

## Asymptotics in poisson order statistics

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# ASYMPTOTICS IN POISSON ORDER STATISTICS

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## ABSTRACT

In order statistics sums involving incomplete gamma functions are met. The asymptotic behaviour of such sums is studied, going beyond the results obtainable by the central limit theorem.

## 1. Introduction

Some colleagues\*) of the author have posed the following problem:  
Determine the asymptotic behaviour for  $\mu \rightarrow \infty$  of the sums

$$(1) \quad S(\mu, m, n) := \sum_{k=0}^{\infty} I^m(\mu, k)(1 - I(\mu, k))^n \quad ,$$

$$(2) \quad T(\mu, m, n) := \sum_{k=0}^{\infty} (\mu - k) I^m(\mu, k)(1 - I(\mu, k))^n \quad ,$$

where  $m$  and  $n$  are positive integers and

$$(3) \quad I(\mu, k) := (k!)^{-1} \int_0^{\mu} e^{-t} t^k dt \quad (\mu > 0, k \in \mathbb{N}_0) \quad .$$

This problem arose in the study of the expectation and variance of the order statistics in a random sample from the Poisson distribution with large mean  $\mu$ . In section 2 we present the results. A brief description of the derivation is given in section 3. The details of the derivation are given in sections 4 to 8.

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\*) F.W. Steutel and D.A. Overdijk, Department of Mathematics, Eindhoven University of Technology.

## 2. Results

The sums  $S$  and  $T$  have the following asymptotic behaviour:

$$(4) \quad S(\mu, m, n) = A(m, n)\mu^{1/2} + B(m, n) + O(\mu^{-1/2}) \quad (\mu \rightarrow \infty) ,$$

$$(5) \quad T(\mu, m, n) = C(m, n)\mu + D(m, n)\mu^{1/2} + O(1) \quad (\mu \rightarrow \infty) ,$$

where

$$(6) \quad A(m, n) = \sqrt{2} \int_{-\infty}^{\infty} f^m(x) f^n(-x) dx ,$$

$$(7) \quad B(m, n) = -2/3 \int_{-\infty}^{\infty} x f^m(x) f^n(-x) dx ,$$

$$(8) \quad C(m, n) = -3B(m, n) ,$$

$$(9) \quad D(m, n) = \sqrt{2} \int_{-\infty}^{\infty} f^m(x) f^n(-x) (2/3 - x^2) dx ,$$

and

$$(10) \quad f(x) := \pi^{-1/2} \int_{-\infty}^x e^{-s^2} ds \quad (x \in \mathbb{R}) .$$

Clearly  $f$  can be expressed in the errorfunction but formulas (6) to (10) do not become simpler in doing so. Some coefficients are

$$(11) \quad A(1, 1) = 2A(1, 2) = 2A(2, 1) = \pi^{-1/2}$$

$$B(n, n) = 0 \quad (n \in \mathbb{N})$$

$$B(1, 2) = -B(2, 1) = B(1, 3) = -B(3, 1) = \frac{\sqrt{3}}{12} \pi^{-1}$$

$$D(1, 1) = 2D(1, 2) = 2D(2, 1) = \frac{1}{4} \pi^{-1/2} .$$

## 3. Sketch of the derivation

The results are obtained by taking the following steps.

- (i) The sums are approximated by sums over  $|u-k| \leq \mu^{2/3}$  with an error of  $O(e^{-c\mu^{1/3}})$  ( $\mu \rightarrow \infty$ ) where  $c$  is some positive constant.
- (ii) For  $k \in [\mu - c\mu^{2/3}, \mu + c\mu^{2/3}]$  the asymptotic behaviour of  $I(\mu, k)$  for  $\mu \rightarrow \infty$  is determined. Let  $x \in \mathbb{R}$  be defined by

$$x = k^{1/2} \left| h(\mu k^{-1} - 1) \right|^{1/2} \text{sgn}(\mu - k)$$

where  $h(s) := -s + \log(1+s)$  ( $s > -1$ ) .

