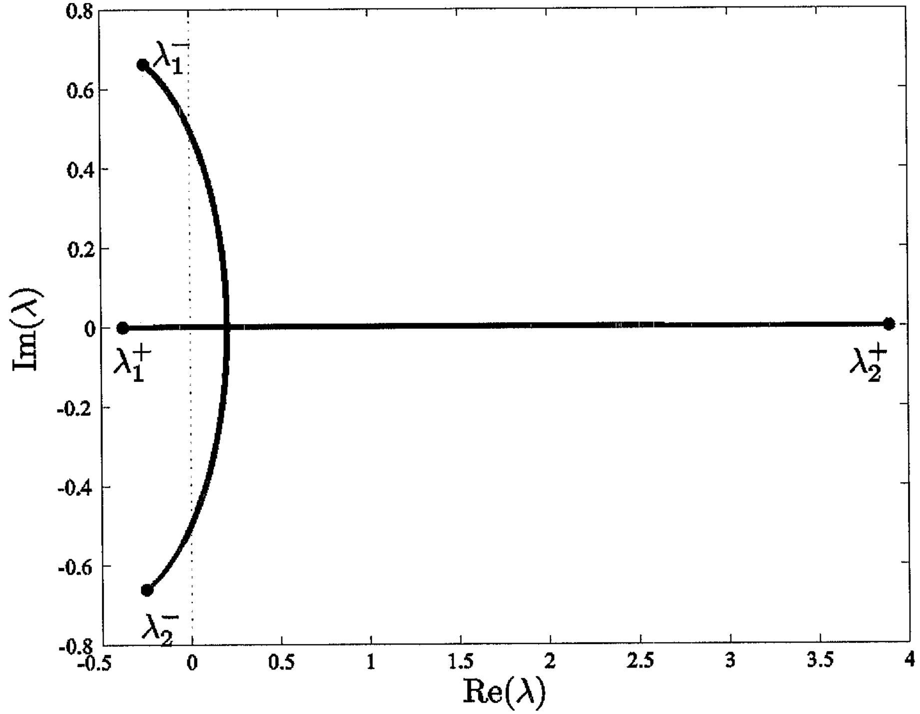
A smooth approximating system

$$\begin{aligned}\dot{x}_1 &= x_1 + \tfrac{4}{\pi}\arctan(\varepsilon x_1)x_1 + x_2,\\ \dot{x}_2 &= x_1 + \tfrac{4}{\pi}\arctan(\varepsilon x_1)x_1 + \tfrac{1}{2}x_2 + \mu.\end{aligned}$$

Example 3: Generalized Jacobian

Generalized Jacobian

$$\boldsymbol{J}(x_1) = \begin{bmatrix} 1 + 2\,\mathrm{Sign}(x_1) & 1 \\ 1 + 2\,\mathrm{Sign}(x_1) & \frac{1}{2} \end{bmatrix}$$



At the bifurcation point:

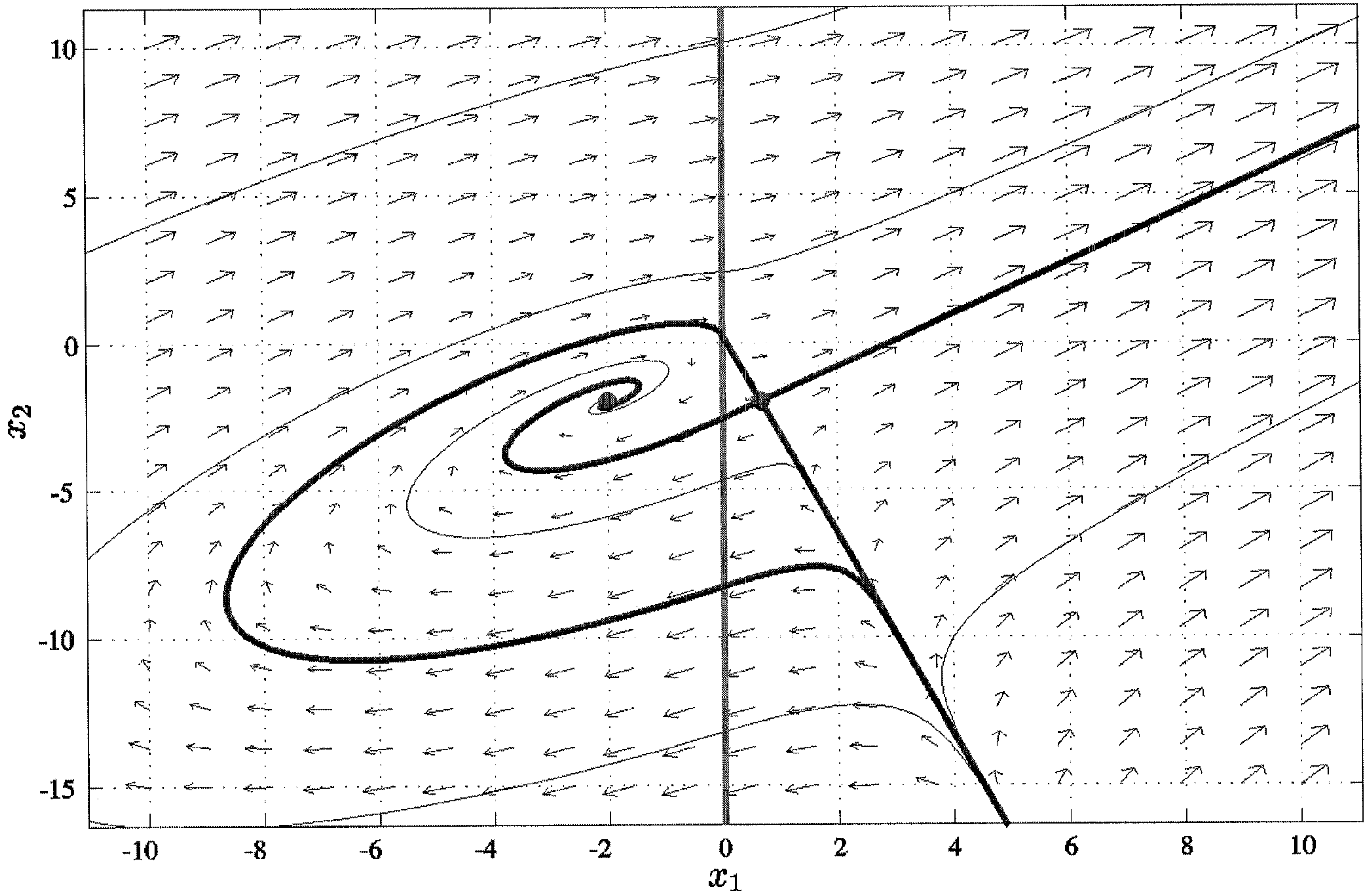
$$\boldsymbol{J}(\boldsymbol{0}) = \overline{\mathrm{co}}(\boldsymbol{J}_-, \boldsymbol{J}_+) = \{(1-q)\boldsymbol{J}_- + q\boldsymbol{J}_+,\ \forall q \in [0,1]\}$$

