

Sequences of points on a circle

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1. Introduction. We consider sequences $\{a\}$ of points a_1, a_2, a_3, \dots on a circle with radius $1/2\pi$, in other words numbers mod 1. The numbers a_1, a_2, \dots, a_n define n intervals with total length 1; denote by $M_n^1(a)$ and $m_n^1(a)$ the largest and the smallest length. Clearly

$$n M_n^1(a) \geq 1 \geq n m_n^1(a).$$

Analogously $M_n^r(a)$ and $m_n^r(a)$ denote the maximum and minimum length of the sum of r consecutive intervals, so that $n M_n^r(a) \geq r \geq n m_n^r(a)$. We put

$$\limsup_{n \rightarrow \infty} n M_n^r(a) = A_r(a)$$

$$\liminf_{n \rightarrow \infty} n m_n^r(a) = \lambda_r(a)$$

$$\limsup_{n \rightarrow \infty} M_n^r(a) / m_n^r(a) = \mu_r(a)$$

and

$$A_r = \text{g.l.b. } A_r(a) \quad , \quad \lambda_r = \text{l.u.b. } \lambda_r(a) \quad , \quad \mu_r = \text{g.l.b. } \mu_r(a).$$

We are able to determine

$$A_1 = 1/\log 2 \quad , \quad \lambda_1 = 1/\log 4 \quad , \quad \mu_1 = 2.$$

The problem of A_r, λ_r, μ_r is closely related to a problem concerning "just distributions" solved by Mrs VAN AARDENNE-EHRENFEST¹⁾. All we can prove is that $\mu_r \geq 1 + 1/r$ (and analogous inequalities for A_r and λ_r); we conjecture that $r(\mu_r - 1)$ is unbounded. From this the theorem of Mrs VAN AARDENNE-EHRENFEST would follow.

2. A sequence which gives the best possible values of $A_1(a), \lambda_1(a), \mu_1(a)$. Take $a_k = {}^2\log(2k-1)$, reduced mod 1. We show that a_1, \dots, a_n occur in the following order

$${}^2\log n, {}^2\log(n+1), \dots, {}^2\log(2n-1). \quad \dots \quad (2.1)$$

Namely, no two of the a_k 's and no two of the numbers (2.1) are congruent mod 1, but each number in (2.1) is congruent to just one a_k .

It follows from (2.1) that the lengths of the intervals defined by a_1, \dots, a_n are

$${}^2\log \frac{n+1}{n}, {}^2\log \frac{n+2}{n+1}, \dots, {}^2\log \frac{2n-1}{2n-2}, {}^2\log \frac{2n}{2n-1},$$

¹⁾ Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam 48, 266—271 (1945) = *Indagationes Mathematicae*, 7, 71—76 (1946).

and so

$$n M_n^1(a) = \frac{n \log \left(1 + \frac{1}{n}\right)}{\log 2}, \quad n m_n^1(a) = \frac{n \log \left(1 - \frac{1}{2n}\right)^{-1}}{\log 2}.$$

For $n \rightarrow \infty$, $n M_n^1(a)$ increases to the limit $1/\log 2$; $n m_n^1(a)$ decreases to the limit $1/\log 4$; $M_n^1(a)/m_n^1(a)$ increases to the limit 2. It follows that $\Delta_1(a) = 1/\log 2$, $\lambda_1(a) = 1/\log 4$, $\mu_1(a) = 2$.

3. Lower bound for $\Delta_r(a)$. *

Let $\{a\}$ be a sequence, n a natural number, and suppose that ϱ is such that

$$k M_n^1(a) < \varrho. \quad (n \leq k < 2n) \quad \dots \quad (3.1)$$

Let the intervals determined by a_1, \dots, a_n be I_1, \dots, I_n , arranged in descending order of length. Denote the length of I_j by a_j ; so that

$$a_1 \geq a_2 \geq \dots \geq a_n; \quad a_1 + \dots + a_n = 1. \quad \dots \quad (3.2)$$

