On contractions and an inequality for nonhamiltonian graphs

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ON CONTRACTIONS AND AN INEQUALITY FOR NONHAMILTONIAN GRAPHS.

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CONTENTS

ABSTRACT

Both a sufficient condition and a necessary one for a 2-connected graph to be nonhamiltonian, as given by Hoede and Veldman in terms of contractibility, are generalized. The results are related to some theorems of Chvátal on weakly hamiltonian graphs.

1. TERMINOLOGY

We use the terminology of Harary's book [1] extended with some notations and definitions used by Chvátal [2] and by Hoede and Veldman [3].

We consider graphs without loops or multiple lines. $V(G) \equiv V$ and $X(G) \equiv X$ denote the set of points and the set of lines respectively of the graph $G = (V,X)$. By $G_1 \leq G_2$ we denote the fact that G_1 is a spanning subgraph of G_2. The graph obtained from a wheel on 2k spokes by deleting a set of k spokes incident to k mutually nonadjacent points is denoted by M_k. The graph $K_k + \overline{K}_{k+1}$ is denoted by N_k. Moreover $K(G)$ denotes the set of components of G, and $k(G) = |K(G)|$. For arbitrary sets $R,T \in V$ we define

$$[R,T] = \{\{r,t\} \mid r \in R,\ t \in T\}$$

$$[R,T]_G = [R,T] \cap X$$

$$q(R,T) = |[R,T] \cap X|$$

$$[R] = [R,R]$$

$$[R]_G = [R,R] \cap X$$

$$\langle R\rangle = \langle R\rangle_G = (R, [R]_G)$$
$$N(R) = \{v \in V \setminus R \mid [\{v\},R]_G \neq \emptyset\}$$
$$E(R) = [R,V \setminus R] \cap X$$
$$[R]^T = [R]_G^T = [R]_G \setminus \{x \in [R]_G \mid \exists K \in K(T) \ ([N(K)] = \{x\})\}$$
$$\langle R\rangle^T = \langle R\rangle_G^T = (R, [R]_G^T).$$

Contraction of a graph G to another graph H is considered to be a map $G \to H$. This explains the terminology of *images* and *inverse images* of points and lines in this context. A graph (V,X) is called T-*sharply contractible* to a graph H if the contraction of (V,X) is sharp and T is a set of points of G such that none of the points of T is contracted to a point outside T. $\lfloor x \rfloor$ denotes the greatest integer k with $k \leq x$.

2. A NECESSARY CONDITION FOR A GRAPH TO BE NONHAMILTONIAN

THEOREM 1.

If $\kappa \geq 2$ *and* G *is a* κ-*connected nonhamiltonian graph, then* G *is contractible to a graph* H_κ *with* $M_\kappa \leq H_\kappa \leq N_\kappa$.

Proof.

Let C be a cycle in G of maximal length. Because G is κ-connected a point w_0 in G not on C is connected to at least κ points $v_1, \ldots, v_\kappa$ of C by point-disjoint pathes $P_1, \ldots, P_\kappa$ (see [4]). No two of these points can be incident to the same line of C, as otherwise C might be enlarged to include w_0, contradicting the maximality of C.

Fix an orientation on C. We may assume that $v_1, \ldots, v_\kappa$ are on C in this order. The points $v_1, \ldots, v_\kappa$ are separated along C by other points. Let v_i be directly followed by the points w_i ($i = 1, \ldots, \kappa$). None of these points is connected to w_0 by a single line or by a path consisting of points not on C or on the pathes P_i, as this would again contradict the maximality of C. Neither is w_i connected to w_j by a single line or by a path consisting of points not on C or on the pathes P_i: if v_j follows v_i with respect to the orientation of C, consider the cycle C' consisting of the pathes P_i and P_j, the part of C from v_i to w_j in opposite direction of the orientation, the supposed connection from w_j to w_i and the part of C from w_i to v_j in the direction of the orientation. The size of C' contradicts the maximality of C.

As G is 2-connected any point not on C or on the pathes P_i is adjacent, or connected by two point-disjoint pathes of similar points, to two points on C or the pathes P_i; at least one of two such points is not one of the points $w_1, \ldots, w_\kappa$. Contraction of the points not on C or the pathes P_i to points different from the points $w_1, \ldots, w_\kappa$ leads to a graph on the points of C and the pathes P_i in which no pair of the points $w_1, \ldots, w_\kappa$ is adjacent.

The next set of contractions consists of contracting the points of path P_i, except point w_0, to v_i, using lines of this path.