$$\boldsymbol{J}_- = \begin{bmatrix} -1 & 1 \\ -1 & \frac{1}{2} \end{bmatrix} \quad \text{for } x_1 < 0, \quad \lambda_{1,2} = -\frac{1}{4} \pm i\sqrt{\frac{7}{16}},$$

$$\boldsymbol{J}_+ = \begin{bmatrix} 3 & 1 \\ 3 & \frac{1}{2} \end{bmatrix} \quad \text{for } x_1 > 0, \quad \lambda_{1,2} = \frac{7}{4} \pm \sqrt{\frac{73}{16}} \approx \{-0.386, 3.886\}.$$

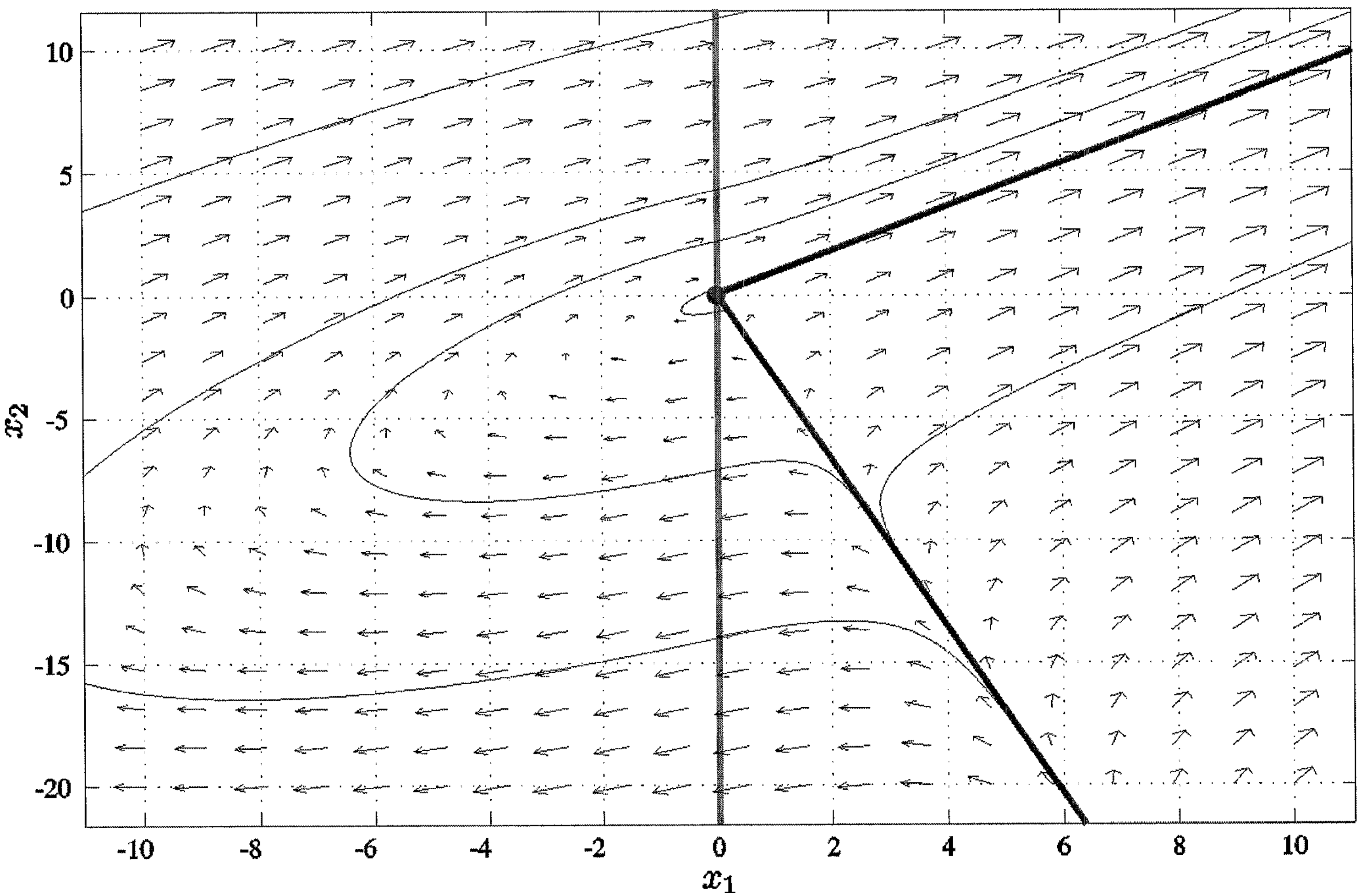
Example 3: Phase Portrait 1 Nonsmooth

