A theory of generalized functions based on holomorphic semi-groups

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DEPARTMENT OF MATHEMATICS

Memorandum 83-08

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A THEORY OF GENERALIZED FUNCTIONS BASED ON HOLOMORPHIC SEMI-GROUPS

Part C: Linear mappings, tensor products and kernel theorems.

CHAPTER 4. Characterization of continuous linear mappings between the spaces $S_{X,A}$, $T_{X,A}$, $S_{Y,B}$ and $T_{Y,B}$

Let B be a nonnegative self-adjoint operator in a separable Hilbert space Y. As before A is a nonnegative self-adjoint operator in a separable Hilbert space X. In this chapter we shall derive conditions implying the continuity of linear mappings $S_{X,A} \to S_{Y,B}$, $S_{X,A} \to T_{Y,B}$, $T_{X,A} \to T_{Y,B}$, $T_{X,A} \to S_{Y,B}$. Further we investigate which linear operators, defined on a subset of X, can be continously extended to operators on $T_{X,A}$. The next theorem is an immediate consequence of the fact that $S_{X,A}$ is bornological. Cf. Theorem 1.11. For completeness we give an ad hoc proof.

Theorem 4.1. A linear map $Q : S_{X,A} \to V$, where V is an arbitrary locally convex topological vector space, is continuous

I. iff for each $t > 0$ the map $Qe^{-tA} : X \to V$ is continuous;

II. iff for each null sequence $\{u_n\} \subset S_{X,A}$, $u_n \to 0$ in $S_{X,A}$, the sequence $\{Qu_n\}$ is a null sequence in V.

Proof

I. $\Rightarrow$) e^{-tA} is an isomorphism from X to X_t. By the definition of the inductive limit, X_t is continuously injected in $S_{X,A}$. So if Q is continuous it follows that Qe^{-tA} is continuous.

$\Leftarrow$) Let Q_t denote the restriction of Q to X_t. From the continuity of Qe^{-tA} on X follows the continuity of Q_t on X_t. Now let $\Omega \ni 0$ be open in V. For each $t > 0$, $Q^{-1}(\Omega) \cap X_t = Q_t^{-1}(\Omega)$ is an open 0-neighbourhood in X_t. Thus $Q^{-1}(\Omega)$ is open in $S_{X,A}$.

II. Follows from I because null sequences in S are always null sequences in some X_t, $t > 0$, and vice versa. □

In the next theorem we characterize continuous linear mappings from $S_{X,A}$ to $S_{Y,B}$.

Theorem 4.2. Suppose $\mathcal{P} : S_{X,A} \to S_{Y,\mathcal{B}}$ is a linear mapping. The following seven conditions are equivalent.

I. $\mathcal{P}$ is continuous with respect to the strong topologies of $S_{X,A}$ and $S_{Y,\mathcal{B}}$.

II. $u_n \to 0$, strongly in $S_{X,A}$, implies $\mathcal{P}u_n \to 0$, strongly in $S_{Y,\mathcal{B}}$.

III. For each $\alpha > 0$ the operator $\mathcal{P}e^{-\alpha A}$ is a bounded linear operator from X into Y.

IV. For each $\alpha > 0$ and each $\psi \in \mathcal{B}$ the operator $\psi(\mathcal{B})\mathcal{P}e^{-\alpha A}$ is a bounded linear operator from X into Y.

V. For each $t > 0$ there exists $\beta > 0$ such that $\mathcal{P}e^{-tA}(X) \subset e^{-\beta\mathcal{B}}(Y)$ and $e^{\beta\mathcal{B}}\,\mathcal{P}e^{-tA}$ is a bounded linear operator from X into Y.

VI. There exists a dense linear subspace $\Xi \subset Y$ such that for each fixed $y \in \Xi$ the linear functional $L_{\mathcal{P},y}(f) = (\mathcal{P}f,y)_Y$ is continuous on $S_{X,A}$.

VII. For each $t > 0$ $(\mathcal{P}e^{-tA})^*$ is a bounded linear operator from Y to X. (Remark: One has $(\mathcal{P}e^{-tA})^*(Y) \subset S_{X,A}$).

Proof.

I $\Leftrightarrow$ II. See Theorem 4.1.

II $\Rightarrow$ III. If $\{x_n\}$ is a null sequence in X then for any $\alpha > 0$ $\{e^{-\alpha A}x_n\}$ is a null sequence in $S_{X,A}$. By II $\{\mathcal{P}e^{-tA}x_n\}$ is a null sequence in $S_{Y,\mathcal{B}}$ and hence in Y.

III $\Rightarrow$ IV. The operator $\psi(\mathcal{B})\mathcal{P}e^{-\alpha A}$ is closed and defined on the whole of X. Therefore it is bounded.

IV $\Rightarrow$ V. Let U denote the unit ball in X. Because of IV the set $\mathcal{P}e^{-\alpha A}(U)$ is bounded in $S_{Y,\mathcal{B}}$. Now apply Theorem 1.6.

III $\Rightarrow$ VI. We can take $\Xi = Y$. Take $y \in Y$ fixed. For each $x \in X$ and each $t > 0$ we have

$$|L_{\mathcal{P},y}(e^{-tA}x)| = |(\mathcal{P}e^{-tA}x,y)_Y| \le c_{t,y}\|x\|_X .$$

Together with Theorem 4.1 the result follows.

VI $\Rightarrow$ VII. According to Riess' theorem for each $y \in \Xi$ and each $t > 0$ there exists $f_t \in X$ such that for each $h \in X$

$$(*) \qquad L_{P,y}(e^{-tA}h) = (Pe^{-tA}h,y)_Y = (h,f_t)_X \,.$$

Replacing h by $e^{-\tau A}x$, $x \in X$, we observe that $f_{t+\tau} = e^{-\tau A}f_t$ so that $f_t \in S_{X,A}$. From (*) we obtain $f_t = (Pe^{-tA})^*y$. So $D((Pe^{-tA})^*) \supset \Xi$ which is dense in Y. Since Pe^{-tA} is defined on the whole X the operator $(Pe^{-tA})^*$ is defined on the whole Y and bounded. Repeating the argument with arbitrary $y \in X$ shows $(Pe^{-tA})^*y \in S_{X,A}$.

VII $\Rightarrow$ III. Pe^{-tA} is bounded because $(Pe^{-tA})^*$ is bounded.

V $\Rightarrow$ II. Trivial. □

The next corollary is important for applications.

