# An extension of a theorem of Cauchy with applications to probability theory

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An extension of a theorem of Cauchy with applications to Probability Theory

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Abstract

The paper deals with an extension of Cauchy's theorem for improper integrals. We use the theorem to solve an integral equation of the Pollaczek type [7] and to study a Markov chain $\{\underline{w}_n, n \geq 0\}$ with state space $[0,\infty[$ closely related to Queueing Theory.

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# 1. An extension of a theorem of Cauchy

## 1.1. Introduction

Let S be the complex $\omega$-plane, $S_\pm := \{\omega \mid \operatorname{Im}\omega \gtrless 0\}$ and $\mathbb{R} := \{\sigma \mid -\infty < \sigma < +\infty\}$.

A. Let $f(\omega)$ be a function, holomorphic in $S_-$ and continuous in $S_- + \mathbb{R}$ including the point at infinity i.e., $f(\omega) = f(\infty) + o(1)$ as $|\omega| \to \infty$ in $S_- + \mathbb{R}$. Then

$$\text{(a)} \qquad \frac{1}{2\pi i} \, C\!\!\!\int_{-\infty}^{+\infty} f(\sigma)\frac{d\sigma}{\sigma - \omega} = \begin{cases} \tfrac{1}{2}f(\infty) & \text{if } \omega \in S_+ , \\ -f(\omega) + \tfrac{1}{2}f(\infty) & \text{if } \omega \in S_- . \end{cases}$$

The symbol C indicates the principal value (Cauchy value) of the integral.
B. Let $f(\omega)$ be a function, holomorphic in $S_+$ and continuous in $S_+ + \mathbb{R}$ including the point at infinity. Then

$$\text{(b)} \qquad \frac{1}{2\pi i} \, C\!\!\!\int_{-\infty}^{+\infty} f(\sigma)\frac{d\sigma}{\sigma - \omega} = \begin{cases} f(\omega) - \tfrac{1}{2}f(\infty) & \text{if } \omega \in S_+ , \\ -\tfrac{1}{2}f(\infty) & \text{if } \omega \in S_- . \end{cases}$$

The formulae (a) and (b) are called "Cauchy formulae" for the regions $S_-$ and $S_+$ respectively.
A simple proof of (a) and (b) is given in [6].

Remark. The condition that $f(\omega) = f(\infty) + o(1)$ as $|\omega| \to \infty$ in $S_- + \mathbb{R}$ (or in $S_+ + \mathbb{R}$) is not necessarily satisfied when $f(\omega)$ is a Fourier-Stieltjes transform (F.S.T.). Therefore, we shall prove the following extension related to Probability Theory.

## 1.2. The theorem

Let $\{\Omega, B, P\}$ be a probability space with $\Omega := ]-\infty, +\infty[$ the sample space, B the smallest $\sigma$-algebra containing all open subsets (Borel sets) of $\Omega$ and $P$ a probability measure on B. Introduce a B-measurable mapping $\underline{x}: \Omega \to \mathbb{R}$ (a random variable) with probability distribution function (p.d.f.)

$$F(x) := P[\underline{x} < x], \quad x \in \mathbb{R}$$

and define

$$E\{e^{-i\sigma\underline{x}}\} := \int_{-\infty}^{+\infty} e^{-i\sigma x} dF(x), \quad \sigma \in \mathbb{R} .$$

Denote by (A) de indicator of the event A, i.e.

$$(A) = \begin{cases} 1 & \text{if A occurs,} \\ 0 & \text{if not .} \end{cases}$$

Theorem 1.

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} \frac{1}{2}E(\underline{x} = 0) - E\{e^{-i\omega\underline{x}}(\underline{x} \geq 0)\} & \text{if } \omega \in S_- , \\ -\frac{1}{2}E(\underline{x} = 0) + E\{e^{-i\omega\underline{x}}(\underline{x} \leq 0)\} & \text{if } \omega \in S_+ . \end{cases}$$

Proof. Since $(\underline{x} < 0) + (\underline{x} = 0) + (\underline{x} > 0) = 1$, we have $\forall\omega \mid \omega \in S_+ \cup S_-$

$$(1.1) \qquad \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}\} \frac{d\sigma}{\sigma - \omega} = \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}(\underline{x} > 0)\} \frac{d\sigma}{\sigma - \omega} +$$

$$+ E(\underline{x} = 0) \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{d\sigma}{\sigma - \omega} + \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}(\underline{x} < 0)\} \frac{d\sigma}{\sigma - \omega} ,$$

provided that all principal values involved exist.