$\mu = -1$



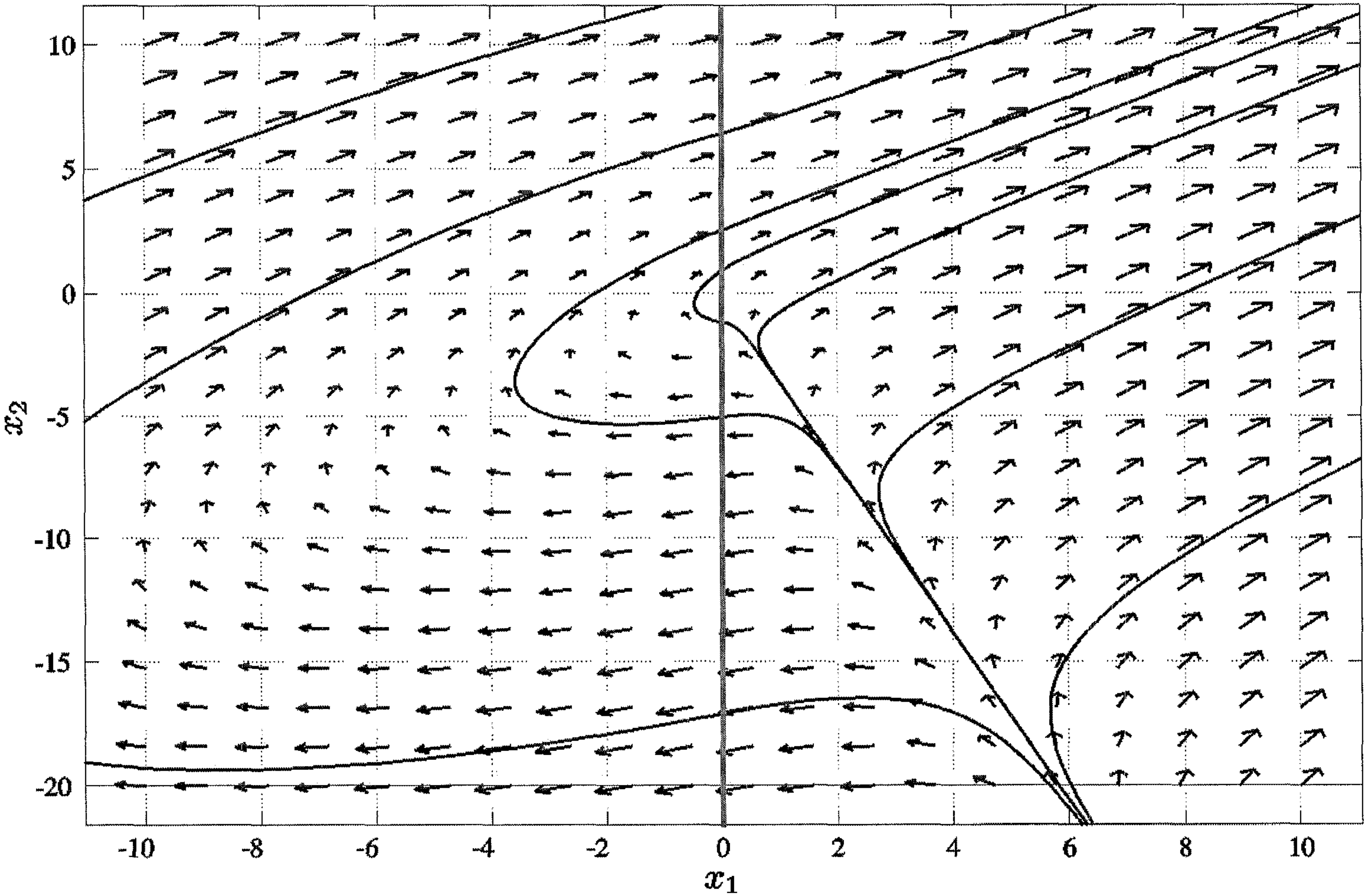
Example 3: Phase Portrait 2 Nonsmooth

$\mu = 0$



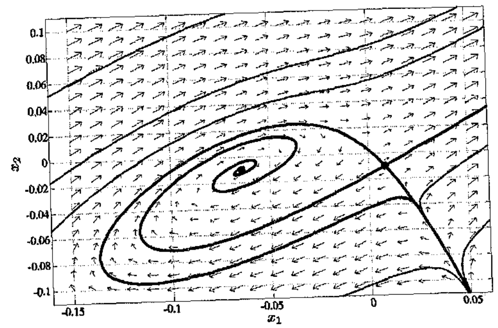
Example 3: Phase Portrait 3 Nonsmooth

$\mu = 1$



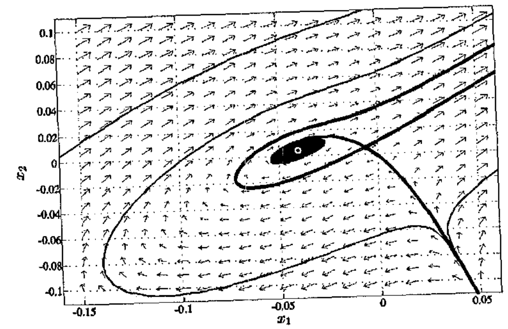
Example 3: Phase Portraits Smooth System

$\mu = -0.005$



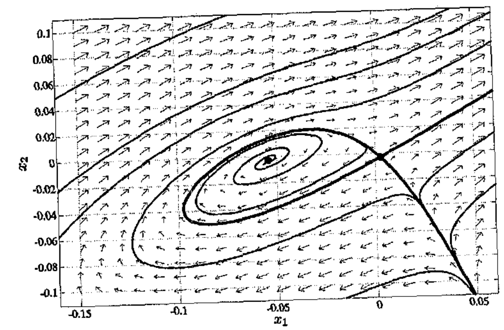
Hopf bifurcation

$\mu = 0.002556$



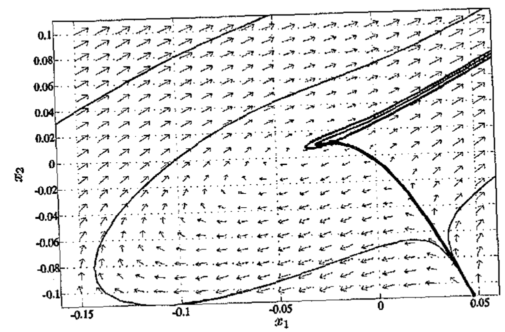