Corollary 4.3. Suppose Q is a densely defined closable operator: $X \to Y$. If $D(Q) \supset S_{X,A}$ and $Q(S_{X,A}) \subset S_{Y,B}$, then Q maps $S_{X,A}$ continuously into $S_{Y,B}$.

Proof. Q is closable iff $D(Q^*)$ is dense in Y. Since $Qe^{-tA} : X \to Y$ is defined on the whole X, its adjoint $(Qe^{-tA})^*$ is bounded. The adjoint, however, is densely defined since on $D(Q^*)$ one has $(Qe^{-tA})^* = e^{-tA}\,Q^*$. Hence $(Qe^{-tA})^*$ is defined on the whole Y and bounded. From this the boundedness of Qe^{-tA} follows. Application of Theorem 4.2 III yields the desired result. □

Theorem 4.4. Let $K : S_{X,A} \to T_{Y,B}$ be a linear mapping. The following three conditions are equivalent.

I. K is continuous with respect to the strong topologies of $S_{X,A}$ and $T_{Y,B}$.

II. For each $t > 0$, $\alpha > 0$, $e^{-tB}\,K e^{-\alpha A}$ is a bounded linear operator from X into Y.

III. For each $t > 0$, $e^{-tB}K$ is a continuous map from $S_{X,A}$ into $S_{Y,B}$.

Proof.

I $\Rightarrow$ II. Let $\{x_n\}$ be a null-sequence in X. Then $\{e^{-\alpha A}x_n\}$ is a null sequence in $S_{X,A}$. Since K is continuous, $\{Ke^{-\alpha A}x_n\}$ is a null sequence in $T_{Y,B}$, which means that for every $t > 0$ $\{(e^{-t\mathcal{B}} Ke^{-\alpha A})x_n\}$ is a null sequence in $\mathcal{Y}$. Hence $e^{-y\mathcal{B}} Ke^{-\alpha A}$ is bounded.

II ⇒ I. Let $\{u_n\}$ be a null-sequence in $S_{X.A}$. Then for some $\alpha > 0$ $\{e^{\alpha A}u_n\}$ is defined and is a null-sequence in X. But then $\{Ku_n\} = \{Ke^{-\alpha A} e^{\alpha A}u_n\}$ is a null-sequence in $T_{Y,\mathcal{B}}$ since for each $t > 0$

$$(Ke^{-\alpha A} e^{\alpha A}u_n)(t) = (e^{-t\mathcal{B}} Ke^{-\alpha A})e^{\alpha A}u_n \to 0 \text{ in } \mathcal{Y}.$$

II ⇔ III. Apply Theorem 4.2 with $P = e^{-t\mathcal{B}}K$. □

Theorem 4.5. Let $\Gamma : T_{X,A} \to S_{Y,\mathcal{B}}$ be a linear mapping. Let $\Gamma_r : X \to \mathcal{Y}$ denote the restriction of Γ to X. The following five conditions are equivalent.

I. Γ is continuous with respect to the strong topologies of $T_{X,A}$ and $S_{Y,\mathcal{B}}$.

II. $\Gamma_r^* : \mathcal{Y} \to X$ is a bounded operator and $\Gamma_r^*(\mathcal{Y}) \subset S_{X,A}$.

III. There exists $t > 0$ such that $\Gamma_r^*(X) \subset e^{-tA}(X)$ and $e^{tA}\Gamma_r^*$ is a bounded operator from $\mathcal{Y}$ into X.

IV. There exists $t > 0$ such that $\Gamma_r e^{tA}$ with domain $X_t \subset X$ is bounded as an operator from X into $\mathcal{Y}$.

V. There exists $t > 0$ and a continuous linear map $\mathcal{Q} : S_{X,A} \to S_{Y,\mathcal{B}}$ such that $\Gamma = \mathcal{Q}e^{-tA}$.

Proof.

I ⇒ II. From the continuity of Γ it immediately follows that Γ_r is continuous. Its adjoint Γ_r^*, defined by $(x,\Gamma_r^*y)_X = (\Gamma_r x,y)_Y$ for all $x \in X$, $y \in \mathcal{Y}$, is a bounded operator as well. Now consider the dual operator $\Gamma' : T_{Y,\mathcal{B}} \to S_{X,A}$ defined by

$$\langle F,\Gamma' G\rangle_X = \langle \Gamma F,G\rangle_Y \quad \text{for all } F \in T_{X,A} .$$

For fixed $G \in T_{Y,\mathcal{B}}$ the right-hand side defines a continuous linear functional on $T_{X,A}$. Γ_r^* is a restriction of Γ' and therefore maps $\mathcal{Y}$ into the set $S_{X,A} \subset X$.

II ⇒ III. For each $\psi \in \mathcal{B}$ the operator $\psi(A)\Gamma_r^* : \mathcal{Y} \to X$ is bounded and $\psi(A)\Gamma_r^*(\mathcal{Y}) \subset S_{X,A}$. Similar to the method employed in Theorem 4.2 the result follows.

III ⇒ IV. $e^{\beta A}T_r^*$ has a bounded adjoint $(e^{\beta A}T_r^*)^*$ but $(e^{\beta A}T_r^*)^* \supset \Gamma_r e^{\beta A}$ with domain $X_\beta \subset X$.

IV ⇒ V. Take $\mathcal{Q} = (e^{\beta A}T_r^*)^*$. For each $\alpha > 0$ $\mathcal{Q}e^{-\alpha A}$ is the extension of $\Gamma_r e^{(\beta-\alpha)A}$ which is bounded. So $\mathcal{Q}$ satisfies condition III of Theorem 4.2. We have $\Gamma = \mathcal{Q}e^{-\beta A}$.

V ⇒ I. If $\Gamma = \mathcal{Q}e^{-tA}$ the mapping Γ is obviously continuous. □

Theorem 4.6. Let $\Phi : T_{X,A} \to T_{Y,B}$ be a linear mapping. Let $\Phi_r : X \to T_{Y,B}$ denote the restriction of Φ to X.

The following six conditions are equivalent.

I. Φ is continuous with respect to the strong topologies of $T_{X,A}$ and $T_{Y,B}$.

II. For each $g \in S_{Y,B}$ the expression $\overline{\langle g, \Phi F\rangle_Y}$, $F \in T_{X,A}$, is a continuous linear functional on $T_{X,A}$.

III. For each $t > 0$ the operator $e^{-tB}\Phi$ is a continuous map from $T_{X,A}$ into $S_{Y,B}$.

IV. For each $t > 0$ $(e^{-tB}\Phi_r)^*(Y) \subset S_{X,A}$.

V. For each $t > 0$ there exists $\beta > 0$ such that $(e^{-tB}\Phi_r)^*(Y) \subset e^{-\beta A}(X)$ and $e^{\beta A}(e^{-tB}\Phi_r)^*$ is a bounded operator from Y into X.