Now put in the points $a_{n+1}, a_{n+2}, \dots, a_{2n-1}$. Since any point "destroys" one I at most, there remains at least one interval of length $\geq a_p$ undisturbed after $a_{n+1}, \dots, a_{n+p-1}$ have been put in ($1 \leq p \leq n$). Hence

$$M_n^1(a) \geq a_1, \quad M_{n+1}^1(a) \geq a_2, \dots, \quad M_{2n-1}^1(a) \geq a_n;$$

consequently, by (3.1) and (3.2),

$$\varrho \left(\frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{2n-1} \right) > 1.$$

It follows that for at least one k ($n \leq k < 2n$) we have

$$k M_k^1(a) \geq \left(\frac{1}{n} + \dots + \frac{1}{2n-1} \right)^{-1} = \sigma_n.$$

We have $\sigma_n < 1/\log 2$, $\sigma_n \rightarrow 1/\log 2$, and so $\Delta_1(a) \geq 1/\log 2$. This holds for any $\{a\}$; the lower bound is attained for the sequence of section 2.

Similarly we can prove that for at least one k ($rn \leq k < (r+1)n$) we have

$$k M_k^r(a) \geq \left(\frac{1}{rn} + \frac{1}{rn+1} + \dots + \frac{1}{rn+n-1} \right)^{-1}$$

and so

$$\Delta_r(a) \geq 1/\log \left(1 + \frac{1}{r} \right) > r.$$

4. Upper bound for $\lambda_r(a)$ ²⁾.

Let $\{a\}$ be a sequence, n a natural number, and suppose that ϱ is such that

$$k m_k^1(a) > \varrho \quad (n < k \leq 2n). \quad \dots \quad (4.1)$$

²⁾ The proof presented in this section was found by Mrs. VAN AARDENNE-EHRENFEST independently.

Let $a_{k_1}, a_{k_2}, \dots, a_{k_{2n}}$ be the cyclic order of the points a_1, \dots, a_{2n} on the circle (k_1, \dots, k_{2n} is a permutation of $1, \dots, 2n$); put $k_{2n+1} = k_1$. If $k_i^* = \text{Max}(k_i, k_{i+1}, n+1)$, then the interval $a_{k_i}, a_{k_{i+1}}$ is one of the intervals determined by $a_1, \dots, a_{k_i^*}$. It follows that its length is less than q/k_i^* . Hence

$$1 > q \sum_{i=1}^{2n} 1/k_i^* \dots \dots \dots (4.2)$$

We have $n < k_i^* \leq 2n$, and any k ($n+1 < k \leq 2n$) occurs ε_k times as a k^* ; $\varepsilon_k = 0, 1$ or 2 . It follows that

$$\sum_{i=1}^{2n} 1/k_i^* = \sum_{n+1}^{2n} \frac{2}{k} + \sum_{n+2}^{2n} (2 - \varepsilon_k) \left\{ \frac{1}{n+1} - \frac{1}{k} \right\} \geq \sum_{n+1}^{2n} \frac{2}{k}$$

Finally, by (4.1) and (4.2) we infer that at least for one ($n < k \leq 2n$) we have

$$k m_k^1(a) \leq \left(\frac{2}{n+1} + \frac{2}{n+2} + \dots + \frac{2}{2n} \right)^{-1} = \tau_n$$

We have $\tau_n > 1/\log 4$, $\tau_n \rightarrow 1/\log 4$, and so $\lambda_1(a) \leq 1/\log 4$. The example of section 2 again shows that $1/\log 4$ is best possible.

Similarly we can show that for at least one k ($r n < k \leq (r+1)n$) we have

$$k m_k^r(a) \leq r \left(\frac{r+1}{nr+1} + \dots + \frac{r+1}{nr+n-1} \right)^{-1}$$

and so

$$\lambda_r(a) \leq \frac{r}{r+1} / \log \left(1 + \frac{1}{r} \right) < r \dots \dots \dots (4.3)$$

5. Lower bound for μ_r .

Let $\{a\}$ be a sequence. We first prove that, for $r \geq 1$, $n \geq 1$ we have

$$M_n^r(a) / m_{n+1}^r(a) \geq 1 + \frac{1}{r} \dots \dots \dots (5.1)$$

We first suppose that $r > 1$. Let I_1, I_2, \dots, I_n be the intervals of the n -th stage, i.e. the intervals determined by a_1, \dots, a_n . Let I_{k_0} be the one into which a_{n+1} falls, and let

$$I_{k_{-r+1}}, I_{k_{-r+2}}, \dots, I_{k_0}, I_{k_1}, \dots, I_{k_{r-1}} \dots \dots \dots (5.2)$$

be consecutive on the circle³⁾.