Homoclinic bifurcation

$\mu = -0.0013987$

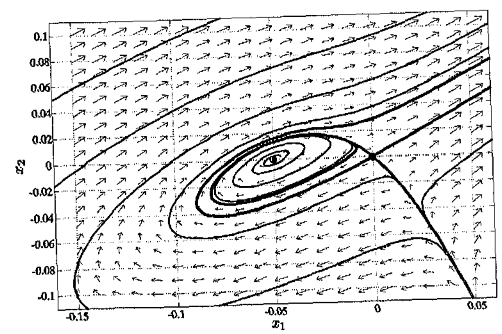


saddle-node bifurcation

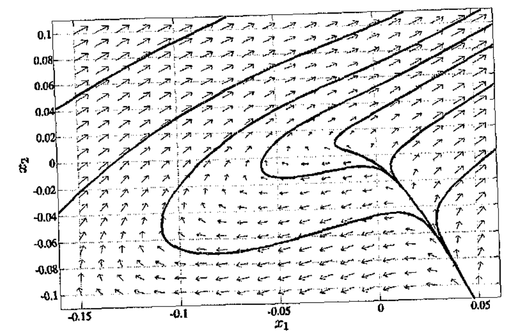
$\mu = 0.0051946$



$\mu = 0$



$\mu = 0.01$



Example 3: Bifurcation Structure

Nonsmooth system

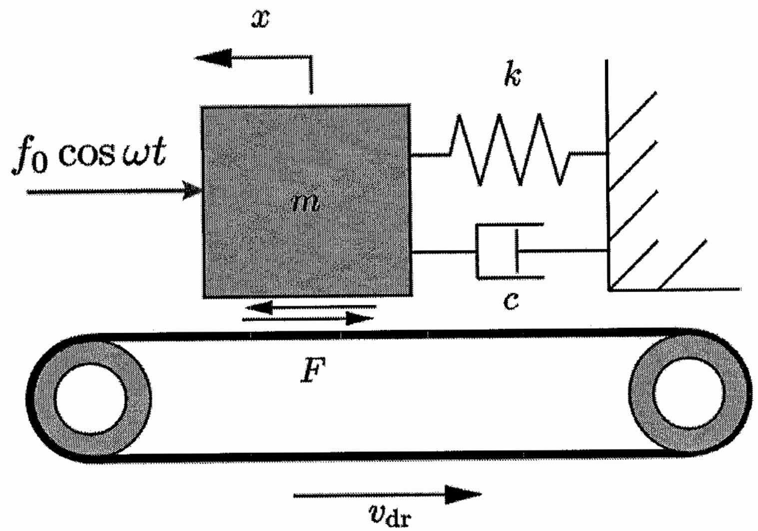
$$\left.\begin{array}{l}\text{stable focus}\\ \text{saddle}\end{array}\right\} \xrightarrow{\text{multiple crossing bifurcation}} \emptyset$$

