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by
A.F.M. ter Elst and Derek W. Robinson

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Department of Mathematics and Computing Science
Eindhoven University of Technology
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The Netherlands
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A.F.M. ter Elst[1] and Derek W. Robinson

Centre for Mathematics and its Applications
School of Mathematical Sciences
Australian National University
Canberra, ACT 0200
Australia

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Home institution:

1. Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Abstract

Let $(\mathcal{X}, G, U)$ be a continuous representation of a Lie group G by bounded operators $g \mapsto U(g)$ on the Banach space $\mathcal{X}$ and let $(\mathcal{X}, \mathfrak{g}, dU)$ denote the representation of the Lie algebra $\mathfrak{g}$ obtained by differentiation. If $a_1, \ldots, a_{d'}$ is a Lie algebra basis of $\mathfrak{g}$, $A_i = dU(a_i)$ and $A^\alpha = A_{i_1} \ldots A_{i_k}$ whenever $\alpha = (i_1, \ldots, i_k)$ we consider the operators

$$H = \sum_\alpha c_\alpha A^\alpha$$

where the c_α are complex coefficients satisfying a weighted strongly elliptic condition in which different directions may have different weights. This condition is such that the class of operators considered encompasses all the standard strongly elliptic operators. We prove that the closure $\overline{H}$ of each such operator H generates a holomorphic semigroup S with holomorphy sector which contains a non-empty subsector determined by the coefficients and independent of the representation. Moreover, the semigroup S has a smooth representation independent kernel and we derive bounds on the kernel and all its derivatives. Finally we establish elliptic regularity properties for the operators and their powers and characterize the analytic and Gevrey vectors. As a corollary we derive optimal growth bounds for the eigenfunctions of the anharmonic oscillators $P^{2m} + Q^{2n}$.

1 Introduction

Within the general theory of elliptic operators the twin notions of subelliptic operator and weighted elliptic operator play related but distinct roles. Subellipticity is based on the idea that there is a certain subset of preferred directions but in the weighted theory all directions are allowed although some have a greater weight, or preference. Both concepts have non-commutative aspects and are naturally described in the setting of Lie groups. In this paper we develop the theory of weighted strongly elliptic operators for Lie groups in analogy with the subelliptic theory. We start with a basis for the Lie algebra, give each direction a weight, define weighted strongly elliptic forms and then introduce weighted strongly elliptic operators associated with an arbitrary strongly continuous or weakly* continuous representation of the Lie group on a Banach space. Under a suitable condition on the Lie algebraic structure we prove that the closure of each such operator generates a continuous semigroup, which has a representation independent kernel. We then derive kernel bounds, regularity and analyticity properties, et cetera.

The simplest illustration of the applicability of the weighted theory involves the Heisenberg group G and the anharmonic oscillator. Let U be the standard irreducible unitary representation of G on $L_2(\mathbf{R})$ and a_1, a_2, a_3 a basis of the Lie algebra $\mathfrak{g}$ of G such that the representatives $A_i = dU(a_i)$ are given by $A_1 = -iP$, $A_2 = iQ$ and $A_3 = iI$, where P and Q are the self-adjoint operators in $L_2(\mathbf{R})$ such that $(Pf)(x) = if'(x)$ and $(Qf)(x) = xf(x)$ for all $f \in C_c^\infty(\mathbf{R})$ and $x \in \mathbf{R}$. Then $[a_1, a_2] = a_3$ and a_3 is central. Now the Laplacian associated with the basis a_1, a_2, a_3 or the sublaplacian associated with the algebraic subbasis a_1, a_2 both give a description of the harmonic oscillator, e.g., $-\sum_{i=1}^2 A_i^2 = P^2 + Q^2$. Therefore the oscillator fits equally well into the theory of strongly elliptic operators or the theory of subelliptic operators on G. But the anharmonic oscillator $P^2 + Q^4$ does not correspond to a subelliptic operator although it can be expressed in this form by the introduction of a different Lie group G_1 and Lie algebra $\mathfrak{g}_1$. Let $\mathfrak{g}_1$ denote the subalgebra of the enveloping Lie algebra of $\mathfrak{g}$ generated by the elements a_1, a_2^2. Thus $\mathfrak{g}_1$ is four-dimensional and has a basis $b_1 = a_1, b_2 = a_2^2, b_3 = a_2 a_3, b_4 = a_3^2$. The corresponding simply connected group G_1 has a unitary representation U_1 on $L_2(\mathbf{R})$ such that if $B_i = dU_1(b_i)$ then $-\sum_{i=1}^2 B_i^2 = P^2 + Q^4$. Nevertheless, marginally more complicated operators such as $P^4 + Q^6$ cannot be incorporated in a subelliptic framework. The subalgebra of the enveloping algebra generated by a_1^2, a_2^3 is not finite-dimensional. All the operators $P^{2j} + Q^{2k}$ with $j, k \in \mathbf{N}$ are, however, weighted strongly elliptic operators in the sense we describe in this paper. Since we establish that all the main features of the usual theory of strongly elliptic operators or subelliptic operators extend to the weighted case the latter theory incorporates results which are not a direct consequence of the standard theory.

Background material for elliptic operators on Lie groups can be found in [Rob2], [VSC] and relevant results for subelliptic operators in [JSC1], [NSW], [RoS], [Sán], [Var]. Related results for weighted operators occur in [DzH], [Heb1], [Heb2], [Dzi].

Throughout the sequel we adopt the general notation of [Rob2]. In particular G denotes a d-dimensional Lie group which we assume to be connected because all analysis takes place on the connected component of the identity. The Lie algebra of G is denoted by $\mathfrak{g}$. Furthermore $(\mathcal{X}, G, U)$ is used for a continuous representation of G on the Banach space $\mathcal{X}$ by bounded operators $g \mapsto U(g)$. Both strong and weak* continuity are considered.

Moreover, if $a_i \in \mathfrak{g}$ then $A_i(= dU(a_i))$ denotes the generator of the one-parameter subgroup $t \mapsto U(e^{-ta_i})$ of the representation. Let $a_1, \ldots, a_d$ a basis for the Lie algebra $\mathfrak{g}$ of G. Let $w_1, \ldots, w_d \in \mathbf{N}$, which we call **weights**. Throughout this paper we assume that the structure constants c_{ij}^k, defined by $[a_i, a_j] = \sum_{k=1}^d c_{ij}^k a_k$, are such that if $c_{ij}^k \neq 0$ then $w_i + w_j - 1 \geq w_k$. So

$$[a_i, a_j] = \sum_{\substack{k \in \{1,\ldots,d\} \\ w_k \leq w_i + w_j - 1}} c_{ij}^k \, a_k \quad .$$

Set

$$w = \operatorname{lcm}(w_1, \ldots, w_d) \quad , \qquad D' = \sum_{i=1}^d w_i \quad .$$

For the multi-indices we need we introduce the following notation. If $n \in \mathbf{N}_0$ let

$$J_n(d) = \bigoplus_{k=0}^n \{1, \ldots, d\}^k$$

and set

$$J(d) = \bigcup_{n=0}^\infty J_n(d) \quad .$$

If $\alpha = (i_1, \ldots, i_n) \in J(d)$ we denote the **length** n of α by $|\alpha|$ and set

$$\|\alpha\| = \sum_{k=1}^n w_{i_k} \quad .$$

If $n \in \mathbf{N}$ we define $\mathcal{X}_n = \bigcap_{\|\alpha\| \leq n} D(A^\alpha)$ and

$$\|x\|_n = \max_{\substack{\alpha \in J(d) \\ \|\alpha\| \leq n}} \|A^\alpha x\| \quad , \quad N_n(x) = \max_{\substack{\alpha \in J(d) \\ \|\alpha\| = n}} \|A^\alpha x\| \quad ,$$

where

$$A^\alpha = A_{i_1} \ldots A_{i_n}$$

if $\alpha = (i_1, \ldots, i_n)$.

Let $m \in \mathbf{N}$ and let $C \colon J(d) \to \mathbf{C}$ be such that $C(\alpha) = 0$ if $\|\alpha\| > m$ and there exists at least one $\alpha \in J(d)$ with $\|\alpha\| = m$ and $C(\alpha) \neq 0$. We write $c_\alpha = C(\alpha)$. The **principal part** P of C is the form

$$P(\alpha) = \begin{cases} C(\alpha) & \text{if } \|\alpha\| = m \quad , \\ 0 & \text{if } \|\alpha\| < m \quad . \end{cases}$$

The **formal adjoint** $C^\dagger$ of C is the function $C^\dagger \colon J_m(d') \to \mathbf{C}$ defined by

$$C^\dagger(\alpha) = (-1)^{|\alpha|} \overline{C(\alpha_*)} \quad ,$$

where $\alpha_* = (i_n, \ldots, i_1)$ if $\alpha = (i_1, \ldots, i_n)$. We call C a **weighted strongly elliptic form of order** m if

$$\operatorname{Re} \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = m}} c_\alpha (i\xi)^\alpha > 0$$

for all $\xi \in \mathbf{R}^d$ with $\xi \neq 0$, where we define $\xi^\alpha = \xi_{i_1} \ldots \xi_{i_n}$ for all $\xi \in \mathbf{C}^d$ and $\alpha = (i_1, \ldots, i_n) \in J(d)$. It easily follows that $2w_i \mid m$ for all $i \in \{1, \ldots, d\}$, and hence $2w \mid m$. We then consider the **weighted strongly elliptic operators**

$$dU(C) = \sum_{\alpha \in J(d)} c_\alpha A^\alpha$$

with domain $D(dU(C)) = \mathcal{X}_m$. If $(\mathcal{F}, G, U_*)$ is the dual representation of $\mathcal{X}, G, U)$ then $dU_*(C^\dagger)$ is called the dual operator and denoted by $H^\dagger$.

Define a **modulus** $|\cdot|$ on $\mathbf{R}^d$ by

$$|\xi|^{2w} = \sum_{i=1}^{d} |\xi_i|^{2w/w_i} \quad .$$

If $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{R}^d$, then

$$\|\xi\| \leq \sqrt{d}|\xi| \quad \text{and} \quad |\xi| \leq \|\xi\|^{1/w}$$

for all $\xi \in \mathbf{R}^d$ with $\|\xi\| \leq 1$. Moreover, for $u > 0$ define a **scaling** $\gamma_u \colon \mathbf{R}^d \to \mathbf{R}^d$ by $\gamma_u(\xi_1, \ldots, \xi_d) = (u^{w_1}\xi_1, \ldots, u^{w_d}\xi_d)$. Then $|\gamma_u(\xi)| = u|\xi|$ and $\gamma_u(\xi^\alpha) = u^{\|\alpha\|}\xi^\alpha$ for all $\alpha \in J(d)$. Let

$$\mu = \min\{\mathrm{Re} \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = m}} c_\alpha (i\xi)^\alpha : \xi \in \mathbf{R}^d, \ |\xi| = 1\}$$

be the **ellipticity constant** of the weighted strongly elliptic form C. Then

$$\mathrm{Re} \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = m}} c_\alpha (i\xi)^\alpha \geq \mu |\xi|^m$$

for all $\xi \in \mathbf{R}^d$. Moreover, let

$$\theta_C = \sup\{\theta \in \langle 0, \pi/2] : \{ \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = m}} c_\alpha \xi^\alpha : |\xi| = 1\} \subset \Lambda(\pi/2 - \theta)\} \quad ,$$

where $\Lambda(\theta = \{z \in \mathbf{C} : |\arg z| < \theta\}$. We say that a quantity depends continuously on the form C if it is majorized by a function which is an increasing function in each of the moduli $|c_\alpha|$ of the coefficients c_α and a decreasing function in the ellipticity constant μ.

One of the main theorems of this paper is the following.

Theorem 1.1 *Let C be a weighted strongly elliptic form of order m. Let $H = dU(C)$ be the associated weighted strongly elliptic operator. Then one has the following.*

I. *The closure $\overline{H}$ of H generates a continuous semigroup S.*

II. *The semigroup S is holomorphic in a sector $\Lambda(\theta) = \{z \in \mathbf{C} : |\arg z| < \theta\}$ where the angle of holomorphy θ satisfies the bounds $\theta_C \leq \theta \leq \pi/2$.*

III. *For all $t > 0$ there exists $K_t \in \bigcap_{\rho > 0} L_1(G; e^{\rho|g|}dg)$ such that*

$$S_t x = \int_G dg \, K_t(g) \, U(g) x$$

for all $x \in \mathcal{X}$. The function K_t is representation independent.

In addition we obtain smoothness properties and 'Gaussian' bounds for the kernel K_t and use these to establish a variety of regularity and analyticity properties of the operators and the representations. In particular we prove that the weighted analytic elements of each continuous representation coincide with those of the fractional powers $\overline{H}^{1/m}$ of the weighted operators. The proof is based on a simplification of the Nelson–Goodman result [Nel] [Goo1], given in the appendix of the latter paper, for regular representations and unweighted operators together with a version of the interpolation argument of [Rob1].

As an illustration reconsider the Heisenberg group on $L_2(\mathbf{R})$. Let $j, k \geq 1$ be integers and assign the weights $k, j, 1$ to the directions a_1, a_2, a_3 in the Heisenberg Lie algebra. Then $(-1)^j A_1^{2j} + (-1)^k A_2^{2k} + (-1)^{jk} A_3^{2jk} = P^{2j} + Q^{2k} + I$ is a weighted strongly elliptic operator of order $2jk$. Therefore the theorem covers all the cases of the anharmonic oscillator mentioned above. Moreover, by specializing our results to the oscillators we are able to derive new properties of certain Gel'fand–Shilov spaces. As a corollary we establish that the operator $P^{2j} + Q^{2k} + I$ has a discrete spectrum and its eigenfunctions $\varphi_1, \varphi_2, \ldots$ can be extended to entire functions on the complex plane satisfying growth bounds

$$|\varphi_n(x+iy)| \leq c_n e^{-a_n|x|^{(j+k)/j} + b_n|y|^{(j+k)/j}}$$

for some $a_n, b_n, c_n > 0$. This fully answers a question raised by Goodman [Goo3].

The outline of this paper is as follows. In Section 2 we prove the above theorem on $\mathbf{R}^d$ for the left regular representation. In Section 3 we prove the first two statements of the theorem and in Section 4 we introduce a distance on G, finish the proof of Theorem 1.1 and prove kernel bounds, including bounds on all the derivatives. In Section 5 we present regularity theorems and in case of unitary representations we obtain optimal regularity. Then in Section 6 we characterize the weighted analytic and Gevrey vectors of a general continuous representation in terms of the weighted operators. Finally, in the last section, we discuss several illustrative examples including the anharmonic oscillators.

2 Semigroups on $L_2(\mathbf{R}^d)$ for pure m-th order operators

Let C be a weighted strongly elliptic form of order m which equals it principal part. In this section we prove that $H = dL(C)$ is the generator of a holomorphic semigroup on $L_2(\mathbf{R}^d)$ where L denotes the left regular representation of $\mathbf{R}^d$ on $L_2(\mathbf{R}^d)$. Let $B: \mathbf{R}^d \to \mathbf{C}$ be defined by

$$B(\xi) = \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| = m}} c_\alpha (i\xi)^\alpha$$

and for $t > 0$ define $K_t: \mathbf{R}^d \to \mathbf{C}$ by

$$K_t(\xi) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} d\zeta \, e^{-i\zeta\cdot\xi} e^{-tB(\zeta)} \quad .$$

Since the action of H corresponds to multiplication of the Fourier transform with the function B and since $\operatorname{Re} B$ is positive definite it follows that H closed.

Proposition 2.1

I. *Define S_t on $L_2(\mathbf{R}^d)$, for all $t > 0$, by $S_t\varphi = K_t * \varphi$. Then S is a holomorphic semigroup with holomorphy sector $\Lambda(\theta)$ where $\theta \in [\theta_C, \pi/2]$ and with generator H.*

II. *The function K_t is infinitely differentiable for each $t > 0$. Moreover, for all $\alpha \in J(d)$ there exist $a, b > 0$ such that*

$$|(A^\alpha K_t)(\xi)| \leq a t^{-D'/m} t^{-\|\alpha\|/m} e^{-b(|\xi|^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t > 0$ and $\xi \in \mathbf{R}^d$. The constants a, b depend continuously on the form C.

Proof The first statement follows because the action of S_t is to multiply the Fourier transform by e^{-tB}. The second statement is a direct analogue of Proposition I.5.3 of [Rob2] but some differences enter because we now use the modulus $\xi \mapsto |\xi|$. Nevertheless, one has estimates such as

$$|(\xi + i\eta)^\beta| \leq C(|\xi|^{\|\beta\|} + |\eta|^{\|\beta\|}) \tag{1}$$

for all $\xi, \eta \in \mathbf{R}^d$ and a suitable $C > 0$, which follow by scaling. Therefore arguing as in [Rob2] one finds the bounds

$$|(A^\alpha K_1)(\xi)| \leq \inf_{\eta \in \mathbf{R}^d} a e^{\xi \cdot \eta} e^{c|\eta|^m}$$

for $t = 1$. Next, for $\xi \in \mathbf{R}^d$ and $c > 0$ define

$$f(\xi, c) = \inf_{\eta \in \mathbf{R}^d} e^{\xi \cdot \eta} e^{c|\eta|^m} \quad .$$

Then obviously

$$f(\xi, c) \leq e^{1-(|\xi|^m c^{-1})^{1/(m-1)}} \tag{2}$$

for all $\xi \in \mathbf{R}^d$ with $|\xi| \leq 1$ and $c = 1$. But $f(\gamma_u(\xi), c) = f(\xi, cu^{-m})$ for all $\xi \in \mathbf{R}^d$, $c > 0$ and $u > 0$. Hence one deduces first that (2) is valid for all $|\xi| \leq 1$ and $0 < c \leq 1$, then for all $\xi \in \mathbf{R}^d$ and $0 < c \leq 1$ and finally for all $\xi \in \mathbf{R}^d$ and $c > 0$. Thus

$$|(A^\alpha K_1)(\xi)| \leq e a e^{-(|\xi|^m c^{-1})^{1/(m-1)}} \quad .$$

Now one can drop the restriction on t by a scaling argument. □

Next we introduce another class of differential operators which turns out to be very useful in the next section in the proof for general Lie groups that weighted strongly elliptic operators generate holomorphic semigroups.

Lemma 2.2 *Let W be an open neighbourhood of the identity element 0 in $\mathbf{R}^d$ and $\varphi: \mathbf{R}^d \to \mathbf{C}$ a C^∞-function. Then for each $n \in \mathbf{N}$ the following are equivalent.*

I. *$(A^\alpha\varphi)(0) = 0$ for all $\alpha \in J(d')$ with $\|\alpha\| < n$.*

II. *For every compact neighbourhood K of 0 such that $K \subset W$ there exists $c > 0$ such that $|\varphi(\xi)| \leq c|\xi|^n$ for all $\xi \in K$.*

Proof By the usual Taylor formula we can write

$$\varphi(\xi) = \sum_{\substack{\alpha\in J(d)\\ |\alpha|\leq n}} (A^\alpha\varphi)(0)\,\xi^\alpha + O(\|\xi\|^n) = \sum_{\substack{\alpha\in J(d)\\ |\alpha|\leq n}} (A^\alpha\varphi)(0)\,\xi^\alpha + O(|\xi|^n) \quad .$$

But if $\alpha \in J(d)$ with $|\alpha| \leq n$ and $\|\alpha\| \geq n$ then $(A^\alpha\varphi)(0)\,\xi^\alpha = O(|\xi|^{|\alpha|}) = O(|\xi|^n)$, where we have used (1). Hence

$$\varphi(\xi) = \sum_{\substack{\alpha\in J(d)\\ \|\alpha\|\leq n}} (A^\alpha\varphi)(0)\,\xi^\alpha + O(|\xi|^n) \quad .$$

Now the implication I⇒II is obvious.