Roughly  $x = (\mu - k) (2\mu)^{-1/2}$ . Then  $I(\mu, k)$  has a complete asymptotic expansion in powers of  $\mu^{-1/2}$  which is uniform with respect to  $x \in \mathbb{R}$ ,  $x = O(\mu^{1/6})$  ( $\mu \rightarrow \infty$ ).

$$I(\mu, k) \approx f(x) - 2^{1/2} 3^{-1} \pi^{-1/2} e^{-x^2} \mu^{-1/2} + \dots (\mu \rightarrow \infty),$$

where  $f$  is defined by (10), i.e.

$$\forall_{N \in \mathbb{N}} \forall_{c > 0} \exists_{A > 0} \exists_{B > 0} \forall_{\mu > A} \forall_{k \in \mathbb{N}} \\ |k - \mu| \leq c \mu^{2/3} \Rightarrow \left| I(\mu, k) - \left\{ f(x) + e^{-x^2} \sum_{l=1}^N q_l(x) \mu^{-l/2} \right\} \right| \leq B \mu^{-(N+1)/2}.$$

(iii) The sums  $\sum_{|\mu-k| \leq \mu^{2/3}}$  are approximated by integrals  $\int_{|\mu-k| \leq \mu^{2/3}} \dots dk$  with an error which, for every positive number  $r$ , is  $O(\mu^{-r})$  ( $\mu \rightarrow \infty$ ). Then these integrals are transformed into integrals over  $x$  and then approximated by integrals  $\int_{-\infty}^{\infty} dx$  with errors of the kind  $O(e^{-c\mu^{1/3}})$  ( $\mu \rightarrow \infty$ ).

#### 4. The truncation of the sum

The function  $I(\mu, k)$  interpreted as a function of the real variable  $k \in [0, \infty)$  is decreasing on  $[0, \infty)$ . This statement follows from

$$\frac{d}{dk} I(\mu, k) = (\Gamma(k+1))^{-2} \int_0^\mu dt \int_\mu^\infty e^{-t-\tau} t^k \tau^k \log(t\tau^{-1}) d\tau < 0 .$$

Let  $k \leq \mu - \mu^{2/3} := a$ . Then substituting  $t = k(1+s)$  we have

$$1 - I(\mu, k) \leq 1 - I(\mu, a) = (\Gamma(a+1))^{-1} e^{-a} a^{a+1} \int_{a^{-1}\mu-1}^\infty e^{ah(s)} ds ,$$

where  $h(s) := -s + \log(1+s)$ . The function  $h$  is concave and negative on  $(-1, \infty)$ , whence  $h(s) \leq h(\mu a^{-1} - 1) + (s - \mu a^{-1} + 1) h'(\mu a^{-1} - 1)$  ( $s \geq \mu a^{-1} - 1$ ).

It follows that

$$0 < e^{-a} a^{a+1} (\Gamma(a+1))^{-1} \int_{a^{-1}\mu-1}^\infty e^{ah(s)} ds \leq e^{-a} a^{a+1} (\Gamma(a+1))^{-1} e^{ah(a^{-1}\mu-1)} (a h'(\mu a^{-1} - 1))^{-1} \\ \leq \mu^{1/2} e^{-1/2\mu^{1/3}} \quad (\mu \geq 8) .$$

Hence, for both of the sums in (1) and (2)

$$0 < \sum_{0 \leq k \leq a} < \mu \cdot \mu^{3/2} e^{-\frac{3a}{2} \mu^{1/3}} = O(e^{-\mu^{1/3}}) \quad (\mu \rightarrow \infty) \quad .$$