We shall now prove that

$$(1.2) \qquad \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma\underline{x}}(\underline{x} > 0)\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} -E\{e^{-i\omega\underline{x}}(\underline{x} > 0)\}, & \omega \in S_- , \\ 0 & \omega \in S_+ . \end{cases}$$

In order to show the statement, let $\omega \in S_-$, i.e., put $\omega = Re^{i\theta}$, $\pi < \theta < 2\pi$, $0 < R < \infty$.

Consider a closed curve $\Gamma$ consisting of a segment $[-T,T]$ of the real line and of a semi-circle $\gamma$ lying in $S_-$, with center at the origin and with radius $T \mid 0 < R < T < \infty$. Finally, let $\omega' = Te^{i\theta'}$, $\pi < \theta' < 2\pi$. The contour integration is clockwise.

From the above definition of $\Gamma$, we obtain for $\omega \in S_-$

$$\frac{1}{2\pi i}\int_{\Gamma} E\{e^{-i\xi \underline{x}}(\underline{x} > 0)\}\frac{d\xi}{\xi - \omega} = \frac{1}{2\pi i}\int_{-T}^{+T} E\{e^{-i\sigma \underline{x}}(\underline{x} > 0)\}\frac{d\sigma}{\sigma - \omega} + \frac{1}{2\pi i}\int_{\gamma} E\{e^{-i\omega' \underline{x}}(\underline{x} > 0)\}\frac{d\omega'}{\omega' - \omega} .$$

But

$$E\{e^{-i\omega' \underline{x}}(\underline{x} > 0)\} = \int_{0+}^{\infty} e^{-i\omega' x} dF(x) .$$

Whence, by Cauchy's theorem, we obtain for $\omega \in S_-$

$$-E\{e^{-i\omega \underline{x}}(\underline{x} > 0)\} = \frac{1}{2\pi i}\int_{-T}^{+T} E\{e^{-i\sigma \underline{x}}(\underline{x} > 0)\}\frac{d\sigma}{\sigma - \omega} + \frac{1}{2\pi i}\int_{\gamma}\int_{0+}^{\infty} e^{-i\omega' x} dF(x)\frac{d\omega'}{\omega' - \omega} .$$

Now

$$\left|\frac{1}{2\pi i}\int_{\gamma}\int_{0+}^{\infty} e^{-i\omega' x} dF(x)\frac{d\omega'}{\omega' - \omega}\right| \le \frac{T}{2\pi}\int_{\pi}^{2\pi}\int_{0+}^{\infty} \frac{e^{Tx\sin\theta'} dF(x)d\theta'}{\min\limits_{\pi<\theta'<2\pi} |Te^{i\theta'} - Re^{i\theta}|}$$

$$\le \frac{T}{T - R}\int_{0+}^{\infty}\int_{0}^{\pi/2} e^{-Tx\sin\varphi} d\varphi dF(x) .$$

The permutation of both integrals in the right-hand side of the first inequality is justified by Fubini's theorem.

It may be verified that for any $\ell \in ]0,\pi/2[$

$$\int_0^{\pi/2} e^{-Tx\sin\varphi} d\varphi \le \sec \ell \int_0^{\ell} e^{-Tx\sin\varphi} \cos\varphi d\varphi + (\pi/2 - \ell) e^{-Tx\sin\ell} .$$

After some algebra we obtain ($\ell$ fixed),

$$\left|\frac{1}{2\pi i}\int_{\gamma}\int_{0+}^{\infty} e^{-i\omega' x} dF(x)\frac{d\omega'}{\omega' - \omega}\right| \le (\pi/2 - \ell)\frac{T}{T - R}\int_{0+}^{\infty} e^{-Tx\sin\ell} dF(x) +$$

$$+ \frac{T}{T-R} \tan \ell \{F(T^{-1} \operatorname{cosec} \ell +) - F(0+)\} + \frac{T}{T-R} \sec \ell \int_{0+}^{\infty} \frac{1}{Tx} \chi_a(x) dF(x) ,$$

where $a := ]T^{-1} \operatorname{cosec} \ell, \infty[$ and

$$\chi_a(x) = \begin{cases} 1 & \text{if } x \in a , \\ 0 & \text{if } x \notin a . \end{cases}$$

