

# An extension of a theorem of Cauchy with applications to probability theory

**Citation for published version (APA):**

Vanderperre, E. J. (1975). *An extension of a theorem of Cauchy with applications to probability theory*. (EUT report. WSK, Dept. of Mathematics and Computing Science; Vol. 75-WSK-05). Eindhoven University of Technology.

**Document status and date:**

Published: 01/01/1975

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

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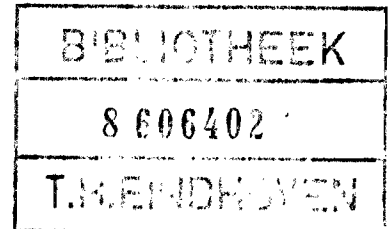
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An extension of a theorem of Cauchy with  
applications to Probability Theory

by

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T.H.-Report 75-WSK-05

August 1975

## Abstract

The paper deals with an extension of Cauchy's theorem for improper integrals. We use the theorem to solve an integral equation of the Pollaczek type [7] and to study a Markov chain  $\{w_n, n \geq 0\}$  with state space  $[0, \infty[$  closely related to Queueing Theory.

1. An extension of a theorem of Cauchy

1.1. Introduction

Let  $S$  be the complex  $\omega$ -plane,  $S_{\pm} := \{\omega \mid \text{Im } \omega \gtrless 0\}$  and  $\mathbb{R} := \{\sigma \mid -\infty < \sigma < +\infty\}$ .

A. Let  $f(\omega)$  be a function, holomorphic in  $S_{-}$  and continuous in  $S_{-} + \mathbb{R}$  including the point at infinity i.e.,  $f(\omega) = f(\infty) + o(1)$  as  $|\omega| \rightarrow \infty$  in  $S_{-} + \mathbb{R}$ . Then

$$(a) \quad \frac{1}{2\pi i} \int_{-\infty}^{+\infty} f(\sigma) \frac{d\sigma}{\sigma - \omega} = \begin{cases} \frac{1}{2}f(\infty) & \text{if } \omega \in S_{+} , \\ -f(\omega) + \frac{1}{2}f(\infty) & \text{if } \omega \in S_{-} . \end{cases}$$

The symbol  $C$  indicates the principal value (Cauchy value) of the integral.

B. Let  $f(\omega)$  be a function, holomorphic in  $S_{+}$  and continuous in  $S_{+} + \mathbb{R}$  including the point at infinity. Then

$$(b) \quad \frac{1}{2\pi i} \int_{-\infty}^{+\infty} f(\sigma) \frac{d\sigma}{\sigma - \omega} = \begin{cases} f(\omega) - \frac{1}{2}f(\infty) & \text{if } \omega \in S_{+} , \\ -\frac{1}{2}f(\infty) & \text{if } \omega \in S_{-} . \end{cases}$$

The formulae (a) and (b) are called "Cauchy formulae" for the regions  $S_{-}$  and  $S_{+}$  respectively.

A simple proof of (a) and (b) is given in [6].

Remark. The condition that  $f(\omega) = f(\infty) + o(1)$  as  $|\omega| \rightarrow \infty$  in  $S_{-} + \mathbb{R}$  (or in  $S_{+} + \mathbb{R}$ ) is not necessarily satisfied when  $f(\omega)$  is a Fourier-Stieltjes transform (F.S.T.). Therefore, we shall prove the following extension related to Probability Theory.

1.2. The theorem

Let  $\{\Omega, B, P\}$  be a probability space with  $\Omega := ]-\infty, +\infty[$  the sample space,  $B$  the smallest  $\sigma$ -algebra containing all open subsets (Borel sets) of  $\Omega$  and  $P$  a probability measure on  $B$ . Introduce a  $B$ -measurable mapping  $\underline{x}: \Omega \rightarrow \mathbb{R}$  (a random variable) with probability distribution function (p.d.f.)

$$F(\underline{x}) := P[\underline{x} < x], \quad x \in \mathbb{R}$$

and define

$$E\{e^{-i\sigma\underline{x}}\} := \int_{-\infty}^{+\infty} e^{-i\sigma x} dF(x), \quad \sigma \in \mathbb{R}.$$

Denote by  $(A)$  de indicator of the event  $A$ , i.e.

$$(A) = \begin{cases} 1 & \text{if } A \text{ occurs,} \\ 0 & \text{if not .} \end{cases}$$

Theorem 1.