No pair of points w_i and w_j has become adjacent in this process. Neither does this occur in the final contractions of those points of C that lie between w_i and v_{i+1} (indices modulo κ) to the point v_{i+1} for $i = 1, \ldots, \kappa$.

The resulting graph H contains the empty graph on the $\kappa + 1$ points $w_0, \ldots, w_\kappa$ and a cycle of 2κ points $v_1, w_1, \ldots, v_\kappa, w_\kappa$. The points $v_1, \ldots, v_\kappa$ are adjacent to w_0 and to at least two other points of $w_1, \ldots, w_\kappa$. In the contraction process they may have become adjacent to any other point as well. Thus $M_\kappa \leqslant H_\kappa \leqslant N_\kappa$.

■

COROLLARY 1.1 ([3], theorem 2).

If G *is a 2-connected nonhamiltonian graph, then* G *is contractible to* M_2 *or to* N_2.

Proof.

For $\kappa = 2$ we have $M_2 \leqslant H_2 \leqslant N_2$. As N_2 has only one line more than M_2 has, H_2 is either M_2 or N_2.

■

COROLLARY 1.2 ([5], theorem 1).

If for a graph G *with* $|V| \geqslant 3$ *we have* $\beta_0(G) \leqslant \kappa$, *then* G *is hamiltonian.*

Proof.

Suppose G is a nonhamiltonian graph with $\beta_0 \leqslant \kappa$. The point-independence number β_0 does not increase when carrying out the contraction of a single line. Thus $\beta_0(H_\kappa) \leqslant \beta_0(G) \leqslant \kappa$. On the other hand $\beta_0(H_\kappa) = \kappa + 1$. This contradiction shows that G must be hamiltonian.

■

REMARKS

1. Corollary 1.2 for $\kappa = 3$ shows that the graph in figure 1 is hamiltonian. Equivalently, the graph is hamiltonian because it is not contractible to any graph H_3 with $M_3 \leqslant H_3 \leqslant N_3$.

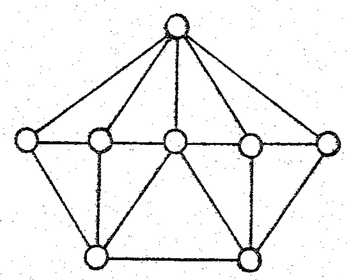


Figure 1.

However, this graph is contractible to N_2. This example shows that theorem 1 is more powerful than its corollary 1.1 when used in proving a graph to be hamiltonian. In this case corollary 1.2 is applicable while corollary 1.1 is not. Inversely, the conclusion that a cycle C_n, with $\lfloor \frac{n}{2} \rfloor = \beta_0 > \kappa = 2$, is hamiltonian may be drawn from corollary 1.1 but not from corollary 1.2.

2. Consider the graph $K_{\kappa+1,\ \kappa+1} - \{x\}$. This is a κ-connected graph with $\beta_0 = \kappa + 1$. That this graph is hamiltonian does not follow from corollary 1.2. It does follow from the fact that the graph is not contractible to a graph H_κ with $M_\kappa \leqslant H_\kappa \leqslant N_\kappa$. Thus theorem 1 turns out to be more powerful than its corollary 1.2 too. On the other hand addition of a line y, not equal to x, to this graph results in a graph that is hamiltonian and κ-connected, but nevertheless contractible to a graph H_κ with $M_\kappa \leqslant H_\kappa \leqslant N_\kappa$.

The example shows that the necessary condition of theorem 1 is not sufficient.

3. SUFFICIENT CONDITIONS FOR A GRAPH TO BE NONHAMILTONIAN

THEOREM 2.

If there is a partition $V = R_0 \cup R_1 \cup R_2 \cup T$ *of the point set of a graph* G *into pairwise disjoint (possibly empty) sets with* $T \neq V$ *and* $|R_2| + \sum_{\Gamma} \lfloor \frac{1}{2}|V(\Gamma) \cap R_1| \rfloor < k(T)$, Γ *ranging over* $K(\langle R_0 \cup R_1 \rangle^T)$, *with* R_1 *containing all* $v \in V \setminus T$ *such that* $q(v,K) > 0$ *for precisely one* $K \in K(T)$, *and* R_2 *containing all* $v \notin V \setminus T$ *such that* $\bar{q}(v,K_1) > 0$, $q(v,K_2) > 0$ *for (at least) two distinct* $K_1, K_2 \in K(T)$, *then* G *is nonhamiltonian.*

Proof.