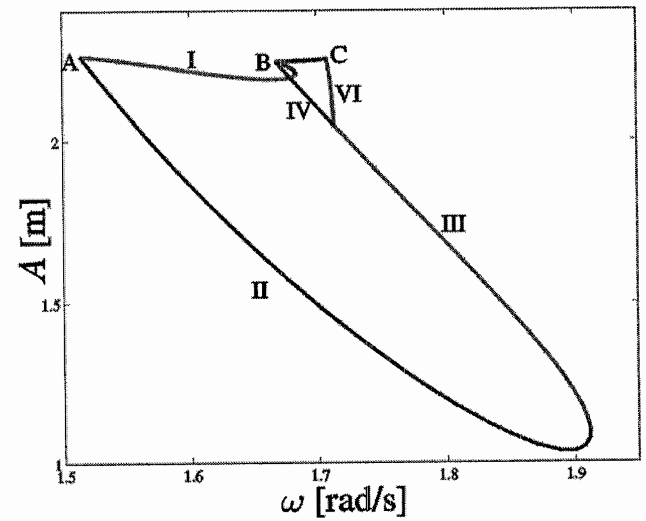
A smooth approximating system

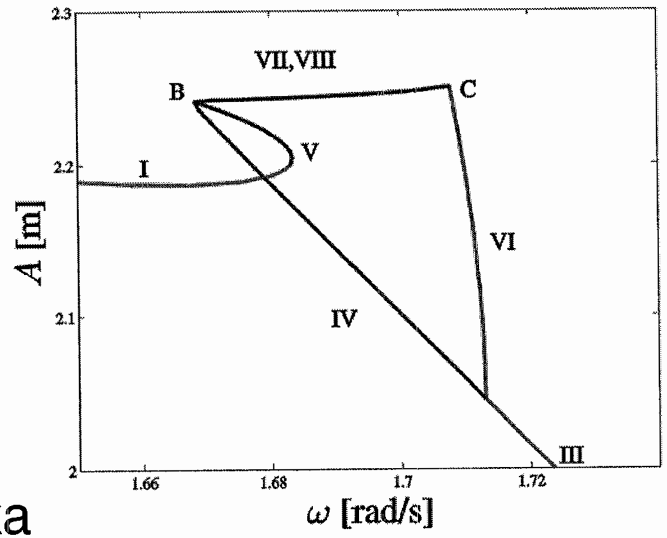
$$\left.\begin{array}{l}\text{stable focus}\\ \\ \text{saddle}\end{array}\right\} \xrightarrow[\text{bifurcation}]{\text{homoclinic}} \begin{array}{l}\text{stable focus}\\ \text{unstable limit cycle}\\ \text{saddle}\end{array}$$

$$\left.\begin{array}{l}\text{stable focus}\\ \text{unstable limit cycle}\end{array}\right\} \xrightarrow[\text{bifurcation}]{\text{Hopf}} \left.\begin{array}{lcl}\text{unstable focus} & \xrightarrow[\text{transition}]{\text{focus–node}} & \text{unstable node}\\ & & \\ \text{saddle} & & \text{saddle}\end{array}\right\} \xrightarrow[\text{bifurcation}]{\text{saddle-node}} \emptyset$$
$$\text{saddle}$$

Block-on-Belt System with Dry Friction







non-standard bifurcations due to dry friction

Yoshitake & Sueoka

Questions (let's not jump to Conclusions...)

- Does the set-valuedness of an eigenvalue, which reaches over the imaginary axis, imply a bifurcation? Or is it a necessary condition for bifurcation?
- Can we classify a (discontinuous) bifurcation by the point(s) where the path of the set-valued eigenvalue (or pair) crosses the imaginary axis?
- How do we analyze bifurcation points on multiple switching boundaries, for which the set-valued eigenvalues form an area in the complex plane?

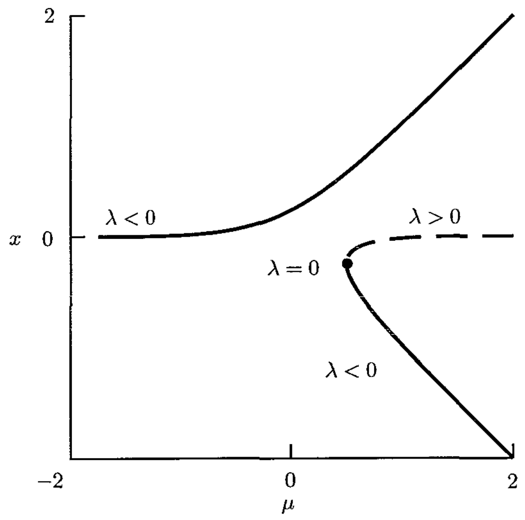


Figure 7.13: Non-symmetric smoothing.

series expansion around ($x = 0$, $\mu = 0$)

$$\begin{aligned}\dot{x} &= -x + |x + \tfrac{1}{2}\mu| - |x - \tfrac{1}{2}\mu| \\ &\approx -x + \tfrac{2}{\pi}\arctan(\varepsilon(x + \tfrac{1}{2}\mu))(x + \tfrac{1}{2}\mu) - \tfrac{2}{\pi}\arctan(\varepsilon(x - \tfrac{1}{2}\mu))(x - \tfrac{1}{2}\mu) \\ &\approx (-1 + \tfrac{4}{\pi}\varepsilon\mu)x - \tfrac{8}{3\pi}\varepsilon^3\mu x^3.\end{aligned}$$

The resulting bifurcation is a continuous pitchfork bifurcation with the bifurcation point at ($x = 0$, $\mu = \frac{\pi}{4\varepsilon}$). The bifurcation point of the smoothened system therefore approaches the origin as ε is increased.