VI. For each $t > 0$ there exists $\beta > 0$ such that $e^{-tB}\Phi_r e^{\beta A} = e^{-tB}\Phi e^{\beta A}$ on the domain $X_\beta \subset X$ is bounded as an operator from X into Y.

Proof.

I ⇒ II. Trivial.

I ⇒ III. Trivial, because $e^{-tB} : T_{Y,B} \to S_{Y,B}$ is continuous.

III ⇒ IV. Theorem 4.5, condition II.

IV ⇒ V. Apply Theorem 4.5 to $e^{-tB}\Phi_r$.

V ⇒ VI. The adjoint of the bounded operator $e^{\beta A}(e^{-tB}\Phi_r)^*$ is an extension of $(e^{-tB}\Phi_r)e^{\beta A}$. Therefore the latter is bounded.

VI ⇒ I. Let $\{F_n\}$ be a null-sequence in $T_{X,A}$. Then for each $t > 0$ there exists $\beta > 0$ such that we can write $(\Phi F_n)(t) = e^{-tB}\Phi_r e^{\beta A}F_n(\beta)$. This converges to zero as $n \to \infty$ because of the boundedness of $e^{-tB}\Phi_r e^{\beta A}$.

II ⇒ V. $\overline{\langle g,\Phi F\rangle_Y}$ has the representation $\overline{\langle f,F\rangle_X}$, see Theorem 3.2.IV, here $f = \Phi' g \in S_{X,A}$. Taking $F = u \in S_{X,A}$ we observe that $(u,\Phi' g)_X = \langle u,\Phi' g\rangle_X = \langle \Phi u,g\rangle_Y$ is, as a function of g, a continuous linear functional on $S_{Y,B}$. Then with Theorem 4.2.VI it follows that Φ' maps $S_{Y,B}$ continuously into $S_{X,A}$. Then, by Theorem 4.2.V for each $t > 0$ there exists $\beta > 0$ such that $e^{\beta A}\Phi' e^{-tB}$ is a bounded operator. But $e^{\beta A}\Phi' e^{-tB} = e^{\beta A}(e^{-tB}\Phi_r)^*$ because for all $x \in X$, $y \in Y$

$$(\Phi' e^{-tB} y,x)_X = \langle e^{-tB} y,\phi_r x\rangle_Y = (y,e^{-tB}\Phi_r x)_Y = ((e^{-tB}\Phi_r)\ y,x)_X \ . \qquad \square$$

Theorem 4.7. Let the linear mappings

$$P : S_{X,A} \to S_{Y,B} \qquad \Phi : T_{X,A} \to T_{Y,B}$$
$$\Gamma : T_{X,A} \to S_{Y,B} \qquad K : S_{X,A} \to T_{Y,B}$$

be continuous with respect to the strong topologies on the mentioned spaces. Then the dual linear mappings

$$P' : T_{Y,B} \to T_{X,A} \qquad \Phi' : S_{Y,B} \to S_{X,A}$$
$$\Gamma' : T_{Y,B} \to S_{X,A} \qquad K' : S_{Y,B} \to T_{X,A}$$

are also continuous with respect to the strong topologies.

Proof.

Compare Theorem 4.2.V with Theorem 4.6.VI.

Compare Theorem 4.5.III with Theorem 4.5.IV.

Look at Theorem 4.4.II. $\square$

The interesting question arises which densely defined (possibly unbounded) operators from X into Y can be extended to a continuous mapping from $T_{X,A}$ into $T_{Y,B}$.

Theorem 4.8. Let E be a linear map $X \supset D(E) \to Y$ with $\overline{D(E)} = X$. E can be extended to a continuous linear map $\overline{E} : T_{X,A} \to T_{Y,B}$ iff E has a densely defined adjoint $E^* : Y \supset D(E^*) \supset S_{Y,B} \to X$ with $E^*(S_{Y,B}) \subset S_{X,A}$.

Proof.

$\Rightarrow$) If $\bar{E}$ exists as a continuous map, its dual operator $\bar{E}'$ maps $S_{Y,B}$ into $S_{X,A}$. For each $x \in D(\bar{E})$ and $g \in S_{Y,B}$ one has $\langle g, Ex\rangle_Y = (g, Ex)_Y = \langle \bar{E}'g, x\rangle_X = (\bar{E}'g, x)_X$. It follows that $E^* \supset \bar{E}'$ and $E^*(S_{Y,B}) \subset S_{X,A}$.

$\Leftarrow$) From Corollary 4.3 it follows that E^* maps $S_{Y,B}$ continuously into $S_{X,A}$. Then by Theorem 4.7 the dual $(E^*)'$ maps $T_{X,A}$ continuously into $T_{Y,B}$. However, $(E^*)'$ is an extension of E. $\square$

Corollary 4.9. A continuous linear map $Q : S_{X,A} \to S_{Y,B}$ can be extended to a continuous linear map $\bar{Q} : T_{X,A} \to T_{Y,B}$ iff Q has a Hilbert space adjoint Q^* with $D(Q^*) \supset S_{Y,B}$ and $Q^*(S_{Y,B}) \subset S_{X,A}$.

CHAPTER 5. Topological tensor products of spaces of type $S_{X,A}$, $T_{X,A}$

For two separable Hilbert spaces X and Y we consider the complex vector space consisting of all Hilbert-Smidt operators Z from X into Y. We shall denote this vector space by $X \otimes Y$. For any $Z \in X \otimes Y$ and any orthonormal basis $\{e_i\} \subset X$ we have

$$|||Z|||^2 = \sum_{i=1}^{\infty} \|Ze_i\|_Y^2 < \infty .$$

The double norm $|||\cdot|||$ does not depend on the choice of the orthonormal basis $\{e_i\}$. We introduce an inner product in $X \otimes Y$ by

$$(Z,K)_{X\otimes Y} = \sum_{i=1}^{\infty} (Ze_i, Ke_i)_Y .$$

Endowed with this inner product $X \otimes Y$ is a Hilbert space. See [RS] Ch. VIII.10. Examples of elements in $X \otimes Y$ are $\xi \otimes \eta$, $\xi \in X$, $\eta \in Y$, defined by $(\xi \otimes \eta)f = (f,\xi)_X\eta$, for all $f \in X$, and finite linear combinations of these: $\Sigma_{j=1}^{N} (\xi_j \otimes \eta_j)$ with $\xi_j \in X$, $\eta_j \in Y$. The linear subspace of $X \otimes Y$ which consists of all HS operators of the type just mentioned will be denoted by $X \otimes_a Y$, i.e. the (sesquilinear) algebraic tensor product of X and Y. $X \otimes Y$ may be regarded as the completion of $X \otimes_a Y$ with respect to the double norm

$|||\cdot|||$. Therefore $X \otimes Y$ is called the completed (sesquilinear) topological tensor product of X and Y. For later reference we mention the following properties taken from [RS] Ch. VIII.