Let G be a hamiltonian graph with hamiltonian cycle C. We estimate the number $k(\langle C \setminus T \rangle)$ of components of $\langle C \setminus T \rangle$. It is evident that $k(\langle C \setminus T \rangle) \geq k(T)$.

In traversing the cycle C from one component of T to another a point of R_2 may be traversed. The number of components of $\langle C \setminus T \rangle$ containing an element of R_2 is $\leq |R_2|$. Other components of $\langle C \setminus T \rangle$ necessarily constitute a path in a component Γ of $\langle R_0 \cup R_1 \rangle^T$. In Γ no more than $\lfloor \frac{1}{2}|V(\Gamma) \cap R_1| \rfloor$ such pathes are possible. Thus

$$k(\langle C \setminus T \rangle) \leq |R_2| + \sum_{\Gamma} \lfloor \frac{1}{2}|V(\Gamma) \cap R_1| \rfloor.$$

We obtain a contradiction whenever the stated inequality holds. ∎

COROLLARY 2.1 ([2], theorem 2.1).

If there is a partition $V = R \cup S \cup T$ *of the point set of a graph* G *into pairwise disjoint (possibly empty) sets with* $T \neq V$ *and* $|S| + \sum_{L} \lfloor \frac{1}{2} q(V(L),T) \rfloor < k(T)$, *where the summation is extended over all components* L *of the induced subgraph* $\langle R \rangle$, *then* G *is nonhamiltonian.*

Proof.

Put $R_2 = \{v \in V \setminus T \mid q(v,K_1) > 0,\ q(v,K_2) > 0$ for at least two distinct $K_1, K_2 \in K(T)\}$, and put $R_1 = \{v \in V \setminus T \mid q(v,K) > 0$ for precisely one $K \in K(T)\}$. R_1 and R_2 will contain a subset of S and a subset of each of the components L of $\langle R \rangle$. By definition $q(v,T) \geq 2$ for any $v \in R_2$. Thus

$$|R_2| \leqslant |S \cap R_2| + \sum_{L} \left\lfloor \frac{1}{2} q(V(L) \cap R_2, T) \right\rfloor .$$

Writing $R_0 = R \setminus (R_1 \cup R_2)$, for any component $\Gamma \in K(\langle R_0 \cup R_1 \rangle^T)$ we have

$$|V(\Gamma) \cap R_1| \leqslant |S \cap R_1 \cap V(\Gamma)| + q(V(L) \cap V(\Gamma) \cap R_1, T).$$

Summing over all Γ and combining both inequalities in a proper way we get

$$|R_2| + \sum_{\Gamma} \left\lfloor \frac{1}{2} |V(\Gamma) \cap R_1| \right\rfloor \leq |S \cap R_2| + |S \cap R_1| +$$

$$+ \sum_{L} \left\lfloor \frac{1}{2} q(V(L),T) \right\rfloor \leq |S| + \sum_{L} \left\lfloor \frac{1}{2} q(V(L),T) \right\rfloor$$

and theorem 2 can be applied. ∎

Corollary 2.1 is weaker than theorem 2. The theorem may also be used to show that the series of graphs in theorems 2.2 and 2.3 of [2] are nonhamiltonian. In the latter case it is essential that Γ ranges over $\langle R_0 \cup R_1\rangle^T$ instead of $\langle R_0 \cup R_1\rangle$.

COROLLARY 2.2.

If a graph is sharply contractible to M_κ $(\kappa \geq 2)$, *then* G *is nonhamiltonian.*

Proof.

Let T be the inverse image in G of the $\kappa + 1$ independent points in M_κ under a given sharp contraction and let R be the complement of T in V. The inverse image U of any point in M_κ satisfies $q(U,T) = 3$. Taking $S = \emptyset$ we may now apply corollary 2.1 noting that

$$\sum_L \tfrac{1}{2}\, q(V(L),T)_\lrcorner = \kappa \text{ and } k(T) = \kappa + 1. \qquad \blacksquare$$

Corollary 2.2 is a partial inverse to theorem 1 and generalizes theorem 3 in [3]. Other nonhamiltonian graphs may be substituted for M_κ, as long as there are no hamiltonian graphs that are also sharply contractible to that graph. However, in general the inverse image of a nonhamiltonian graph may be hamiltonian. In order to illustrate these difficulties in constructing a more general inverse to theorem 1, we present some special cases in the next proposition.

PROPOSITION.

If G *is a hamiltonian graph and* χ_2 *and* ψ_2 *are graphs as depicted in figures 2a and 2b respectively, the following holds.*

i) If G *is sharply contractible to a graph* H *with all degrees* ≤ 3, *then* H *is hamiltonian.*

ii) If G *is sharply contractible to* N_2 *respectively* $K_{2,4}$, *then* G *contains a point subset* T *such that* G *is* T*-sharply contractible to* χ_2 *respectively* ψ_2.