However, not every smoothing function gives a pitchfork bifurcation. Consider for instance the following non-symmetric smoothing:

$$\begin{aligned}|x + \tfrac{1}{2}\mu| &\approx \tfrac{2}{\pi}\arctan(\varepsilon(x + \tfrac{1}{2}\mu))(x + \tfrac{1}{2}\mu) + \tfrac{1}{\varepsilon} \\ |x - \tfrac{1}{2}\mu| &\approx \tfrac{2}{\pi}\arctan(\varepsilon(x - \tfrac{1}{2}\mu))(x - \tfrac{1}{2}\mu)\end{aligned} \tag{7.50}$$

which gives

$$\dot{x} \approx (-1 + \frac{4}{\pi}\varepsilon\mu)x - \frac{8}{3\pi}\varepsilon^3\mu x^3 + \frac{1}{\varepsilon} \tag{7.51}$$

for $|x| \ll 1$ and $\varepsilon \gg 1$. Equation (7.51) has two branches close to the origin in the bifurcation diagram for varying μ, but the branches do not intersect (Figure 7.13). Only a saddle-node bifurcation exists for (7.51).

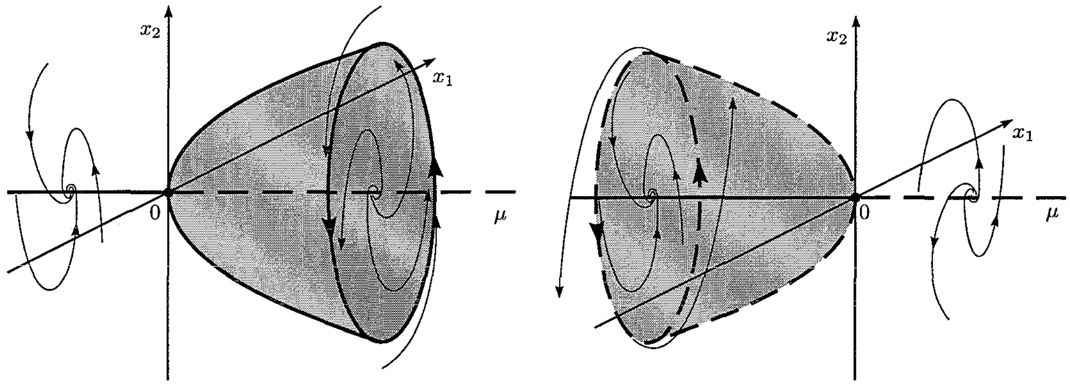
7.6.4 Hopf Bifurcation

At a Hopf bifurcation point the equilibrium loses its stability and a periodic solution is born (or vice-versa). First, we consider the smooth planar system

$$\begin{aligned}\dot{x}_1 &= \mu x_1 - \omega x_2 + (\alpha x_1 - \beta x_2)(x_1^2 + x_2^2), \\ \dot{x}_2 &= \omega x_1 + \mu x_2 + (\beta x_1 + \alpha x_2)(x_1^2 + x_2^2),\end{aligned} \tag{7.52}$$

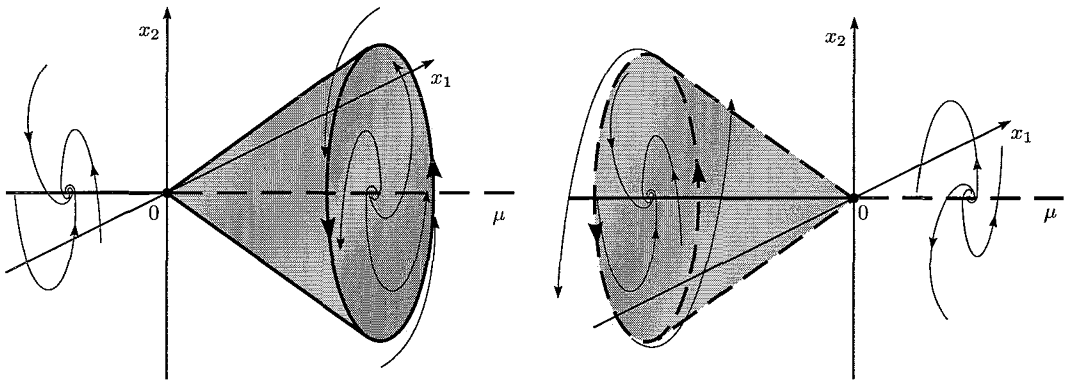
where μ, ω, α and β are constants. We will study the equilibria and periodic solutions of system (7.52) for different values of μ. This system has an equilibrium $\boldsymbol{x} = [x_1, x_2]^{\mathrm{T}} = [0, 0]^{\mathrm{T}}$ for all values of μ and the Jacobian matrix of the linearized system around the equilibrium is

$$\boldsymbol{J} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}$$



(a) supercritical, $\alpha < 0$ (b) subcritical, $\alpha > 0$

Figure 7.14: Hopf bifurcation, continuous.