II⇒I Suppose that $|\varphi(\xi)| \leq c|\xi|^n$ for all $\xi \in K$. Then $\varphi(0) = 0$. We shall prove by induction on k that $(A^\alpha\varphi)(0) = 0$ for all $\alpha \in J(d')$ with $\|\alpha\| = k$. Let $k \in \{1,\ldots,n\}$ and suppose that $(A^\alpha\varphi)(0) = 0$ for all α with $\|\alpha\| \leq k-1$. Then

$$\sum_{\substack{\alpha\in J(d)\\ \|\alpha\|=k}} (A^\alpha\varphi)(0)\,\xi^\alpha = \varphi(\xi) + \sum_{\substack{\alpha\in J(d)\\ n\geq\|\alpha\|>k}} (A^\alpha\varphi)(0)\,\xi^\alpha = O(|\xi|^{k+1}) \quad .$$

Next fix $\xi \in \mathbf{R}^d$. Then for all $u > 0$ one has

$$u^k \sum_{\substack{\alpha\in J(d)\\ \|\alpha\|=k}} (A^\alpha\varphi)(0)\,\xi^\alpha = \sum_{\substack{\alpha\in J(d)\\ \|\alpha\|=k}} (A^\alpha\varphi)(0)\,\gamma_u(\xi)^\alpha = O(|\gamma_u(\xi)|^{k+1}) = O(u^{k+1}) \quad .$$

Therefore $\sum_{\alpha\in J(d),\,\|\alpha\|=k}(A^\alpha\varphi)(0)\,\xi^\alpha = 0$. This is valid for all ξ and since the A_i commute one obtains by differentiating that $(A^\alpha\varphi)(0) = 0$ for all α with $\|\alpha\| = k$. □

Let M_f denote the operator of multiplication with the function f. An n-th order differential operator with variable C^∞-coefficients f_α,

$$L = \sum_{\substack{\alpha\in J(d)\\ \|\alpha\|\leq n}} M_{f_\alpha} A^\alpha \quad ,$$

on an open set W containing the identity element 0, is defined to be an **operator of actual order** N if there exist $a, s > 0$ such that

$$|f_\alpha(\xi)| \leq a|\xi|^{n_\alpha(N)}$$

for all $\alpha \in J(d)$ with $\|\alpha\| \leq n$ and $\xi \in B_s \cap W$ where $n_\alpha(N) = (\|\alpha\| - N) \vee 0$ and B_s is the ball with radius s. It follows immediately from the previous lemma that if L_1 and L_2 are differential operators with variable coefficients and actual order N_1 and N_2 then $L_1 \circ L_2$ is an operator of actual order $N_1 + N_2$.

The resolvent of the closed operator H is defined by the Laplace transform

$$(\lambda I + H)^{-1} = \int_0^\infty dt\, e^{-\lambda t} S_t$$

for $\lambda \in \mathbf{C}$ with $\mathrm{Re}\,\lambda$ sufficiently large. But then it follows from the estimates of Proposition 2.1 that $(\lambda I + H)^{-1}$ has a kernel R_λ given by

$$R_\lambda(\xi) = \int_0^\infty dt\, e^{-\lambda t} K_t(\xi)$$

and $R_\lambda \in L_1$ with

$$\|R_\lambda\|_1 \leq a(\operatorname{Re}\lambda)^{-1}$$

for a suitable $a > 0$ and all $\lambda \in \mathbf{C}$ such that $\operatorname{Re}\lambda > 0$. Since $K_t \in C^\infty(\mathbf{R}^d)$ for $t > 0$ it follows that $R_\lambda \in C^\infty(\mathbf{R}^d\backslash\{0\})$. Moreover, $A^\alpha R_\lambda \in L_1$ for all $\alpha \in J(d)$ with $\|\alpha\| < m$ and

$$\|A^\alpha R_\lambda\|_1 \leq a'(\operatorname{Re}\lambda)^{-(m-|\alpha|)/m}$$

for $\operatorname{Re}\lambda > 0$. (For a related detailed discussion of strongly elliptic operators see [Rob2] Section III.6b and Appendix A of [ElR4].) Higher derivatives of R_λ are, however, not in L_1 because of singularities at the identity e. Nevertheless we now note that differential operators of order larger or equal to m but with actual order less than m do map R_λ into L_1.

Lemma 2.3 *If L is a differential operator on W of actual order N with $N \in \{0, \ldots, m-1\}$ and B is a compact subset of W then $1_B L R_\lambda \in L_1$ and there are $a > 0$ and $\omega \geq 0$ such that*

$$\|1_B L R_\lambda\|_1 \leq a(\operatorname{Re}\lambda)^{-(m-N)/m}$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega$.

Proof Given the Gaussian bounds of Proposition 2.1.II the proof is almost identical to that of Lemma 4.13 in [ElR2]. Therefore we omit further details. □

3 Weighted strongly elliptic operators

In this section we want to extend the generator theorem of the previous section to general groups and arbitrary continuous representations. Let $(\mathcal{X}, G, U)$ be a continuous representation of a Lie group G and $a_1, \ldots, a_d$ a basis of the Lie algebra $\mathfrak{g}$ of G. Let C be a weighted strongly elliptic form of order m, P the principal part of C and $H = dU(C)$ the corresponding weighted strongly elliptic operator. Our aim is to construct a local parametrix for $\lambda I + H$, i.e., an approximate inverse of the form

$$U(\varphi) = \int_G dg\, \varphi(g) U(g)$$

where $\varphi \in L_1$ has compact support, and hence deduce that $\overline{H}$ generates a holomorphic semigroup. The parametrix is constructed by first replacing the group G by the group $\mathbf{R}^d$ considered in the previous section and then approximating $(\lambda I + \overline{H})^{-1}$ by the resolvent of a comparable operator associated with $\mathbf{R}^d$. The proofs are very similar to those of Sections 5 and 7 in [ElR3] but somewhat simpler.

First, if φ has compact support then $U(\varphi)\mathcal{X}_\infty \subseteq \mathcal{X}_\infty$ and if φ is smooth

$$(f, (\lambda I + H)U(\varphi)x) = (f, U((\lambda I + H_l)\varphi)x)$$

for all $x \in \mathcal{X}_\infty$ and $f \in \mathcal{F}$ where H_l denotes the operator expressed in terms of the generators A_i of left translations on $C_0(G)$ and $\mathcal{F}$ is the dual, or predual, of $\mathcal{X}$. Note that this could also be written as

$$(f, (\lambda I + H)U(\varphi)x) = \int_G dg\, \varphi(g)\overline{\big((\overline{\lambda} I + H_l^\dagger)\overline{\psi}\big)(g)} \tag{3}$$

where $H_l^\dagger$ is the dual operator, also on $C_0(G)$, $\psi(g) = (f, U(g)x)\,\varphi'(g)$ and $\varphi' \in C_c^\infty(G)$ with $\varphi' = 1$ on a neighbourhood of $\operatorname{supp}\varphi$. Thus since $g \mapsto (f, U(g)x)$ is a C^∞-function for $x \in \mathcal{X}_\infty$ the equality (3) is valid for all $\varphi \in L_1$ with compact support and $\varphi' \in C_c^\infty(G)$ with $\varphi' = 1$ on a neighbourhood of $\operatorname{supp}\varphi$, by continuity. Therefore the construction of a local approximation to $(\lambda I + H)$ is reduced to the analogous problem for $(\lambda I + H_l)$ on $C_0(G)$ and this can be solved by reduction to a calculation on $\mathbf{R}^d$ by use of the exponential map.

Let $\Omega \subset G$ be an open relatively compact neighbourhood of the identity $e \in G$ and W_0 an open ball in $\mathfrak{g}$ centred at the origin such that $\exp|_{W_0} : W_0 \to \Omega$ is an analytic diffeomorphism. Set $a_\xi = \sum_{i=1}^d \xi_i a_i$ for $\xi \in \mathbf{R}^d$, set $W = \{\xi \in \mathbf{R}^d : a_\xi \in W_0\}$ and for $\varphi : \Omega \to \mathbf{C}$ define $\hat{\varphi} : W \to \mathbf{C}$ by $\hat{\varphi}(\xi) = \varphi(\exp a_\xi)$. Then there are C^∞-vector fields $X_1, \ldots, X_d$ on W such that

$$(X_i\hat{\varphi})(\xi) = (A_i\varphi)(\exp a_\xi)$$

for all $\varphi \in C_c^\infty(\Omega)$, where the $A_1, \ldots, A_d$ are generators of left translations. Set $\hat{U}(\xi) = U(\exp a_\xi)$. If Ω is small enough, there exists a positive C^∞-function σ on W, bounded from below by a strictly positive constant, such that all derivatives are bounded on W and

$$\int_\Omega dg\, \varphi(g) = \int_W d\xi\, \hat{\varphi}(\xi)\, \sigma(\xi)$$

for all $\varphi \in L_1(\Omega\,; dg)$. We normalize the Haar measure on G such that $\sigma(0) = 1$. Then, with H_X and $H_X^\dagger$ the elliptic operators constructed from H and $H^\dagger$ with the vector fields X_i replacing the generators A_i, one has

$$\begin{aligned}(H_X^\dagger\hat{\psi}, \sigma\hat{\varphi}) &= \int_W d\xi\, \sigma(\xi)\hat{\varphi}(\xi)\overline{(H_X^\dagger\hat{\psi})(\xi)} \\ &= (H_l^\dagger\psi, \varphi) = (\psi, H\varphi) = (\sigma\hat{\psi}, H_X\hat{\varphi})\end{aligned}$$

for all $\varphi, \psi \in C_c^\infty(\Omega)$. So by continuity

$$(H_X^\dagger\hat{\psi}, \sigma\hat{\varphi}) = (\sigma\hat{\psi}, H_X\hat{\varphi})$$

is also valid in the sense of distributions for all $\varphi \in L_1(G\,; dg)$ with $\operatorname{supp}\varphi \subset \Omega$ and for all $\psi \in C_c^\infty(\Omega)$. Then (3) can be reexpressed as

$$\begin{aligned}(f, (\lambda I + H)U(\varphi)x) &= \int_{\mathbf{R}^d} d\xi\, \sigma(\xi)\, \hat{\varphi}(\xi)\overline{\big((\overline{\lambda} I + H_X^\dagger)\hat{\psi}\big)(\xi)} \\ &= (\sigma\overline{\hat{\psi}}, (\lambda I + H_X)\hat{\varphi}) \qquad (4)\end{aligned}$$

where $\psi(g) = (f, U(g)x)\,\varphi'(g)$ and $\varphi' \in C_c^\infty(\Omega)$ with $\varphi' = 1$ on a neighbourhood of $\operatorname{supp}\varphi$, valid for all $\varphi \in L_1(G\,; dg)$ with $\operatorname{supp}\varphi \subset \Omega$.

In the case $G = \mathbf{R}^d$ we are precisely in the situation of Section 2. Let $\widetilde{X}_i$ denote the vector field on $\mathbf{R}^d$ by left differentiation in the i-th direction. So

$$(\widetilde{X}_i\hat{\varphi})(\xi) = \frac{d}{dt}\hat{\varphi}(\xi - te_i)\Big|_{t=0} \quad ,$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the i-th basis vector. We identify a vector field on $\mathbf{R}^d$ with the corresponding differential operator which, in general, has variable coefficients.

The key observation is the next lemma which expresses the fact that the $\widetilde{X}_i$ and X_i are very similar. It is this lemma that requires the conditions on the structure constants:

$$[a_i, a_j] = \sum_{\substack{k\in\{1,\ldots,d\}\\ w_k\le w_i+w_j-1}} c_{ij}^k a_k \quad .$$

Lemma 3.1 *For all $i \in \{1,\ldots,d\}$ the differential operator $X_i - \widetilde{X}_i$ is of actual order $w_i - 1$.*

Proof Let Φ be the restriction of the map $\xi \mapsto \exp a_\xi$ to W. There exist $s > 0$ such that $\{\xi : \|\xi\| < s\} \subseteq W_0$. The function

$$\Psi(\xi,\eta) = \Phi^{-1}(\exp a_\xi \exp a_\eta)$$

is analytic on a neighbourhood of $(0,0)$ and by decreasing s we may assume that this neighbourhood contains $\{\xi : \|\xi\| < s\}^2$. We may also assume that the series for Ψ in the Campbell–Baker–Hausdorff formula ([Hoc], page 112) is absolutely convergent on $\{\xi : \|\xi\| < s\}^2$. Let $\Psi = \sum_{n=1}^\infty \Psi_n$ be the series expansion of Ψ in this formula, with each Ψ_n homogeneous of order n, where homogeneity is with respect the Euclidean norm.

Now fix $i \in \{1,\ldots,d\}$. Then for all $\xi \in W_0$ with $\|\xi\| < s$ and $|\xi| \le 1$ one has

$$\begin{aligned}
\big((X_i - \widetilde{X}_i)\hat\varphi\big)(\xi) &= \frac{d}{dt}\varphi(\exp(-ta_i)\exp a_\xi)\Big|_{t=0} - \frac{d}{dt}\varphi(\exp(a_\xi - ta_i))\Big|_{t=0} \\
&= \frac{d}{dt}\hat\varphi(\Phi(\sum_{n=1}^{\infty}\Psi_n(-te_i,\xi)))\Big|_{t=0} - \frac{d}{dt}\varphi(\exp(a_\xi - ta_i))\Big|_{t=0} \\
&= \sum_{k=1}^d \Big(\sum_{n=2}^\infty \frac{d}{dt}[\Psi_n(-te_i,\xi)]_k\Big|_{t=0}\Big)(D_k\hat\varphi)(\xi)
\end{aligned}$$

for all $\varphi \in C^\infty(\Omega)$. So for all $k \in \{1,\ldots,d\}$ with $w_k - (w_i - 1) > 0$ we have to examine the expression

$$\sum_{n=2}^\infty \frac{d}{dt}[\Psi_n(-te_i,\xi)]_k\Big|_{t=0} \quad .$$

Let $k \in \{1,\ldots,d\}$ and suppose that $w_k - w_i + 1 > 0$. Now Ψ_n is a homogeneous polynomial of order n so by the Campbell–Baker–Hausdorff formula there exist $c_{l,i_1,\ldots,i_{n-1}} \in \mathbf{R}$ such that

$$\sum_{n=2}^\infty \sum_{l=0}^{n-1} \sum_{i_1,\ldots,i_{n-1}=1}^d |c_{l,i_1,\ldots,i_{n-1}}| < \infty$$

and

$$\begin{aligned}
&\sum_{k=1}^d \frac{d}{dt}[\Psi_n(-te_i,\xi)]_k\, a_k\Big|_{t=0} \\
&\qquad = \sum c_{l,i_1,\ldots,i_{n-1}} \xi_{i_1}\ldots\xi_{i_{n-1}} [a_{i_1},[\ldots,[a_{i_l},[a_i,[a_{i_{l+1}},\ldots,[a_{i_{n-2}},a_{i_{n-1}}]\ldots]]]\ldots]] \quad ,
\end{aligned}$$

where the sum is over all $l \in \{0,\ldots,n-1\}$ and $i_1,\ldots,i_{n-1} \in \{1,\ldots,d\}$. Now the commutator is an element of $\mathfrak{g}$, so there exist $\lambda_{k,l,i_1,\ldots,i_{n-1}} \in \mathbf{R}$, $|\lambda_{k,l,i_1,\ldots,i_{n-1}}| \le c_1^n$ for some $c_1 > 0$ such that

$$[a_{i_1},[\ldots,[a_{i_l},[a_i,[a_{i_{l+1}},\ldots,[a_{i_{n-2}},a_{i_{n-1}}]\ldots]]]\ldots]] = \sum_{k=1}^d \lambda_{k,l,i_1,\ldots,i_{n-1}}\, a_k \quad .$$

(For example take $c_1 = \max c_{ij}^k$.) But by assumption on the structure constants one has $\lambda_{k,l,i_1,\ldots,i_{n-1}} = 0$ if $w_{i_1} + \ldots + w_{i_{n-1}} + w_i < w_k + n - 1$. Moreover, for all $j \in \{1, \ldots, d\}$ one has $|\xi_j| \leq |\xi|^{w_j} \leq |\xi|$. Combining these estimates one obtains

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \frac{d}{dt}[\Psi_n(-te_i,\xi)]_k \Big|_{t=0} \\
&\leq \sum_{n=2}^{\infty}\sum_{l=0}^{n-1} \sum_{\substack{i_1,\ldots,i_{n-1} \\ w_{i_1}+\ldots+w_{i_{n-1}}+w_i \geq w_k+n-1}} |c_{l,i_1,\ldots,i_{n-1}}| \cdot |\xi_{i_1}| \ldots |\xi_{i_{n-1}}| \, |\lambda_{k,l,i_1,\ldots,i_{n-1}}| \\
&\leq \sum_{n=2}^{\infty}\sum_{l=0}^{n-1} \sum_{\substack{i_1,\ldots,i_{n-1} \\ w_{i_1}+\ldots+w_{i_{n-1}}+w_i \geq w_k+n-1}} |c_{l,i_1,\ldots,i_{n-1}}| \, |\xi|^{w_{i_1}+\ldots+w_{i_{n-1}}} c_1^n \\
&\leq \sum_{n=2}^{w_k-w_i+1}\sum_{l=0}^{n-1}\sum_{i_1,\ldots,i_{n-1}} |\xi|^{w_k-w_i-1}|c_{l,i_1,\ldots,i_{n-1}}| \, c_1^n + \\
&\qquad + \sum_{n=w_k-w_i+2}^{\infty}\sum_{l=0}^{n-1}\sum_{i_1,\ldots,i_{n-1}} |\xi|^{n-1}|c_{l,i_1,\ldots,i_{n-1}}| \, c_1^n \\
&\leq |\xi|^{w_k-w_i-1}\Bigg(\sum_{n=2}^{w_k-w_i+1}\sum_{l=0}^{n-1}\sum_{i_1,\ldots,i_{n-1}} |c_{l,i_1,\ldots,i_{n-1}}| \, c_1^n + \\
&\qquad + c_1^{w_k-w_i+2} \sum_{n=w_k-w_i+2}^{\infty}\sum_{l=0}^{n-1}\sum_{i_1,\ldots,i_{n-1}} |c_{l,i_1,\ldots,i_{n-1}}| \, (c_1|\xi|)^{n-w_k+w_i-2}\Bigg) \quad .
\end{aligned}
$$

Since

$$\sum_{n=w_k-w_i+2}^{\infty}\sum_{l=0}^{n-1}\sum_{i_1,\ldots,i_{n-1}} |c_{l,i_1,\ldots,i_{n-1}}| \, (c_1|\xi|)^{n-w_k+w_i-2} < \infty$$

for all $|\xi| < c_1^{-1}$ the lemma is proved. □

It follows immediately from this lemma together with the remark following Lemma 2.2 that

$$H_X = \widetilde{H^P} + H'$$

where $\widetilde{H^P} = dL_{\mathbf{R}^d}(P)$ and H' is an operator with actual order $m-1$. Then the results of Section 2 apply to $\widetilde{H^P}$. In particular the action of the inverse of $\lambda I + \widetilde{H^P}$ is given by a kernel $R_\lambda \in L_1(\mathbf{R}^d)$ which satisfies the estimates of Lemma 2.3:

$$\int_W d\xi \, 1_B(\xi) \, |(LR_\lambda)(\xi)| \leq c(\mathrm{Re}\,\lambda)^{-(m-N)/m}$$

for any compact $B \subset W$ and a differential operator L with actual order $N < m$.

Next let $\varphi, \varphi' \in C_c^\infty(W)$ with $\varphi(e) = 1$ and $\varphi'(g) = 1$ for all $g \in \operatorname{supp}\varphi$ and introduce the integrable function φ_λ with compact support $\operatorname{supp}\varphi_\lambda \subseteq \Omega$ by

$$\hat{\varphi}_\lambda = R_\lambda \cdot \hat{\varphi} \quad .$$

Then we evaluate (4) with φ replaced by φ_λ. One has

$$
\begin{aligned}
&(f, (\lambda I + H)U(\varphi_\lambda)x) \\
&= \int_W d\xi \, \sigma(\xi)\Big((\lambda I + H_X)(R_\lambda \cdot \hat{\varphi})\Big)(\xi) \, (f, \hat{U}(\xi)x) \, \hat{\varphi}'(\xi) \\
&= \int_W d\xi \, \sigma(\xi)\Big((\lambda I + \widetilde{H^P} + H')(R_\lambda \cdot \hat{\varphi})\Big)(\xi) \, (f, \hat{U}(\xi)x) \, \hat{\varphi}'(\xi) \quad . \qquad (5)
\end{aligned}
$$

Now we analyze the two contributions in (5).