Since $\frac{1}{Tx} \chi_a(x) \leq 1$, we have by a well-known dominated convergence theorem

$$\frac{1}{2\pi i} \int_{\gamma} \int_{0+}^{\infty} e^{-i\omega' x} dF(x) \frac{d\omega'}{\omega' - \omega} = o_T(1) \quad \text{as } T \to +\infty .$$

Finally, we remark that

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{d\sigma}{\sigma - \omega} = \begin{cases} -\frac{1}{2} & \text{if } \omega \in S_- , \\ +\frac{1}{2} & \text{if } \omega \in S_+ . \end{cases}$$

In a similar way, we can prove that

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{i\sigma \underline{x}'} (\underline{x}' > 0)\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} E\{e^{i\omega \underline{x}'} (\underline{x}' > 0)\}, & \omega \in S_+ , \\ 0 & \omega \in S_- . \end{cases}$$

Hence, for $\underline{x}' = -\underline{x}$, we have

$$(1.3) \qquad \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma \underline{x}} (\underline{x} < 0)\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} E\{e^{-i\omega \underline{x}} (\underline{x} < 0)\}, & \omega \in S_+ , \\ 0 & \omega \in S_- . \end{cases}$$

By (1.1), (1.2) and (1.3) we obtain

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-i\sigma \underline{x}}\} \frac{d\sigma}{\sigma - \omega} = \begin{cases} -\frac{1}{2} E(\underline{x} = 0) - E\{e^{-i\omega \underline{x}} (\underline{x} > 0)\}, & \text{if } \omega \in S_- , \\ +\frac{1}{2} E(\underline{x} = 0) + E\{e^{-i\omega \underline{x}} (\underline{x} < 0)\}, & \text{if } \omega \in S_+ . \end{cases}$$

<u>Corollary 1</u>. Let $\underline{x}$ be a nonnegative random variable (r.v.) with characteristic function (c.f.), $E\{e^{-i\omega \underline{x}}\}$, Im $\omega \leq 0$. Then

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{-it\underline{x}}\} \frac{dt}{t - \omega} = \begin{cases} \frac{1}{2} P[\underline{x} = 0] - E\{e^{-i\omega \underline{x}}\}, & \omega \in S_- , \\ \frac{1}{2} P[\underline{x} = 0], & \omega \in S_+ . \end{cases}$$

(1.4)

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} E\{e^{it\underline{x}}\} \frac{dt}{t-\omega} = \begin{cases} -\tfrac{1}{2}P[\underline{x} = 0] + E\{e^{i\omega\underline{x}}\}, & \omega \in S_+ , \\ -\tfrac{1}{2}P[\underline{x} = 0], & \omega \in S_- . \end{cases}$$

<u>Corollary 2.</u> Let $\langle \underline{x}_n \rangle_{n=0}^{\infty}$ be a sequence of nonnegative r.v.'s with corresponding sequence of c.f.'s $\langle E\{e^{-i\omega\underline{x}_n}\} \rangle_{n=0}^{\infty}$, Im $\omega \le 0$.
For $\omega \in S_-$ and $z \in [0,1[$ we have

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \sum_{n=0}^{\infty} z^n E\{e^{-it\underline{x}_n}\} \frac{dt}{t-\omega} = \tfrac{1}{2} \sum_{n=0}^{\infty} z^n P[\underline{x}_n = 0] - \sum_{n=0}^{\infty} z^n E\{e^{-i\omega\underline{x}_n}\},$$

$$\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \sum_{n=0}^{\infty} z^n E\{e^{it\underline{x}_n}\} \frac{dt}{t-\omega} = -\tfrac{1}{2} \sum_{n=0}^{\infty} z^n P[\underline{x}_n = 0] .$$

<u>Proof</u>. Let $\langle p_n \rangle_{n=0}^{\infty}$ be a discrete proper probability distribution, then $\sum_{n=0}^{\infty} p_n E\{e^{-i\omega\underline{x}_n}\}$, Im $\omega \le 0$, is a c.f. of a nonnegative r.v. and we may apply (1.4). Since $p_n$ is arbitrary, we may put $p_n = (1-z)z^n$, $\forall z \mid z \in [0,1[$.