(a) supercritical (b) subcritical

Figure 7.15: Hopf bifurcation, discontinuous.

with the eigenvalues $\lambda_{1,2} = \mu \pm i\omega$. For $\mu < 0$ the equilibrium is asymptotically stable. When μ is increased to $\mu = 0$ the equilibrium becomes non-hyperbolic, and for $\mu > 0$ the equilibrium becomes unstable. By using the transformation

$$x_1 = r\cos\theta \quad \text{and} \quad x_2 = r\sin\theta, \tag{7.53}$$

we transform (7.52) into

$$\dot{r} = \mu r + \alpha r^3, \tag{7.54}$$

$$\dot{\theta} = \omega + \beta r^2. \tag{7.55}$$

The trivial equilibrium of (7.54) corresponds to the equilibrium of (7.52), and the nontrivial equilibrium ($r \neq 0$) of (7.54) corresponds to a periodic solution of (7.52). In the latter case, r is the amplitude and $\dot{\theta}$ is the frequency of the periodic solution that is created by the Hopf bifurcation. The transformation (7.53) therefore transforms the Hopf bifurcation into a pitchfork bifurcation. The bifurcation diagram for the Hopf bifurcation is depicted in Figure 7.14 and the bifurcation diagram for the transformed system (7.54) is identical to Figure 7.11 where x should be replaced by r.

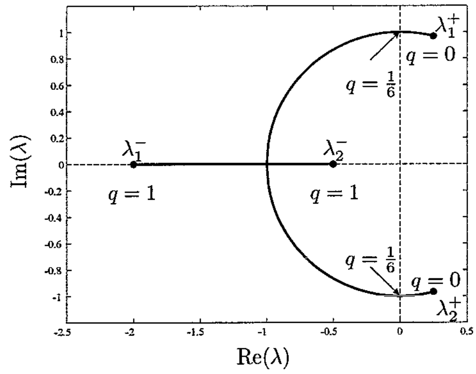


Figure 7.16: Eigenvalue-path of the discontinuous Hopf bifurcation of system (7.59).

We now study the following non-smooth continuous system

$$\begin{aligned} \dot{x}_1 &= -x_1 - \omega x_2 + \frac{x_1}{\sqrt{x_1^2+x_2^2}}(|\sqrt{x_1^2+x_2^2} + \tfrac{1}{2}\mu| - |\sqrt{x_1^2+x_2^2} - \tfrac{1}{2}\mu|), \\ \dot{x}_2 &= \omega x_1 - x_2 + \frac{x_2}{\sqrt{x_1^2+x_2^2}}(|\sqrt{x_1^2+x_2^2} + \tfrac{1}{2}\mu| - |\sqrt{x_1^2+x_2^2} - \tfrac{1}{2}\mu|), \end{aligned} \tag{7.56}$$

which is dependent on the parameters μ and ω. We will study the equilibria and periodic solutions of system (7.56) for different values of μ. The non-smooth system (7.56) has the same equilibrium as the smooth system with the same stability. We transform the system (7.56) with the transformation (7.53) into

$$\dot{r} = -r + |r + \frac{1}{2}\mu| - |r - \frac{1}{2}\mu| \tag{7.57}$$

$$\dot{\theta} = \omega \tag{7.58}$$

The one-dimensional system (7.57) is identical to the non-smooth system (7.45) exposing a discontinuous pitchfork bifurcation. The scenario for the discontinuous Hopf bifurcation is depicted in Figure 7.15 and the scenario for (7.57) is identical to Figure 7.12.

The discontinuous Hopf bifurcation of the preceding example occurred in a system with two switching boundaries. A discontinuous Hopf bifurcation can, however, also occur in a system with a single switching boundary. Consider the second order system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_2 - \tfrac{3}{2}|x_2 - \mu| - x_1, \end{aligned} \tag{7.59}$$

which has only one switching boundary $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 - \mu = 0\}$. Note that system (7.59) can easily be transformed to be of the type (7.25). The system has only one equilibrium

$$x_1 = -\frac{3}{2}|\mu|, \quad x_2 = 0, \tag{7.60}$$

which exists for all $\mu \in \mathbb{R}$. The generalized Jacobian of the system is

$$\boldsymbol{J}(\boldsymbol{x}, \mu) = \begin{bmatrix} 0 & 1 \\ -1 & -1 - \frac{3}{2}\operatorname{Sign}(x_2 - \mu) \end{bmatrix}. \tag{7.61}$$

The generalized Jacobian at the equilibrium jumps from $\boldsymbol{J}_-$ to $\boldsymbol{J}_+$ when μ is increased from $\mu < 0$ to $\mu > 0$ with

$$\begin{aligned} \boldsymbol{J}_+ &= \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{bmatrix}, \quad \lambda_{1,2} = \tfrac{1}{4} \pm \tfrac{1}{4}\sqrt{15}, \\ \boldsymbol{J}_- &= \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \quad \lambda_1 = -\tfrac{1}{2}, \ \lambda_2 = -2. \end{aligned} \tag{7.62}$$