<u>Properties 5.1.</u>

(a) $\forall x,\xi \in X \ \forall y,\eta \in Y \ (\xi \otimes \eta, x \otimes y)_{X\otimes Y} = (x,\xi)_X (\eta,y)_Y$.

(b) $\forall \lambda \in \mathbb{C} \ \forall \xi \in X \ \forall \eta \in Y \ \lambda(\xi \otimes \eta) = (\bar{\lambda}\xi) \otimes \eta = \xi \otimes (\lambda\eta)$.

Thus the canonical mapping $X \times Y \to X \otimes Y$, defined by $[x;y] \to x \otimes y$ is anti-linear in x and linear in y.

(c) $\forall Z \in X\otimes Y \ \forall x \in X \ \forall y \in Y \ (Z, x\otimes y)_{X\otimes Y} = (Zx,y)_Y$.

Let H respectively J denote bounded linear operators on X, respectively Y into themselves. $H \otimes J$ denotes the linear mapping of $X \otimes Y$ into itself defined by $(H \otimes J)(x \otimes y) = (Hx) \otimes (Jy)$ and linear extension, followed by continuous extension.

(d) The uniform operator norms of H, J and $H \otimes J$ are related by $\|H \otimes J\| = \|H\|\,\|J\|$.

(e) $\forall Z \in X\otimes Y \ (H \otimes J)Z = JZH^*$.

(f) H and J injective $\Rightarrow H \otimes J$ is injective.

The theory of closable tensor products of unbounded closable operators and the description of their properties in terms of corresponding properties of their factors presents greater difficulties. Only rather recently significant results have been attained [T].

<u>Definition 5.2.</u> Let A with domain $D(A)$ be a densely defined closed linear operator in X. Let B with domain $D(B)$ be the same in Y. On $D(A) \otimes_a (B) \subset X \otimes Y$ we introduce the operator $A \otimes_a I + I \otimes_a B$ by $(A \otimes_a I + I \otimes_a B)(x \otimes y) = (Ax) \otimes y + x \otimes (By)$ and linear extension. This extension is well defined and closable, [RS] p. 298.

Lemma 5.3. Let A respectively B be self-adjoint operators in X respectively Y.

I. $A \otimes_a I + I \otimes_a B$ is essentially self-adjoint in $X \otimes Y$. We denote the unique self-adjoint extension by $A \otimes I + I \otimes B$ or, briefly, $A \boxdot B$.

II. $A \geq 0$ and $B \geq 0$ implies $A \boxdot B \geq 0$.

Proof. As in [W] section 8.5.

Theorem 5.4. On $X \otimes Y$ we have for $t \geq 0$

$$e^{-t(A \boxdot B)} = e^{-tA} \otimes e^{-tB} .$$

Proof. As in [W] section 8.5.

Applying the results of the preceding chapters we can introduce the spaces $S_{X\otimes Y, A\boxdot B}$, $T_{X\otimes Y, A\boxdot B}$ and, by taking $A = 0$ or $B = 0$, the spaces $S_{X\otimes Y, A\boxdot I}$, $T_{X\otimes Y, I\boxdot B}$, etc.

Definition 5.5. The canonical sesquilinear map $\otimes : S_{X,A} \times S_{Y,B} \to S_{X\otimes Y, A\boxdot B}$ is defined by $[u;v] \to u \otimes v$. Here the symbol $\otimes$ is the same as in Properties 5.1. This definition is consistent because for $u \in S_{X,A}$, $v \in S_{Y,B}$ there exist $x \in X$, $y \in Y$ and $t > 0$ such that $u = e^{-tA}x$, $v = e^{-tA}y$. Further, $u \otimes v = (e^{-tA} \otimes e^{-tB})(x \otimes y) = (e^{-tA}x) \otimes (e^{-tB}y)$, so that $u \otimes v \in S_{X\otimes Y, A\boxdot B}$.

Theorem 5.6. $S_{X\otimes Y, A\boxdot B}$ is a complete topological tensor product of $S_{X,A}$ and $S_{Y,B}$. By this we mean:

I. $S_{X\otimes Y, A\boxdot B}$ is complete.

II. The canonical sesquilinear map $\otimes : S_{X,A} \times S_{Y,B} \to S_{X\otimes Y, A\boxdot B}$ is continuous.

III. The span of the image of $\otimes$, i.e. the algebraic tensor product $S_{X,A} \otimes_a S_{Y,B}$, is dense in $S_{X\otimes Y, A\boxdot B}$.

Proof.

I. The completeness follows from Theorem 1.11.

II. It is enough to check the continuity of $\otimes$ at $[0;0]$. Let W be a convex open neighbourhood of 0 in $S_{X\otimes Y, A\boxdot B}$. Then for each $t > 0$, $W \cap (X \otimes Y)_t$

is an open 0-neighbourhood in $(X \otimes Y)_t$ and it contains an open ball centered at 0 and radius r_t, $0 < r_t < 1$. In X_t respectively Y_t we introduce open balls A_t respectively B_t, centered at 0 and with radius both r_t. Let

$$A = \bigcup_{t>0} A_t \subset S_{X,A} \quad \text{and} \quad B = \bigcup_{t>0} B_t \subset S_{Y,B} .$$

Then $\otimes$ maps $A \times B$ in W since

$$\|x \otimes y\|_{\min(t,\tau)} \le \|x\|_t \|y\|_\tau \le r_{\min(t,\tau)}$$

whenever $x \in A$, $y \in B$. Let $\hat{A}$ respectively $\hat{B}$ denote the convex hulls of A respectively B. Then $\otimes$ maps $\hat{A} \times \hat{B}$ in W. The set $\hat{A}$ is convex and $\hat{A} \cap X_t$ contains an open neighbourhood in X_t. From Theorem 1.4.II it follows that $\hat{A}$ contains an open set $U_{\psi,\varepsilon}$.
Similarly $\hat{B}$ contains an open set $V_{\chi,\delta}$. We conclude that $\otimes$ maps $U_{\psi,\varepsilon} \times V_{\chi,\delta}$ into W.