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} \frac{1}{2}E(\underline{x} = 0) - E\{e^{-i\omega\underline{x}}(\underline{x} \geq 0)\} & \text{if } \omega \in S_- , \\ -\frac{1}{2}E(\underline{x} = 0) + E\{e^{-i\omega\underline{x}}(\underline{x} \leq 0)\} & \text{if } \omega \in S_+ . \end{cases}$$

Proof. Since  $(\underline{x} < 0) + (\underline{x} = 0) + (\underline{x} > 0) = 1$ , we have  $\forall \omega \mid \omega \in S_+ \cup S_-$

$$(1.1) \quad \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}\} \frac{d\sigma}{\sigma - \omega} = \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}(\underline{x} > 0)\} \frac{d\sigma}{\sigma - \omega} + \\ + E(\underline{x} = 0) \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{d\sigma}{\sigma - \omega} + \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}(\underline{x} < 0)\} \frac{d\sigma}{\sigma - \omega},$$

provided that all principal values involved exist.

We shall now prove that

$$(1.2) \quad \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}(\underline{x} > 0)\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} -E\{e^{-i\omega\underline{x}}(\underline{x} > 0)\}, & \omega \in S_- , \\ 0 & \omega \in S_+ . \end{cases}$$

In order to show the statement, let  $\omega \in S_-$ , i.e., put  $\omega = Re^{i\theta}$ ,  $\pi < \theta < 2\pi$ ,  $0 < R < \infty$ .

Consider a closed curve  $\Gamma$  consisting of a segment  $[-T, T]$  of the real line and of a semi-circle  $\gamma$  lying in  $S_-$ , with center at the origin and with radius  $T \mid 0 < R < T < \infty$ . Finally, let  $\omega' = Te^{i\theta'}$ ,  $\pi < \theta' < 2\pi$ . The contour integration is clockwise.

From the above definition of  $\Gamma$ , we obtain for  $\omega \in S_-$

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma} E\{e^{-i\xi \underline{x}}(\underline{x} > 0)\} \frac{d\xi}{\xi - \omega} &= \frac{1}{2\pi i} \int_{-T}^{+T} E\{e^{-i\sigma \underline{x}}(\underline{x} > 0)\} \frac{d\sigma}{\sigma - \omega} + \\ &+ \frac{1}{2\pi i} \int_{\gamma} E\{e^{-i\omega' \underline{x}}(\underline{x} > 0)\} \frac{d\omega'}{\omega' - \omega} . \end{aligned}$$

But

$$E\{e^{-i\omega' \underline{x}}(\underline{x} > 0)\} = \int_{0+}^{\infty} e^{-i\omega' x} dF(x) .$$

Whence, by Cauchy's theorem, we obtain for  $\omega \in S_-$

$$\begin{aligned} -E\{e^{-i\omega \underline{x}}(\underline{x} > 0)\} &= \frac{1}{2\pi i} \int_{-T}^{+T} E\{e^{-i\sigma \underline{x}}(\underline{x} > 0)\} \frac{d\sigma}{\sigma - \omega} \\ &+ \frac{1}{2\pi i} \int_{\gamma} \int_{0+}^{\infty} e^{-i\omega' x} dF(x) \frac{d\omega'}{\omega' - \omega} . \end{aligned}$$

Now

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\gamma} \int_{0+}^{\infty} e^{-i\omega' x} dF(x) \frac{d\omega'}{\omega' - \omega} \right| &\leq \frac{T}{2\pi} \int_{\pi}^{2\pi} \int_{0+}^{\infty} \frac{e^{-Tx \sin \theta'} dF(x) d\theta'}{\min_{\pi < \theta' < 2\pi} |Te^{i\theta'} - Re^{i\theta}|} \\ &\leq \frac{T}{T - R} \int_{0+}^{\infty} \int_0^{\pi/2} e^{-Tx \sin \varphi} d\varphi dF(x) . \end{aligned}$$

The permutation of both integrals in the right-hand side of the first inequality is justified by Fubini's theorem.

It may be verified that for any  $\ell \in ]0, \pi/2[$

$$\int_0^{\pi/2} e^{-Tx \sin \varphi} d\varphi \leq \sec \ell \int_0^{\ell} e^{-Tx \sin \varphi} \cos \varphi d\varphi + (\pi/2 - \ell) e^{-Tx \sin \ell} .$$

After some algebra we obtain ( $\ell$  fixed),

$$\left| \frac{1}{2\pi i} \int_{\gamma} \int_{0+}^{\infty} e^{-i\omega' x} dF(x) \frac{d\omega'}{\omega' - \omega} \right| \leq (\pi/2 - \ell) \frac{T}{T - R} \int_{0+}^{\infty} e^{-Tx \sin \ell} dF(x) +$$

$$+ \frac{T}{T-R} \tan \ell \{F(T^{-1} \operatorname{cosec} \ell +) - F(0+)\} + \frac{T}{T-R} \sec \ell \int_{0+}^{\infty} \frac{1}{Tx} \chi_a(x) dF(x),$$

where  $a := ]T^{-1} \operatorname{cosec} \ell, \infty[$  and

$$\chi_a(x) = \begin{cases} 1 & \text{if } x \in a, \\ 0 & \text{if } x \notin a. \end{cases}$$