## 2. <u>An integral equation of the Pollaczek type</u>

In this section, we use the results of section 1 to solve an integral equation of the Pollaczek type [7].
In order to introduce the equation, let $\{\underline{\tau}_n, n \in \mathbb{N}\}$ be a family of independent, identically distributed r.v.'s with p.d.f. $A(\tau)$ concentrated on $\mathbb{R}$ and with c.f. $a(\sigma) = E\{e^{-i\sigma\underline{\tau}}\}$, $\sigma \in \mathbb{R}$.
Define

$$\underline{s}_n := \underline{\tau}_1 + \underline{\tau}_2 + \ldots + \underline{\tau}_n, \quad n \in \mathbb{N} .$$

$\{\underline{s}_n, n \in \mathbb{N}\}$ is a random walk generated by $A(\cdot)$. Let us introduce the following functions for $z \in [0,1[$

(2.1) $$\gamma_+(\omega,z) = e^{-\sum_{n\ge1} \frac{z^n}{n} E\{e^{-i\omega\underline{s}_n}(\underline{s}_n \le 0)\}}, \quad \text{Im } \omega \ge 0 ,$$

$$\gamma_-(\omega,z) = e^{+\sum_{n\geq 1} \frac{z^n}{n} E\{e^{-i\omega \underline{s}_n}(\underline{s}_n > 0)\}}, \quad \mathrm{Im}\,\omega \leq 0 . \tag{2.2}$$

By (2.1) and (2.2), we obtain for $z \in [0,1[$ and $\sigma \in \mathbb{R}$

$$\gamma_+(\sigma,z) - [1 - za(\sigma)]\gamma_-(\sigma,z) = 0 . \tag{2.3}$$

Define

$$\begin{cases} \underline{x}_n := \max\{0,\underline{s}_1,\underline{s}_2,\dots,\underline{s}_{n-1}\}, & n \geq 2 , \\ \underline{x}_1 := 0 . \end{cases}$$

From the results of Fluctuation Theory, (see e.g. Cohen [1], pag. 151) we have for $z \in [0,1[$ and $\mathrm{Im}\,\omega \leq 0$

$$\sum_{n\geq 1} z^n E\{e^{-i\omega \underline{x}_n}\} = \frac{z}{1-z} e^{-\sum_{n\geq 1} \frac{z^n}{n} E(\underline{s}_n > 0)} \gamma_-(\omega,z) . \tag{2.4}$$

A similar expression can be given with respect to $\gamma_+$.
To simplify notation, we define

$$\alpha(z) := \tfrac{1}{2} \sum_{n\geq 1} \frac{z^n}{n} E(\underline{s}_n = 0) .$$

By (2.1), (2.2) we have

$$\lim_{\substack{\mathrm{Im}\,\omega\to+\infty \\ \mathrm{Re}\,\omega=0}} \gamma_+(\omega,z) = e^{-2\alpha(z)} , \tag{2.5}$$

$$\lim_{\substack{\mathrm{Im}\,\omega\to-\infty \\ \mathrm{Re}\,\omega=0}} \gamma_-(\omega,z) = 1 . \tag{2.6}$$

Hence by corollary 1 and 2, (2.4), (2.5) and (2.6) we obtain after simplification

$$\gamma_-(\omega,z) = \tfrac{1}{2} - \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \gamma_-(\sigma,z)\frac{d\sigma}{\sigma-\omega} , \quad \omega \in S_- , \tag{2.7}$$

$$-\tfrac{1}{2}e^{-2\alpha(z)} = \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \gamma_+(\sigma,z)\frac{d\sigma}{\sigma-\omega} , \quad \omega \in S_- . \tag{2.8}$$

Rewriting (2.3) in the form

$$\gamma_+(\sigma,z) - \gamma_-(\sigma,z) = za(\sigma)\gamma_-(\sigma,z)$$

yields by (2.7) and (2.8),

$$(2.9) \qquad \gamma_-(\omega,z) = \tfrac{1}{2}[1 + e^{-2\alpha(z)}] - \frac{z}{2\pi i} \oint_{-\infty}^{+\infty} a(\sigma)\gamma_-(\sigma,z)\frac{d\sigma}{\sigma-\omega} .$$

Define

$$(2.10) \qquad u_-(\omega,z) := \frac{\gamma_-(\omega,z)}{\tfrac{1}{2}[1 + e^{-2\alpha(z)}]} , \quad \operatorname{Im}\omega \le 0 .$$

By (2.9) and (2.10) we have that $u_-(\omega,z)$ is the solution of the integral equation

$$(2.11) \qquad u_-(\omega,z) = 1 - \frac{z}{2\pi i} \oint_{-\infty}^{+\infty} a(\sigma)u_-(\sigma,z)\frac{d\sigma}{\sigma-\omega} , \ \omega \in S_- .$$