III. For each $t > 0$, $X_t \otimes_a Y_t$ is dense in $(X \otimes Y)_t$. From this the desired result follows. □

Remark. Our strong topology on $S_{X\otimes Y, A\boxplus B}$ is, generally speaking, not the projective tensor product topology. Cf. [SCH] p. 93. Therefore the universal factorization property for continuous sesquilinear maps on this space does not hold in general.

Definition 5.7. The canonical sesquilinear map $\otimes : T_{X,A} \times T_{Y,B} \to T_{X\otimes Y, A\boxplus B}$, $[F;G] \to F \otimes G$, is defined by $(F \otimes G)(t) = F(t) \otimes G(t)$.
Here $\otimes$ is the same as in Properties 5.1. The definition is consistent because

$$(F \otimes G)(t + \tau) = e^{-\tau A} F(t) \otimes e^{-\tau B} G(t) =$$
$$= (e^{-\tau A} \otimes e^{-\tau B})(F(t) \otimes G(t)) = (e^{-\tau A} \otimes e^{-\tau B})(F \otimes G)(t) .$$

Theorem 5.8. $T_{X\otimes Y, A\boxplus B}$ is a complete topological tensor product of $T_{X,A}$ and $T_{Y,B}$. By this we mean:

I. $T_{X\otimes Y, A\boxplus B}$ is complete.

II. The canonical sesquilinear map $\otimes : T_{X,A} \times T_{Y,\mathcal{B}} \to T_{X\otimes Y, A\otimes\mathcal{B}}$ is continuous.

III. The span of the image of $\otimes$, i.e. the algebraic tensor product $T_{X,A} \otimes_a T_{Y,\mathcal{B}}$, is dense in $T_{X\otimes Y, A\otimes\mathcal{B}}$.

Proof.

I. The completeness follows from Theorem 2.5.

II. For each $t > 0$ we have $\|F(t) \otimes G(t)\|_{X\otimes Y} = \|F(t)\|_X \|G(t)\|_Y$. From this the continuity at $[0;0]$ follows.

III. $X \otimes_a Y$ is dense in $X \otimes Y$ which is dense in $T_{X\otimes Y, A\otimes\mathcal{B}}$. □

Now mixed sesquilinear topological tensor products of type $S_{X,A} \otimes T_{Y,\mathcal{B}}$, $T_{X,A} \otimes S_{Y,\mathcal{B}}$ will be considered. The notation of [ETh], Ch. II, will be used.

Definition 5.9. We introduce the following linear subspace of $T_{X\otimes Y, I\otimes\mathcal{B}}$:

$$T(S_{X\otimes Y, A\otimes I}, I \otimes \mathcal{B}) = \{\Phi \in T_{X\otimes Y, I\otimes\mathcal{B}} \mid \forall t>0 \ \Phi(t) \in S_{X\otimes Y, A\otimes I}\} .$$

In this space we take the topology generated by the semi-norms

$$\rho_{t,\psi}(\Phi) = \|(\psi(A) \otimes I)\Phi(t)\|_{X\otimes Y} , \quad t > 0 , \quad \psi \in B_+ .$$

Definition 5.10. The canonical sesquilinear map

$$\otimes : S_{X,A} \times T_{Y,\mathcal{B}} \to T_{X\otimes Y, I\otimes\mathcal{B}}$$

is defined by

$$f \otimes G : t \mapsto f \otimes G(t) .$$

(It is clear that $\forall t>0$ $f \otimes G(t) \in S_{X\otimes Y, A\otimes I}$.)

Theorem 5.11. $T(S_{X\otimes Y, A\otimes I}, I \otimes \mathcal{B})$ is a complete topological tensor product of $S_{X,A}$ and $T_{Y,\mathcal{B}}$. By this I mean

I. $T(S_{X\otimes Y, A\otimes I}, I \otimes \mathcal{B})$ is complete.

II. The canonical sesquilinear mapping

$$\otimes : S_{X,A} \times T_{Y,\mathcal{B}} \to T(S_{X\otimes Y, A\otimes I}, I \otimes \mathcal{B})$$

is continuous.

III. $S_{X,A} \otimes_a T_{Y,B}$ is dense in $T(S_{X\otimes Y,A\otimes I}, I \otimes B)$.

Proof

I. Let $\{\Phi_\alpha\}$ be a Cauchy net. The α's belong to a directed set D. First take $\psi = 1$. Φ_α tends to a limit point $\Phi \in T_{X\otimes Y,I\otimes B}$ because the latter is complete. It remains to show that, for each $t > 0$, $\Phi(t) \in S_{X\otimes Y,A\otimes I}$. For each $\psi \in B_+$ and each $t > 0$, $(\psi(A) \otimes I)\Phi_\alpha = \psi(A \otimes I)\Phi_\alpha$ converges in $X \otimes Y$. From the closedness of $\psi(A \otimes I)$ it follows that $\Phi(t) \in D(\psi(A \otimes I))$. This is true for each $\psi \in B_+$ and therefore by Theorem 1.10, for each $t > 0$, $\Phi(t) \in S_{X\otimes Y,A\otimes I}$.

II. Let $\psi \in B_+$ and let $t > 0$. Then for $f \in S_{X,A}$ and $G \in T_{Y,B}$

$$\|(\psi(A) \otimes I)(f \otimes G(t))\|_{X\otimes Y} \leq \|\psi(A)f\|_X \|G(t)\|_Y .$$

From this inequality the continuity of $\otimes$ follows.

III. Since $S_{X\otimes Y,A\otimes B}$ is dense in $T(S_{X\otimes Y,A\otimes I}, I \otimes B)$ and since $S_{X,A} \otimes_a S_{Y,B}$ is dense in $S_{X\otimes Y,A\otimes B}$, the assertion follows. □

Definition 5.12. We introduce the following linear subspace of $T_{X\otimes Y,A\otimes I}$:

$$T(S_{X\otimes Y,I\otimes B}, A \otimes I) = \{P \in T_{X\otimes Y,A\otimes I} \mid \forall t>0 \; P(t) \in S_{X\otimes Y,I\otimes B}\} .$$

In this space we take the topology generated by the semi-norms

$$\sigma_{t,\varphi}(P) = \|(I \otimes \varphi(B))P(t)\|_{X\otimes Y} , \quad t > 0 , \quad \varphi \in B_+ .$$

Definition 5.13. The canonical sesquilinear map

$$\otimes : T_{X,A} \times S_{Y,B} \to T_{X\otimes Y,A\otimes I}$$

is defined by

$$F \otimes g : t \to F(t) \otimes g .$$

(It is clear that $\forall t>0 \; F(t) \otimes g \in S_{X\otimes Y,I\otimes B}$.)