Let  $k \geq \mu + \mu^{2/3}$ . Then  $I(\mu, k) = e^{-k} k^{k+1} (\Gamma(k+1))^{-1} \int_{-1}^{k^{-1}\mu-1} e^{kh(s)} ds$ . Since  $h(s) \leq -1/2s^2$  on  $(-1, 0]$  we have

$$\begin{aligned} I(\mu, k) &\leq e^{-k} k^{k+1} (\Gamma(k+1))^{-1} \int_{-1}^{k^{-1}\mu-1} e^{-1/2ks^2} ds \\ &\leq e^{-k} k^{k+1} (\Gamma(k+1))^{-1} \int_{1-k^{-1}\mu}^{\infty} e^{-1/2ks^2} ds \\ &\leq e^k k^{k+1} (\Gamma(k+1))^{-1} \int_{1-k^{-1}\mu}^{\infty} e^{-1/2k(1-k^{-1}\mu)s} ds \\ &\leq \mu^{-1/6} e^{-1/2k(\mu^{1/3}+1)^{-2}} \quad (\mu \geq 8) \quad . \end{aligned}$$

Hence, for both of the sums

$$0 < \sum_{k \geq \mu + \mu^{2/3}} \leq \mu^{-1/6} \sum_{k \geq \mu + \mu^{2/3}} k e^{-1/2km(\mu^{1/3}+1)^{-2}} \leq 2\mu e^{-1/3m\mu^{1/3}} \quad (\mu \geq 8) \quad .$$

Hence, in both cases

$$\sum_{k=0}^{\infty} = \sum_{|\mu-k| \leq \mu^{2/3}} + O(e^{-c\mu^{1/3}}) \quad (\mu \rightarrow \infty)$$

where  $c$  is a positive number.

### 5. The incomplete gamma function

As we have seen already the substitution

$$(12) \quad t = k(1+s)$$

in the integral representation of  $I(\mu, k)$  gives

$$(13) \quad I(\mu, k) = (k!)^{-1} e^{-k} k^{k+1} \int_{-1}^{k^{-1}\mu-1} e^{kh(s)} ds \quad ,$$

where

$$(14) \quad h(s) := -s + \log(1+s) \quad (s > -1).$$

We introduce a new integration variable  $y$  by

$$(15) \quad y := k^{1/2} |h(s)|^{1/2} \operatorname{sgn}(s) \quad (s > -1).$$

Then we get

$$(16) \quad I(\mu, k) = (k!)^{-1} e^{-k} k^{k+1/2} \int_{-\infty}^x e^{-y^2} \frac{ds}{dy} dy \quad ,$$

where

$$(17) \quad x := k^{1/2} |h(k^{-1}\mu - 1)|^{1/2} \operatorname{sgn}(\mu - k) \quad .$$

To study the transformation (15) we introduce first in (13)

$$(18) \quad \tau = |h(s)|^{1/2} \operatorname{sgn}(s) \quad .$$

Then

$$(19) \quad \begin{aligned} \sqrt{2} \tau &= s(1 - 2/3s + 2/4s^2 - 2/5s^3 + \dots)^{1/2} \quad (|s| < 1) \quad , \\ &= s - 1/3s^2 + 7/36s^3 + \dots \quad (|s| < 1) \quad , \end{aligned}$$

the radius of convergence being one since  $h(s)/s^2$  has no zeros inside the unit circle. By the Bürmann-Lagrange theorem we can expand  $s$  as a powerseries in  $\tau$  with a positive radius of convergence, say  $\rho$ .

We calculate

$$(20) \quad s = \sqrt{2} \tau + 2/3 \tau^2 + \frac{\sqrt{2}}{18} \tau^3 - 2/135 \tau^4 + \dots \quad (|\tau| < \rho) \quad .$$

$$(21) \quad \frac{ds}{d\tau} = \sum_{l=0}^{\infty} c_l \tau^l = \sqrt{2} + 4/3 \tau + \frac{\sqrt{2}}{6} \tau^2 - 8/135 \tau^3 + \dots \quad (|\tau| < \rho) \quad .$$

The transformation (18) changes the integral (13) into

$$(22) \quad I(\mu, k) = (k!)^{-1} e^{-k} k^{k+1} \int_{-\infty}^{k^{-1/2} x} e^{-k\tau^2} \frac{ds}{d\tau} d\tau \quad .$$

We shall study the asymptotic behaviour of this integral for  $k \rightarrow \infty$  and fixed  $x \in \mathbb{R}$ . Therefore we shall denote the right hand of (22) by  $\tilde{I}(x, k)$ .