Since  $\frac{1}{Tx} \chi_a(x) \leq 1$ , we have by a well-known dominated convergence theorem

$$\frac{1}{2\pi i} \int_{\gamma} \int_{0+}^{\infty} e^{-i\omega'x} dF(x) \frac{d\omega'}{\omega' - \omega} = o_T(1) \quad \text{as } T \rightarrow +\infty.$$

Finally, we remark that

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\sigma}{\sigma - \omega} = \begin{cases} -\frac{1}{2} & \text{if } \omega \in S_-, \\ +\frac{1}{2} & \text{if } \omega \in S_+. \end{cases}$$

In a similar way, we can prove that

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} E\{e^{i\sigma \underline{x}'}(\underline{x}' > 0)\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} E\{e^{i\omega \underline{x}'}(\underline{x}' > 0)\}, & \omega \in S_+, \\ 0 & \omega \in S_-. \end{cases}$$

Hence, for  $\underline{x}' = -\underline{x}$ , we have

$$(1.3) \quad \frac{1}{2\pi i} \int_{-\infty}^{+\infty} E\{e^{-i\sigma \underline{x}}(\underline{x} < 0)\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} E\{e^{-i\omega \underline{x}}(\underline{x} < 0)\}, & \omega \in S_+, \\ 0 & \omega \in S_-. \end{cases}$$

By (1.1), (1.2) and (1.3) we obtain

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} E\{e^{-i\sigma \underline{x}}\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} -\frac{1}{2}E(\underline{x} = 0) - E\{e^{-i\omega \underline{x}}(\underline{x} > 0)\}, & \text{if } \omega \in S_-, \\ +\frac{1}{2}E(\underline{x} = 0) + E\{e^{-i\omega \underline{x}}(\underline{x} < 0)\}, & \text{if } \omega \in S_+. \end{cases}$$

Corollary 1. Let  $\underline{x}$  be a nonnegative random variable (r.v.) with characteristic function (c.f.),  $E\{e^{-i\omega \underline{x}}\}$ ,  $\operatorname{Im} \omega \leq 0$ . Then

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} E\{e^{-it\underline{x}}\} \frac{dt}{t - \omega} = \begin{cases} \frac{1}{2}P[\underline{x} = 0] - E\{e^{-i\omega \underline{x}}\}, & \omega \in S_-, \\ \frac{1}{2}P[\underline{x} = 0], & \omega \in S_+. \end{cases}$$

(1.4)

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} E\{e^{it\underline{x}}\} \frac{dt}{t-\omega} = \begin{cases} -\frac{1}{2}P[\underline{x} = 0] + E\{e^{i\omega\underline{x}}\}, & \omega \in S_+ , \\ -\frac{1}{2}P[\underline{x} = 0], & \omega \in S_- . \end{cases}$$

Corollary 2. Let  $\langle \underline{x}_n \rangle_{n=0}^{\infty}$  be a sequence of nonnegative r.v.'s with corresponding sequence of c.f.'s  $\langle E\{e^{-i\omega\underline{x}_n}\} \rangle_{n=0}^{\infty}$ ,  $\text{Im } \omega \leq 0$ .

For  $\omega \in S_-$  and  $z \in [0, 1[$  we have

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} z^n E\{e^{-it\underline{x}_n}\} \frac{dt}{t-\omega} = \frac{1}{2} \sum_{n=0}^{\infty} z^n P[\underline{x}_n = 0] - \sum_{n=0}^{\infty} z^n E\{e^{-i\omega\underline{x}_n}\},$$

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} z^n E\{e^{it\underline{x}_n}\} \frac{dt}{t-\omega} = -\frac{1}{2} \sum_{n=0}^{\infty} z^n P[\underline{x}_n = 0] .$$

Proof. Let  $\langle p_n \rangle_{n=0}^{\infty}$  be a discrete proper probability distribution, then

$\sum_{n=0}^{\infty} p_n E\{e^{-i\omega\underline{x}_n}\}$ ,  $\text{Im } \omega \leq 0$ , is a c.f. of a nonnegative r.v. and we may apply

(1.4). Since  $p_n$  is arbitrary, we may put  $p_n = (1-z)z^n$ ,  $\forall z \mid z \in [0, 1[$ .