The proof of the following theorem runs the same as the proof of Theorem 5.11.

Theorem 5.14. $T(S_{X\otimes Y,I\otimes B}, A\otimes I)$ is a complete topological tensor product of $T_{X,A}$ and $S_{Y,B}$. Hence

I. $T(S_{X\otimes Y,I\otimes B}, A\otimes I)$ is complete.

II. The canonical sesquilinear mapping

$$\otimes : T_{X,A} \times S_{Y,B} \to T(S_{X\otimes Y,I\otimes B}, A\otimes I)$$

is continuous.

III. $T_{X,A} \otimes_a S_{Y,B}$ is dense in $T(S_{X\otimes Y,I\otimes B}, A\otimes I)$.

Next I introduce a second type of mixed topological tensor product.

Definition 5.15. We introduce the following linear subspace of $T_{X\otimes Y,A\otimes I}$:

$$S(T_{X\otimes Y,A\otimes I}, I\otimes B) = \bigcup_{t>0} (I\otimes e^{-tB})(T_{X\otimes Y,A\otimes I}) .$$

Each of the spaces $(I\otimes e^{-tB})(T_{X\otimes Y,A\otimes I})$ can be written as $T_{X\otimes e^{-tB}(Y),A\otimes I}$. According to Chapter II they are Fréchet spaces. Their semi-norms are given by

$$\eta \to \| (I\otimes e^{tB})\eta(\tfrac{1}{n})\|_{X\otimes Y} , \quad \eta \in (I\otimes e^{-tB})(T_{X\otimes Y,A\otimes I}) , \quad n \in \mathbb{N} .$$

The space $S(T_{X\otimes Y,A\otimes I}, I\otimes B)$ is an inductive limit of the spaces $(I\otimes e^{-tB})(T_{X\otimes Y,A\otimes I})$. For its topology we take the inductive limit topology. On $T(S_{X\otimes Y,A\otimes I}, I\otimes B) \times S(T_{X\otimes Y,A\otimes I}, I\otimes B)$ we introduce the pairing

$$\langle\langle \Phi, F\rangle\rangle_B = \langle \Phi(\varepsilon), (I\otimes e^{\varepsilon B})F\rangle_{X\otimes Y}$$

for $\varepsilon > 0$ sufficiently small. Finally, we define the embedding of $S(T_{X\otimes Y,A\otimes I}, I\otimes B)$ into $T(S_{X\otimes Y,I\otimes B}, A\otimes I)$ by

$$(\text{emb } F)(t) = (e^{-tA}\otimes I)F .$$

Theorem 5.16.

I. $T_{X,A} \otimes_a S_{Y,B}$ is dense in $S(T_{X\otimes Y,A\otimes I}, I \otimes B)$.

II. For each fixed $\Phi \in T(S_{X\otimes Y,A\otimes I}, I \otimes B)$ the linear functional $F \to \langle\langle \Phi, F\rangle\rangle_B$ is continuous on $S(T_{X\otimes Y,A\otimes I}, I \otimes B)$.

III. The embedding described in Definitions 5.15 is continuous.

Proof.

I. $S_{X,A} \otimes_a S_{Y,B}$ is dense in $S_{X\otimes Y,A\otimes B}$, which is dense in $S(T_{X\otimes Y,A\otimes I}, I \otimes B)$.

II. It is sufficient to prove the continuity for restrictions to each space $(I \otimes e^{-tB})(T_{X\otimes Y,A\otimes I})$. We have

$$\langle\langle \Phi, F\rangle\rangle_B = \langle (I \otimes e^{-tB})\Phi, (I \otimes e^{tB})F\rangle_{X\otimes Y} .$$

From this the continuity follows.

III. Let $t > 0$. Let $\psi \in B_+$. Let $\alpha > 0$ be fixed. Then for $W \in T_{X\otimes Y,A\otimes I}$

$$\|(e^{-tA} \otimes \psi(B))\, \mathrm{emb}(I \otimes e^{-\alpha B})W\|_{X\otimes Y} \leq \|I \otimes \psi(B)e^{-\alpha B}\| \|W(t)\|_{X\otimes Y} .$$

So emb $\circ$ $(I \otimes e^{-\alpha B})$ is continuous from $T_{X\otimes Y,A\otimes I}$ into $T(S_{X\otimes Y,I\otimes B}, A \otimes I)$. From this the continuity of emb follows. □

Definition 5.17. Similar to Definition 5.15 we introduce the space

$$S(T_{X\otimes Y,I\otimes B}, A \otimes I) = \bigcup_{t>0} (e^{-tA} \otimes I)(T_{X\otimes Y,I\otimes B})$$

with the appropriate inductive limit topology.
On $T(S_{X\otimes Y,I\otimes B}, A \otimes I) \times S(T_{X\otimes Y,I\otimes B}, A \otimes I)$ we introduce the pairing

$$\langle\langle P, T\rangle\rangle_A = \langle P(\varepsilon), (e^{\varepsilon A} \otimes I)T\rangle_{X\otimes Y} .$$

The embedding of $S(T_{X\otimes Y,I\otimes B}, A \otimes I)$ into $T(S_{X\otimes Y,A\otimes I}, I \otimes B)$ is defined by

$$(\mathrm{emb}\ T)(t) = (I \otimes e^{-tB})T .$$

Analogous to Theorem 5.16 we have

Theorem 5.18.

I. $S_{X,A} \otimes_a T_{Y,\mathcal{B}}$ is dense in $S(T_{X\otimes Y,A\otimes I}, I \otimes \mathcal{B})$.

II. For each fixed $P \in T(S_{X\otimes Y,A\otimes I}, I \otimes \mathcal{B})$ the linear functional $T \to \langle\langle P,T\rangle\rangle_A$ is continuous on $S(T_{X\otimes Y,A\otimes I}, I \otimes \mathcal{B})$.

III. The embedding described in Definition 5.17 is continuous.

Remark. For more details on the topological properties of the spaces $T(S_{X\otimes Y,A\otimes I}, I \otimes \mathcal{B})$, etc., see [ETh].

CHAPTER 6. Kerneltheorems

In this final chapter we show that the elements of the completed sesquilinear topological tensor products of the preceding chapter can, in a very natural way, be interpreted as linear maps of the types we discussed in Chapter 4. We give necessary and sufficient conditions on the semi-groups e^{-tA} and $e^{-t\mathcal{B}}$ which ensure that the topological tensor product comprise all continuous linear maps. In this case we say that a kernel theorem holds.