In order to use (21) we must truncate the interval of integration. Now it is easily shown that

$$\left| \int_{|\tau| \geq 1/2\rho} e^{-k\tau^2} \frac{ds}{d\tau} d\tau \right| = O(e^{-ck^{1/3}}) \quad (k \rightarrow \infty) \quad ,$$

where  $c$  is a positive number.

Inside the circle  $|\tau| \leq 1/2\rho$  the powerseries in (21) is also an asymptotic series, i.e. for every  $N \in \mathbb{N}$  we have

$$(23) \quad \frac{ds}{d\tau} = \sqrt{2} + 4/3\tau + \frac{\sqrt{2}}{6}\tau^2 - 8/135\tau^3 + \dots + c_{N-1}\tau^{N-1} + O(\tau^N) \quad (|\tau| \leq 1/2\rho)$$

Hence

$$(24) \quad \tilde{I}(x, k) = (k!)^{-1} e^{-k} k^{k+1} \int_{|\tau| \leq 1/2\rho, \tau < k^{-1/2}x} e^{-k\tau^2} \left( \sqrt{2} + \frac{4}{3}\tau + \dots + c_{N-1}\tau^{N-1} + O(\tau^N) \right) d\tau \\ + O\left[ e^{-ck^{1/3}} \right] \quad (k \rightarrow \infty) .$$

Now we change the lower bound into  $-\infty$  and, eventually, the upper bound into  $k^{-1/2}x$ , thereby making an error of the kind  $O(e^{-ck^{1/3}})$  ( $k \rightarrow \infty$ ) uniformly in  $x \in \mathbb{R}$ .

Then we substitute  $\tau = k^{-1/2}y$  and we get

$$(25) \quad \tilde{I}(x, k) = (k!)^{-1} e^{-k} k^{k+1/2} \int_{-\infty}^x e^{-y^2} \left( \sqrt{2} + 4/3k^{-1/2}y + \frac{\sqrt{2}}{6}k^{-1}y^2 + \dots \right. \\ \left. + c_{N-1}k^{-\frac{N-1}{2}}y^{N-1} + O(k^{-N/2}y^N) \right) dy \\ + O(e^{-ck^{1/3}}) \quad (k \rightarrow \infty) \quad \text{uniformly in } x \in \mathbb{R}.$$

Since  $\int_{-\infty}^x e^{-y^2} O(k^{-N/2}y^N) dy = O(k^{-N/2})$  ( $k \rightarrow \infty$ ) uniformly in  $x \in \mathbb{R}$ , we have

$$(26) \quad \tilde{I}(x, k) = \frac{g(k)}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2} \left( \sqrt{2} + 4/3k^{-1/2}y + \dots + c_{N-1}k^{-(N-1)/2}y^{N-1} \right) dy \\ + O(k^{-N/2}) \quad (k \rightarrow \infty)$$

uniformly in  $x \in \mathbb{R}$ , where

$$(27) \quad g(k) := (2\pi)^{1/2} (k!)^1 e^{-k} k^{k+1/2} .$$

Hence

$$(28) \quad \tilde{I}(x, k) = g(k) \sum_{l=0}^{N-1} c_l f_l(x) k^{-l/2} + O(k^{-N/2}) \quad (k \rightarrow \infty) \quad \text{uniformly in } x \in \mathbb{R},$$

where the  $c_l$ 's are given by (21) and

$$(29) \quad f_l(x) := (2\pi)^{-1/2} \int_{-\infty}^x y^l e^{-y^2} dy \quad (l \in \mathbb{N}_0) .$$



Integration by parts gives

$$(30) \quad f_l(x) = e^{-x^2} p_l(x) + \frac{1+(-1)^l}{2} \Gamma\left(\frac{l+1}{2}\right) (2\pi)^{-1/2} f(x) \quad (l \in \mathbb{N}) \quad .$$

where  $f$  is defined by (10) and

$$(31) \quad p_l(x) := -2^{-3/2} \pi^{-1/2} \Gamma\left(\frac{l+1}{2}\right) \left[ \sum_{s=0}^{\lfloor (l-1)/2 \rfloor} \left(\Gamma\left(\frac{l+1}{2} - s\right)\right)^{-1} x^{l-1-2s} \right] \quad (l \in \mathbb{N}) \quad .$$

Letting  $x \rightarrow \infty$  we find that  $f_l(\infty) = 0$  if  $l$  is odd and  $f_l(\infty) = \Gamma\left(\frac{l+1}{2}\right) (2\pi)^{-1/2}$  if  $l$  is even.