## 2. An integral equation of the Pollaczek type

In this section, we use the results of section 1 to solve an integral equation of the Pollaczek type [7].

In order to introduce the equation, let  $\{\tau_n, n \in \mathbb{N}\}$  be a family of independent, identically distributed r.v.'s with p.d.f.  $A(\tau)$  concentrated on  $\mathbb{R}$  and with c.f.  $a(\sigma) = E\{e^{-i\sigma\underline{\tau}}\}$ ,  $\sigma \in \mathbb{R}$ .

Define

$$\underline{s}_n := \tau_1 + \tau_2 + \dots + \tau_n, \quad n \in \mathbb{N} .$$

$\{\underline{s}_n, n \in \mathbb{N}\}$  is a random walk generated by  $A(\cdot)$ . Let us introduce the following functions for  $z \in [0, 1[$

$$(2.1) \quad \gamma_+(\omega, z) = e^{-\sum_{n \geq 1} \frac{z^n}{n} E\{e^{-i\omega\underline{s}_n}(\underline{s}_n \leq 0)\}}, \quad \text{Im } \omega \geq 0 ,$$



$$(2.2) \quad \gamma_-(\omega, z) = e^{+\sum_{n \geq 1} \frac{z^n}{n} E\{e^{-i\omega \underline{s}_n} (\underline{s}_n > 0)\}}, \quad \text{Im } \omega \leq 0 .$$

By (2.1) and (2.2), we obtain for  $z \in [0, 1[$  and  $\sigma \in \mathbb{R}$

$$(2.3) \quad \gamma_+(\sigma, z) - [1 - za(\sigma)]\gamma_-(\sigma, z) = 0 .$$

Define

$$\begin{cases} \underline{x}_n := \max\{0, \underline{s}_1, \underline{s}_2, \dots, \underline{s}_{n-1}\}, & n \geq 2, \\ \underline{x}_1 := 0. \end{cases}$$

From the results of Fluctuation Theory, (see e.g. Cohen [1], pag. 151) we have for  $z \in [0, 1[$  and  $\text{Im } \omega \leq 0$

$$(2.4) \quad \sum_{n \geq 1} z^n E\{e^{-i\omega \underline{x}_n}\} = \frac{z}{1-z} e^{-\sum_{n \geq 1} \frac{z^n}{n} E(\underline{s}_n > 0)} \gamma_-(\omega, z) .$$

A similar expression can be given with respect to  $\gamma_+$ .

To simplify notation, we define

$$\alpha(z) := \frac{1}{2} \sum_{n \geq 1} \frac{z^n}{n} E(\underline{s}_n = 0) .$$

By (2.1), (2.2) we have

$$(2.5) \quad \lim_{\substack{\text{Im } \omega \rightarrow +\infty \\ \text{Re } \omega = 0}} \gamma_+(\omega, z) = e^{-2\alpha(z)} ,$$

$$(2.6) \quad \lim_{\substack{\text{Im } \omega \rightarrow -\infty \\ \text{Re } \omega = 0}} \gamma_-(\omega, z) = 1 .$$

Hence by corollary 1 and 2, (2.4), (2.5) and (2.6) we obtain after simplification

$$(2.7) \quad \gamma_-(\omega, z) = \frac{1}{2} - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \gamma_-(\sigma, z) \frac{d\sigma}{\sigma - \omega} , \quad \omega \in S_- ,$$

$$(2.8) \quad -\frac{1}{2} e^{-2\alpha(z)} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \gamma_+(\sigma, z) \frac{d\sigma}{\sigma - \omega} , \quad \omega \in S_- .$$

Rewriting (2.3) in the form

$$\gamma_+(\sigma, z) - \gamma_-(\sigma, z) = za(\sigma)\gamma_-(\sigma, z)$$

yields by (2.7) and (2.8),

$$(2.9) \quad \gamma_-(\omega, z) = \frac{1}{2}[1 + e^{-2\alpha(z)}] - \frac{z}{2\pi i} \int_{-\infty}^{+\infty} a(\sigma)\gamma_-(\sigma, z) \frac{d\sigma}{\sigma - \omega} .$$

Define

$$(2.10) \quad u_-(\omega, z) := \frac{\gamma_-(\omega, z)}{\frac{1}{2}[1 + e^{-2\alpha(z)}]} , \quad \text{Im } \omega \leq 0 .$$

By (2.9) and (2.10) we have that  $u_-(\omega, z)$  is the solution of the integral equation

$$(2.11) \quad u_-(\omega, z) = 1 - \frac{z}{2\pi i} \int_{-\infty}^{+\infty} a(\sigma)u_-(\sigma, z) \frac{d\sigma}{\sigma - \omega} , \quad \omega \in S_- .$$