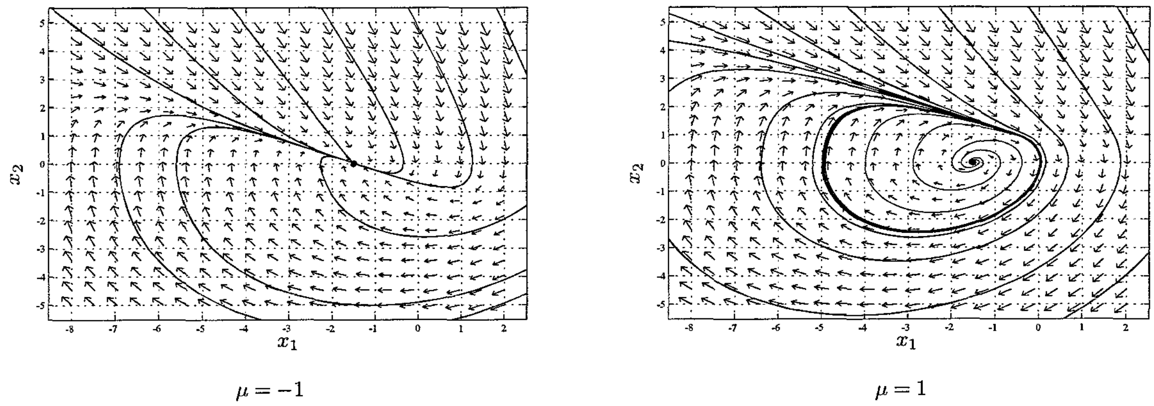


Figure 7.17: Discontinuous Hopf bifurcation of system (7.59).

The equilibrium is therefore a stable node for $\mu < 0$ and an unstable focus for $\mu > 0$ and therefore loses stability at $\mu = 0$. The set of the generalized Jacobian at the equilibrium point $x_1 = x_2 = 0$ for $\mu = 0$ can be expressed as $\boldsymbol{J}(\boldsymbol{0}) = \{\boldsymbol{J}_q, q \in [0,1]\}$ with $\boldsymbol{J}_q = q\boldsymbol{J}_+ + (1-q)\boldsymbol{J}_-$. The eigenvalues are therefore set-valued at the the equilibrium point $x_1 = x_2 = 0$ for $\mu = 0$ and form a one-dimensional path in the complex plane given by

$$\lambda = \begin{cases} \frac{1}{4} - \frac{3}{2}q \pm \frac{1}{4}i\sqrt{-(1-6q)^2+16} & q \leq \frac{5}{6}, \\ \frac{1}{4} - \frac{3}{2}q \pm \frac{1}{4}\sqrt{(1-6q)^2-16} & q > \frac{5}{6}, \end{cases} \tag{7.63}$$

with $q \in [0,1]$. The path of the eigenvalues is depicted in Figure 7.16. The eigenvalues cross the imaginary axis for $q = \frac{1}{6}$, which causes a discontinuous Hopf bifurcation. The phase plane of system (7.59) is shown in Figure 7.17 for $\mu = -1$ and for $\mu = 1$. The phase plane for $\mu = -1$ shows the equilibrium as a stable node, while the phase plane for $\mu = 1$ shows an unstable focus as well as a stable limit cycle, created by the discontinuous Hopf bifurcation.

In Section 7.3, a criterion is given for the co-existence of equilibria for a special class of piecewise linear systems to which system (7.59) belongs. Equilibria do not co-exist for system (7.59) according to condition (7.18), using the fact that

$$\det(\boldsymbol{J}_+)\det(\boldsymbol{J}_-) > 0, \tag{7.64}$$

which indeed agrees with the bifurcation scenario of system (7.59). Furthermore, Proposition 7.2 and the Hopf Bifurcation Theorem 7.2, presented in Section 7.5, apply for system (7.59). Indeed, it holds that

$$\mathrm{trace}(\boldsymbol{J}_+)\,\mathrm{trace}(\boldsymbol{J}_-) < 0, \tag{7.65}$$

and the set-valued eigenvalues therefore contain a crossing with the imaginary axis as a complex conjugated pair.

7.7 Multiple Crossing Bifurcations

The bifurcations in the previous section where all characterized by a single crossing of the eigenvalue(s) through the imaginary axis. If the eigenvalues were set-valued, which was the case for the discontinuous bifurcations, then set of eigenvalues formed a one-dimensional path in the complex plane. The eigenvalue(s) either moved continuously through the imaginary axis under the variation of a parameter (being a continuous bifurcation) or a one-dimensional path of eigenvalues crossed the imaginary axis during a jump (leading to a discontinuous bifurcation). Non-smooth continuous systems can also exhibit bifurcations of equilibria for which a one-dimensional path of eigenvalue(s) crosses multiple times the imaginary axis, as was already pointed out in Section 7.4. Equilibria of non-smooth continuous systems with multiple switching boundaries can have set-valued eigenvalues which do not form a one-dimensional path but a