Since  $\lim_{\mu \rightarrow \infty} I(\mu, k) = 1$  we get the asymptotic series for  $(g(k))^{-1}$  by letting  $x \rightarrow \infty$  in (28). We have

$$(32) \quad 1 \approx g(k) \sum_{l=0}^{\infty} c_l f_l(\infty) k^{-l/2} \quad (k \rightarrow \infty) \quad .$$

Comparing (28) (30) and (32) we see that the factor with which  $f(x)$  occurs in (28) has the same asymptotic expansion as the factor  $g(k)$  in (32). Hence

$$(33) \quad \bar{I}(x, k) \approx f(x) + g(k) \sum_{l=1}^{\infty} c_l e^{-x^2} p_l(x) k^{-l/2} \quad (k \rightarrow \infty) \text{ uniformly in } x \in \mathbb{R}.$$

(i.e. after truncation the (absolute) error is smaller than  $c k^{-(N+1)/2}$  with  $c$  independent of  $x$ ).

According to the results of section 4 we restrict ourselves to values of  $k \in [\mu - \mu^{2/3}, \mu + \mu^{2/3}]$ . Then  $x = O(\mu^{1/6})$  ( $\mu \rightarrow \infty$ ).

In order to get an asymptotic series for  $I(\mu, k)$  for  $\mu \rightarrow \infty$  we have to express  $k$  as a function of  $\mu$  and  $x$ . From (17) we have  $kh(\mu k^{-1} - 1) = -x^2$ ,  $\text{sgn} x = \text{sgn}(\mu - k)$ . Putting  $k = \mu(1 - \psi)$  and  $\mu^{-1/2} x = z$  we get

$$(34) \quad \psi + (1 - \psi) \log(1 - \psi) = z^2$$

whence

$$(35) \quad \sum_{i=2}^{\infty} \frac{\psi^i}{(i-1)i} = z^2 \quad (|\psi| < 1), \text{sgn } z = \text{sgn } \psi.$$

Then

$$(36) \quad \psi(1 + 2 \sum_{i=3}^{\infty} \frac{\psi^i}{(i-1)i})^{1/2} = \sqrt{2} z \quad .$$

By the Bürmann-Lagrange inversion theorem there is a positive number  $r$  such that

$$(37) \quad \psi = \sum_{i=1}^{\infty} d_i z^i \quad (|z| < r) \quad .$$

Hence

$$(38) \quad k = \mu - \sum_{i=1}^{\infty} d_i \mu^{1-i/2} x^i \quad (|x| < r\sqrt{\mu}).$$

Clearly, for  $\mu$  sufficiently large,  $x$  is within the range of convergence since  $x = O(\mu^{1/6})$  ( $\mu \rightarrow \infty$ ). A few coefficients  $d_i$  are calculated.

$$(39) \quad d_1 = \sqrt{2}, d_2 = -1/3, d_3 = \frac{-\sqrt{2}}{36}, d_4 = -\frac{23}{90} \quad .$$

We need also expansions for  $k^{-1/2}$ ,  $k^{-1}$  and  $k^{-3/2}$ :

Clearly, there is a positive number  $r_0$  such that for all  $\alpha \in \mathbb{R}$  the function  $(1-\psi)^\alpha$  has a power-series expansion in  $z$  which converges for  $|z| < r_0$ ; if

$$(40) \quad (1-\psi)^\alpha = \sum_{j=0}^{\infty} a_j(\alpha) z^j \quad (|z| < r_0)$$

then

$$(41) \quad k^\alpha = \mu^\alpha (1-\psi)^\alpha = \sum_{j=0}^{\infty} a_j(\alpha) \mu^{\alpha-i/2} x^j \quad (|x| < r_0 \mu^{1/2}) \quad .$$