Since $\widetilde{H^P}$ is an m-th order operator the first term can be reexpressed in the form

$$\begin{aligned}(\lambda I + \widetilde{H^P})(R_\lambda \cdot \hat{\varphi}) &= (\lambda I + \widetilde{H^P})R_\lambda \cdot \hat{\varphi} + \sum_i (L_i R_\lambda) \cdot \varphi_i \\ &= \delta + \sum_i (L_i R_\lambda) \cdot \varphi_i\end{aligned}$$

where the sum is a finite sum and the L_i are operators of order at most $(m-1)$ given by linear combinations of products of the vector fields $\widetilde{X}_j$ and $\varphi_i \in C_c^\infty(W)$, both independent of λ, and $\operatorname{supp}\varphi_i \subseteq \operatorname{supp}\hat{\varphi}$. Therefore

$$\int_W d\xi\, \sigma(\xi)\Big((\lambda I + \widetilde{H})(R_\lambda \cdot \hat{\varphi})\Big)(\xi)\, (f, \hat{U}(\xi)x)\, \hat{\varphi}'(\xi) = (f,x) + (f, U(\psi_\lambda^{(1)})x)$$

where

$$\hat{\psi}_\lambda^{(1)} = \sum_i (L_i \hat{R}_\lambda)\, \varphi_i\, \hat{\varphi}'$$

is an integrable function with compact support which by Lemma 2.3 satisfies estimates

$$\|\psi_\lambda^{(1)}\|_1 \le a(\operatorname{Re}\lambda)^{-1/m}$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega$ for a suitable $\omega \ge 0$.

The second term in (5) is handled similarly. Now one has

$$H'(R_\lambda \cdot \hat{\varphi}) = (H'R_\lambda) \cdot \hat{\varphi} + \sum_i (L_i' R_\lambda) \cdot \varphi_i' \quad .$$

The important fact is that H' and L_i' are operators with actual order at most $m-1$ so one again has

$$\int_W d\xi\, \sigma(\xi)\big(H'(R_\lambda \cdot \hat{\varphi})\big)(\xi)\, (f, \hat{U}(\xi)x)\, \hat{\varphi}'(\xi) = (f, U(\psi_\lambda^{(2)})x)$$

where $\psi_\lambda^{(2)}$ is an integrable function with compact support which by Lemma 2.3 satisfies estimates

$$\|\psi_\lambda^{(2)}\|_1 \le a(\operatorname{Re}\lambda)^{-1/m}$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega$.

Combination of all these observations now allows one to conclude from (5) that

$$(f, (\lambda I + H)U(\varphi_\lambda)x) = (f,x) + (f, U(\psi_\lambda)x)$$

where ψ_λ is an integrable function with compact support in Ω satisfying estimates

$$\|\psi_\lambda\|_1 \le a(\operatorname{Re}\lambda)^{-1/m}$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega$. This suffices to give the analogue for weighted operators of Proposition 5.1 of [ElR3] for subcoercive operators.

Proposition 3.2 *Let $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega \geq 0$.*

I. *If ω is sufficiently large then there exist $\varphi_\lambda, \psi_\lambda \in L_1(G; dg)$ with compact support such that $U(\varphi_\lambda)\mathcal{X} \subseteq D(\overline{H})$,*

$$(\lambda I + \overline{H})U(\varphi_\lambda)x = x + U(\psi_\lambda)x$$

for all $x \in \mathcal{X}$, and

$$\|\varphi_\lambda\|_1 \leq a(\operatorname{Re}\lambda)^{-1} \quad , \quad \|\psi_\lambda\|_1 \leq a(\operatorname{Re}\lambda)^{-1/m}$$

for a suitable $a > 0$ independent of λ. Moreover, there is a compact set $K \subset G$ such that $\operatorname{supp}\varphi_\lambda \subseteq K$ and $\operatorname{supp}\psi_\lambda \subseteq K$ for all λ.

II. *One may also find $\tilde{\varphi}_\lambda, \tilde{\psi}_\lambda \in L_1(G; dg)$ with compact support such that*

$$U(\tilde{\varphi}_\lambda)(\lambda I + \overline{H})x = x + U(\tilde{\psi}_\lambda)x$$

for all $x \in D(\overline{H})$ and

$$\|\tilde{\varphi}_\lambda\|_1 \leq a'(\operatorname{Re}\lambda)^{-1} \quad , \quad \|\tilde{\psi}_\lambda\|_1 \leq a'(\operatorname{Re}\lambda)^{-1/m}$$

for a suitable $a' > 0$ independent of λ. There is a compact set $K \subset G$ such that $\operatorname{supp}\tilde{\varphi}_\lambda \subseteq K$ and $\operatorname{supp}\tilde{\psi}_\lambda \subseteq K$ for all λ.

Proof The first statement follows for $x \in \mathcal{X}_\infty$ by the foregoing discussion and extends to the closure $\mathcal{X}$ by density.

The second statement follows by considering the first statement for $H^\dagger$ acting on $\mathcal{F}$ as in the proof of Proposition 5.1 of [ElR3]. □

Although the proposition is only formulated for λ with $\operatorname{Re}\lambda > \omega$ it is valid for a larger set. Since

$$(\lambda e^{i\theta} I + H) = e^{i\theta}(\lambda I + e^{-i\theta} H)$$

and $e^{-i\theta}H$ is an operator of the type under consideration for $|\theta| < \theta_C$ Proposition 3.2 can be applied to each $e^{-i\theta}H$. Moreover, since the parameters ω, a and a' can be chosen uniformly for this family of operators the parametrices for $(\lambda I + H_\theta)$ give parametrices for $(\lambda e^{i\theta} I + H)$ by a simple transformation. These observations allow us to prove the first two statements of Theorem 1.1.

Theorem 3.3 *Let $(\mathcal{X}, G, U)$ be a continuous representation, $a_1, \ldots, a_d$ a basis in the Lie algebra $\mathfrak{g}$ of G and C a weighted strongly elliptic form of order m. Suppose the structure constants c_{ij}^k, defined by $[a_i, a_j] = \sum_{k=1}^d c_{ij}^k a_k$, are such that $w_i + w_j - 1 \geq w_k$ if $c_{ij}^k \neq 0$. Let $H = dU(C)$ be the associated weighted strongly elliptic operator. Then one has the following.*

I. *The closure $\overline{H}$ of H generates a continuous semigroup S.*

II. *The semigroup S is holomorphic in a sector $\Lambda(\theta) = \{z \in \mathbf{C} : |\arg z| < \theta\}$ where the angle of holomorphy θ satisfies the bounds $\theta_C \leq \theta \leq \pi/2$.*

III. $\overline{H} = H^{\dagger *}$.

Proof The proof is based directly on the use of the parametrices constructed above and is a direct repetition of the proof of Theorem 5.2 of [ElR2]. We refer to the latter paper for details. □

The parametrix method makes it possible to compare the domain of $\overline{H}$ with the differential structure of the representation associated with the basis $a_1, \ldots, a_d$, i.e., the spaces $\mathcal{X}_n$. This in turn allows one to deduce smoothness properties for the action of the corresponding semigroup S relative to the differential structure.

Theorem 3.4

I. *If* $1 \leq k < mn$ *then* $D(\overline{H}^n) \subseteq \mathcal{X}_k$ *and there exists* $c > 0$ *such that*

$$\|x\|_k \leq \varepsilon^{mn-k}\|\overline{H}^n x\| + c\varepsilon^{-k}\|x\| \tag{6}$$

for all $x \in D(\overline{H}^n)$ *and* $\varepsilon \in \langle 0, 1]$. *In particular*

$$\mathcal{X}_\infty = \bigcap_{n=1}^{\infty} D(\overline{H}^n)$$

and the semigroup S *generated by* $\overline{H}$ *maps into the smooth elements, i.e.,*

$$S_t \mathcal{X} \subseteq \mathcal{X}_\infty$$

for all $t > 0$.

II. *If* $k \in \mathbf{N}$ *then there exists* $c_k > 0$ *such that*

$$\|S_t x\|_k \leq c_k t^{-k/m}\|x\|$$

for all $t \in \langle 0, 1]$ *and* $x \in \mathcal{X}$.

Proof If $n = 1$ we can use the previous parametrix, if $n > 1$ we first have to construct a parametrix for $(\lambda I + \overline{H})^n$. We use all the notation introduced above. Moreover, we let $\widetilde{K}$ be the kernel of the semigroup generated by the closed operator $\widetilde{H^P} = dL_{\mathbf{R}^d}(P)$ associated with the left translations on $L_2(\mathbf{R}^d)$ as in Section 2. Then it follows by the estimates of Proposition 2.1 that $(\lambda I + \widetilde{H^P})^{-n}$ has a kernel $R_\lambda^{(n)}$ defined by

$$R_\lambda^{(n)}(\xi) = (n-1)!^{-1} \int_0^\infty dt\, e^{-\lambda t} t^{n-1} \widetilde{K}_t(\xi)$$

for all sufficiently large $\lambda \in \mathbf{R}$ and all $\xi \in \mathbf{R}^d$. Also $A^\alpha R_\lambda^{(n)} \in L_1(\mathbf{R}^d)$ for every multi-index α with $\|\alpha\| \leq mn - 1$ and we have the estimates

$$\|A^\alpha R_\lambda^{(n)}\|_1 \leq a\lambda^{-(mn-|\alpha|)/m}$$

for some $a > 0$ and all large $\lambda > 0$. So Lemma 2.3 extends to $R_\lambda^{(n)}$ in the following way. If L is a differential operator on W of actual order N with $N \in \{0, \ldots, mn-1\}$ and B is a compact subset of W then $1_B L R_\lambda^{(n)} \in L_1(\mathbf{R}^d)$ and there are $a > 0$ and $\omega \geq 0$ such that

$$\|1_B L R_\lambda^{(n)}\|_1 \leq a\lambda^{-(mn-w)/m} \tag{7}$$

for all $\lambda \in \langle\omega, \infty\rangle$. Moreover, $(\lambda I + \widetilde{H^P})^n R_\lambda^{(n)} = \delta$ as a distribution on $\mathbf{R}^d$. For λ sufficiently large define the integrable function $\varphi_\lambda^{(n)}$ with compact support $\operatorname{supp}\varphi_\lambda^{(n)} \subseteq \Omega$ by

$$\hat{\varphi}_\lambda^{(n)} = R_\lambda^{(n)} \cdot \hat{\varphi} \quad .$$

If $H' = H_X - \widetilde{H^P}$ as before, with H' an operator of actual order $m-1$ we can write, using the fact that $\widetilde{H^P}$ has order m,

$$(\lambda I + H_X)^n = (\lambda I + \widetilde{H^P})^n + \sum \lambda^{k_1} L_{k_2,k_3}$$

where the sum is over all $k_1, k_2, k_3 \in \mathbf{N}_0$ such that $k_1 + k_2 + k_3 = n$ and $k_3 \geq 1$. The operator L_{k_2,k_3} has actual order $mk_2 + (m-1)k_3$. Using the inequality

$$\|1_{\operatorname{supp}\varphi} \lambda^{k_1} L_{k_2,k_3} R_\lambda^{(n)}\|_1 \leq a\lambda^{k_1}\lambda^{-(mn-mk_2-(m-1)k_3)/m} = a\lambda^{-k_3/m} \leq a\lambda^{-1/m}$$

and arguing as in the proof of Proposition 3.2 we see that there exist $\psi_\lambda^{(n)} \in L_1(G\,;dg)$ with $\operatorname{supp}\psi_\lambda^{(n)} \subset \Omega$ such that

$$(\lambda I + \overline{H})^n U(\varphi_\lambda^{(n)}) = I + U(\psi_\lambda^{(n)})$$

and

$$\|\psi_\lambda^{(n)}\|_1 \leq a\lambda^{-1/m}$$

for all $\lambda > \omega$ with ω sufficiently large.

Next, let $\alpha = (i_1, \ldots, i_k) \in J(d)$ and suppose that $\|\alpha\| \leq mn-1$. Then for all $\xi \in W$

$$\begin{aligned}(A_{i_1} \ldots A_{i_k}\varphi_\lambda)(\exp a_\xi) &= (X_{i_1} \ldots X_{i_k}\hat{\varphi}_\lambda)(\xi) \\ &= \Big((\widetilde{X}_{i_1} + (X_{i_1} - \widetilde{X}_{i_1}))\ldots(\widetilde{X}_{i_k} + (X_{i_k} - \widetilde{X}_{i_k}))(R_\lambda \cdot \hat{\varphi})\Big)(\xi) \quad .\end{aligned}$$

Now $(\widetilde{X}_{i_1} + (X_{i_1} - \widetilde{X}_{i_1}))\ldots(\widetilde{X}_{i_k} + (X_{i_k} - \widetilde{X}_{i_k}))$ is a differential operator of actual order $\|\alpha\|$ by Lemma 3.1. Arguing as in the proof of Proposition 3.2 and using the inequalities (7) we obtain that $A_{i_1} \ldots A_{i_k}\varphi_\lambda^{(n)} \in L_1$ and

$$\|A_{i_1} \ldots A_{i_k}\varphi_\lambda^{(n)}\|_1 \leq a\lambda^{-(mn-\|\alpha\|)/m}$$

for all $\lambda > \omega$ with ω sufficiently large.

Now the proof of Statement I is based on the representation

$$(\lambda I + \overline{H})^{-n} = U(\varphi_\lambda^{(n)})(I + U(\psi_\lambda^{(n)}))^{-1}$$

which leads to estimates

$$\|A^\alpha(\lambda I + \overline{H})^{-n}x\| \leq c'\lambda^{-(mn-\|\alpha\|)/m}\|x\|$$

for all $\alpha \in J(d)$ with $\|\alpha\| \leq mn-1$ and for all $x \in \mathcal{X}$ and large λ. Taking ε proportional to $\lambda^{-1/m}$ and rearranging it follows that

$$\|A^\alpha x\| \leq \varepsilon^{mn-\|\alpha\|}\|\overline{H}^n x\| + c''\varepsilon^{-\|\alpha\|}\|x\|$$

for all $x \in D(\overline{H}^n)$ and for small positive values of ε. Statement I then follows.

Next, since S is a holomorphic semigroup, there exists $c_1 > 0$ such that $\|\overline{H}^n S_t\| \leq c_1 t^{-n}$ for all $t \in \langle 0, 1]$ and Statement II follows from Statement I by replacing x by $S_t x$ and taking ε proportional to $t^{1/m}$,

We refer to [ElR3], and in particular the proof of Theorem 5.3, for details. □

Corollary 3.5 *The space $\mathcal{X}_\infty$ is a core for $\overline{H}$.*

Proof By Theorem 3.4.I the space $\mathcal{X}_\infty$ is invariant under the semigroup S. So $\mathcal{X}_\infty$ is a core for $\overline{H}$ by [BrR1] Corollary 3.1.7 since $\mathcal{X}_\infty$ is dense in $\mathcal{X}$. □

Note that the property $S_t\mathcal{X} \subseteq \mathcal{X}_\infty$, $t > 0$, immediately implies that S leaves the weighted C^n-subspaces invariant. But in general the representation U does not leave $\mathcal{X}_n$ invariant even if the representation is unitary (see Example 7.7).

4 Kernels

In this section we prove that the semigroup S constructed in the previous section has a representation independent kernel K and we derive 'Gaussian' type bounds on the kernel and its derivatives. But we first need to introduce a suitable distance on the connected group G and for this we proceed as in [NSW] Definition 1.1..

Let B_i be the left invariant vector field which corresponds to a_i, i.e.,

$$(B_i\psi)(g) = \frac{d}{dt}\psi(g\exp(ta_i))\Big|_{t=0}$$

for $i \in \{1,\ldots,d\}$ and $\psi \in C^\infty(G)$. Then for $\delta > 0$ let $C(\delta)$ be the set of all absolutely continuous functions $\varphi\colon [0,1] \to G$ which satisfy the differential equation

$$\dot{\varphi}(t) = \sum_{i=1}^{d} \varphi_i(t)\, B_i\Big|_{\varphi(t)} \tag{8}$$

almost everywhere with

$$|\varphi_i(t)| < \delta^{w_i}$$

for all $i \in \{1,\ldots,d\}$ and $t \in [0,1]$. Now we define the **distance** $d(g\,;h)$ between two elements $g, h \in G$ by

$$d(g\,;h) = \inf\{\delta > 0 : \exists_{\varphi\in C(\delta)}[\varphi(0) = g \text{ and } \varphi(1) = h]\} \quad .$$

Then $d(kg\,;kh) = d(g\,;h)$ for all $g, h, k \in G$. Moreover, we define the **modulus** $|\cdot|'$ on G by $|g|' = d(g\,;e)$.

Remark This definition gives an L_∞-version of the distance in contrast to the L_2-version used in [Rob2]. In the current context the L_2 distance would correspond to

$$d_2(g\,;h) = \inf_{\varphi} \int_0^1 dt\, |\tilde{\varphi}(t)| \quad ,$$

where $\tilde{\varphi}(t) = (\varphi_1(t),\ldots,\varphi_d(t)) \in \mathbf{R}^d$, with the $\varphi_i(t)$ as in (8) and the infimum is over all absolutely continuous paths $\varphi\colon [0,1] \to G$ with $\varphi(0) = g$ and $\varphi(1) = h$. Then obviously $d(g\,;h) \leq c d_2(g\,;h)$, with $c = d^{1/(2w)}$, and it is probable that the distances d and d_2 are equivalent. In the usual case without weights the corresponding distances are actually equivalent [JSC2] but the proof of the equivalence relies on the homogeneous property of the Euclidean distance.

Next, for $s > 0$ let

$$B'_s = \{g \in G : |g|' < s\}$$

be the ball with radius s. If $|B'_s|$ denotes the measure of B'_s with respect to a fixed (left-)Haar measure on G, then it follows from Theorem 1 of [NSW] that there exist $c, C > 0$ such that

$$cs^{D'} \leq |B'_s| \leq Cs^{D'}$$

uniformly for all small $s > 0$ and hence for all $s \in \langle 0, 1]$. Using Proposition III.4.2 of [VSC] one deduces that the modulus $|\cdot|'$ and the usual modulus $|\cdot|$ are comparable for large distances. In particular, there exists $c' > 0$ such that

$$c'^{-1}|g| \leq |g|' \leq c'|g|$$

for all $g \in G$ such that $|g| \geq 1$, or $|g|' \geq 1$.

Now we state the main theorem of this section.

Theorem 4.1 *Let $(\mathcal{X}, G, U)$ be a continuous representation of the Lie group G and C a weighted strongly elliptic form of order m. Then for all $t > 0$ there exists $K_t \in C^\infty(G)$ such that $K_t \in L_1(G; e^{\rho|g|}dg)$ for all $\rho > 0$ and*

$$S_t x = \int_G dg\, K_t(g)U(g)x \quad ,$$

where S is the semigroup generated by the closure of the $dU(C)$.

Moreover, for each $\alpha \in J(d)$ there exist $a, b > 0$ and $\omega \geq 0$ such that

$$|K_t(g)| \leq a t^{-D'/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

$$|(A^\alpha K_t)(g)| \leq a t^{-(D'+\|\alpha\|)/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

for all $g \in G$ and $t > 0$.

Proof The idea is to construct a candidate for the kernel by approximation with the kernel of a corresponding operator on $\mathbf{R}^d$. Then we apply the parametrix expansion for the kernel.

Let $\widetilde{K}$ be the kernel on $\mathbf{R}^d$ corresponding to the operator $\widetilde{H^P} = dL_{\mathbf{R}^d}(P)$, the weighted strongly elliptic operator with respect to the left regular representation on $\mathbf{R}^d$ and the principal part P of the form C. If $t \leq 0$ we define $\widetilde{K_t} = 0$. It follows by a calculation similar to the one in Section 3 that

$$\Big((\partial_t + H_X)(\widetilde{K_t} \cdot \hat{\varphi})\Big)(\xi) = \delta(t)\delta(\xi) + \widehat{M_t}(\xi)$$

as distributions on $\mathbf{R} \times \mathbf{R}^d$, where

$$\widehat{M_t} = \sum_i (L_i \widetilde{K_t}) \cdot \varphi_i \quad ,$$

the sum is finite, the operators L_i are operators with actual order at most $m - 1$ and the $\varphi_i \in C_c^\infty(W)$. Now for $t \in \mathbf{R}$ define the function $K_t^{(0)}$ on G with compact support $\operatorname{supp} K_t^{(0)} \subseteq \Omega$ by

$$\widehat{K_t^{(0)}} = \widetilde{K_t} \cdot \hat{\varphi} \quad .$$

Then

$$\Big((\partial_t + H)(K_t^{(0)})\Big)(g) = \delta(t)\delta(g) + M_t(g)$$

as distribution on $\mathbf{R} \times G$.