Finally, we define

$$p_n := -\frac{z^n}{n}\frac{1}{\log(1-z)} , \quad n \in \mathbb{N},\ z \in ]0,1[ .$$

Since

$$\sum_{n=1}^{\infty} p_n a^n(\sigma) = \frac{\log[1 - za(\sigma)]}{\log(1-z)}$$

is the c.f. of the p.d.f. $\sum_{n=1}^{\infty} p_n A^{n^*}(t)$, $t \in \mathbb{R}$, we have for $\omega \in S_-$

$$(2.12) \qquad \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \log[1 - za(\sigma)]\frac{d\sigma}{\sigma-\omega} = \alpha(z) + \log \gamma_-(\omega,z) .$$

By (2.10) and (2.12) we finally obtain for $\omega \in S_-$, $z \in [0,1[$

$$u_-(\omega,z) = \operatorname{sech}\alpha(z) e^{\frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \log[1 - za(\sigma)]\frac{d\sigma}{\sigma-\omega}} .$$

Remarks. Integral equations similar to (2.11) have been studied in [7] and [6] using different methods. Pollaczek assumed that $a(\omega)$ is analytic in some strip $|\mathrm{Im}\ \omega| < \delta$ for some $\delta > 0$, whereas Keilson and Kooharian made the assumption that $a(\sigma) = o(1)$ as $|\sigma| \to \infty$ . (In this case sech $\alpha(z) = 1$.)
By theorem 1 and a combination of some of the techniques presented in [1], [2] and [7], we have shown that the above assumptions concerning $a(\sigma)$ are no longer necessary to solve equations similar to (2.11).
As a second application of theorem 1, we study:

## 3. The transient behaviour of a Markov chain $\{\underline{w}_n;\ n \geq 0\}$ with state space $[0,\infty[$

### 3.1. Introduction

Let $\{\underline{w}_n,\ n \geq 0\}$ be a Markov chain with state space $[0,\infty[$ recursively determined by

$$(3.1)\qquad \begin{cases} \underline{w}_0 \overset{a.s.}{=} 0\ , \\ \underline{w}_{n+1} = \begin{cases} \max\{0,\underline{\sigma}_H^{(n)} - \underline{\sigma}_G^{(n)}\}, & \text{if } \underline{w}_n = 0\ , \\ \max\{0,\underline{w}_n + \underline{\sigma}_F^{(n)} - \underline{\sigma}_G^{(n)}\}, & \text{if } \underline{w}_n \neq 0\ , \end{cases} \end{cases}$$

where $\underline{\sigma}_G^{(n)}$, $n = 0,1,2,\ldots$ are positive independent r.v.'s with p.d.f. $P[\underline{\sigma}_G^{(n)} < t] = G(t)$ and $P[\underline{\sigma}_G^{(n)} \geq 0] = 1$ (ditto $\underline{\sigma}_H^{(n)},\underline{\sigma}_F^{(n)}$).
We also assume that $\underline{\sigma}_G^{(n)}$, $\underline{\sigma}_H^{(n)}$, $\underline{\sigma}_F^{(n)}$ are mutually independent for each n.

Remark. The transient behaviour of the Markov chain $\{\underline{w}_n\}$ has been studied by J. Loris-Teghem [4], [5] using an algebraic method based on a theorem due to G. Baxter [5]. Our approach is based on a Wiener-Hopf integral equation of the second kind and its reduction to a functional equation on $\mathbb{R}$ in terms of F.S.T.'s.

### 3.2. A Wiener-Hopf integral equation for the Markov chain $\{\underline{w}_n\}$

For $t > 0$ and $n = 0,1,2,\ldots$ we define

$$(3.2)\qquad W_n(t) := P[\underline{w}_n < t]$$

and $W_n(0+) := \lim_{t\downarrow 0} W_n(t)$.