We calculate

$$(42) \quad k^{-1/2} = \mu^{-1/2} + 2^{-1/2} \mu^{-1} x + \frac{7}{12} \mu^{-3/2} x^2 + 13 \frac{\sqrt{2}}{36} \mu^{-2} x^3 + \dots$$

$$(43) \quad k^{-1} = \mu^{-1} + 2^{1/2} \mu^{-3/2} x + 5/3 \mu^{-2} x^2 + \dots$$

$$(44) \quad k^{-3/2} = \mu^{-3/2} + 3 \cdot 2^{-1/2} \mu^{-2} x + \dots$$

An asymptotic expansion for  $g(k)$  can be determined from (32).

$$(45) \quad g(k) \approx \sum_{l=0}^{\infty} g_l k^{-l/2} \quad (k \rightarrow \infty)$$

Some coefficients  $g_l$  are

$$(46) \quad g_0 = 1, g_2 = -1/12 \text{ and } g_l = 0 \text{ if } l = \text{odd}.$$

Substituting (46) into (33) and then using (41) we get a complete asymptotic series for  $I(\mu, k)$ .

$$(47) \quad I(\mu, k) \approx f(x) + e^{-x^2} \sum_{l=1}^{\infty} q_l(x) \mu^{-l/2} \quad (\mu \rightarrow \infty)$$

uniformly in  $x = O(\mu^{1/6}) \quad (\mu \rightarrow \infty)$ .

The  $q_l$ 's are polynomials. We calculate

$$(48) \quad q_1(x) = -2^{1/2} 3^{-1} \pi^{-1/2} \quad .$$

$$(49) \quad q_2(x) = -\frac{5}{12} \pi^{-1/2} x \quad .$$

### 6. The replacement of the sum by an integral

We have already

$$(50) \quad \frac{d}{dk} I(\mu, k) = (\Gamma(k+1))^{-2} \int_0^{\mu} dt \int_0^{\infty} e^{-t-\tau} (t\tau)^k \log(t\tau^{-1}) d\tau \quad .$$

By induction one can prove easily that

$$(51) \quad \frac{d^l}{dk^l} I(\mu, k) = (\Gamma(k+1))^{-l-1} \int_0^{\mu} dt \int_0^{\infty} \dots$$

$$\int_0^{\infty} e^{-t-\tau_1-\dots-\tau_l} (t\tau_1 \dots \tau_l)^k L(t, \tau_1, \dots, \tau_l) d\tau_1 \dots d\tau_l \quad ,$$

where the functions  $L(\tau_0, \tau_1, \dots, \tau_l)$  are defined by

$$(52) \quad L(\tau_0) = 1$$

$$(53) \quad L(\tau_0, \tau_1, \dots, \tau_{l+1}) = L(\tau_0, \tau_1, \dots, \tau_l) \log(\tau_0 \tau_1 \dots \tau_l \tau_{l+1}^{-l-1}) \quad (l=0, 1, \dots) \quad .$$

With methods similar to those used in the treatment of  $I(\mu, k)$  in section 5 we can prove easily that

$$(54) \quad \left| \frac{d^l}{dk^l} I(\mu, k) \right| = O(e^{-c\mu^{1/3}}) \quad (\mu \rightarrow \infty, |\mu - k| \geq \mu^{2/3}) \quad .$$

Furthermore, if we restrict ourselves to values of  $k$  such that  $|\mu - k| \leq \mu^{2/3}$ , then the integrals in

$$(51) \text{ can be replaced by } \int_{\mu-2\mu^{2/3}}^{\mu} \int_{\mu+2\mu^{2/3}}^{\mu+2\mu^{2/3}} \dots \int_{\mu-2\mu^{2/3}}^{\mu+2\mu^{2/3}}$$

with an error of the kind  $O(e^{-c\mu^{1/3}}) \quad (\mu \rightarrow \infty)$ .