By Proposition 2.1 we have the following estimates for $\widetilde{K}$ and its derivatives: for all $\alpha \in J(d)$ there exist $a, b > 0$ such that

$$|(A^\alpha \widetilde{K}_t)(\xi)| \leq a t^{-D'/m} t^{-\|\alpha\|/m} e^{-b(|\xi|^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t > 0$ and $\xi \in \mathbf{R}^d$. Then one also obtains

$$|(L\widetilde{K}_t)(\xi)| \leq a t^{-D'/m} t^{-N/m} e^{-b(|\xi|^m t^{-1})^{1/(m-1)}} \tag{9}$$

uniformly for all $t > 0$ and ξ in a compact subset on which L is defined. for any operator L of actual order N.

Our next step is to translate the bounds on $\widetilde{K}_t$ to bounds on $K_t^{(0)}$.

Lemma 4.2 *There exists* $c > 0$ *such that*

$$c|\xi| \leq |\exp(a_\xi)|' \leq |\xi|$$

for all $\xi \in \mathbf{R}^d$ *with* $|\xi| \leq 1$.

Proof Let $\xi \in \mathbf{R}^d$. Define $\varphi\colon [0,1] \to G$ by $\varphi(t) = \exp(a_{t\xi})$. Let B be the left invariant vector field on G corresponding to $a_\xi \in \mathfrak{g}$. Then by definition of the exponential map one has

$$\dot{\varphi}(t) = B\Big|_{\varphi(t)} = \sum_{i=1}^d \xi_i B_i\Big|_{\varphi(t)}$$

for all $t \in [0,1]$. Since $|\xi_i| \leq |\xi|^{w_i}$ for all $i \in \{1, \ldots, d\}$ it follows that $|\exp(a_\xi)|' \leq |\xi|$.

The converse inequality is valid only locally and is more difficult. For $\delta > 0$ let $C_2(\delta)$ be the set of all absolutely continuous functions $\varphi\colon [0,1] \to G$ which almost everywhere satisfy the differential equation

$$\dot{\varphi}(t) = \sum_{i=1}^d \varphi_i \, B_i\Big|_{\varphi(t)}$$

with

$$|\varphi_i| < \delta^{w_i}$$

for all $i \in \{1, \ldots, d\}$ and $t \in [0,1]$ where the φ_i are constants. We define the quasi-distance $d'(g\,;h)$ between two elements $g, h \in G$ by

$$d'(g\,;h) = \inf\{\delta > 0 : \exists_{\varphi \in C_2(\delta)}[\varphi(0) = g \text{ and } \varphi(1) = h]\} \quad .$$

Then d_2 is locally equivalent to d by [NSW] Theorem 2. So there exist $c, s > 0$ such that $d(g\,;h) \geq c d_2(g\,;h)$ for all $g, h \in B'_s$. In particular, $|g|' \geq c d_2(g\,;e)$ for all $g \in B'_s$.

Now let $\xi \in \mathbf{R}^d$ be such that $|\xi| \leq s$. Let $\varphi\colon [0,1] \to G$ be an absolutely continuous path such that $\varphi(0) = e$, $\varphi(1) = \exp(a_\xi)$ and

$$\dot{\varphi}(t) = \sum_{i=1}^d \varphi_i \, B_i\Big|_{\varphi(t)}$$

with

$$|\varphi_i| < \delta^{w_i}$$

for all $i \in \{1,\ldots,d\}$ and $t \in [0,1]$, where the φ_i are constants. Let $a = \sum_{i=1}^{d} \varphi_i a_i \in \mathfrak{g}$ and B be the left invariant vector field corresponding to a. Then $\dot{\varphi}(t) = B\Big|_{\varphi(t)}$ for all $t \in [0,1]$. So φ is a C^∞ function and by the unicity theorem for integral curves (see, for example, [SaW] Theorem 2.37) it follows that $\varphi(t) = \exp(ta)$ for all $t \in [0,1]$, if s is small enough. In particular, $\exp(a_\xi) = \varphi(1) = \exp(a)$. Hence, if s is small enough, it follows that $\varphi_i = \xi_i$ for all $i \in \{1,\ldots,d\}$. Thus $|\xi_i| < \delta^{w_i}$ for all $i \in \{1,\ldots,d\}$. But there is a $j \in \{1,\ldots,d\}$ such that

$$|\xi_j|^{2w/w_j} \geq d^{-1}|\xi|^{2w} \quad ,$$

so

$$\delta^{w_j} > |\xi_j| \geq d^{-w_j/(2w)}|\xi|^{w_j} \quad .$$

Thus $\delta > d^{-w_j/(2w)}|\xi|$ and $d_2(\exp(a_\xi)\,;e) \geq d^{-w_j/(2w)}|\xi|$. This proves the other part of the lemma. □

We continue with the proof of Theorem 4.1. It follows from the previous lemma, Lemma 3.1 and the estimates (9) that for each $\alpha \in J(d)$ there are $a, b > 0$ such that

$$|(A^\alpha K_t^{(0)})(g)| \leq a t^{-(D'+\|\alpha\|)/m} e^{-b((|g|')^m t^{-1})^{1/(m-1)}} \quad ,$$

$$|(A^\alpha M_t)(g)| \leq a t^{-(D'+\|\alpha\|+m-1)/m} e^{-b((|g|')^m t^{-1})^{1/(m-1)}} \quad .$$

uniformly for all $t > 0$ and $g \in G$, since $K_t^{(0)}$ has compact support.

Let $\rho \geq 0$ and define L_1^ρ and $L_{\hat{1}}^\rho$ to be the L_1-spaces with respect to the measures $dg\, e^{\rho|g|'}$ and $d\hat{g}\, e^{\rho|g|'}$, where $d\hat{g}$ is the right-Haar measure on G, with norms

$$\|\varphi\|_1^\rho = \int_G dg\, e^{\rho|g|'}\, |\varphi(g)| \quad , \quad \|\varphi\|_{\hat{1}}^\rho = \int_G d\hat{g}\, e^{\rho|g|'}\, |\varphi(g)| \quad .$$

Similarly let L_∞^ρ be the space of measurable functions for which $g \mapsto e^{\rho|g|'}|\varphi(g)|$ is essentially bounded with norm

$$\|\varphi\|_\infty^\rho = \operatorname*{ess\,sup}_{g\in G} e^{\rho|g|'}|\varphi(g)| \quad .$$

Then $A^\alpha K_t^{(0)} \in L_1^\rho$ if $\|\alpha\| \leq m-1$ and $M_t \in L_1^\rho$ for all $\rho \in \mathbf{R}$ by a quadrature estimate.

We need one more technical lemma which is a weighted version of Lemma 7.3 of [ElR3].

Lemma 4.3 *For each $i, j \in \{1,\ldots,d\}$ there exists a function $c_{i,j}\colon G \to \mathbf{R}$ such that*

$$L(k^{-1})A_i L(k) - A_i = \sum_{j=1}^{d} c_{i,j}(k) A_j$$

and $|c_{i,j}(k)| \leq M(|k|')^{(w_j - w_i)\vee 0} e^{\sigma|k|'}$ for all $k \in G$ where $M, \sigma \geq 0$.

Proof Let $s > 0$. Since Ad is a continuous homomorphism from G into $\mathcal{L}(\mathfrak{g})$ the lemma is easily established if $|k|' \geq s$. So let $k \in G$, $|k|' \leq s$. We may take $s < 1$ and assume that

$B'_s \subseteq \{\exp a_\xi : |\xi| \leq 1\}$. Let $i \in \{1, \ldots, d\}$ and $\xi \in \mathbf{R}^d$ be such that $|\xi| \leq 1$. Then

$$\begin{aligned}
\mathrm{Ad}((\exp a_\xi)^{-1})a_i - a_i &= \int_0^1 dt_1 \, \tfrac{d}{dt_1} \mathrm{Ad}(\exp a_{-t_1\xi})a_i \\
&= \int_0^1 dt_1 \, \tfrac{d}{dt_1} e^{-t_1 \, \mathrm{ad}\, a_\xi} a_i \\
&= \int_0^1 dt_1 \sum_{i_1=1}^d \xi_{i_1} e^{-t_1 \, \mathrm{ad}\, a_\xi}[a_{i_1}, a_i] \\
&= \sum_{i_1=1}^d \int_0^1 dt_1 \, \xi_{i_1}[a_{i_1}, a_i] + \ldots \\
&\quad + \sum_{i_1=1}^d \cdots \sum_{i_{w-1}=1}^d \int_0^1 dt_1 \ldots \int_0^1 dt_{w-1} \, \xi_{i_1} \ldots \xi_{i_{w-1}} \cdot \\
&\qquad \cdot t_1^{w-2} \ldots t_{w-2}[a_{i_{w-1}}, [\ldots, [a_{i_1}, a_i] \ldots]] \\
&\quad + \sum_{i_1=1}^d \cdots \sum_{i_w=1}^d \int_0^1 dt_1 \ldots \int_0^1 dt_w \, \xi_{i_1} \ldots \xi_{i_w} \cdot \\
&\qquad \cdot t_1^{w-1} \ldots t_{w-1} e^{-t_1 \ldots t_w \, \mathrm{ad}\, a_\xi}[a_{i_w}, [\ldots, [a_{i_1}, a_i] \ldots]] \quad .
\end{aligned}$$

Since $\mathrm{Ad}(g)$ maps $\mathfrak{g}$ into $\mathfrak{g}$ with uniform bounded norm if g is in a bounded set and $|\xi_n| \leq |\xi|^{w_n}$ for all $n \in \{1, \ldots, d\}$, the last term gives a contribution $\sum_{j=1}^d c'_{ij}(\xi)a_j$, with $|c'_{ij}(k)| \leq M|\xi|^w$. Similarly one has for the n-th term the contribution

$$n!^{-1} \sum_{i_1=1}^d \cdots \sum_{i_n=1}^d \xi_{i_1} \ldots \xi_{i_n}[a_{i_n}, [\ldots, [a_{i_1}, a_i] \ldots]] \quad .$$

But $[a_{i_n}, [\ldots, [a_{i_1}, a_i] \ldots]] = \sum_j c_j a_j$ where the j is such that $w_j \leq w_{i_1} + \ldots + w_{i_n} + w_i - n$. On the other hand $|\xi_{i_1} \ldots \xi_{i_n}| \leq |\xi|^{w_{i_1} + \ldots + w_{i_n}}$. From these two observations it follows that there exists $c_{i,j}(k) \in \mathbf{R}$, such that

$$L((\exp a_\xi)^{-1})A_i L(\exp a_\xi) - A_i = dL((\mathrm{Ad}(\exp a_\xi)^{-1})a_i - a_i) = \sum_{j=1}^d c_{i,j}(\exp a_\xi)Aj \quad .$$

Moreover, there exists $M > 0$ such that

$$|c_{i,j}(\exp a_\xi)| \leq M(|\xi|)^{(w_j - w_i) \vee 0}$$

for all $\xi \in \mathbf{R}^d$ with $|\xi| \leq 1$. This proves the lemma since $|\xi| \leq c|\exp a_\xi|'$ for some $c > 0$ by Lemma 4.2. □

We continue with the proof of Theorem 4.1. For $n \in \mathbf{N}$ define inductively $K_t^{(n)} = -(K^{(n-1)} \hat{*} M)_t$, where the convolution product $\hat{*}$ is defined on $\mathbf{R} \times G$ by

$$\begin{aligned}
(\varphi \hat{*} \psi)_t(g) &= \int_{\mathbf{R}} ds \int_G dh \, \varphi_s(h)\psi_{t-s}(h^{-1}g) \\
&= \int_{\mathbf{R}} ds \int_G dh \, \varphi_{t-s}(h)\psi_s(h^{-1}g) \quad .
\end{aligned}$$

Now one can argue as in the appendix of [BrR2] and in Section 7 of [ElR3] to show that the definition of $K^{(n)}$ indeed makes sense, the function $K_t^{(n)}$ is continuous for all $t > 0$ and $n \in \mathbf{N}_0$, $K_t^{(n)} \in L_1^\rho \cap L_1^\rho \cap L_\infty^\rho$ and, moreover, the sum

$$K_t = \sum_{n=0}^{\infty} K_t^{(n)}$$

converges in $L_1^\rho \cap L_\infty^\rho$. The function K_t then satisfies the bounds

$$|K_t(g)| \leq a t^{-D'/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t > 0$ and $g \in G$. In addition, since for fixed $\alpha \in J(d)$

$$(A^\alpha K_t^{(n)})(h) = -\int_0^{t/2} ds \int_G dk\, (A^\alpha K_{t-s}^{(n-1)})(k)\, M_s(k^{-1}h) \\ - \int_{t/2}^t ds \int_G dk\, K_{t-s}^{(n-1)}(k)\, (A^\alpha L(k) M_s)(h) \quad ,$$

one can apply the above argument, together with Lemma 4.3, to deduce that

$$|(A^\alpha K_t)(g)| \leq a t^{-(D'+\|\alpha\|)/m} e^{\omega t} e^{-b((|g|')^m t^{-1})^{1/(m-1)}}$$

uniformly for all $t > 0$ and $g \in G$.

Next we prove that K is the kernel of the semigroup S. By definition the function K_t satisfies

$$\Big((\partial_t + H)K_t\Big)(g) = \delta(t)\delta(g)$$

as distribution on $\mathbf{R} \times G$. We use K to define a family of bounded operators. For $t \geq 0$ and $x \in \mathcal{X}$ define $T_t x$ by

$$T_t x = U(K_t)x = \int_G dg\, K_t(g) U(g) x \quad .$$

Then by the 'Gaussian' bounds on K_t we see that T_t maps $\mathcal{X}$ continuously into $\mathcal{X}$. Moreover, one has bounds $\|T_t\| \leq a e^{\omega t}$ for some $a, \omega > 0$. Our aim is to prove that $S_t = T_t$ for all $t > 0$.

First let $\rho > 0$ and consider the left regular representation in the weighted space $L_1^\rho = L_1(G; e^{\rho|g|} dg)$. Since K satisfies 'Gaussian' bounds one may define the kernel R_λ by

$$R_\lambda(g) = \int_0^\infty dt\, e^{-\lambda t} K_t(g)$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda$ large enough and introduce the corresponding operator r_λ by

$$r_\lambda x = \int_G dg\, R_\lambda(g)\, U(g) x \quad .$$

Then r_λ is continuous. Now let $\varphi, \psi \in C_c^\infty(G)$. For $N \in \mathbf{N}$ let $\chi_N \in C_c^\infty(\mathbf{R})$ satisfy $0 \leq \chi_N \leq 1$, $\chi_N(t) = 1$ if $t \in [0, N]$, $\operatorname{supp} \chi_N \subseteq [-N^{-1}, N+1]$ and $\chi_N(N+t) = \chi_1(1+t)$ for all $t \in [0,1]$. Moreover, set $e_\lambda(t) = e^{-\lambda t}$ and $\Phi_N(t, g, h) = \psi(g)\, \overline{\varphi(h^{-1}g)}\, (e_\lambda \chi_N)(t)$. In the next calculation the operators act with respect to the g-variable, except the operator $H^\dagger_h$, which acts on the h. Then

$$
\begin{aligned}
&((\overline{\lambda}I + H^\dagger)\psi, r_\lambda\varphi)\\
&\quad = \int_G dg \int_G d\hat{h} \int_0^\infty dt\, \overline{\big((\overline{\lambda}I + H^\dagger)\psi\big)(g)}\, e_\lambda(t)\, K_t(gh^{-1})\, \varphi(h)\\
&\quad = \lim_{N\to\infty} \int_G dg \int_G d\hat{h} \int_0^\infty dt\, \overline{\big((\overline{\lambda}I + H^\dagger)\psi\big)(g)}\, e_\lambda(t)\, \chi_N(t)\, K_t(gh^{-1})\, \varphi(h)\\
&\quad = \lim_{N\to\infty} \int_G dg \int_G d\hat{h} \int_0^\infty dt\, \overline{\big((-\partial_t + H^\dagger)(\psi \otimes \overline{\varphi} \otimes (e_{\overline{\lambda}}\chi_N)\big)(g,h,t)}\, K_t(gh^{-1})\\
&\qquad\qquad + \lim_{N\to\infty} \int_G dg \int_G d\hat{h} \int_0^\infty dt\, \overline{\psi(g)}\, e_\lambda(t)\, \chi_N'(t)\, K_t(gh^{-1})\, \varphi(h)\\
&\quad = \lim_{N\to\infty} \int_G dg \int_G d\hat{h} \int_0^\infty dt\, \overline{\big(\psi \otimes \overline{\varphi} \otimes (e_{\overline{\lambda}}\chi_N)\big)(g,h,t)}\, (HK_t)(gh^{-1})\\
&\qquad\qquad + \lim_{N\to\infty} \int_G dg \int_G d\hat{h} \int_0^\infty dt\, \overline{\big(-\partial_t(\psi \otimes \overline{\varphi} \otimes (e_{\overline{\lambda}}\chi_N))\big)(g,h,t)}\, K_t(gh^{-1})\\
&\quad = \lim_{N\to\infty} \int_G dg \int_G dh \int_0^\infty dt\, \overline{\Phi_N(g,h,t)}\, (HK_t)(h)\\
&\qquad\qquad + \lim_{N\to\infty} \int_G dg \int_G dh \int_0^\infty dt\, \overline{-(\partial_t\Phi_N)(g,h,t)}\, K_t(gh^{-1})\\
&\quad = \lim_{N\to\infty} \int_G dg \int_G dh \int_0^\infty dt\, \overline{\big(-\partial_t + H^\dagger_h)\Phi_N\big)(g,h,t)}\, K_t(h)\\
&\quad = \lim_{N\to\infty} \int_G dg\, \overline{\Phi_N(0,g,e)}\\
&\quad = (\psi, \varphi) \quad .
\end{aligned}
$$

So since r_λ is continuous it follows that

$$((\overline{\lambda}I + H^\dagger)\psi, r_\lambda\varphi) = (\psi, \varphi) \tag{10}$$

for all $\varphi \in L_1^\rho$ and $\psi \in C_c^\infty(G)$. But the space $C_c^\infty(G)$ is weakly* dense in $L^\rho_{\infty;m}$ by a line by line extension of Lemma 2.4 of [ElR3] to weighted differentiable vectors. So (10) is valid for all $\varphi \in L_1^\rho$ and $\psi \in L^\rho_{\infty;m}$. Therefore $r_\lambda\varphi \in D((\overline{\lambda}I + H^\dagger)^*)$ and $(\lambda I + H^{\dagger *})r_\lambda\varphi = \varphi$. But $H^{\dagger *} = \overline{H}$, so if $\operatorname{Re}\lambda$ is large enough one has $r_\lambda = (\lambda I + \overline{H})^{-1}$. Then for all $\varphi, \psi \in C_c^\infty(G)$ and $\operatorname{Re}\lambda$ large enough

$$
\begin{aligned}
\int_0^\infty dt\, e^{-\lambda t}(\psi, S_t\varphi) &= (\psi, (\lambda I + \overline{H})^{-1}\varphi)\\
&= \int_G d\hat{h} \int_0^\infty dt\, e^{-\lambda t}\, \overline{\psi(g)}\, K_t(g;\, h)\, \varphi(h)\\
&= \int_0^\infty dt\, e^{-\lambda t}(\psi, T_t\varphi) \quad .
\end{aligned}
$$

So $(\psi, S_t\varphi) = (\psi, T_t\varphi)$ and $S_t = T_t$. Therefore S has a kernel and this kernel is K. Since S is a semigroup it then follows that the kernel is a convolution semigroup, e.g., $K_t * K_s = K_{t+s}$ for all $s, t > 0$.