For $t \in \mathbb{R}$, we define

$$A_F(t) := P[\underline{\sigma}_F^{(0)} - \underline{\sigma}_G^{(0)} < t] ,$$

$$A_H(t) := P[\underline{\sigma}_H^{(0)} - \underline{\sigma}_G^{(0)} < t] .$$

Whence for $t \in \mathbb{R}$

$$(3.3) \qquad A_F(t) = \int_0^\infty F(t + \tau)dG(\tau) ,$$

$$(3.4) \qquad A_H(t) = \int_0^\infty H(t + \tau)dG(\tau) .$$

By (3.1) we have for $t > 0$ and $n \geq 1$

$$(\underline{w}_n < t) = (\max\{0,\underline{\sigma}_H^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t)(\underline{w}_{n-1} = 0) +$$

$$+ (\max\{0,\underline{w}_{n-1} + \underline{\sigma}_F^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t)(\underline{w}_{n-1} > 0) .$$

Since $(\underline{w}_{n-1} > 0) = 1 - (\underline{w}_{n-1} = 0)$, we have

$$(\underline{w}_n < t) = (\max\{0,\underline{\sigma}_H^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t)(\underline{w}_{n-1} = 0) +$$

$$- (\max\{0,\underline{\sigma}_F^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t)(\underline{w}_{n-1} = 0) +$$

$$+ (\max\{0,\underline{w}_{n-1} + \underline{\sigma}_F^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t) .$$

or

$$P[\underline{w}_n < t] = \{P[\max\{0,\underline{\sigma}_H^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t]$$

$$- P[\max\{0,\underline{\sigma}_F^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t]\}W_{n-1}(0+)$$

$$+ P[\max\{0,\underline{w}_{n-1} + \underline{\sigma}_F^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t] .$$

But for $t > 0$, we have

$$P[\max\{0, \underline{w}_{n-1} + \underline{\sigma}_F^{(n-1)} - \underline{\sigma}_G^{(n-1)}\} < t]$$

$$= P[\underline{w}_{n-1} + \underline{\sigma}_F^{(n-1)} - \underline{\sigma}_G^{(n-1)} < t] ,$$

$$= \int_{-\infty}^{+\infty} P[\underline{w}_{n-1} < t - \tau] dP[\underline{\sigma}_F^{(0)} - \underline{\sigma}_G^{(0)} < \tau] .$$

Whence for $t > 0$ and $n \geq 1$ we obtain

$$W_n(t) = \{A_H(t) - A_F(t)\} W_{n-1}(0+) + \int_{-\infty}^{+\infty} W_{n-1}(t - \tau) dA_F(t) . \tag{3.5}$$

Since $W(0+) = 1$ and $W_0(t) = U_0(t)$ where $U_0(t)$ is the Heaviside unit-step function at $t = 0$, we have that equation (3.5) is also valid for $n = 1$ ($W_1(t) = A_H(t)$ for $t > 0$).

Remark. If $H \equiv F$, then (3.5) reduces to Lindley's integral equation for the Markov chain $\{\tilde{\underline{w}}_n, n \geq 0\}$ recursively determined by

$$\begin{cases} \tilde{\underline{w}}_0 \overset{a.\underline{s}.}{=} 0 , \\ \tilde{\underline{w}}_{n+1} = \max\{0, \tilde{\underline{w}}_n + \underline{\sigma}_F^{(n)} - \underline{\sigma}_G^{(n)}\} . \end{cases}$$

## 3.3. Solution of the integral equation (3.5)

In order to apply a Wiener-Hopf technique to equation (3.5), we define for $t \geq 0$, $n \in \mathbb{N}$; $W_n^+(t) := W_n(t)$ (note that $W_n^+(0) = 0$ or $W_n^+(0) \neq W_n^+(0+) = W_n(0+)$), and for $t \leq 0$, $n \in \mathbb{N}$; $W_n^-(t) := 0$.
For $t \leq 0$, $n \in \mathbb{N}$ we define

$$W_n^-(t) := -\{A_H(t) - A_F(t)\} W_{n-1}(0+) - \int_{-\infty}^{+\infty} W_{n-1}(t - \tau) dA_F(\tau) .$$

Finally we define for $t \in \mathbb{R}$

$$W_0^+(t) := W_0^-(t) := W_0(t) \quad (= U_0(t)).$$

By (3.5) and the above definitions we have for $t \in \mathbb{R}$ and $n \geq 1$

$$(3.6)\qquad W_n^+(t) = W_{n-1}(0+)\{A_H(t) - A_F(t)\} + \int_{-\infty}^{+\infty} W_{n-1}^+(t-\tau)dA_F(\tau) + W_n^-(t) .$$