CASE a: Continuous linear maps $T_{X,A} \to S_{Y,\mathcal{B}}$. We consider an element $\theta \in S_{X\otimes Y,A\boxtimes\mathcal{B}}$ as a linear operator $T_{X,A} \to S_{Y,\mathcal{B}}$ in the following way. Let $F \in T_{X,A}$. We define θF by

(a) $$\theta F = e^{-\varepsilon\mathcal{B}}(e^{\varepsilon\mathcal{B}}\,\theta\, e^{\varepsilon A})F(\varepsilon) \; .$$

For $\varepsilon > 0$ and sufficiently small this definition makes sense and does not depend on ε.

Theorem 6.1.

I. For each $\theta \in S_{X\otimes Y,A\boxtimes\mathcal{B}}$ the linear operator $\theta : T_{X,A} \to S_{Y,\mathcal{B}}$, as defined by (a), is continuous.

II. For each $\theta \in S_{X\otimes Y, A\boxtimes B}$, $F \in T_{X,A}$, $G \in T_{Y,B}$,

$$\langle \theta F, G\rangle_Y = \langle \theta, F \otimes G\rangle_{X\otimes Y} .$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB}, is HS, then $S_{X\otimes Y, A\boxtimes B}$ comprises all continuous linear operators from $T_{X,A}$ into $S_{Y,B}$.

IV. $S_{X\otimes X, A\boxtimes A}$ comprises all continuous linear operators from $T_{X,A}$ into $S_{X,A}$ iff for each $t > 0$ the operator e^{-tA} is HS.

<u>Proof.</u>

I. We shall prove that θ satisfies condition IV of Theorem 4.5. Since $\theta \in X \otimes Y$ we have $\theta_r = \theta$. Since $\theta \in S_{X\otimes Y, A\boxtimes B}$ we have for t sufficiently small $e^{tB}\,\theta e^{tA} \in X \otimes Y$. Therefore, $\theta e^{tA} = e^{-tB}(e^{tB}\,\theta e^{tA})$ is bounded.

II. For ε sufficiently small $e^{\varepsilon B}\,\theta e^{\varepsilon A} \in X \otimes Y$, therefore by Properties 5.1.c

$$\langle \theta F, G\rangle_Y = \langle e^{-\varepsilon B}(e^{\varepsilon B}\,\theta e^{\varepsilon A})F(\varepsilon), G\rangle_Y = ((e^{\varepsilon B}\,\theta e^{\varepsilon A})F(\varepsilon), G(\varepsilon))_Y =$$

$$= (e^{\varepsilon B}\,\theta e^{\varepsilon A}, F(\varepsilon) \otimes G(\varepsilon))_{X\otimes Y} = \langle \theta, F \otimes G\rangle_{X\otimes Y} .$$

III. Let $\Gamma : T_{X,A} \to S_{Y,B}$ be continuous. By Theorem 4.5.V there exists $\tau > 0$ and continuous $Q : S_{X,A} \to S_{Y,B}$ such that $\Gamma = Qe^{-\tau A}$. By Theorem 4.2.V there exists $\beta > 0$ such that $e^{\beta B}\,Qe^{-\frac{1}{2}\tau A}$ is a bounded operator. Put $\alpha = \frac{1}{2}\min(\beta, \frac{1}{2}\tau)$, then

$$= e^{-\alpha B}\{e^{-(\beta-\alpha)B}(e^{\beta B}\,Qe^{-\frac{1}{2}\tau A})e^{-(\frac{1}{2}\tau-\alpha)A}\}e^{-\alpha A} .$$

The operator between ( ) is bounded. Further the operator between { } is HS since $e^{-(\beta-\alpha)B}$ or $e^{-(\frac{1}{2}\tau-\alpha)A}$ is HS.
It follows that $\Gamma \in S_{X\otimes Y, A\boxtimes B}$.

IV. The if-partis a special case of III. For the only-if-part consider the special map $\Gamma = e^{-\alpha A} : T_{X,A} \to S_{X,A}$ for some $\alpha > 0$. In order that $\Gamma \in S_{X\otimes X, A\boxtimes A}$ it has to be HS. □

<u>CASE b</u>: Continuous linear maps $S_{X,A} \to T_{Y,B}$.
Let $K \in T_{X\otimes Y, A\boxtimes B}$. For $f \in S_{X,A}$ we define $Kf \in T_{Y,B}$ by

(b) $$(Kf)(t) = e^{-(t-\varepsilon)B} K(\varepsilon) e^{\varepsilon A} f \,.$$

For any $f \in S_{X,A}$ and $t > 0$ this definition makes sense for $\varepsilon > 0$ sufficiently small. Moreover, $(Kf)(t)$ does not depend on ε.

<u>Theorem 6.2.</u>

I. For each $K \in T_{X\otimes Y, A\boxtimes B}$ the linear operator $K : S_{X,A} \to T_{Y,B}$ defined by (b) is continuous.

II. For each $K \in T_{X\otimes Y, A\boxtimes B}$, $f \in S_{X,A}$, $g \in S_{Y,B}$,

$$\langle g, Kf\rangle = \langle f \otimes g, K\rangle_{X\otimes Y} \,.$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is HS, then $T_{X\otimes Y, A\boxtimes B}$ comprises all continuous linear operators from $S_{X,A}$ into $T_{Y,B}$.

IV. $T_{X\otimes X, A\boxtimes A}$ comprises all continuous linear operators from $S_{X,A}$ into $T_{X,A}$ iff for each $t > 0$ the operator e^{-tA} is HS.

<u>Proof.</u>

I. We use condition II of Theorem 4.4.
For each $t > 0$, $\alpha > 0$, $e^{-tB} K e^{-\alpha A}$ is a bounded operator from X into Y because for ε sufficiently small

$$e^{-tB} K e^{-\alpha A} = e^{-(t-\varepsilon)B} K(\varepsilon) e^{-(\alpha-\varepsilon)A} \,.$$

All operators in the last expression are bounded.