Next we deduce that K is the kernel of the semigroup S generated by the closure of the operator $dU(C)$ for a general representation U. Let $\rho > 0$ be so large that $\|S_t\| \leq Me^{t\rho}$

uniformly for all $t > 0$, for some $M > 0$. Let

$$T_t x = \int_G dg\, K_t(g) U(g) x$$

as before. It follows from the convolution property of K that T is a semigroup, which is continuous because of the 'Gaussian' bounds. Let H^L and S^L be the operator and semigroup corresponding to the left regular representation in L_1^ρ. Let $\varphi \in L_{1;\infty}^\rho$ and $x \in \mathcal{X}$. Then for all $t > 0$ one has

$$T_t U(\varphi) x = U(K_t * \varphi) x = U(S_t^L \varphi) x \quad .$$

Hence by the Duhamel formula

$$\begin{aligned} T_t U(\varphi) x - U(\varphi) x &= U(S_t^L \varphi - \varphi) x \\ &= -U\Big(\int_0^t ds\, H^L S_s^L \varphi \Big) x \\ &= -\int_0^t ds\, \big(U(S_s^L H^L \varphi) x \big) \quad . \end{aligned}$$

Therefore

$$\begin{aligned} \Big\| t^{-1} \big(T_t U(\varphi) x - U(\varphi) x \big) - U(H^L \varphi) x \Big\| &= \Big\| t^{-1} \int_0^t ds\, U(S_s^L H^L \varphi - H^L \varphi) x \Big\| \\ &\leq \sup_{0 < s \leq t} \| S_s^L H^L \varphi - H^L \varphi \|_1^\rho \, \|x\| \quad . \end{aligned}$$

Since S^L is a continuous semigroup, it follows that $U(\varphi)x$ is in the domain of the generator H^T of T and

$$H^T U(\varphi) x = U(H^L \varphi) x = H U(\varphi) x \quad .$$

Let $\mathcal{D} = \{U(\varphi)x : \varphi \in L_{1;\infty}^\rho,\ x \in \mathcal{X}\}$. Then $H^T \supseteq H|_{\mathcal{D}}$. But $\mathcal{D}$ is invariant under S and dense in $\mathcal{X}$. Therefore $\mathcal{D}$ is a core for H by the same argument used in the proof of Corollary 3.5. So $H^T \supseteq \overline{H|_{\mathcal{D}}} = \overline{H}$. Since a semigroup generator does not have a strict generator extension, it follows that $H^T = \overline{H}$ and hence $S_t = T_t$ for all $t > 0$. This proves that K_t is the kernel of S_t. □

Although we have derived upper bounds for the kernel, there are no lower bounds for the kernel. In fact, the kernel corresponding to a weighted strongly elliptic operator is positive only in the trivial situation that one has a real second order operator in the usual, Euclidean, sense.

Proposition 4.4 *The kernel K_t is positive for all $t > 0$ if, and only if, all weights $w_1, \ldots, w_d$ are equal, $m = 2w$ and the coefficients are real.*

Proof It follows from the proof of Theorem III.5.1 in [Rob2] that H is a real second order operator, the order counted in the Euclidean sense. Since the form is weighted strongly elliptic, this implies that $2w_1 = \ldots = 2w_d = m$. Thus all weights are equal. □

5 Regularity

In this section we prove some regularity theorems for weighted strongly elliptic operators which are analogous to those for unweighted operators [ElR1] but there are some restrictions placed by the weighting. We use the standard methods of interpolation theory and adopt the notation of [BuB] for the intermediate spaces, interpolation functions, etc.

Proposition 5.1 *Let* $p \in [1, \infty]$.

I. *If* $0 < \gamma < n$ *then*

$$(\mathcal{X}, D(\overline{H}^n))_{\frac{\gamma}{nm},p;K} \subseteq (\mathcal{X}, \mathcal{X}_n)_{\frac{\gamma}{n},p;K} \quad .$$

II. *If* $0 < \gamma < nw$ *then*

$$(\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;K} = (\mathcal{X}, D(\overline{H}^n))_{\frac{\gamma}{nm},p;K} \quad . \tag{11}$$

Proof By the inequalities (6) it follows that $\mathcal{X}_n \in J(1/m, \mathcal{X}, D(\overline{H}^n))$. Statement I then follows by the reiteration theorem, [BuB] Proposition 3.2.19.

Obviously $\mathcal{X}_{nm}$ is continuously embedded in $D(\overline{H}^n)$. Hence $(\mathcal{X}, \mathcal{X}_{nm})_{\frac{\gamma}{nm},p;K}$ is included in $(\mathcal{X}, D(\overline{H}^n))_{\frac{\gamma}{nm},p;K}$ by interpolation. Combination with Statement I gives II if n is a multiple of m/w. The space on the right hand side of (11) is independent of n, if n is large enough, so the same is valid for the space on the left hand side, as long as n is a multiple of m/w. But the left hand side does not depend on H, so $(\mathcal{X}, D(\overline{H_1}^n))_{\frac{\gamma}{nm_1},p;K} = (\mathcal{X}, D(\overline{H_2}^n))_{\frac{\gamma}{nm_2},p;K}$ if H_1 and H_2 are weighted strongly elliptic operators of order m_1 and m_2, respectively and n is large enough. So we may suppose that $H = \sum_{i=1}^d (-1)^{w/w_i} A_i^{2w/w_i}$ and $m = 2w$.

Assume n is odd. The case n is even can be handled by a similar but easier argument. Let $n = 2k + 1$, with $k \in \mathbf{N}_0$, and let $c_1 > 0$ be such that $\|(H^\dagger)^{n-k-1} S_t^\dagger\| \leq c_1 t^{-n+k+1}$ for all $t \in \langle 0, 2]$. Moreover, there exists $c_2 > 0$ such that $\|A_*^{w/w_i} S_t^\dagger\| \leq c_2 t^{-1/2}$ for all $t \in \langle 0, 2]$ and $i \in \{1, \ldots, d\}$. Now let $x \in \mathcal{X}_{nw}$. Then for all $t \in \langle 0, 1]$ and for all $f \in \mathcal{F}$, the dual, or predual, of $\mathcal{X}$ one has

$$\begin{aligned} |(f, H^n S_t x)| &= \sum_{i=1}^d |(A_{i*}^{w/w_i} S_{t/2}^\dagger (H^\dagger)^{n-k-1} S_{t/2}^\dagger f, A_i^{w/w_i} H^k x)| \\ &\leq \sum_{i=1}^d 2^{n/2} c_1 c_2 t^{-n/2} \|f\| \, \|A_i^{w/w_i} H^k x\| \\ &\leq c t^{-n/2} \|f\| \, \|x\|_{nw} \end{aligned}$$

for some $c > 0$. Therefore $\|H^n S_t x\| \leq c t^{-n/2} \|x\|_{nw}$. But this implies that the space $\mathcal{X}_{nw}$ is continuously embedded in the interpolation space $(\mathcal{X}, D(\overline{H}^n))_{\frac{nw}{2nw},\infty;K}$. Hence by the reiteration theorem, [BuB], Proposition 3.2.18, it follows that for all $0 < \gamma < nw$ one has $(\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;K} \subseteq (\mathcal{X}, D(\overline{H}^n))_{\frac{\gamma}{nm},p;K}$. This completes the proof of the proposition. □

Corollary 5.2 *Let* $p \in [1, \infty]$. *If* $n_1, n_2 \in \mathbf{N}$ *and* $0 < \gamma < n_1 \wedge n_2$ *then* $(\mathcal{X}, \mathcal{X}_{n_1 w})_{\frac{\gamma}{n_1 w},p;K} = (\mathcal{X}, \mathcal{X}_{n_2 w})_{\frac{\gamma}{n_2 w},p;K}$.

Proof Take $H = \sum_{i=1}^{d}(-1)^{w/w_i} A_i^{2w/w_i}$ with $D(H) = \mathcal{X}_{2w}$ and apply Proposition 5.1.II together with [BuB] Chapter 3. □

Corollary 5.3 *If $k, n \in \mathbf{N}$ and $k < nw$ then there exists $c > 0$ such that*

$$\|x\|_k \leq \varepsilon^{nw-k}\|x\|_{nw} + c\varepsilon^{-k}\|x\|$$

for all $\varepsilon > 0$ and $x \in \mathcal{X}_{nw}$.

Proof It follows from the inequalities (6) and [Tri] Lemma 1.10.1(a) that

$$(\mathcal{X}, \mathcal{X}_{nw})_{\frac{k}{nw},1;\mathrm{K}} = (\mathcal{X}, D(\overline{H}^n))_{\frac{k}{nm},1,\mathrm{K}} \subseteq \mathcal{X}_k \quad .$$

This implies the corollary by the same lemma of [Tri]. □

If $n, k \in \mathbf{N}$, $\gamma \in \langle 0, n\rangle$ and $p \in [1, \infty]$ define the space $(\mathcal{X}, \mathcal{X}_n)_{\gamma/n,p;\mathrm{K};k}$ of k times differentiable vectors for the Lipschitz space $(\mathcal{X}, \mathcal{X}_n)_{\gamma/n,p;\mathrm{K}}$ by

$$(\mathcal{X}, \mathcal{X}_n)_{\gamma/n,p;\mathrm{K};k} = \{x \in \mathcal{X}_k : A^\alpha x \in (\mathcal{X}, \mathcal{X}_n)_{\gamma/n,p;\mathrm{K}} \text{ for all } \alpha \in J(d) \text{ with } \|\alpha\| \leq k\}$$

with norm

$$\|x\|_{\gamma/n,p;\mathrm{K};k} = \max_{\|\alpha\|\leq k} \|A^\alpha x\|_{(\mathcal{X},\mathcal{X}_n)_{\gamma/n,p;\mathrm{K}}} \quad .$$

Then $(\mathcal{X}, \mathcal{X}_n)_{\gamma/n,p;\mathrm{K};k}$ is a Banach space. Note that the spaces $(\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K};k}$ are independent of n, if $nw > \gamma$, by Corollary 5.2.

Lemma 5.4 *If $n, k \in \mathbf{N}$, $\gamma \in \langle 0, nw\rangle$ and $p \in [1, \infty]$ then*

$$(\mathcal{X}, \mathcal{X}_{(n+k)w})_{\frac{\gamma+kw}{(n+k)w},p;\mathrm{K}} \subseteq (\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K};kw}$$

and the inclusion is continuous.

Proof It follows from Corollary 5.3 that $\mathcal{X}_{kw} \in J(\frac{k}{n+k}, \mathcal{X}, \mathcal{X}_{(n+k)w})$. Therefore by reiteration, Proposition 3.2.19 of [BuB],

$$(\mathcal{X}, \mathcal{X}_{(n+k)w})_{\frac{\gamma+kw}{(n+k)w},p;\mathrm{K}} \subseteq (\mathcal{X}_{kw}, \mathcal{X}_{(n+k)w})_{\frac{\gamma}{nw},p;\mathrm{K}} \subseteq \mathcal{X}_{kw} \quad ,$$

with continuous embeddings. Now let $\alpha \in J(d)$ with $\|\alpha\| \leq kw$. Then A^α maps $\mathcal{X}_{kw}$ continuously into $\mathcal{X}$ and $\mathcal{X}_{(n+k)w}$ continuously into $\mathcal{X}_{nw}$. So by interpolation, A^α maps $(\mathcal{X}_{kw}, \mathcal{X}_{(n+k)w})_{\frac{\gamma}{nw},p;\mathrm{K}}$ continuously into $(\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K}}$. Hence $(\mathcal{X}_{kw}, \mathcal{X}_{(n+k)w})_{\frac{\gamma}{nw},p;\mathrm{K}}$ is continuously embedded in $(\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K};kw}$ and the lemma follows. □

The next corollary shows that the C^{km}-norm is dominated by the operator norm of $\overline{H}^k$, with respect to the Lipschitz spaces. We use $\|x\|_E$ to denote the norm of a vector x in a Banach space E.

Corollary 5.5 *If $n, k \in \mathbf{N}$, $\gamma \in \langle 0, nw\rangle$, $p \in [1, \infty]$ and H is a weighted strongly elliptic operator of order m then*

$$\{x \in D(\overline{H}^k) : \overline{H}^k x \in (\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K}}\} \subseteq (\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K};km} \quad .$$

Moreover, if λ is large enough then there exists $c > 0$ such that

$$\|x\|_{(\mathcal{X},\mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K};km}} \leq c\|(\overline{H} + \lambda I)^k x\|_{(\mathcal{X},\mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K}}}$$

for all $x \in \{x \in D(\overline{H}^k) : \overline{H}^k x \in (\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K}}\}$.

Proof This follows immediately from Lemma 5.4 and the reduction theorem [BuB], Theorem 3.4.6. □

There are two more types of interpolation spaces, which depend more explicitly on the representation (see also [ElR1]). Let $\mathcal{O}$ be a bounded open neighbourhood of the identity e of G, $p \in [1, \infty]$ and $n \in \mathbf{N}$ then for each $\gamma \in \langle 0, n\rangle$ define $\|\cdot\|_\gamma^{n,p,U} : \mathcal{X} \to [0, \infty]$ by

$$\|x\|_\gamma^{n,p,U} = \|x\| + \left(\int_{\mathcal{O}^n} d\mu_n(\mathbf{g}) \left(|\mathbf{g}|^{-\gamma} \|(I - U(g_1)) \ldots (I - U(g_n))x\|\right)^p\right)^{1/p} ,$$

where $\mathbf{g} = (g_1, \ldots, g_n)$ and $|\mathbf{g}| = |g_1|' + \ldots + |g_n|'$. Moreover, μ_n is the absolutely continuous measure with respect to the left Haar measure on G^n with density $\mathbf{g} \mapsto |\mathbf{g}|^{-nD'}$. The usual changes are needed in the case $p = \infty$. Then the Lipschitz space $\mathcal{X}_\gamma^{n,p}(U)$ is defined by

$$\mathcal{X}_\gamma^{n,p}(U) = \{x \in \mathcal{X} : \|x\|_\gamma^{n,p,U} < \infty\} .$$

It is a Banach space with respect to the norm $\|\cdot\|_\gamma^{n,p,U}$. Note that as the space is independent of the choice of $\mathcal{O}$, up to equivalence of norms, we have omitted it from the notation.

Next we introduce a uniform version of the Lipschitz spaces. First, for each $x \in \mathcal{X}$ and $n \in \mathbf{N}_0$ define $\omega_x^{(n)} : \langle 0, \infty\rangle \to [0, \infty\rangle$ by $\omega_x^{(0)}(t) = \|x\|$ and

$$\omega_x^{(n)}(t) = \sup_{\substack{g_1, \ldots, g_n \in G \\ |g_j|' \leq t}} \|(I - U(g_1)) \ldots (I - U(g_n))x\|$$

for $n \in \mathbf{N}$. Secondly, for $\gamma \in \langle 0, n\rangle$ define $\|\cdot\|_\gamma^{n,p,\omega} : \mathcal{X} \to [0, \infty]$ by

$$\|x\|_\gamma^{n,p,\omega} = \|x\| + \left(\int_0^1 dt\, t^{-1} \left(t^{-\gamma} \omega_x^{(n)}(t)\right)^p\right)^{1/p} .$$

Then the space

$$\mathcal{X}_\gamma^{n,p,\omega} = \{x \in \mathcal{X} : \|x\|_\gamma^{n,p,\omega} < \infty\}$$

is a Banach space with respect to the norm $\|\cdot\|_\gamma^{n,p,\omega}$.

If n is a multiple of w we shall prove that these Lipschitz spaces coincide with the corresponding interpolation spaces of Proposition 5.1. We need the following lemma.

Lemma 5.6 *If $k, n \in \mathbf{N}$ with $kw < n$ then there exists $c > 0$ such that*

$$\|S_t x\|_n \leq c t^{-(n-kw)/m} \|x\|_{kw}$$

for all $x \in \mathcal{X}_{kw}$ and $t \in \langle 0, 1]$.

Proof First, it follows from the inequality (6) that the space $(\mathcal{X}, D(\overline{H}^n))_{\frac{n}{nm},1;K}$ is continuously embedded in $\mathcal{X}_n$ (see [Tri] Lemma 1.10.1(a)). Hence there exists $c_1 > 0$ such that

$$\|x\|_n \leq c_1 \|x\|_{(\mathcal{X}, D(\overline{H}^n))_{\frac{n}{nm},1;K}}$$

for all $x \in (\mathcal{X}, D(\overline{H}^n))_{\frac{n}{nm},1;K}$.

Secondly, it follows as in the proof of Proposition 4.4 of [ElR1] that there exists $c_2 > 0$ such that

$$\|S_t x\|_{(\mathcal{X}, D(\overline{H}^n))_{\frac{n}{nm},1;K}} \leq c_2 t^{-(n-kw)/m} \|x\|_{(\mathcal{X}, D(\overline{H}^n))_{\frac{kw}{nm},\infty;K}}$$

for all $t \in \langle 0,1]$ and $x \in (\mathcal{X}, D(\overline{H}^n))_{\frac{kw}{nm},\infty;\mathrm{K}}$.

Thirdly, in the proof of Proposition 5.1 it has been shown that the space $\mathcal{X}_{kw}$ is continuously embedded in the interpolation space $(\mathcal{X}, D(\overline{H}^n))_{\frac{kw}{nm},\infty;\mathrm{K}}$. So there exists $c_3 > 0$ such that

$$\|x\|_{(\mathcal{X},D(\overline{H}^n))_{\frac{kw}{nm},\infty;\mathrm{K}}} \leq c_3\|x\|_{kw}$$

Combination of these three inequalities then gives $\|S_t x\|_n \leq c_1c_2c_3t^{-(n-kw)/m}\|x\|_{kw}$ for all $t \in \langle 0,1]$. □

Now we are prepared to identify these various Lipschitz spaces in the manner of Theorem 3.2 [ElR1].

Theorem 5.7 *If $n \in \mathbf{N}$ and $0 < \gamma < nw$ then*

$$(\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K}} = \mathcal{X}_\gamma^{\prime nw,p,\omega} = \mathcal{X}_\gamma^{nw,p}(U)$$

as Banach spaces.

Proof The proof of the inclusions

$$\mathcal{X}_\gamma^{nw,p,\omega} \subseteq \mathcal{X}_\gamma^{nw,p}(U) \subseteq (\mathcal{X}, D(\overline{H}^n))_{\frac{\gamma}{nm},p;\mathrm{K}}$$

is precisely the same as in Steps 2 and 3 of the proof of Theorem 3.2 of [ElR1], so by Proposition 5.1 we only have to prove that $(\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;\mathrm{K}} \subseteq \mathcal{X}_\gamma^{nw,p,\omega}$. This is established as in Step 1 of the proof of Theorem 3.2 of [ElR1], but there is a complication.

If $x \in \mathcal{X}$ and $x = x_0 + x_{nw}$ with $x_0 \in \mathcal{X}$ and $x_{nw} \in \mathcal{X}_{nw}$ then

$$\omega_x^{(nw)}(t) \leq \omega_{x_0}^{(nw)}(t) + \omega_{x_{nw}}^{(nw)}(t) \quad .$$

for all $t \in \langle 0,1]$. Let $M \geq 1$ be such that $\|U(g)\| \leq M$ for all $g \in G$ with $|g|' \leq 4$. Then $\omega_{x_0}^{(nw)}(t) \leq (1+M)^{nw}\|x_0\|$ for all $t \in \langle 0,1]$.

Similarly, if S is the semigroup generated by the closure of a weighted strongly elliptic operator H of order $m = 2nw$ then

$$\omega_{x_{nw}}^{(nw)}(t) \leq \omega_{(I-S_{t^m})x_{nw}}^{(nw)}(t) + \omega_{S_{t^m}x_{nw}}^{(nw)}(t) \quad .$$

There exists $c > 0$ such that $\|S_s Hy\| \leq cs^{-1/2}\|y\|_{nw}$ for all $s \in \langle 0,1]$ and $y \in \mathcal{X}_{nw}$. Then by a Duhamel estimate one obtains

$$\begin{aligned}\omega_{(I-S_{t^m})x_{nw}}^{(nw)}(t) &\leq (1+M)^{nw}\|(I-S_{t^m})x_{nw}\| \\ &\leq (1+M)^{nw}\int_0^{t^m} ds\,\|S_sHx_{nw}\| \\ &\leq (1+M)^{nw}\int_0^{t^m} ds\,cs^{-1/2}\|x_{nw}\|_{nw} \\ &= 2c(1+M)^{nw}t^{nw}\|x_{nw}\|_{nw} \quad .\end{aligned}$$

So it remains to estimate $\omega_{S_{t^m}x_{nw}}^{(nw)}(t)$.