We will now reduce equation (3.6) into a functional equation on $\mathbb{R}$ in terms of Fourier-Stieltjes transforms. Therefore, we define for $z \in [0,1[$

$$w_-(\omega,z) := \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-i\omega t} dW_n^+(t), \qquad \mathrm{Im}\,\omega \le 0 ,$$

$$w_+(\omega,z) := \sum_{n=0}^{\infty} z^n \int_{-\infty}^{0} e^{-i\omega t} dW_n^-(t), \qquad \mathrm{Im}\,\omega \ge 0 ,$$

$$W(z) := \sum_{n=0}^{\infty} z^n W_n(0+),$$

$$a_F(\omega) := \int_{-\infty}^{+\infty} e^{-i\omega t} dA_F(t), \qquad \mathrm{Im}\,\omega = 0 ,$$

$$a_H(\omega) := \int_{-\infty}^{+\infty} e^{-i\omega t} dA_H(t), \qquad \mathrm{Im}\,\omega = 0 .$$

Put $\sigma = \mathrm{Re}\,\omega$. For $\sigma \in \mathbb{R}$ and $z \in [0,1[$ we obtain by (3.6) and the above definitions

$$(3.7)\qquad w_+(\sigma,z) - \{1 - za_F(\sigma)\}w_-(\sigma,z) = zW(z)\{a_F(\sigma) - a_H(\sigma)\} .$$

Note that

$$a_F(\sigma) = E\{e^{-i\sigma \underline{\sigma}_F^{(n)}}\}E\{e^{i\sigma \underline{\sigma}_G^{(n)}}\} ,$$

$$a_H(\sigma) = E\{e^{-i\sigma \underline{\sigma}_H^{(n)}}\}E\{e^{i\sigma \underline{\sigma}_G^{(n)}}\} .$$

In order to simplify (3.7), we introduce a sequence of r.v.'s $\langle \underline{\tau}_n \rangle_{n=1}^{\infty}$ with the property that $\underline{\tau}_n \overset{\mathcal{D}}{=} \underline{\sigma}_F^{(n)} - \underline{\sigma}_G^{(n)}$, $n \in \mathbb{N}$. Since $E\{e^{-i\sigma\underline{\tau}_n}\} = a_F(\sigma)$, we have by (2.1), (2.2), (2.3)

$$1 - za_F(\sigma) = \frac{\gamma_+(\sigma,z)}{\gamma_-(\sigma,z)} , \quad \sigma \in \mathbb{R} .$$

Substituting in (3.7) yields

$$(3.8) \qquad \frac{w_+(\sigma,z)}{\gamma_+(\sigma,z)} - \frac{w_-(\sigma,z)}{\gamma_-(\sigma,z)} = zW(z)\,\frac{a_F(\sigma) - a_H(\sigma)}{\gamma_+(\sigma,z)} .$$

Since $\dfrac{a_F(\sigma) - a_H(\sigma)}{\gamma_+(\sigma,z)}$ is a F.S.T. of a function of bounded variation, we have that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \frac{a_F(\sigma) - a_H(\sigma)}{\gamma_+(\sigma,z)}\, d\sigma$$

exists and is finite. Hence we may put

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \frac{a_F(\sigma) - a_H(\sigma)}{\gamma_+(\sigma,z)}\, d\sigma = M(z) .$$

Moreover

$$(3.9) \qquad \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \frac{w_-(\sigma,z)}{\gamma_-(\sigma,z)}\, d\sigma = \lim_{\substack{\mathrm{Im}\ \omega\to-\infty \\ \mathrm{Re}\ \omega=0}} \frac{w_-(\sigma,z)}{\gamma_-(\sigma,z)} = W(z) .$$

Whence

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \frac{w_+(\sigma,z)}{\gamma_+(\sigma,z)}\, d\sigma = W(z)\{1 + zM(z)\} .$$

Or equivalently

$$(3.10) \qquad \lim_{\substack{\mathrm{Im}\ \omega\to+\infty \\ \mathrm{Re}\ \omega=0}} \frac{w_+(\sigma,z)}{\gamma_+(\sigma,z)} = W(z)\{1 + zM(z)\} .$$

By (3.8), (3.9), (3.10) and corollary (2), we obtain for $\omega \in S_-$, $z \in [0,1[$

$$(3.11) \qquad \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{w_+(\sigma,z)}{\gamma_+(\sigma,z)}\, \frac{d\sigma}{\sigma-\omega} = -\tfrac{1}{2}W(z)\{1 + zM(z)\}$$

$$(3.12) \qquad \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{w_-(\sigma,z)}{\gamma_-(\sigma,z)}\, \frac{d\sigma}{\sigma-\omega} = \tfrac{1}{2}W(z) - \frac{w_-(\sigma,z)}{\gamma_-(\sigma,z)} .$$