Put $M = M_n^r(a)$, $m = m_{n+1}^r(a)$ and denote by M_1 the maximum length of the sum of r consecutive intervals from the set (5.2). Denote the length of I_{k_i} by β_i . Let γ_1 and γ_2 be the lengths of the parts into which I_{k_0} is divided by a_{n+1} .

³⁾ If $2r - 1 > n$ the k_i are not all different.

Clearly at least one of the numbers $\beta_{-r+1}, \dots, \beta_{-1}, \beta_1, \dots, \beta_{r-1}$ (β_j say) is $\geq (M_1 - \beta_0)/(r-1)$; we may suppose that $j > 0$. Now we have

$$m \leq \beta_{j-r+1} + \dots + \beta_{-1} + \gamma_1 + \gamma_2 + \dots + \beta_{j-1}$$

and hence

$$m \leq M_1 - \beta_j \leq \frac{r-2}{r-1} M_1 + \frac{\beta_0}{r-1} \dots \dots \dots (5.3)$$

On the other hand it follows from

$$\begin{aligned} m &\leq \gamma_2 + \beta_1 + \dots + \beta_{r-1} \leq M_1 - \gamma_1 \\ m &\leq \beta_{-r+1} + \dots + \beta_{-1} + \gamma_1 \leq M_1 - \gamma_2 \end{aligned}$$

that

$$m \leq M_1 - \frac{1}{2} \beta_0 \dots \dots \dots (5.4)$$

Trivially we have $M_1 \leq M$. If $\beta_0 \leq 2M_1/(r+1)$ we infer $m \leq M_1 r/(1+r) \leq M r/(1+r)$ from (5.3); if $\beta_0 \geq 2M_1/(r+1)$ we deduce the same result from (5.4). This proves (5.1) for $r > 1$.

If $r = 1$, (5.1) immediately follows from

$$m \leq \text{Min}(\gamma_1, \gamma_2) \leq \frac{1}{2} \beta_0 \leq \frac{1}{2} M.$$

Now suppose that n is a natural number and that for $nr \leq k \leq n(r+1)$ we have

$$M'_k(a)/m'_k(a) < \left(1 + \frac{1}{r}\right) / \left(1 + \frac{1}{k}\right)^2 \dots \dots \dots (5.5)$$

It follows, by (5.1) that

$$m'_{k+1}/m'_k < k^2/(k+1)^2 \quad (nr \leq k < nr+n)$$

and also

$$m'_{rn+n}/m'_{rn} < r^2/(1+r)^2 \dots \dots \dots (5.6)$$

Trivially we have $m'_{rn} \leq 1/n$; on the other hand, by (5.5)

$$m'_{rn+n} > \frac{r}{1+r} M'_{rn+n} \geq \frac{r}{1+r} \cdot \frac{r}{rn+n-1} \geq \frac{r^2}{(r+1)^2} \cdot \frac{1}{n}.$$

This contradicts (5.6). Hence for at least one k ($nr \leq k \leq nr+n$) (5.5) is not true. It follows that

$$\mu_r \geq 1 + \frac{1}{r} \dots \dots \dots (5.7)$$

6. The inequalities (3.3), (4.3) and (5.7) are probably not best possible if $r \geq 2$. We conjecture that the expressions

$$r(A_r - 1) \quad , \quad r(1 - \lambda_r) \quad , \quad r(\mu_r - 1)$$

tend to infinity if $r \rightarrow \infty$.

We owe some useful remarks to Mrs. T. VAN AARDENNE-EHRENFEST and Mr. J. KOREVAAR with whom we first discussed the above problems.