Following an idea of Pesenson [Pes], we first prove that

$$\omega_{x_\infty}^{(l)}(t) \leq 2(M+1)^w \sum_{k=1}^d t^{w_k}\omega_{A_kx_\infty}^{((l-w_k)\vee 0)}((1+\varepsilon)t)$$

for all $l \in \mathbf{N}$, all $x_\infty \in \mathcal{X}_\infty$, all $t \in \langle 0, 2]$ and all $\varepsilon \in \langle 0, 1]$, where M is as above. For $a_i \in \mathfrak{g}$ let $\widetilde{X}_i$ be the corresponding left invariant vector field on G. So

$$(\widetilde{X}_i \psi)(g) = \frac{d}{dt}\psi(g \exp(ta_i))\Big|_{t=0}$$

for every C^∞-function ψ on G. In particular if

$$\psi(h) = (f, U(h)x_\infty)$$

where $x_\infty \in \mathcal{X}_\infty$ and $f \in \mathcal{F}_\infty$, the space of C^∞-elements for the adjoint representation on the dual, or predual, $\mathcal{F}$, then

$$(\widetilde{X}_i \psi)(h) = (f, U(h)A_i x_\infty) \quad .$$

Now if $\varphi : [0,1] \to G$ is an absolutely continuous path from e to g_l which satisfies the differential equation

$$\dot{\varphi}(t) = \sum_{i=1}^d \varphi_i(t)\, \widetilde{X}_i\Big|_{\varphi(t)}$$

almost everywhere with

$$|\varphi_i(t)| \le ((1+\varepsilon)|g|')^{w_i}$$

for all $i \in \{1, \ldots, d\}$ and $t \in [0,1]$ then one has

$$\begin{aligned}(f, (I - U(g_l))x_\infty) &= \psi(e) - \psi(g_l) \\ &= -\int_0^1 ds \sum_{i=1}^d \varphi_i(s)\, (f, U(\varphi(s))A_i x_\infty) \\ &= -\int_0^1 ds \sum_{i=1}^d \varphi_i(s) \Big((f, (I - U(\varphi(s)))A_i x_\infty) + (f, A_i x_\infty) \Big) \quad .\end{aligned}$$

Moreover, $|\varphi(s)|' \le (1+\varepsilon)|g_l|'$ for all $s \in [0,1]$. Hence

$$\begin{aligned}&|(f, (I - U(g_l))x_\infty)| \\ &\quad \le \sum_{i=1}^d ((1+\varepsilon)|g_l|')^{w_i} \sup\{|(f, (I - U(\varphi(s))A_i x_\infty)| + |(f, A_i x_\infty)| : s \in [0,1]\} \quad .\end{aligned}$$

Now replacing f by $((I - U(g_1)) \ldots (I - U(g_{l-1})))^* f$ and taking the supremum over f with $\|f\| \le 1$ and $|g_i|' \le t$ one finds

$$\begin{aligned}\omega_{x_\infty}^{(l)}(t) &\le \sum_{i=1}^d ((1+\varepsilon)t)^{w_i} \Big(\omega_{A_i x_\infty}^{(l)}((1+\varepsilon)t) + \omega_{A_i x_\infty}^{(l-1)}(t) \Big) \\ &\le \sum_{i=1}^d ((1+\varepsilon)t)^{w_i} 2(M+1)^{w_i} \omega_{A_i x_\infty}^{((l-w_i)\vee 0)}((1+\varepsilon)t) \quad .\end{aligned}$$

It now follows by iteration of this inequality that

$$\omega_{x_\infty}^{(nw)}(t) \le c \sum_{\substack{\alpha \in J(d) \\ nw^2 \ge \|\alpha\| \ge nw}} t^{\|\alpha\|} \|A^\alpha x_\infty\|$$

for all $t \in \langle 0,1]$. Consequently

$$\begin{aligned}\omega^{(nw)}_{S_{t^m}x_{nw}}(t) &\leq c \sum_{\substack{\alpha\in J(d)\\ nw^2\geq\|\alpha\|\geq nw}} t^{\|\alpha\|}\|A^\alpha S_{t^m}x_{nw}\| \\ &\leq c' \sum_{\substack{\alpha\in J(d)\\ nw^2\geq\|\alpha\|\geq nw}} t^{\|\alpha\|}(t^m)^{-(\|\alpha\|-nw)/m}\|x_{nw}\|_{nw} \\ &= c'' t^{nw}\|x_{nw}\|_{nw}\end{aligned}$$

for some c', c'', by Lemma 5.6. Thus

$$\omega^{(nw)}_x(t) \leq c(\|x_0\| + t^{nw}\|x_{nw}\|_{nw})$$

for some $c > 0$, independent of x and t and by taking the infimum over all decompositions $x = x_0 + x_n$ with $x_n \in \mathcal{X}_{nw}$ one concludes that

$$\omega^{(nw)}_x(t) \leq c\kappa^{(0,nw)}_x(t^{nw})$$

for all $t \in \langle 0,1]$, where

$$\kappa^{(0,nw)}_x(t) = \inf\{\|x_0\| + t\|x_{nw}\|_{nw} : x = x_0 + x_{nw},\ x_0 \in \mathcal{X},\ x_{nw} \in \mathcal{X}_{nw}\} \quad .$$

Therefore $(\mathcal{X}, \mathcal{X}_{nw})_{\frac{\gamma}{nw},p;K}$ is continuously embedded in $\mathcal{X}^{n,p,\omega}_\gamma$. □

Next we consider unitary representations.

Theorem 5.8 *Suppose $(\mathcal{X}, G, U)$ is a unitary representation. Then one has the following.*

I. *The operator H is closed.*

II. *For all $n \in \mathbf{N}$ and all large $\lambda > 0$*

$$D((\lambda I + H)^{nw/m}) = \mathcal{X}_{nw}$$

with equivalent norms.

III. *There exist $p > 0$ and $q \in \mathbf{R}$, independent of the representation U, such that*

$$\mathrm{Re}(Hx, x) \geq p(\|x\|_{m/2})^2 - q\|x\|^2$$

for all $x \in \mathcal{X}_\infty$.

IV. *If $n \in \mathbf{N}$ then*

$$\mathcal{X}_{nw} = \bigcap_{i=1}^d D(A_i^{nw/w_i}) \quad .$$

Proof First let $n \in \mathbf{N}$ and consider the operator

$$H = \sum_{\substack{\alpha\in J(d)\\ \|\alpha\|\leq nw}} (-1)^{|\alpha|} A^{(\alpha_*,\alpha)}$$

with domain $D(H) = \mathcal{X}_{2nw}$ where $\langle \alpha_*, \alpha \rangle$ denotes the multi-index formed by composition of α and the index α_* obtained by reversing the order of α. Then H is a weighted strongly elliptic of order $2nw$. Clearly H is positive symmetric and by Theorem 3.3 the operator $\overline{H}$ is a generator of a semigroup. Therefore $\overline{H}$ is self-adjoint. Hence

$$D(\overline{H}^{1/2}) = (\mathcal{X}, D(\overline{H}))_{1/2,2;\mathrm{K}} = (\mathcal{X}, \mathcal{X}_{2nw})_{1/2,2;\mathrm{K}}$$

by Proposition 5.1.

Alternatively, define $h\colon \mathcal{X}_{nw} \to \mathbf{C}$ by

$$h(x) = \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| \le nw}} \|A^\alpha x\|^2 \quad .$$

Then h is a closed quadratic form and by the second representation theorem for quadratic forms (see [Kat2], Theorem VI.2.23) there exists a positive self-adjoint operator K such that $D(K^{1/2}) = \mathcal{X}_{nw}$ and $h(x) = \|K^{1/2}x\|^2$ for all $x \in \mathcal{X}_{nw}$. But one straightforwardly verifies that K extends $\overline{H}$ and since both operators are self-adjoint $K = \overline{H}$. Consequently

$$\|\overline{H}^{1/2} x\|^2 = \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| \le nw}} \|A^\alpha x\|^2$$

for all $x \in \mathcal{X}_{nw}$. Therefore

$$\mathcal{X}_{nw} = D(\overline{H}^{1/2}) = (\mathcal{X}, \mathcal{X}_{2nw})_{\frac{nw}{2nw},2;\mathrm{K}}$$

and the norms are equivalent.

Secondly, if C_0 is the form such that

$$H_0 = dU(C_0) = \sum_{i=1}^{d} (-1)^{w/w_i} A_i^{2w/w_i}$$

then it follows from Proposition 5.1 that

$$D(\overline{H_0}^{n/2}) = (\mathcal{X}, D(\overline{H_0}^{n}))_{\frac{1}{2},2;\mathrm{K}} = (\mathcal{X}, \mathcal{X}_{2nw})_{\frac{nw}{2nw},2;\mathrm{K}} = \mathcal{X}_{nw}$$

for all $n \in \mathbf{N}$ with equivalent norms.

Thirdly, we prove the Gårding inequality, Statement III. If μ is the ellipticity constant of the weighted form C, then the form $\Re C - 2^{-1}\mu C_0^{m/(2w)}$ is also a weighted strongly elliptic form, for which the corresponding operator is symmetric and its closure is self-adjoint and a generator of a semigroup. Hence the closure of $dU(\Re C - 2^{-1}\mu\, C_0^{m/(2w)})$ is lower bounded by spectral theory. Let $-\rho$ denote the lower bound. Then

$$\mathrm{Re}(dU(C)x, x) = (dU(\Re C)x, x) \ge 2^{-1}\mu \|dU(C_0)^{m/(2w)} x\|^2 - \rho\|x\|^2 \ge p(\|x\|_{m/2})^2 - q\|x\|^2$$

for some $p > 0$ and $q \ge 0$, uniformly in $x \in \mathcal{X}_\infty$. This proves Statement III. The independence of the constants on the representation follows from the fact that the semigroup generated by $dU(C)$ has a representation independent kernel and all constants involved can be expressed in terms of this kernel.

Fourthly, if C is a weighted strongly elliptic form then so is $C^\dagger C$. Therefore by the above there exist $p, q > 0$ such that

$$(dU(C^\dagger C)x, x) \geq p(\|x\|_m)^2 - q\|x\|^2$$

for all $x \in \mathcal{X}_\infty$. Then

$$(\|x\|_m)^2 \leq p^{-1}\|dU(C)x\|^2 + qp^{-1}\|x\|^2$$

for all $x \in \mathcal{X}_\infty$. Since $\mathcal{X}_\infty$ is dense in $D(\overline{dU(C)})$ and $\mathcal{X}_m$ is complete, it follows that $D(\overline{dU(C)}) \subseteq \mathcal{X}_m$ and in particular $dU(C)$ is closed. This proves Statement I.

Fifthly, let $H = dU(C)$. By the Gårding inequality both the operators $\lambda I + H$ and $H_0^{m/(2w)}$ are closed maximal accretive operators with the same domain if λ is large enough. So by [Kat1] it follows that $D((\lambda I + H)^{nw/m}) = D(H_0^{n/2}) = \mathcal{X}_{nw}$ for all $n \in \{1, \ldots, m/w - 1\}$.

Sixthly, let $n \in \mathbf{N}$ and suppose that $D((\lambda I + H)^{nw/m}) = \mathcal{X}_{nw}$. Then

$$\begin{aligned}
D((\lambda I + H)^{(nw+m)/m}) &= \{x \in D(H) : (\lambda I + H)x \in D((\lambda I + H)^{nw/m})\} \\
&= \{x \in D(H) : (\lambda I + H)x \in \mathcal{X}_{nw}\} \\
&= \{x \in D(H) : (H + \lambda I)x \in (\mathcal{X}, D(H^{n+1}))_{\frac{nw}{(n+1)m},2;\mathrm{K}}\} \\
&= \{x \in D(H) : x \in (\mathcal{X}, D(H^{n+1}))_{\frac{nw+m}{(n+1)m},2;\mathrm{K}}\} \\
&= (\mathcal{X}, D(H^{n+1}))_{\frac{nw+m}{(n+1)m},2;\mathrm{K}} \\
&= \mathcal{X}_{nw+m} \quad .
\end{aligned}$$

Now Statement II follows by induction.

Finally we prove Statement IV. Let $x \in \bigcap_{i=1}^d D(A_i^{nw/w_i})$ and set $c_1 = \sum_{i=1}^d \|A_i^{nw/w_i}x\| + \|x\|$. Let

$$H = \sum_{i=1}^d (-1)^{nw/w_i} A_i^{2nw/w_i}$$

which is weighted strongly elliptic of order $2nw$. Then for all $y \in \mathcal{X}_\infty$

$$\begin{aligned}
|(x, (H + I)y)| &= |(-1)^n(x, \sum_{i=1}^d A_i^{2nw/w_i}y) + (x, y)| \\
&= |\sum_{i=1}^d (A_i^{nw/w_i}x, A_i^{nw/w_i}y) + (x, y)| \\
&\leq c_1\|y\|_{nw} \quad .
\end{aligned}$$

By Statement II there exists $c_2 > 0$ such that

$$\|y\|_{nw} \leq c_2\|(H + I)^{1/2}y\|$$

for all $y \in \mathcal{X}_\infty$. Since $(H + I)^{1/2}$ maps $\mathcal{X}_\infty$ onto $\mathcal{X}_\infty$ it follows that

$$|(x, (dU(C_{2n}) + I)^{1/2}y)| \leq c_1c_2\|y\|$$

for all $y \in \mathcal{X}_\infty$ and, by continuity, for all $y \in D((H + I)^{1/2})$. So $x \in D(((H + I)^{1/2})^*) = D((H + I)^{1/2}) = \mathcal{X}_{nw}$ by Statement II again. □

It is also possible to obtain regularity results for the left regular representation on the L_p-spaces with respect to left Haar measure if $p \in \langle 1, \infty \rangle$. These are basically a result of the good kernel bounds and the regularity on L_2.

Corollary 5.9 *Let L be the left regular representation on L_p, where $p \in \langle 1, \infty \rangle$ and let $H = dL(C)$. Then*

I. *The operator H is closed.*

II. *For all $n \in \mathbf{N}$ one has*

$$D((\lambda I + H)^{nw/m}) = L_{p;nw}$$

with equivalent norms, if $\lambda > 0$ is large enough.

III. *If $n \in \mathbf{N}$ then*

$$L_{p;nw} = \bigcap_{i=1}^{d} D(A_i^{nw/w_i}) \quad .$$

Similar statements are valid on the space L_p spaces with respect to right Haar measure, $L_{\hat{p}}$.

Proof The proof is precisely the same as that for the unweighted operators in [BER]. □

Corollary 5.10 *Let L be the left regular representation on L_p, where $p \in \langle 1, \infty \rangle$ and let $H = dL(C)$. If $\theta \in \langle 0, \theta_C \rangle$ then there is a $\nu_0 \geq 0$, independent of p, such that the operators $\nu I + H$, $\nu > \nu_0$, have a bounded functional analysis over the functions which are bounded and holomorphic in the sector $\Lambda(\varphi)$ with $\varphi \in \langle \pi/2 - \theta, \pi]$.*

Proof The proof is precisely the same as in [ElR4]. □

6 Weighted Gevrey vectors

For all $\lambda > 0$ define the vector space $\mathcal{X}^\lambda(U)$ of **weighted Gevrey vectors of order λ for U** by

$$\mathcal{X}^\lambda(U) = \{x \in \mathcal{X}_\infty : \exists_{c,t>0} \forall_{\alpha \in J(d)} [\|A^\alpha x\| \leq c t^{\|\alpha\|} \|\alpha\|!^\lambda]\} \quad .$$

For many groups the space $\mathcal{X}^\lambda(U)$ consists only of the zero vector if λ is small but for other groups $\mathcal{X}^\lambda(U)$ is dense even if $\lambda = 0$. In the Euclidean case with all weights equal one and $\lambda = 1$ the space $G^1(U)$ is precisely the space $\mathcal{X}_a(U)$ of analytic vectors for U, see [Rob2] Section II.2. Moreover, the space $\mathcal{X}_a(U)$ is dense in $\mathcal{X}$ by Nelson's theorem [Nel] (see also [Rob2] Theorem II.2.2). Therefore it follows in the weighted case that $\mathcal{X}_a(U) \subseteq \mathcal{X}^{1/n}(U)$, where $n = \min(w_1, \ldots, w_d)$. So the space $\mathcal{X}^\lambda(U)$ is dense in $\mathcal{X}$ if $\lambda \geq 1/n$.

Next, if T is an operator on $\mathcal{X}$ we define the vector space $G_\lambda(T)$ of **Gevrey vectors of order λ for T** by

$$G_\lambda(T) = \{x \in D^\infty(T) : \exists_{c,t>0} \forall_{n \in \mathbf{N}_0} [\|T^n x\| \leq c t^n n!^\lambda]\} \quad .$$

Note that if $m \in \mathbf{N}$ and $\lambda > 0$ then

$$G_{m\lambda}(T) = \{x \in D^\infty(T) : \exists_{c,t>0} \forall_{n \in \mathbf{N}_0} [\|T^n x\| \leq c t^n (mn)!^\lambda]\}$$

since $n!k! \leq (n+k)! \leq 2^{n+k} n!k!$ for all $n, k \in \mathbf{N}_0$. Now consider the Gevrey vectors for weighted strongly elliptic operators H of order m. Since $D^\infty(\overline{H}) = \mathcal{X}_\infty$ by Theorem 3.4, one has $G_\lambda(\overline{H}) = G_\lambda(H)$ and we can omit the closure in the subsequent discussion.

Because H is an operator of order m it then follows by an elementary counting argument that

$$\mathcal{X}^\lambda(U) \subseteq G_{m\lambda}(H)$$

for all $\lambda > 0$. The converse inclusion

$$G_{m\lambda}(H) \subseteq \mathcal{X}^\lambda(U) \tag{12}$$

is not valid in general. One can provide simple counterexamples by taking λ so small that the space $\mathcal{X}^\lambda(U)$ is trivial. More interestingly, in Example 7.8 we construct a unitary representation, a self-adjoint weighted strongly elliptic operator H and a $\lambda > 0$ such that both spaces $G_{m\lambda}(H)$ and $\mathcal{X}^\lambda(U)$ are dense in $\mathcal{X}$, but unequal. We next prove, however, that for all $\lambda \geq 1$ the converse inclusion (12) is valid.

Theorem 6.1 *If* $(\mathcal{X}, G, U)$ *is a continuous representation then*

$$\mathcal{X}^\lambda(U) = G_{m\lambda}(\overline{H})$$

for all $\lambda \geq 1$.

Proof We only have to prove (12). Following [Rob1] we first reduce it to a problem on Lipschitz spaces.

First, note that $G_{m\lambda}(H) \subseteq D^\infty(\overline{H}) = \mathcal{X}_\infty$ by Theorem 3.4.

Next fix $p \in [1, \infty\rangle$ and $\gamma \in \langle 0, m\rangle$. We temporarily denote the space $(\mathcal{X}, \mathcal{X}_m)_{\gamma/m,p;\mathrm{K}}$ by $\mathcal{X}_\gamma$, with norm $\|\cdot\|_\gamma$, and the corresponding subspace of C^n-vectors by $\mathcal{X}_{\gamma;n}$ with norm $\|\cdot\|_{\gamma;n}$. Let H_γ be the restriction of $\overline{H}$ to $\mathcal{X}_\gamma$ and let

$$\mathcal{X}_\gamma^\lambda = \{x \in \mathcal{X}_\infty : \exists_{c,t>0} \forall_{\alpha \in J(d)} [\|A^\alpha x\|_\gamma \leq c t^{\|\alpha\|} \|\alpha\|!^\lambda]\} \quad .$$

Then $\mathcal{X}_\gamma^\lambda \subseteq \mathcal{X}^\lambda(U)$ since $\|y\| \leq \|y\|_\gamma$ for all $y \in \mathcal{X}_\gamma$. Moreover, since $\mathcal{X}_m$ is continuously embedded in $\mathcal{X}_\gamma$ there exists $c_1 > 0$ such that $\|y\|_\gamma \leq c_1 \|y\|_m$ for all $y \in \mathcal{X}_m$. By adding a large constant to H, if necessary, it follows from Theorem 3.4 that there exists $c_2 > 0$ such that $\|y\|_m \leq c_2 \|H^2 y\|$ for all $y \in D(H^2)$. So if $x \in G_{m\lambda}(H)$ and $c, t > 0$ are such that $\|H^n x\| \leq c t^n n!^{m\lambda}$ for all $n \in \mathbf{N}_0$ it then follows that

$$\|H_\gamma^n x\|_\gamma \leq c_1 c_2 \|H^{n+2} x\| \leq c_1 c_2 c t^{n+2} (n+2)!^{m\lambda} \leq c_1 c_2 c t^{n+2} (2^{n+2} 2!)^{m\lambda} n!^{m\lambda} \quad .$$

So $G_{m\lambda}(H) \subseteq G_{m\lambda}(H_\gamma)$. Therefore the theorem is proved once we establish that

$$G_{m\lambda}(H_\gamma) \subseteq \mathcal{X}_\gamma^\lambda \quad .$$

The main problem in the proof of the weighted case compared with the proof of the theorem in the Euclidean case, [Rob2] Theorem II.3.1, is that for a general multi-index α with $\|\alpha\| > m$ one cannot find multi-indices β and γ with $\|\beta\| = m$ such that $\alpha = \langle \beta, \gamma \rangle$, the concatenation of β and γ. If, however, the length $|\alpha|$ is large enough, then there must be an $i \in \{1, \ldots, d\}$ such that the multi-index α contains at least m/w_i times the index i. The key observation is that one can move these indices to the left. This will introduce a number of commutators and structure constants from

$$[a_i, a_j] = \sum_{\substack{k \in \{1,\ldots,d\} \\ w_k \leq w_i + w_j - 1}} c_{ij}^k \, a_k \quad .$$

Let $N_1 = (m/w_1 - 1) + \ldots + (m/w_d - 1) + 1$ and $N_2 = dN_1 \max(m/w_1, \ldots, m/w_d)$. Then it follows by taking into account the first N_1 indices that for all α with $|\alpha| \geq N_1$ there exists $i_\alpha \in \{1, \ldots, d\}$ and for all $n \in \{1, \ldots, N_2\}$ there exist $c_{\alpha,n} \in \{c_{ij}^k : i, j, k \in \{1, \ldots, d\}\} \cup \{0\}$ and $\beta_{\alpha,n} \in J(d)$ with $\|\beta_{\alpha,n}\| \leq \|\alpha\| - 1$ such that

$$A^\alpha = A_{i_\alpha}^{m/w_{i_\alpha}} A^{\gamma_\alpha} + \sum_{n=1}^{N_2} c_{\alpha,n} A^{\beta_{\alpha,n}} \quad . \tag{13}$$

Let $M_1 = \max\{|c_{ij}^k| : i, j, k \in \{1, \ldots, d\}\}$. We may as well assume that $M_1 \geq 1$. Moreover, let $N_3 = N_1 \max(w_1, \ldots, w_d)$. Then $|\alpha| \geq N_1$ for all α with $\|\alpha\| \geq N_3$. Note that $N_3 \geq m$.