By (3.11), (4.11) and (3.8) we have

$$(3.13)\qquad w_-(\omega,z) = \gamma_-(\omega,z)W(z)\{1 + \frac{z}{2}M(z) + \frac{z}{2\pi i}\oint_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau,z)}\frac{d\tau}{\tau-\omega}\} .$$

Finally, we have

$$\lim_{\substack{\omega\to 0\\ \omega\in S_-}} w_-(\omega,z) = (1-z)^{-1} .$$

Whence

$$(3.15)\qquad W(z) = \frac{1}{1-z}\,\frac{1}{\gamma_-(0,z)[1 + \frac{z}{2}M(z) + \lim\limits_{\substack{\omega\to 0\\ \omega\in S_-}} \frac{z}{2\pi i}\oint_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau,z)}\frac{d\tau}{\tau-\omega}]} .$$

By (3.13) and (3.15) we have the following:

<u>Theorem 2.</u>

$$(3.16)\qquad w_-(\omega,z) = \tilde{w}_-(\omega,z)\frac{1 + \frac{z}{2}M(z) + \frac{z}{2\pi i}\oint_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau,z)}\frac{d\tau}{\tau-\omega}}{1 + \frac{z}{2}M(z) + \lim\limits_{\substack{\omega\to 0\\ \omega\in S_-}} \frac{z}{2\pi i}\oint_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau,z)}\frac{d\tau}{\tau-\omega}}$$

where

$$\tilde{w}_-(\omega,z) = \frac{1}{1-z}\, e^{-\sum_{n\ge 1} \frac{z^n}{n} E\{(1 - e^{-i\omega \underline{s}_n})(\underline{s}_n > 0)\}} .$$

<u>Corollary 3.</u> If $a_F(\sigma)$, $a_H(\sigma)$ are $o(1)$ as $|\sigma| \to \infty$, then

$$(3.17)\qquad w_-(\omega,z) = \tilde{w}_-(\omega,z)\frac{1 + \frac{z}{2\pi i}\oint_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau,z)}\frac{d\tau}{\tau-\omega}}{1 + \lim\limits_{\substack{\omega\to 0\\ \omega\in S_-}} \frac{z}{2\pi i}\oint_{-\infty}^{+\infty} \frac{a_F(\tau) - a_H(\tau)}{\gamma_+(\tau,z)}\frac{d\tau}{\tau-\omega}} .$$

Proof. Since $\lim_{|\sigma|\to\infty} a_F(\sigma) = \lim_{|\sigma|\to\infty} a_H(\sigma) = 0$, we have by (3.7),

$$\lim_{|\sigma|\to\infty} w_+(\sigma,z) = \lim_{|\sigma|\to\infty} w_-(\sigma,z) = W(z) .$$

Hence $M(z) = 0$.

Remark 1. If $a_F(\sigma)$ and $a_H(\sigma)$ are Hölder continuous at infinity, then (3.17) can be obtained by the methods of the classical Hilbert problem [8], [10].

Remark 2. Finally, we remark that J. Loris-Teghem's result [5] can be obtained from (3.13). In order to show this we remark that for any $\varepsilon > 0$,

$$\frac{w_-(\omega,z)}{\tilde{w}_-(\omega,z)} - \frac{w_-(i\varepsilon,z)}{\tilde{w}_-(i\varepsilon,z)} = \frac{1}{2\pi i}\oint_{-\infty}^{+\infty} \frac{zW(z)\{a_F(\tau) - a_H(\tau)\}}{\{1 - za_F(\tau)\}w_-(\tau,z)}\left\{\frac{1}{\tau-\omega} - \frac{1}{\tau - i\varepsilon}\right\}d\tau .$$

Since $\lim_{\varepsilon\downarrow 0} \frac{w_-(i\varepsilon,z)}{\tilde{w}_-(i\varepsilon,z)} = 1$, we have

$$w_-(\omega,z) = \tilde{w}_-(\omega,z)\left[1 + \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\oint_{-\infty}^{+\infty} \frac{zW(z)\{a_F(\tau) - a_H(\tau)\}}{\{1 - za_F(\tau)\}w_-(\tau,z)}\left\{\frac{1}{\tau-\omega} - \frac{1}{\tau - i\varepsilon}\right\}d\tau\right]$$

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