Then it follows from (51) that for  $|\mu - k| \leq \mu^{2/3}$

$$(55) \quad \left| \frac{d^l}{dk^l} I(\mu, k) \right| < M^l := \max \{ |L(t, \tau_1, \dots, \tau_l)| \mid \mu - 2\mu^{2/3} \leq t, \dots, \tau_l \leq \mu + 2\mu^{2/3} \}$$

$$+ O(e^{c\mu^{1/3}}) \quad (\mu \rightarrow \infty)$$

From (52) and (53) it follows that

$$M_{l+1} \leq M_l \log \left[ \frac{\mu + 2\mu^{2/3}}{\mu - 2\mu^{2/3}} \right]^{l+1} \quad (l \in \mathbb{N}_0),$$

whence, by (54) and (55), we have for all  $k \in \mathbb{N}$

$$(56) \quad \left| \frac{d^l}{dk^l} I(\mu, k) \right| = O(\mu^{-l/3}) \quad (\mu \rightarrow \infty) \quad (l \in \mathbb{N}_0).$$

Now we apply the Euler-Maclaurin sumformula: For every fixed  $r \in \mathbb{N}$  we have

$$(57) \quad \sum_{k=p}^q f(k) = \int_p^q f(x) dx + 1/2 f(q) + 1/2 f(p) \\ + \sum_{l=1}^r \frac{B_{2l}}{(2l)!} \left[ f^{(2l-1)}(q) - f^{(2l-1)}(p) \right] \\ + O \left[ \int_p^q |f^{(2r)}(x)| dx \right] \quad (q \geq p) \quad .$$

Taking  $f(k) = I^m(\mu, k) (1 - I(\mu, k))^n$ ,  $p = \left\lfloor \mu - \mu^{2/3} \right\rfloor$ ,  $q = \left\lfloor \mu + \mu^{2/3} \right\rfloor$ ,

we find, for every  $r \in \mathbb{N}$ , using (54), (56) and (57)

$$(58) \quad \sum_{|k-\mu| \leq \mu^{2/3}} I^m(\mu, k) (1 - I(\mu, k))^n = \int_{\mu - \mu^{2/3}}^{\mu + \mu^{2/3}} I^m(\mu, k) (1 - I(\mu, k))^n dk \\ + O(\mu^{-2/3r+2/3}) \quad (\mu \rightarrow \infty)$$

Similarly

$$(59) \quad \sum_{|k-\mu| \leq \mu^{2/3}} (\mu - k) I^m(\mu, k) (1 - I(\mu, k))^n = \int_{\mu - \mu^{2/3}}^{\mu + \mu^{2/3}} (\mu - k) I^m(\mu, k) (1 - I(\mu, k))^n dk \\ + O(\mu^{-2/3r+4/3}) \quad (\mu \rightarrow \infty) \quad .$$

## 7. The asymptotic behaviour of S and T

According to section 6 the sums  $\sum_{|k-\mu| \leq \mu^{2/3}}$  can be replaced by integrals with small errors which are, for every  $r > 0$ ,  $O(\mu^{-r}) (\mu \rightarrow \infty)$ . Then these integrals are transformed into integrals

$$\int_{-A(\mu)}^{B(\mu)} \dots \frac{dk}{dx} dx \quad \text{where } A(\mu) \text{ and } B(\mu) \text{ are asymptotically equivalent with } 1/2\sqrt{2} \mu^{1/6} \quad (\mu \rightarrow \infty)$$

By means of (38) and (47) we get complete asymptotic expansions for the two integrands.