By adding a large constant to H we may assume that $\|y\|_\gamma \leq \|Hy\|_\gamma$ for all $y \in \mathcal{X}_\infty$. Let $M, M_2 > 0$ be such that

$$M^{-1}\|Hy\|_\gamma \leq \|y\|_{\gamma;m} \leq M\|Hy\|_\gamma \quad , \quad \|y\|_{\gamma;N_3+m} \leq M_2\|H^{N_3+m}y\|_\gamma$$

for all $y \in \mathcal{X}_\infty$. These constants exist because of Corollary 5.5. Now let $x \in G_{m\lambda}(H_\lambda)$. There exist $a, c > 0$ such that

$$\|H^n x\|_\gamma \leq ac^n (mn)^\lambda$$

for all $n \in \mathbf{N}_0$. For all $p, q \in \mathbf{N}_0$ define

$$c_{p,q} = \max_{\|\alpha\| \leq p} \|A^\alpha H^q x\|_\gamma \quad ,$$

$$d_{p,q} = \max_{\|\alpha\| \leq p} \|HA^\alpha H^q x\|_\gamma \quad .$$

Then $c_{p,q} \leq d_{p,q}$ for all $p, q \in \mathbf{N}_0$. For all $\alpha \in J(d)$ with $\|\alpha\| \leq N_3$ and all $q \in \mathbf{N}$ we have

$$\begin{aligned}
\|HA^\alpha H^q x\|_\gamma &\leq M\|H^q x\|_{\gamma;N_3+m} \\
&\leq MM_2\|H^{q+N_3+m}x\|_\gamma \\
&\leq aMM_2c^{q+N_3+m}(m(q+N_3+m))!^\lambda \\
&\leq aMM_2c^{q+N_3+m}(mq)!^\lambda(mN_3+m^2)!^\lambda 2^{(mq+mN_3+m^2)\lambda} \\
&= a_1c_1^q(mq)!^\lambda \quad ,
\end{aligned}$$

for some constants $a_1, c_1 > 1$, independent of q and α. So

$$c_{p,q} \leq d_{p,q} \leq a_1c_1^q(mq)!^\lambda \tag{14}$$

for all $p \leq N_3$ and $q \in \mathbf{N}_0$.

We shall prove that there exists $b_1 \geq 2$ such that

$$d_{p,q} \leq a_1b_1^pc_1^q(p+mq)!^\lambda \tag{15}$$

for all $p, q \in \mathbf{N}_0$. The proof is by induction on p. If we have proved this then

$$\|A^\alpha x\|_\gamma \leq c_{\|\alpha\|,0} \leq d_{\|\alpha\|,0} \leq a_1b_1^{\|\alpha\|}\|\alpha\|!^\lambda$$

for all $\alpha \in J(d)$ and $x \in \mathcal{X}_\gamma^\lambda$.

Let $\alpha \in J(d)$, $q \in \mathbf{N}$ and suppose that $p = \|\alpha\| \geq N_3$. Then it follows from (13) that

$$\begin{aligned}
\|A^\alpha H^q x\|_\gamma &\leq \|A_{i_\alpha}^{m/w_{i_\alpha}} A^{\gamma_\alpha} H^q x\|_\gamma + \sum_{n=1}^{N_2} |c_{\alpha,n}| \, \|A^{\beta_{\alpha,n}} H^q x\|_\gamma \\
&\leq M \|H A^{\gamma_\alpha} H^q x\|_\gamma + \sum_{n=1}^{N_2} M_1 \|A^{\beta_{\alpha,n}} H^q x\|_\gamma \\
&\leq M d_{p-m,q} + N_2 M_1 c_{p-1,q} \quad .
\end{aligned}$$

So

$$c_{p,q} \leq M d_{p-m,q} + N_2 M_1 c_{p-1,q} \tag{16}$$

for all $p \geq N_3$ and $q \in \mathbf{N}_0$. By iteration of this inequality it follows that

$$c_{p,q} \leq M \sum_{n=0}^{p-N_3-1} (N_2 M_1)^n d_{p-m-n,q} + (N_2 M_1)^{p-N_3} c_{N_3,q} \tag{17}$$

for all $p \geq N_3$.

Now the operator H consists of a finite number of terms, say N_4, and each of these terms consists of at most $N_5 = \max(m/w_1, \ldots, m/w_d)$ operators A_i, so for all $\alpha \in J(d)$ it follows by the condition on the structure constants that $[H, A^\alpha]$ can be written as a linear combination of A^β's with $\|\beta\| \leq \|\alpha\| + m - 1$, the sum consists of at most $N_4 N_5 |\alpha| d$ terms and the only constants involved are the structure constants. So $[H, A^\alpha] = \sum_{k=1}^{N_4 N_5 |\alpha| d} b_{\alpha,k} A^{\gamma_{\alpha,k}}$ for some $b_{\alpha,k}$ with $|b_{\alpha,k}| \leq M_1$ and $\|\gamma_{\alpha,k}\| \leq \|\alpha\| + m - 1$. Hence

$$\begin{aligned}
\|H A^\alpha H^q x\|_\gamma &\leq \|A^\alpha H^{q+1} x\|_\gamma + \sum_{k=1}^{|\alpha| d N_4 N_5} |b_{\alpha,k}| \|A^{\gamma_{\alpha,k}} H^q x\|_\gamma \\
&\leq c_{\|\alpha\|,q+1} + |\alpha| d M_1 N_4 N_5 c_{\|\alpha\|+m-1,q} \\
&\leq c_{\|\alpha\|,q+1} + \|\alpha\| d M_1 N_4 N_5 c_{\|\alpha\|+m-1,q}
\end{aligned}$$

for all $q \in \mathbf{N}_0$. So

$$d_{p,q} \leq c_{p,q+1} + p d M_1 N_4 N_5 c_{p+m-1,q} \tag{18}$$

for all $p, q \in \mathbf{N}_0$.

Now we prove (15) for all $p, q \in \mathbf{N}_0$. If $p \leq N_3$ then (15) is valid because of the estimates (14). Let $p \in \mathbf{N}$, $p > N_3$ and suppose that (15) is valid for all smaller p and all $q \in \mathbf{N}_0$. Then it follows from (17), the induction hypothesis and (14) that

$$\begin{aligned}
d_{p,q} &\leq c_{p,q+1} + p d M_1 N_4 N_5 c_{p+m-1,q} \\
&\leq M \sum_{n=0}^{p-N_3-1} (N_2 M_1)^n d_{p-m-n,q+1} + (N_2 M_1)^{p-N_3} c_{N_3,q+1} \\
&\quad + p d M_1 M N_4 N_5 \sum_{n=0}^{p+m-N_3-2} (N_2 M_1)^n d_{p-n-1,q} + p d M_1 N_4 N_5 (N_2 M_1)^{p+m-1-N_3} c_{N_3,q}
\end{aligned}$$

$$\leq M \sum_{n=0}^{p-N_3-1} e^{N_2M_1} n! a_1 b_1^{p-m-n} c_1^{q+1}(p+mq-n)!^\lambda + e^{N_2M_1}(p-N_3)! a_1 c_1^{N_3}(mq+m)!^\lambda$$

$$+ pdM_1MN_4N_5 \sum_{n=0}^{p+m-N_3-2} e^{N_2M_1} n! a_1 b_1^{p-n-1} c_1^q (p+mq-n-1)!^\lambda$$

$$+ pdM_1N_4N_5 e^{N_2M_1}(p+m-1-N_3)! a_1 c_1^q (mq)!^\lambda$$

where we have used the inequality $a^n \leq e^a n!$. Since $\lambda \geq 1$ we have $n!(p+mq-n)!^\lambda \leq (p+mq)!^\lambda$. Also $(p-N_3)!(mq+m)!^\lambda \leq (p+mq-(N_3-m))!^\lambda \leq (p+mq)!^\lambda$ since $N_3 \geq m$. Similarly $pn!(p+mq-n-1)!^\lambda \leq (p+mq)!^\lambda$ and $p(p+m-1-N_3)!(mq)!^\lambda \leq (p+mq)!^\lambda$. So

$$d_{p,q} \leq a_1 b_1^p c_1^q (p+mq)!^\lambda \Big(M e^{N_2M_1} c_1 b_1^{-m} \sum_{n=0}^{\infty} b_1^{-n} + e^{N_2M_1} b_1^{-p}$$

$$+ dM_1MN_4N_5 e^{N_2M_1} b_1^{-1} \sum_{n=0}^{\infty} b_1^{-n} + dM_1N_4N_5 e^{N_2M_1} b_1^{-p} \Big)$$

$$\leq a_1 b_1^p c_1^q (p+mq)!^\lambda b_1^{-1} \left(2M e^{N_2M_1} c_1 + e^{N_2M_1} + 2dM_1MN_4N_5 e^{N_2M_1} + dM_1N_4N_5 e^{N_2M_1} \right) .$$

If we take b_1 larger than the quantity between the brackets it follows that

$$d_{p,q} \leq a_1 b_1^p c_1^q (p+mq)!^\lambda$$

and the theorem has been proved. □

7 Examples

In this section we illustrate the theorems with some examples. We begin with an application of the foregoing results to the anharmonic oscillators mentioned in the introduction.

Example 7.1 Let G be the simply connected Heisenberg group, U the standard irreducible unitary representation of G in $\mathcal{X} = L_2(\mathbf{R})$ and a_1, a_2, a_3 a basis in the Lie algebra $\mathfrak{g}$ of G such that $A_1 = -iP$, $A_2 = iQ$ and $A_3 = iI$, where P and Q are the self-adjoint operators in $L_2(\mathbf{R})$ given by $(Pf)(x) = if'(x)$ and $(Qf)(x) = xf(x)$ for all $f \in C_c^\infty(\mathbf{R})$ and $x \in \mathbf{R}$. Then $[a_1, a_2] = a_3$ and a_3 is central. It is well known that $\mathcal{X}_\infty$ is equal to the Schwartz' space $\mathcal{S}$.

Let $j, k \in \mathbf{N}$. We consider the operator

$$P^{2j} + Q^{2k}$$

on several domains Two natural domains are the Schwartz space $\mathcal{S}$ and $D(Q^{2k}) \cap D(P^{2j})$. Let $d_0 = \gcd(k, j)$. Define the weights $w_1 = k/d_0$, $w_2 = j/d_0$ and $w_3 = 1$. Then $w = kjd_0^{-2}$. Moreover, the condition on the structure constants in Theorem 3.3 is satisfied. Let C be the weighted strongly elliptic form such that

$$H = dU(C) = (-1)^j A_1^{2j} + (-1)^k A_2^{2k} + (-1)^{kj/d_0} A_3^{2kj/d_0}$$

with domain $D(H) = \mathcal{X}_{2kj/d_0}$. Then it follows from Theorem 3.3 that the operator $\overline{H}$ is self-adjoint. But Theorem 5.8.I states that H is closed, so in fact the operator H is self-adjoint. Moreover,

$$D(H) = D(P^{2j}) \cap D(Q^{2k}) \cap D(I^{2kj/d_0}) = D(P^{2j}) \cap D(Q^{2k})$$

by Theorem 5.8.IV. Thus

$$H = P^{2j} + Q^{2k} + I \quad .$$

An application of Corollary 3.5 gives that the Schwartz space $\mathcal{S} = \mathcal{X}_\infty$ is a core for H. So the operator

$$P^{2j} + Q^{2k}\Big|_{\mathcal{S}}$$

is essentially self-adjoint and its closure equals $P^{2j} + Q^{2k}$. Next, Theorem 5.8.II together with Theorem 5.8.IV gives

$$D((P^{2j} + Q^{2k})^{l/(2d_0)}) = D(P^{lj/d_0}) \cap D(Q^{lk/d_0})$$

for all $l \in \mathbf{N}$. Proposition 4.4 gives that the kernel on the Heisenberg group corresponding to the semigroup generated by the operator H is positive if, and only if, the operator H is the harmonic oscillator, i.e., $j = k = 1$. (However, if $j = 1$, or $k = 1$, or $j = k = 2$, then there exist other groups G' such that the kernel of the semigroup generated by H has a positive kernel on G', e.g., if $j = 1$ then the Lie algebra of G' is generated by a_1 and a_2^k.)

Next we argue that the spectrum of $P^{2j} + Q^{2k} + I$ is a countable discrete set, with a possible accumulation point at infinity, and each point in the spectrum corresponds to an eigenvalue with finite multiplicity. Let $B = \{\varphi \in L_2(\mathbf{R}) : \|\varphi\| \le 1\}$. By Theorem 5.8.II there exists $c > 0$ such that $\|H^{-1}x\|_{2kj/d_0} \le c\|x\|$ for all $x \in \mathcal{X}$. Hence the set $(P^2 + Q^2)(P^{2j} + Q^{2k} + I)^{-1}(B)$ is a bounded set in L_2. Since the inverse $(P^2 + Q^2)^{-1}$ of the harmonic oscillator operator is compact, it follows that the set the closure of the set

$$(P^{2j} + Q^{2k} + I)^{-1}(B) = (P^2 + Q^2)^{-1}(P^2 + Q^2)(P^{2j} + Q^{2k} + I)^{-1}(B)$$

is compact in L_2. Therefore, the operator $(P^{2j} + Q^{2k} + I)^{-1}$ is compact. From this the above claim follows.

Finally let $\varphi_1, \varphi_2, \ldots$ be an orthonormal basis in L_2 of eigenvectors of $P^{2j} + Q^{2k} + I$. Then

$$\varphi_n \in D^\infty(P^{2j} + Q^{2k} + I) = \mathcal{X}_\infty = \mathcal{S}$$

for all $n \in \mathbf{N}$ by Theorem 3.4.I. We can, however, obtain much more information about the smoothness of the eigenfunctions by use of Gevrey vectors.

For the rest of this example it is preferable to change the notation and take

$$w_1 = k \quad , \quad w_2 = j \quad , \quad w_1 = 1 \quad .$$

Then $H = P^{2j} + Q^{2k} + I$ is a weighted elliptic operator of order $m = 2jk$. For all $\alpha, \beta > 0$ the Gel'fand–Shilov space S_α^β is defined by

$$S_\alpha^\beta = \{\varphi \in \mathcal{S} : \exists_{c,t>0} \forall_{l,n \in \mathbf{N}_0} \sup_{x \in \mathbf{R}} |x^n \varphi^{(l)}(x)| \le c t^{l+n} n!^\alpha l!^\beta\}$$

(see [GeS] Section IV.3.3). These spaces are dense in $L_2(\mathbf{R})$ if, and only if, $\alpha + \beta \geq 1$ ([GeS] Sections IV.8.2 and IV.8.4). Moreover, one can replace the supremum norm in this definition by an L_2 norm ([Wlo] §29.5) and one is allowed to change the order of the individual Q and $D = -iP$'s in $Q^n D^l$ (see [Els] Theorem 11). Therefore we obtain the following characterization for the weighted Gevrey vectors:

$$\mathcal{X}^\lambda(U) = S_{j\lambda}^{k\lambda}$$

for all $\lambda > 0$. So the space $\mathcal{X}^\lambda(U)$ is dense in $\mathcal{X}$ if, and only if, $\lambda \geq (j+k)^{-1}$. It follows from Theorem 6.1 that

$$S_{j\lambda}^{k\lambda} = G_{2jk\lambda}(P^{2j} + Q^{2k})$$

for all $\lambda \geq 1$. (If we did not change the weights the identity would be valid for all $\lambda \geq (\gcd(j,k))^{-1}$.) Equalities of this kind were obtained before in the various special cases:

- $j = k = 1$ and $\lambda \geq 1/2$, (see [Zha]),
- $j = 1$ or $k = 1$ and $\lambda = (j+k)^{-1}$, (see [EGP]),
- $j/k \in \mathbf{N}$ or $k/j \in \mathbf{N}$ and $\lambda = \min(1/j, 1/k)$, (see [Goo3], Theorem 6.1).

If one replaces the operator $P^{2j} + Q^{2k}$ by its restriction to the Schwartz space $\mathcal{S}$ then the equality has also been established in the following cases, (see [ElE]);

- $\lambda \geq \max(1/j, 1/k)$,
- $j = 1$ or $k = 1$ and $\lambda \geq \min(1/j, 1/k)$.

The equality of the two spaces has been conjectured by Van Eijndhoven and De Graaf, [EiG], Conjecture II.2.7, in all non trivial cases, i.e., $\lambda \geq (j+k)^{-1}$. (Obviously equality fails if $\lambda < (j+k)^{-1}$.) The proof of Theorem 6.1 enables us to verify this conjecture.

Theorem 7.2 *If $j, k \in \mathbf{N}$ then*

$$S_{j\lambda}^{k\lambda} = G_{2jk\lambda}(P^{2j} + Q^{2k})$$

for all $\lambda \geq (j+k)^{-1}$.

Proof We show that $G_{2jk\lambda}(P^{2j} + Q^{2k}) \subseteq \mathcal{X}^\lambda(U)$. Most of the proof is the same as the argument used to establish Theorem 6.1, so we describe the differences. In fact, in the present situation one does not need to translate the problem to the Lipschitz spaces since a unitary representation has the optimal regularity property, Theorem 5.8.II.