Finally, we define

$$p_n := -\frac{z^n}{n} \frac{1}{\log(1-z)} , \quad n \in \mathbb{N}, z \in ]0, 1[ .$$

Since

$$\sum_{n=1}^{\infty} p_n a^n(\sigma) = \frac{\log[1 - za(\sigma)]}{\log(1-z)}$$

is the c.f. of the p.d.f.  $\sum_{n=1}^{\infty} p_n A_n^*(t)$ ,  $t \in \mathbb{R}$ , we have for  $\omega \in S_-$

$$(2.12) \quad \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \log[1 - za(\sigma)] \frac{d\sigma}{\sigma - \omega} = \alpha(z) + \log \gamma_-(\omega, z) .$$

By (2.10) and (2.12) we finally obtain for  $\omega \in S_-$ ,  $z \in [0, 1[$

$$u_-(\omega, z) = \text{sech } \alpha(z) e^{\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \log[1 - za(\sigma)] \frac{d\sigma}{\sigma - \omega}} .$$

Remarks. Integral equations similar to (2.11) have been studied in [7] and [6] using different methods. Pollaczek assumed that  $a(\omega)$  is analytic in some strip  $|\text{Im } \omega| < \delta$  for some  $\delta > 0$ , whereas Keilson and Kooharian made the assumption that  $a(\sigma) = o(1)$  as  $|\sigma| \rightarrow \infty$ . (In this case  $\text{sech } \alpha(z) = 1$ .)

By theorem 1 and a combination of some of the techniques presented in [1], [2] and [7], we have shown that the above assumptions concerning  $a(\sigma)$  are no longer necessary to solve equations similar to (2.11).

As a second application of theorem 1, we study:

### 3. The transient behaviour of a Markov chain $\{\underline{w}_n; n \geq 0\}$ with state space $[0, \infty[$

#### 3.1. Introduction

Let  $\{\underline{w}_n, n \geq 0\}$  be a Markov chain with state space  $[0, \infty[$  recursively determined by

$$(3.1) \quad \begin{cases} \underline{w}_0 \stackrel{\text{a.s.}}{=} 0, \\ \underline{w}_{n+1} = \begin{cases} \max\{0, \underline{\sigma}_H^{(n)} - \underline{\sigma}_G^{(n)}\}, & \text{if } \underline{w}_n = 0, \\ \max\{0, \underline{w}_n + \underline{\sigma}_F^{(n)} - \underline{\sigma}_G^{(n)}\}, & \text{if } \underline{w}_n \neq 0, \end{cases} \end{cases}$$

where  $\underline{\sigma}_G^{(n)}$ ,  $n = 0, 1, 2, \dots$  are positive independent r.v.'s with p.d.f.

$P[\underline{\sigma}_G^{(n)} < t] = G(t)$  and  $P[\underline{\sigma}_G^{(n)} \geq 0] = 1$  (ditto  $\underline{\sigma}_H^{(n)}, \underline{\sigma}_F^{(n)}$ ).

We also assume that  $\underline{\sigma}_G^{(n)}, \underline{\sigma}_H^{(n)}, \underline{\sigma}_F^{(n)}$  are mutually independent for each  $n$ .

Remark. The transient behaviour of the Markov chain  $\{\underline{w}_n\}$  has been studied by J. Loris-Teghem [4], [5] using an algebraic method based on a theorem due to G. Baxter [5]. Our approach is based on a Wiener-Hopf integral equation of the second kind and its reduction to a functional equation on  $\mathbb{R}$  in terms of F.S.T.'s.

#### 3.2. A Wiener-Hopf integral equation for the Markov chain $\{\underline{w}_n\}$

For  $t > 0$  and  $n = 0, 1, 2, \dots$  we define

$$(3.2) \quad W_n(t) := P[\underline{w}_n < t]$$

and  $W_n(0+) := \lim_{t \downarrow 0} W_n(t)$ .

For  $t \in \mathbb{R}$ , we define

$$A_{\underline{F}}(t) := P[\underline{\sigma}_{\underline{F}}^{(0)} - \underline{\sigma}_{\underline{G}}^{(0)} < t] ,$$

$$A_{\underline{H}}(t) := P[\underline{\sigma}_{\underline{H}}^{(0)} - \underline{\sigma}_{\underline{G}}^{(0)} < t] .$$