Proof.

i) For the inverse image T of any point of H we have $|[C]_G \cap [V \setminus T,T]| = 2$. Since these numbers are preserved after the contraction for each point of H, it is clear that the contraction of the hamiltonian cycle is a hamiltonian cycle in H. ∎

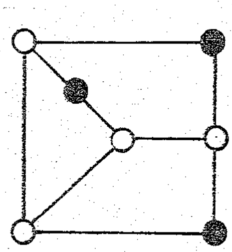
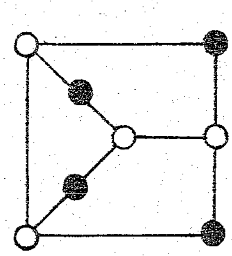

a)

Figure 2.


b)

ii) The case of $K_{2,4}$ being rather similar to the case of N_2, we shall only establish the result for N_2. Let T be the inverse image of G of the subgraph $\overline{K}_3$ and let R and R' be the inverse images of the two distinct points of the subgraph K_2 of $N_2 = K_2 + \overline{K}_3$ under the given sharp contraction. With R_1, R_2 and

Γ defined as in theorem 2, we get from this theorem

$$|R_2| + \lfloor \tfrac{1}{2}|R_1| \rfloor \geq |R_2| + \sum_{\Gamma} \lfloor \tfrac{1}{2}|V(\Gamma) \cap R_1| \rfloor \geq 3.$$

As the contraction is sharp, we have $|R_2| \leq 2$, whence $\lfloor \frac{1}{2}|R_1| \rfloor \geq 1$; so $|R_1| \geq 2$. If $|R_1| \geq 3$, then $q(V \setminus T, T) = 6$ implies that $|R_2| \leq 1$ and in R or in R' the hamiltonian cycle C connects two points of R_1. Contraction of this part of the cycle leads to a hamiltonian graph for which $|R_2|$ is enlarged by 1. Thus we can reduce the proof to the case $|R_1| = 2$. But then $|R_2| \geq 2$, so $|R_2| = 2$.

Evidently $|[V \setminus T, T] \cap C| = 6$ and $|R_1 \cap R| = |R_1 \cap R'| = |R_2 \cap R| = |R_2 \cap R'| = 1$. Moreover we must have $|[R,R'] \cap C| = 1$. Let $r_1 \in R_1 \cap R$, $r_1' \in R_1 \cap R'$ and $r \in R$, $r' \in R'$ with $\{\{r,r'\}\} = [R,R']_G$. There are parts of C from r_1 to r and from r_1' to r' that are completely within R and R' respectively. Contraction of C along these pathes gives χ_2. ∎

REMARKS

1. Theorem 2 provides a sufficient condition for a graph to be nonhamiltonian that is powerful enough to prove the results following from Chvátal's characterization of non-weakly-hamiltonian graphs. No use is made of results from the theory of linear programming and an interpretation is given to the sets arising in the partitioning of V. We conjecture, however,

that no graph satisfying the hypotheses of theorem 2 is weakly hamiltonian.

2. Proposition (i) shows that in corollary 2.2 one may substitute the Petersen graph for M_κ. However, proposition (ii) shows, that a similar statement with N_2 substituted for M_κ is not valid unless one adds the statement that the graph G is not sharply contractible to X_2.
The single "obstruction" X_2 for sharp contractibility of a graph G to a graph H_2 with $M_2 \leq H_2 \leq N_2$, tends to a large number of obstructions when passing from 2 to arbitrary κ.

3. In view of theorem 2 it is natural to define the *crossing quotient* to be

$$\tau(G) = \min_{\emptyset \neq T \neq V} \frac{|R_2| + \sum_{\Gamma} \lfloor \frac{1}{2} |V(\Gamma) \cap R_1| \rfloor}{k(T)},$$

where R_1, R_2 and the range of Γ are defined as in the theorem. Now the theorem can be restated as

$$\tau(G) < 1 \Rightarrow G \text{ is nonhamiltonian.}$$

In fact τ is a refinement of the toughness $t(G)$ defined in [2]: $\tau(G) \leq t(G)$. If G is the nonhamiltonian graph obtained from the Petersen graph by replacing each point by a triangle such that the resulting graph is regular of degree 3, then $\tau(G) = \frac{4}{3}$ holds, while $t(G) = \frac{3}{2}$.

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