Since $[A_1, A_2] = iI$ and a_3 is central in the Lie algebra one obtains

$$A^\alpha [A_i, A_l] A^\beta = c_{il} A^\gamma$$

for all $\alpha, \beta \in J(d)$ and $i, l \in \{1, \ldots, d\}$ with $|c_{il}| \leq 1$ and $\|\gamma\| = \|\alpha\| + \|\beta\|$. This implies that one actually has $\|\beta_{\alpha,n}\| \leq \|\alpha\| - j - k$ in (13) instead of merely $\|\beta_{\alpha,n}\| \leq \|\alpha\| - 1$. As a result one obtains the recurrence relation

$$c_{p,q} \leq M d_{p-m,q} + N_2 M_1 c_{p-j-k,q}$$

instead of (16). So by iteration one obtains that

$$c_{p,q} \le M \sum_{n=0}^{N} (N_2 M_1)^n d_{p-m-n(j+k),q} + (N_2 M_1)^{N+1} c_{N_3,q}$$

where $N = [(p - N_3)/(j+k)]$ for all $p \ge N_3$, we now assume, in addition, that $N_3 \ge j + k + m$ and where $[s]$ denotes the integer part of a real number s. Since $[H, A^\alpha]$ now can be written as a sum of A^β with $\|\beta\| \le \|\alpha\| + m - j - k$, instead of the condition $\|\beta\| \le \|\alpha\| + m - 1$, one has the recurrence relation

$$d_{p,q} \le c_{p,q+1} + pdM_1 N_4 N_5 c_{p+m-j-k,q}$$

instead of (18). Using the induction hypothesis one deduces that

$$\begin{aligned} d_{p,q} \le\, & M \sum_{n=0}^{N} e^{N_2 M_1} n! a_1 b_1^{p-m-n(j+k)} c_1^{q+1} (p + mq - n(j+k))!^\lambda \\ & + e^{N_2 M_1} (N+1)! a_1 c_1^{N_3} (mq + m)!^\lambda \\ & + pdM_1 M N_4 N_5 \sum_{n=0}^{N'} e^{N_2 M_1} n! a_1 b_1^{p-n(j+k)-j-k} c_1^q (p + mq - n(j+k) - (j+k))!^\lambda \\ & + pdM_1 N_4 N_5 e^{N_2 M_1} (N'+1)! a_1 c_1^q (mq)!^\lambda \end{aligned}$$

with $N' = [(p + m - (j+k) - N_3)/(j+k)]$. Now consider the factorials in the four terms. In the first term one has

$$\begin{aligned} n!(p + mq - n(j+k))!^\lambda &= (n!^{j+k})^{1/(j+k)} (p + mq - n(j+k))!^\lambda \\ &\le (n(j+k))!^{1/(j+k)} (p + mq - n(j+k))!^\lambda \\ &\le \Big((n(j+k))! (p + mq - n(j+k))! \Big)^\lambda \le (p + mq)!^\lambda \quad , \end{aligned}$$

where we have used that $\lambda \ge (j+k)^{-1}$. Similarly,

$$\begin{aligned} (N+1)!(mq+m)!^\lambda &\le ((j+k)(N+1) + mq + m)!^\lambda \\ &\le (p + mq - (N_3 - m - j - k))!^\lambda \le (p+mq)!^\lambda \quad . \end{aligned}$$

Here we have again used $N_3 \ge m + j + k$. The factorials together with the p in the third term can be estimated as follows;

$$\begin{aligned} & pn!(p + mq - n(j+k) - (j+k))!^\lambda \\ & \qquad \le \Big(p^{j+k} (p + mq - (j+k))! \Big)^\lambda \\ & \qquad \le \Big(p(p-1)\ldots(p-j-k+1) 2^{j+k} (p + mq - (j+k))! \Big)^\lambda \\ & \qquad \le 2^{(j+k)\lambda} (p+mq)!^\lambda \quad , \end{aligned}$$

since $p > N_3 \geq j + k + m \geq 2(j + k)$. Finally,

$$\begin{aligned} p(N'+1)!(mq)!^\lambda &\leq (p^{j+k}\big(p + mq - (j+k) - (N_3 - j - k - m)\big)!)^\lambda \\ &\leq (p(p-1)\dots(p-j-k+1)2^{j+k}\big(p+mq-(j+k)\big)!)^\lambda \\ &\leq 2^{(j+k)\lambda}(p+mq)!^\lambda \quad . \end{aligned}$$

Then one deduces as before that $d_{p,q} \leq a_1 b_1^p c_1^q (p+mq)!^\lambda$ if b_1 is large enough. □

Corollary 7.3 *If $j, k \in \mathbf{N}$ then the eigenfunctions $\varphi_1, \varphi_2, \dots$ of the operator $P^{2j} + Q^{2k}$ belong to the Gel'fand-Shilov space $S_{j/(j+k)}^{k/(j+k)}$. In particular the eigenfunctions φ_n can be extended to entire functions into the complex plane and they satisfy the growth bound*

$$|\varphi_n(x+iy)| \leq c e^{-a|x|^{(j+k)/j} + b|y|^{(j+k)/j}}$$

for all $x, y \in \mathbf{R}$, for some constants $a, b, c > 0$, depending on n, or, equivalently,

$$|\varphi_n(z)| \leq c e^{b|z|^{(j+k)/j}} \quad , \quad |\varphi_n(x)| \leq c e^{-a|x|^{(j+k)/j}}$$

for all $z \in \mathbf{C}$ and $x \in \mathbf{R}$.

Proof The Gel'fand-Shilov space $S_{j/(j+k)}^{k/(j+k)}$ consists precisely of all functions with these analyticity properties and growth bounds, see [GeS] Section IV.2.3 and pp. 220–221. □

This corollary fully answers positively a question in [Goo3], p. 468: each eigenfunction is an entire vector (see [Goo2]) for the group G_d in [Goo3]. The next corollary answers several questions raised by Gundersen [Gun78], p. 311, in case $j = 1$ and n a natural number.

Corollary 7.4 *For $n > 0$ let P be the semigroup generated by the operator $H^{1/n}$. Then the range $\mathcal{R} = \bigcup_{t>0} P_t(L_2(\mathbf{R}))$ of P equals $S_{j/(j+k)}^{k/(j+k)}$ if $n \leq 2jk/(j+k)$ and it equals $S_{n/(2k)}^{n/(2j)}$ if $n \geq 2jk/(j+k)$. In particular, if $n < 2j$ then any element of $\mathcal{R}$ can be extended to an entire function in the complex plane and if $n = 2j$ then any function in $\mathcal{R}$ can be extended analytically to a horizontal strip $\{z \in \mathbf{C} : |\operatorname{Im} z| \leq T\}$.*

Proof The range $\mathcal{R}$ of P is equal to the space $G_1(H^{1/n})$ of analytic vectors for $H^{1/n}$ (see [Rob2] Lemma II.2.1). But $G_1(H^{1/n}) = G_n(H)$. So if $n \leq 2jk/(j+k)$ then

$$\mathcal{R} = G_n(H) \subseteq G_{2jk/(j+k)}(H) = S_{j/(j+k)}^{k/(j+k)}$$

by Theorem 7.2. If n is large one can apply immediately Theorem 7.2.

The statements about analytic continuation into the complex plane follow from [GeS] Section IV.2. □

The next corollary generalizes [Goo3] Theorem 6.1(iii)

Corollary 7.5 *Let $\varphi_1, \varphi_2, \dots$ be an orthonormal basis of eigenfunctions of the operator $P^{2j} + Q^{2k}$ with eigenvalues $\mu_1 \leq \mu_2 \leq \dots$, respectively, and let $\lambda \geq (j+k)^{-1}$. Then a function $\varphi \in L_2(\mathbf{R})$ belongs to $S_{j\lambda}^{k\lambda}$ if, and only if, there exists $t > 0$ such that*

$$|(\varphi_n, \varphi)| = O(e^{-t\mu_n^{1/(2jk\lambda)}}) \quad .$$

Proof This follows from Corollary 7.4, the semigroup property of the semigroup P in that corollary and the spectral theorem. □

It is also possible to estimate the eigenvalues μ_n. If $j = k = 1$ then $\mu_n = n$. But if $j = k > 1$ then for each $\varepsilon > 0$ there is a $\lambda_\varepsilon \geq 0$ such that

$$(P^{2j} + Q^{2j}) \geq (2^{-(j-1)} - \varepsilon)(P^2 + Q^2)^j - \lambda_\varepsilon I$$

and a $\lambda'_\varepsilon \geq 0$ such that

$$(P^2 + Q^2)^j \geq (1 - \varepsilon)(P^{2j} + Q^{2j}) - \lambda'_\varepsilon I$$

by the Gårding inequality. Therefore, by the minimax theorem,

$$(1 - \varepsilon)^{-1} n^j + \lambda'_\varepsilon \geq \mu_n \geq (2^{-(j-1)} - \varepsilon) n^j - \lambda_\varepsilon$$

which establishes that μ_n grows asymptotically like n^j. Similarly, if j is an integer multiple of k, or vice-versa, one may use the estimates of [Tit], page 144, to deduce that μ_n grows like $n^{2jk/(j+k)}$. It is expected that this conclusion is valid for all $j, k \in \mathbf{N}$.

Remark Theorem 7.2 and its corollaries remain valid if the operator $P^{2j} + Q^{2k}$ is replaced by any other weighted strongly elliptic operator of order $2jk$.

Now we present an example which shows that a condition on the structure constants is necessary, e.g., a theorem like Theorem 5.8 is wrong if one replaces the condition

$$[a_i, a_j] = \sum_{\substack{k \in \{1,\ldots d\}, \\ w_k \leq w_i + w_j - 1}} c_{ij}^k a_k$$

by

$$[a_i, a_j] = \sum_{\substack{k \in \{1,\ldots d\}, \\ w_k \leq w_i + w_j}} c_{ij}^k a_k \quad .$$

Example 7.6 Consider the simply connected Heisenberg group G as in Example 7.1 but with the weights $w_1 = w_2 = 1$ and $w_3 = 2$. Then G is stratified. Let $\lambda \in \mathbf{R}$ be in the spectrum of the operator $P^4 + Q^4 + I$. Then the operator

$$H = dV(C) = A_1^4 + A_2^4 - A_3^2 - \lambda(-i[A_1, A_2])^2 \quad ,$$

is a weighted strongly elliptic operator. It is not hypoelliptic, however, if V is the left regular representation in $L_2(G)$ by the Helffer–Nourigat theorem, [Hel] Theorem 2.1, since the operator $dU(C)$, with U the irreducible unitary representation as above, is not injective. Therefore the norms $\varphi \mapsto \|dV(C)\varphi\|$ and $\varphi \mapsto N_4(\varphi)$ are not equivalent on $C_c^\infty(G)$ by the same theorem of Helffer and Nourigat. Therefore the operator $dV(C)$ cannot satisfy Theorem 5.8.II, by a scaling argument.

Next we consider in more detail the non-simply connected Heisenberg group $G = A(\mathbf{R})$ and show that the space $\mathcal{X}_n$ is not generally invariant under the representation, even if the representation is unitary and n is an even multiple of w.

Example 7.7 Let $G = \mathbf{R} \times \mathbf{R} \times \mathbf{T}$ with multiplication given by

$$(a_1, b_2, z_1) \circ (a_2, b_2, z_2) = (a_1 + a_2, b_1 + b_2, z_1 z_2 e^{ia_1 b_2}) \quad .$$

Next let

$$\mathcal{X} = \bigoplus_{n=1}^{\infty} \mathcal{Y}_n$$

where $\mathcal{Y}_n = L_2(\mathbf{R}\,; dx)$. Then for each $(a, b, z) \in G$ and $f = (f_n)_{n \in \mathbf{N}} \in \mathcal{X}$ define $U(a, b, z)f \in \mathcal{X}$ by

$$(U(a, b, z)f)_n(x) = z^n e^{ibnx} f_n(a + x) \quad .$$

It follows that $(\mathcal{X}, G, U)$ is a strongly continuous unitary representation. Let a_1, a_2, a_3 be the standard basis in the Lie algebra $\mathfrak{g}$ of G. Then $[a_1, a_2] = a_3$. We take the weights $w_1 = w_2 = 2$ and $w_3 = 3$. Then $w = 6$.

If P, and Q, are the self-adjoint operators in $L_2(\mathbf{R}\,; dx)$ such that

$$(Pf)(x) = if'(x) \quad ,$$
$$(Qf)(x) = ixf(x) \quad ,$$

for all $f \in C_c^\infty(\mathbf{R})$ and $x \in \mathbf{R}$ then

$$(A_1 f)_n = -iPf_n \quad ,$$
$$(A_2 f)_n = inQf_n \quad ,$$
$$(A_3 f)_n = inf_n \quad ,$$

where $A_i = dU(a_i)$. Next consider the weighted strongly elliptic operator

$$H = -A_1^6 - A_2^6 + A_3^4$$

and remark that

$$(Hf)_n = (P^6 + n^6 Q^6 + n^4 I) f_n$$

for all $f \in D(H)$.

By Example 7.1, the operator $P^6 + Q^6 + I$ has a discrete spectrum, so let $\psi \in L_2(\mathbf{R}\,; dx)$ be an eigenvector of $P^6 + Q^6 + I$ with eigenvalue α. For all $n \in \mathbf{N}$ define $\varphi_n \in L_2(\mathbf{R}\,; dx)$ by setting

$$\varphi_n(x) = n^{1/4} \psi(n^{1/2} x) \quad .$$

Then $\|\varphi_n\| = 1$ and

$$(P^6 + n^6 Q^6 + n^4 I)\varphi_n = \Big(n^3(\alpha - 1) + n^4\Big)\varphi_n$$

for all $n \in \mathbf{N}$. Next let $(\lambda_n)_{n \in \mathbf{N}} \in l^2(\mathbf{N})$ and consider $f = (f_n)_{n \in \mathbf{N}} \in \mathcal{X}$ where $f_n = \lambda_n \varphi_n$. It follows from the above that $f \in \mathcal{X}_{12}$ if, and only if, $f \in D(H)$, i.e., if, and only if,

$$\sum_{n=1}^{\infty} |n^4 \lambda_n|^2 < \infty \quad .$$

But if $t > 0$ and $g = \exp(ta_1)$ then $(U(g)f)_n(x) = f_n(x + t)$ and

$$(A_2^6 U(g) f)_n(x) = -n^6 x^6 f_n(x + t) = -n^6 x^6 \lambda_n n^{1/4} \psi(n^{1/2}(x + t)) \quad .$$

Moreover

$$\int_{-\infty}^{\infty} dx\, |-n^6x^6\lambda_n n^{1/4}\psi(n^{1/2}(x+t))|^2 = \int_{-\infty}^{\infty} dx\, |n^6(n^{-1/2}x-t)\lambda_n\psi(x)|^2$$
$$= \int_{-\infty}^{\infty} dx\, |(n^6t^6+\ldots+n^3x^6)^6\lambda_n\psi(x)|^2 \quad .$$

Hence $U(g)f \in D(A_2^6)$ if, and only if,

$$\sum_{n=1}^{\infty} |n^6\lambda_n|^2 < \infty \quad .$$

Thus by appropriate choice of $\lambda_1, \lambda_2, \ldots$ one can arrange that $f \in \mathcal{X}_{12}$ but $U(g)f \notin \mathcal{X}_{12}$. Consequently $\mathcal{X}_{2w}$ is not U-invariant.

Finally we give an example which establishes that the inclusion (12) is not generally valid.

Example 7.8 Let $G = SU(2)$ and choose a basis a_1, a_2, a_3 of $\mathfrak{g}$ such that $[a_1, a_2] = a_3$, $[a_2, a_3] = a_1$ and $[a_3, a_1] = a_2$. We take the weights $w_1 = w_2 = 2$ and $w_3 = 3$. Further define

$$L_+ = a_1 - ia_2 \quad , \quad L_- = a_1 + ia_2$$

in the complex enveloping algebra of $\mathfrak{g}$. For every $l \in \{0, \frac{1}{2}, 1, \ldots\}$ there exists a unique irreducible unitary representation π_l of G in a Hilbert space $\mathcal{X}_l$ of dimension $2l+1$. Let $e_{l,-l}, e_{l,-l+1}, \ldots, e_{l,l}$ be an orthonormal basis for $\mathcal{X}_l$ such that

$$\begin{aligned} d\pi_l(L_+)e_{l,m} &= \sqrt{l(l+1) - m(m+1)}\, e_{l,m+1}, \\ d\pi_l(L_-)e_{l,m} &= \sqrt{l(l+1) - m(m-1)}\, e_{l,m-1}, \\ d\pi_l(a_3)e_{l,m} &= im\, e_{l,m} \end{aligned}$$

for all $m \in \{-l, \ldots, l\}$. Now let

$$\mathcal{X} = \bigoplus_{l \in \{0, \frac{1}{2}, \ldots\}} \mathcal{X}_l$$

and

$$U = \bigoplus_{l \in \{0, \frac{1}{2}, \ldots\}} \pi_l \quad .$$

Then U is a unitary representation of G in $\mathcal{X}$. Next let $A_i = dU(a_i)$ and introduce the weighted strongly elliptic operator

$$H = (-A_1^2 - A_2^2)^3 + A_3^4$$

of order $m = 12$. Then H is self-adjoint and it follows from Theorem 6.1 that

$$G_{12\lambda}(\overline{H}) = \mathcal{X}^\lambda(U)$$

for all $\lambda \geq 1$. Moreover, since all weights are at least 2 it follows from Nelson's theorem that the space $\mathcal{X}^\lambda(U)$ is dense in $\mathcal{X}$ for all $\lambda \geq 1/2$. Obviously $G_{12\lambda}(\overline{H})$ is dense in $\mathcal{X}$ and

$\mathcal{X}^\lambda(U) \subseteq G_{12\lambda}(\overline{H})$ for all $\lambda > 0$. Now for each $\lambda < 1$ we construct an $x \in G_{12\lambda}(\overline{H})$ such that $x \notin \mathcal{X}^\lambda(U)$. This shows that

$$G_{m\lambda}(\overline{H}) \nsubseteq \mathcal{X}^\lambda(U)$$

for all $\lambda < 1$.

Let $\lambda < 1$. For all $t \in \mathbf{R}$ let $[t]$ denote the integer part of t, i.e., the largest integer which is less than or equal to t. Let

$$x = \sum_{N=0}^{\infty} e^{-N} e_{[N^{3\lambda}],[N^{3\lambda}]} \quad .$$

It follows from

$$He_{l,m} = \Big((l(l+1) - m^2)^3 + m^4\Big) e_{l,m}$$

that

$$\|H^n x\|^2 = \sum_{N=0}^{\infty} e^{-2N}([N^{3\lambda}]^3 + [N^{3\lambda}]^4)^{2n} \leq 2^{2n} \sum_{N=0}^{\infty} e^{-2N} N^{24n\lambda} \leq 2^{2n}(24n)!^\lambda \sum_{N=0}^{\infty} e^{-(2-\lambda)N}$$

$$\leq (12n)!^{2\lambda} 2^{24n\lambda+2n} \sum_{N=0}^{\infty} e^{-(2-\lambda)N}$$

for all $n \in \mathbf{N}_0$. So $x \in G_{12\lambda}(\overline{H})$.

Now suppose that $x \in \mathcal{X}^\lambda(U)$. Then there exist $c, t > 0$ such that

$$\|x\|_n \leq ct^n n!^\lambda$$

for all $n \in \mathbf{N}_0$. We have to distinguish two cases.

Case 1: $\lambda > 1/3$. Let $k \in \mathbf{N}$ be such that $(9\lambda + 3)k > (12k + 2)\lambda$.

The operator H can be written as a sum of 9 monomials A^α with $\|\alpha\| = 12$. So by an elementary counting argument one obtains that

$$\|H^{kN} dU(L_-)^N x\| \leq 9^{kN} 2^N \|x\|_{(12k+2)N} \leq c 9^{kN} 2^N t^{(12k+2)N}((12k+2)N)!^\lambda$$

$$\leq c\Big(2 \cdot 9^k (12k+2)^{(12k+2)\lambda} t^{12k+2}\Big)^N (N^N)^{(12k+2)\lambda}$$

for all $N \in \mathbf{N}$. However,

$$\|H^{kN} dU(L_-)^N x\| \geq e^{-N} \|H^{kN} dU(L_-)^N e_{[N^{3\lambda}],[N^{3\lambda}]}\|$$

and

$$dU(L_-)^N e_{[N^{3\lambda}],[N^{3\lambda}]} = \beta_N \, e_{[N^{3\lambda}],[N^{3\lambda}]-N}$$

for some $\beta_N \in \mathbf{R}$, with $\beta_N \geq 1$. Moreover,

$$He_{[N^{3\lambda}],[N^{3\lambda}]-N} = \Big(([N^{3\lambda}] + 2[N^{3\lambda}]N - N^2)^3 + ([N^{3\lambda}] - N)^4\Big) e_{[N^{3\lambda}],[N^{3\lambda}]-N} \quad .$$

So

$$\|H^{kN} dU(L_-)^N x\| \geq e^{-N}\Big((N^{3\lambda+1})^3\Big)^{kN} = e^{-N}(N^N)^{(9\lambda+3)k} \quad .$$

But

$$e^{-N}(N^N)^{(9\lambda+3)k} > c\Big(2 \cdot 9^k (12k+2)^{(12k+2)\lambda} t^{12k+2}\Big)^N (N^N)^{(12k+2)\lambda}$$

for large N and so by contradiction one concludes that $x \notin \mathcal{X}^\lambda(U)$.

Case 2: $\lambda \leq 1/3$. This case can be treated as before if one uses $H^N dU(L_-)^{[N^{3\lambda}]}$ instead of $H^{kN} dU(L_-)^N$.

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