$$(60) \quad I^m(\mu, k)(1-I(\mu, k))^n \frac{dk}{dx} \approx \sum_{l=1}^{\infty} s_l(x) \mu^{-l/2} \quad (\mu \rightarrow \infty) \quad ,$$

$$(61) \quad (\mu-k)I^m(\mu, k)(1-I(\mu, k))^n \frac{dk}{dx} \approx \sum_{l=2}^{\infty} t_l(x) \mu^{-l/2} \quad (\mu \rightarrow \infty) \quad ,$$

both uniformly in  $x \in \mathbb{R}$ ,  $x = O(\mu^{1/6})$  ( $\mu \rightarrow \infty$ ). The functions  $s_l(x)$  and  $t_l(x)$  are absolutely integrable over  $(-\infty, \infty)$ . Now we proceed as follows. Let  $N \in \mathbb{N}$ . Then

$$\begin{aligned} & \int_{\mu-\mu^{2/3}}^{\mu+\mu^{2/3}} I^m(\mu, k)(1-I(\mu, k))^n \frac{dk}{dx} dx \\ &= \sum_{l=1}^N \mu^{-l/2} \int_{-A(\mu)}^{B(\mu)} s_l(x) dx + O\left[\mu^{-\frac{N}{2} - \frac{1}{2} + \frac{1}{6}}\right] \quad (\mu \rightarrow \infty). \end{aligned}$$

It is easily seen that the functions  $s_l(x)$  are of the form  $\bar{p}(x, e^{-x^2}, f(x)) e^{-x^2} + \bar{q}(x, e^{-x^2}, f(x)) f(x)(1-f(x))$ , where  $\bar{p}$  and  $\bar{q}$  are polynomials. Hence  $\int_{x < -A(\mu)} |s_l(x)| dx$  and  $\int_{x > B(\mu)} |s_l(x)| dx$  are  $O(e^{-c\mu^{1/3}})$  ( $\mu \rightarrow \infty$ ).

It follows that  $S(\mu, m, n)$  has a complete asymptotic expansion

$$(62) \quad S(\mu, m, n) \approx \sum_{l=1}^{\infty} \mu^{-l/2} \int_{-\infty}^{\infty} s_l(x) dx \quad (\mu \rightarrow \infty)$$

A similar argument holds for  $T(\mu, m, n)$ .

$$(63) \quad T(\mu, m, n) \approx \sum_{l=2}^{\infty} \mu^{-l/2} \int_{-\infty}^{\infty} t_l(x) dx \quad (\mu \rightarrow \infty).$$

We calculate

$$(64) \quad s_{-1}(x) = \sqrt{2} f^m(x) f^n(-x)$$

$$(65) \quad s_0(x) = 2/3\pi^{-1/2} \left[ n f^m(x) f^{n-1}(-x) - m f^{m-1}(x) f^n(-x) \right] e^{-x^2} - 2/3x f^m(x) f^n(-x)$$

$$(66) \quad t_{-2}(x) = 2x f^m(x) f^n(-x)$$

$$(67) \quad t_{-1}(x) =$$

$$2^{3/2} 3^{-1} \pi^{-1/2} \left[ n f^m(x) f^{n-1}(-x) - m f^{m-1}(x) f^n(-x) \right] x e^{x^2} - 2^{1/2} x^2 f^m(x) f^n(-x)$$

The term with factor  $e^{-x^2}$  in the right hand of (65) gives 0 upon integration.

Integration by parts of the term with factor  $x e^{-x^2}$  in the right hand of (67) gives

$$2/3\sqrt{2} \int_{-\infty}^{\infty} f^m(x) f^n(-x) dx.$$

$$=2/3A(1,1)-2\frac{\sqrt{2}}{3}\pi^{-1/2}\int_{-\infty}^{\infty}xe^{-2x^2}f(x)dx-\frac{\sqrt{2}}{3}\pi^{-1}\int_{-\infty}^{\infty}x^2e^{-2x^2}dx$$

$$=2/3A(1,1)-\frac{5}{12}\pi^{-1/2}=1/4\pi^{-1/2}.$$

$$D(1,2)=D(2,1)=2/3A(1,2)-1/2\sqrt{2}\int_{-\infty}^{\infty}x^2(f(x)f^2(-x)+f^2(x)f(-x))dx=$$

$$=2/3A(1,2)-1/2\sqrt{2}\int_{-\infty}^{\infty}x^2f(x)f(-x)dx$$

$$=2/3A(1,2)+1/2D(1,1)-1/3A(1,1)=1/8\pi^{-1/2}.$$