Whence for  $t \in \mathbb{R}$

$$(3.3) \quad A_{\underline{F}}(t) = \int_0^{\infty} F(t + \tau) dG(\tau) ,$$

$$(3.4) \quad A_{\underline{H}}(t) = \int_0^{\infty} H(t + \tau) dG(\tau) .$$

By (3.1) we have for  $t > 0$  and  $n \geq 1$

$$\begin{aligned} (\underline{w}_{\underline{n}} < t) &= (\max\{0, \underline{\sigma}_{\underline{H}}^{(n-1)} - \underline{\sigma}_{\underline{G}}^{(n-1)}\} < t) (\underline{w}_{\underline{n-1}} = 0) + \\ &\quad + (\max\{0, \underline{w}_{\underline{n-1}} + \underline{\sigma}_{\underline{F}}^{(n-1)} - \underline{\sigma}_{\underline{G}}^{(n-1)}\} < t) (\underline{w}_{\underline{n-1}} > 0) . \end{aligned}$$

Since  $(\underline{w}_{\underline{n-1}} > 0) = 1 - (\underline{w}_{\underline{n-1}} = 0)$ , we have

$$\begin{aligned} (\underline{w}_{\underline{n}} < t) &= (\max\{0, \underline{\sigma}_{\underline{H}}^{(n-1)} - \underline{\sigma}_{\underline{G}}^{(n-1)}\} < t) (\underline{w}_{\underline{n-1}} = 0) + \\ &\quad - (\max\{0, \underline{\sigma}_{\underline{F}}^{(n-1)} - \underline{\sigma}_{\underline{G}}^{(n-1)}\} < t) (\underline{w}_{\underline{n-1}} = 0) + \\ &\quad + (\max\{0, \underline{w}_{\underline{n-1}} + \underline{\sigma}_{\underline{F}}^{(n-1)} - \underline{\sigma}_{\underline{G}}^{(n-1)}\} < t) . \end{aligned}$$

or

$$\begin{aligned} P[\underline{w}_{\underline{n}} < t] &= \{P[\max\{0, \underline{\sigma}_{\underline{H}}^{(n-1)} - \underline{\sigma}_{\underline{G}}^{(n-1)}\} < t] \\ &\quad - P[\max\{0, \underline{\sigma}_{\underline{F}}^{(n-1)} - \underline{\sigma}_{\underline{G}}^{(n-1)}\} < t]\} W_{\underline{n-1}}(0+) \\ &\quad + P[\max\{0, \underline{w}_{\underline{n-1}} + \underline{\sigma}_{\underline{F}}^{(n-1)} - \underline{\sigma}_{\underline{G}}^{(n-1)}\} < t] . \end{aligned}$$

But for  $t > 0$ , we have

$$\begin{aligned}
 & P[\max\{0, \underline{w}_{n-1} + \sigma_{-F}^{(n-1)} - \sigma_{-G}^{(n-1)}\} < t] \\
 &= P[\underline{w}_{n-1} + \sigma_{-F}^{(n-1)} - \sigma_{-G}^{(n-1)} < t] , \\
 &= \int_{-\infty}^{+\infty} P[\underline{w}_{n-1} < t - \tau] dP[\sigma_{-F}^{(0)} - \sigma_{-G}^{(0)} < \tau] .
 \end{aligned}$$

Whence for  $t > 0$  and  $n \geq 1$  we obtain

$$(3.5) \quad W_n(t) = \{A_H(t) - A_F(t)\}W_{n-1}(0+) + \int_{-\infty}^{+\infty} W_{n-1}(t - \tau) dA_F(t) .$$

Since  $W(0+) = 1$  and  $W_0(t) = U_0(t)$  where  $U_0(t)$  is the Heaviside unit-step function at  $t = 0$ , we have that equation (3.5) is also valid for  $n = 1$  ( $W_1(t) = A_H(t)$  for  $t > 0$ ).

Remark. If  $H \equiv F$ , then (3.5) reduces to Lindley's integral equation for the Markov chain  $\{\tilde{w}_n, n \geq 0\}$  recursively determined by

$$\begin{cases} \tilde{w}_0 \text{ a.s. } 0 , \\ \tilde{w}_{n+1} = \max\{0, \tilde{w}_n + \sigma_{-F}^{(n)} - \sigma_{-G}^{(n)}\} . \end{cases}$$

### 3.3. Solution of the integral equation (3.5)

In order to apply a Wiener-Hopf technique to equation (3.5), we define for  $t \geq 0, n \in \mathbb{N}$ ;  $W_n^+(t) := W_n(t)$  (note that  $W_n^+(0) = 0$  or  $W_n^+(0) \neq W_n^+(0+) = W_n(0+)$ ), and for  $t \leq 0, n \in \mathbb{N}$ ;  $W_n^-(t) := 0$ .

For  $t \leq 0, n \in \mathbb{N}$  we define

$$W_n^-(t) := -\{A_H(t) - A_F(t)\}W_{n-1}(0+) - \int_{-\infty}^{+\infty} W_{n-1}(t - \tau) dA_F(\tau) .$$

Finally we define for  $t \in \mathbb{R}$

$$W_0^+(t) := W_0^-(t) := W_0(t) \quad (= U_0(t)).$$

By (3.5) and the above definitions we have for  $t \in \mathbb{R}$  and  $n \geq 1$

$$(3.6) \quad W_n^+(t) = W_{n-1}(0+)\{A_H(t) - A_F(t)\} + \int_{-\infty}^{+\infty} W_{n-1}^+(t-\tau)dA_F(\tau) + W_n^-(t) .$$

We will now reduce equation (3.6) into a functional equation on  $\mathbb{R}$  in terms of Fourier-Stieltjes transforms. Therefore, we define for  $z \in [0,1[$

$$w_-(\omega, z) := \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-i\omega t} dW_n^+(t), \quad \text{Im } \omega \leq 0 ,$$

$$w_+(\omega, z) := \sum_{n=0}^{\infty} z^n \int_{-\infty}^0 e^{-i\omega t} dW_n^-(t), \quad \text{Im } \omega \geq 0 ,$$

$$W(z) := \sum_{n=0}^{\infty} z^n W_n(0+),$$

$$a_F(\omega) := \int_{-\infty}^{+\infty} e^{-i\omega t} dA_F(t), \quad \text{Im } \omega = 0 ,$$

$$a_H(\omega) := \int_{-\infty}^{+\infty} e^{-i\omega t} dA_H(t), \quad \text{Im } \omega = 0 .$$

Put  $\sigma = \text{Re } \omega$ . For  $\sigma \in \mathbb{R}$  and  $z \in [0,1[$  we obtain by (3.6) and the above definitions

$$(3.7) \quad w_+(\sigma, z) - \{1 - za_F(\sigma)\}w_-(\sigma, z) = zW(z)\{a_F(\sigma) - a_H(\sigma)\} .$$

Note that

$$a_F(\sigma) = E\{e^{-i\sigma\tau_{-F}^{(n)}}\}E\{e^{i\sigma\tau_{-G}^{(n)}}\} ,$$

$$a_H(\sigma) = E\{e^{-i\sigma\tau_{-H}^{(n)}}\}E\{e^{i\sigma\tau_{-G}^{(n)}}\} .$$

In order to simplify (3.7), we introduce a sequence of r.v.'s  $\langle \tau_n \rangle_{n=1}^{\infty}$  with the property that  $\tau_n \stackrel{D}{=} \tau_{-F}^{(n)} - \tau_{-G}^{(n)}$ ,  $n \in \mathbb{N}$ . Since  $E\{e^{-i\sigma\tau_n}\} = a_F(\sigma)$ , we have by (2.1), (2.2), (2.3)

$$1 - za_F(\sigma) = \frac{\gamma_+(\sigma, z)}{\gamma_-(\sigma, z)}, \quad \sigma \in \mathbb{R} .$$

Substituting in (3.7) yields

$$(3.8) \quad \frac{w_+(\sigma, z)}{\gamma_+(\sigma, z)} - \frac{w_-(\sigma, z)}{\gamma_-(\sigma, z)} = zW(z) \frac{a_F(\sigma) - a_H(\sigma)}{\gamma_+(\sigma, z)} .$$

Since  $\frac{a_F(\sigma) - a_H(\sigma)}{\gamma_+(\sigma, z)}$  is a F.S.T. of a function of bounded variation, we have that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \frac{a_F(\sigma) - a_H(\sigma)}{\gamma_+(\sigma, z)} d\sigma$$

exists and is finite. Hence we may put

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \frac{a_F(\sigma) - a_H(\sigma)}{\gamma_+(\sigma, z)} d\sigma = M(z) .$$

Moreover

$$(3.9) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \frac{w_-(\sigma, z)}{\gamma_-(\sigma, z)} d\sigma = \lim_{\substack{\text{Im } \omega \rightarrow \infty \\ \text{Re } \omega = 0}} \frac{w_-(\sigma, z)}{\gamma_-(\sigma, z)} = W(z) .$$

Whence

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \frac{w_+(\sigma, z)}{\gamma_+(\sigma, z)} d\sigma = W(z)\{1 + zM(z)\} .$$

Or equivalently

$$(3.10) \quad \lim_{\substack{\text{Im } \omega \rightarrow +\infty \\ \text{Re } \omega = 0}} \frac{w_+(\sigma, z)}{\gamma_+(\sigma, z)} = W(z)\{1 + zM(z)\} .$$

By (3.8), (3.9), (3.10) and corollary (2), we obtain for  $\omega \in S_-$ ,  $z \in [0, 1[$

$$(3.11) \quad \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{w_+(\sigma, z)}{\gamma_+(\sigma, z)} \frac{d\sigma}{\sigma - \omega} = -\frac{1}{2}W(z)\{1 + zM(z)\}$$

$$(3.12) \quad \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{w_-(\sigma, z)}{\gamma_-(\sigma, z)} \frac{d\sigma}{\sigma - \omega} = \frac{1}{2}W(z) - \frac{w_-(\sigma, z)}{\gamma_-(\sigma, z)} .$$

By (3.11), (4.11) and (3.8) we have

$$(3.13) \quad w_-(\omega, z) = \gamma_-(\omega, z)W(z)\left\{1 + \frac{z}{2} M(z) + \frac{z}{2\pi i} \int_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau, z)} \frac{d\tau}{\tau - \omega}\right\}.$$

Finally, we have

$$\lim_{\substack{\omega \rightarrow 0 \\ \omega \in S_-}} w_-(\omega, z) = (1 - z)^{-1}.$$

Whence

$$(3.15) \quad W(z) = \frac{1}{1 - z} \frac{1}{\gamma_-(0, z)\left[1 + \frac{z}{2} M(z) + \lim_{\substack{\omega \rightarrow 0 \\ \omega \in S_-}} \frac{z}{2\pi i} \int_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau, z)} \frac{d\tau}{\tau - \omega}\right]}.$$

By (3.13) and (3.15) we have the following:

Theorem 2.

$$(3.16) \quad w_-(\omega, z) = \tilde{w}_-(\omega, z) \frac{1 + \frac{z}{2} M(z) + \frac{z}{2\pi i} \int_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau, z)} \frac{d\tau}{\tau - \omega}}{1 + \frac{z}{2} M(z) + \lim_{\substack{\omega \rightarrow 0 \\ \omega \in S_-}} \frac{z}{2\pi i} \int_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau, z)} \frac{d\tau}{\tau - \omega}}$$

where

$$\tilde{w}_-(\omega, z) = \frac{1}{1 - z} e^{-\sum_{n \geq 1} \frac{z^n}{n} E\{(1 - e^{-i\omega s_n})(s_n > 0)\}}.$$

Corollary 3. If  $a_F(\sigma)$ ,  $a_H(\sigma)$  are  $o(1)$  as  $|\sigma| \rightarrow \infty$ , then

$$(3.17) \quad w_-(\omega, z) = \tilde{w}_-(\omega, z) \frac{1 + \frac{z}{2\pi i} \int_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau, z)} \frac{d\tau}{\tau - \omega}}{1 + \lim_{\substack{\omega \rightarrow 0 \\ \omega \in S_-}} \frac{z}{2\pi i} \int_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau, z)} \frac{d\tau}{\tau - \omega}}.$$



Proof. Since  $\lim_{|\sigma| \rightarrow \infty} a_F(\sigma) = \lim_{|\sigma| \rightarrow \infty} a_H(\sigma) = 0$ , we have by (3.7),

$$\lim_{|\sigma| \rightarrow \infty} w_+(\sigma, z) = \lim_{|\sigma| \rightarrow \infty} w_-(\sigma, z) = W(z) .$$

Hence  $M(z) = 0$ .

Remark 1. If  $a_F(\sigma)$  and  $a_H(\sigma)$  are Hölder continuous at infinity, then (3.17) can be obtained by the methods of the classical Hilbert problem [8], [10].

Remark 2. Finally, we remark that J. Loris-Teghem's result [5] can be obtained from (3.13). In order to show this we remark that for any  $\varepsilon > 0$ ,

$$\frac{w_-(\omega, z)}{\tilde{w}_-(\omega, z)} - \frac{w_-(i\varepsilon, z)}{\tilde{w}_-(i\varepsilon, z)} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{zW(z)\{a_F(\tau) - a_H(\tau)\}}{\{1 - za_F(\tau)\}w_-(\tau, z)} \left\{ \frac{1}{\tau - \omega} - \frac{1}{\tau - i\varepsilon} \right\} d\tau .$$

Since  $\lim_{\varepsilon \rightarrow 0} \frac{w_-(i\varepsilon, z)}{\tilde{w}_-(i\varepsilon, z)} = 1$ , we have

$$w_-(\omega, z) = \tilde{w}_-(\omega, z) \left[ 1 + \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{zW(z)\{a_F(\tau) - a_H(\tau)\}}{\{1 - za_F(\tau)\}w_-(\tau, z)} \left\{ \frac{1}{\tau - \omega} - \frac{1}{\tau - i\varepsilon} \right\} d\tau \right]$$

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