II. For each $\tau > 0$, $K(\tau)$ is a HS-map. For ε sufficiently small, with Properties 5.1.c

$$\langle g, Kf\rangle_Y = (e^{\varepsilon B} g, (e^{-\varepsilon B} K e^{-\varepsilon A})(e^{\varepsilon A} f))_Y$$

$$(e^{\varepsilon A} f \otimes e^{\varepsilon B} g, K(\varepsilon))_{X\otimes Y} = ((e^{\varepsilon A} \otimes e^{\varepsilon B})(f \otimes g), K(\varepsilon))_{X\otimes Y} = \langle f \otimes g, K\rangle_{X\otimes Y} \,.$$

III. Let $L : S_{X,A} \to T_{Y,B}$ be continuous. According to Theorem 4.4.II the operators $e^{-tB} L e^{-tA}$ are bounded for each $t > 0$. However, if e^{-tB} or e^{-tA} is HS for each $t > 0$, then $e^{-tB} L e^{-tA}$ is HS for each $t > 0$ and it defines an element in $T_{X\otimes Y, A\otimes B}$. A simple verification shows that this element reproduces L by recipe (b).

IV. The if-part is a special case of III. For the only-if-part consider the special map $L = \text{emb} = I$. Here I is the identity map. $e^{-tA} I e^{-tA} = e^{-2tA}$ can be considered as an element of $S_{X\otimes Y, A\otimes B}$ iff e^{-tA} is HS for all $t > 0$. □

CASE c: Continuous linear maps: $S_{X,A} \to S_{Y,B}$.
Let $P \in T(S_{X\otimes Y, I\otimes B}, A \otimes I)$. For $f \in S_{X,A}$ we define $Pf \in S_{Y,B}$ by

(c) $$Pf = P(\varepsilon) e^{\varepsilon A} f \, .$$

$Pf \in S_{Y,B}$ since $P(\varepsilon) \in S_{X\otimes Y, I\otimes B}$. The definition makes sense for ε sufficiently small and does not depend on the choice of ε.

Theorem 6.3.

I. For each $P \in T(S_{X\otimes Y, I\otimes B}, A \otimes I)$ the linear operator $P : S_{X,A} \to S_{Y,B}$ defined by (c) is continuous.

II. For each $P \in T(S_{X\otimes Y, I\otimes B}, A \otimes I)$, each $f \in S_{X,A}$, each $G \in T_{Y,B}$,

$$\langle Pf, G\rangle_X = \langle P, f \otimes G\rangle_A \, .$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is HS, then $T(S_{X\otimes Y, I\otimes B}, A \otimes I)$ comprises all continuous linear operators from $S_{X,A}$ into $S_{Y,B}$.

IV. Consider the special case $X = Y$ and $B = A$. The space $T(S_{X\otimes X, I\otimes A}, A \otimes I)$ comprises all continuous linear operators from $S_{X,A}$ into itself iff for each $t > 0$ the operator e^{-tA} is HS.

Proof.

I. We use condition II of Theorem 4.2.
Let $f_n \to 0$ strongly in $S_{X,A}$. For some $\varepsilon > 0$, $e^{\varepsilon A} f_n \to 0$ in X.
$P(\varepsilon) \in S_{X\otimes Y, I\otimes B}$, therefore there exists $\delta > 0$ such that $e^{\delta B} P(\varepsilon)$ is a bounded operator. But then $e^{\delta B} P f_n = (e^{\delta B} P(\varepsilon)) e^{\varepsilon A} f_n \to 0$ in Y, which shows that $P f_n \to 0$ strongly in $S_{Y,B}$.

II. For β and δ sufficiently small and positive we have

$$\langle\!\langle P, f \otimes G \rangle\!\rangle_A = \langle P(\beta), (e^{\beta A} \otimes I)(f \otimes G)\rangle_{X\otimes Y}$$

$$= \langle P(\beta), e^{\beta A} f \otimes G\rangle_{X\otimes Y} = (e^{\delta B} P(\beta) e^{\delta A} (e^{(\beta-\delta)A} f) \otimes G(\delta))_{X\otimes Y}$$

$$= (e^{\delta B} P(\beta) e^{\delta A} e^{(\beta-\delta)A} f, G(\delta))_Y$$

$$= \langle P(\beta) e^{\beta A} f, G\rangle_Y = \langle P f, G\rangle_Y .$$

III. Let $Q : S_{X,A} \to S_{Y,B}$ be continuous. According to Theorem 4.2.V for each $t > 0$ there exist $\beta(t) > 0$ such that $e^{\beta(t)B} Q e^{-tA}$ is a bounded map from X into Y. Now because of the assumption on $e^{-\alpha A}$, $e^{-\alpha B}$, we find that $Q e^{-tA} = e^{-\beta(t)B} (e^{\beta(t)B} Q e^{-tA})$ is an element of $T(S_{X\otimes Y, I\otimes B}, A \otimes I)$. It reproduces the operator Q if the recipe (c) is applied.

IV. The if-part is a special case of III. For the only-if-part consider the identity map $I : S_{X,A} \to S_{X,A}$. In order that $I e^{-tA}$ as a function of t is an element of $T(S_{X\otimes X, I\otimes A}, A \otimes I)$ the operator e^{-tA} has to be HS for all $t > 0$. □

CASE d: Continuous linear maps: $T_{X,A} \to T_{Y,B}$.
Let $\Phi \in T(S_{X\otimes Y, A\otimes I}, I \otimes B)$. For $F \in T_{X,A}$ we define $\Phi F \in T_{Y,B}$ by

(d) $$(\Phi F)(t) = \Phi(t) e^{\varepsilon(t)A} F(\varepsilon(t)) .$$

This definition makes sense for $t > 0$ and $\varepsilon(t) > 0$ sufficiently small.
$(\Phi F)(t) \in S_{Y,B}$ since $\Phi(t) \in S_{X\otimes Y, A\otimes I}$. Moreover

$$e^{-\tau B}(\Phi F)(t) = e^{-\tau B} \Phi(t) e^{\varepsilon(t)A} F(\varepsilon(t)) = \Phi(t+\tau) e^{\varepsilon(t)A} F(\varepsilon(t)) = (\Phi F)(t+\tau) .$$

Theorem 6.4.

I. For each $\Phi \in T(S_{X\otimes Y,A\otimes I}, I \otimes B)$ the linear operator $\Phi : T_{X,A} \to T_{Y,B}$ defined by (d) is continuous.

II. For each $\Phi \in T(S_{X\otimes Y,A\otimes I}, I \otimes B)$, $F \in T_{X,A}$, $g \in S_{Y,B}$,

$$\langle g, \Phi F\rangle_Y = \overline{\langle\!\langle \Phi, F \otimes g\rangle\!\rangle_B} .$$

III. If for each $t > 0$ at least one of the operators e^{-tA}, e^{-tB} is HS, then $T(S_{X\otimes Y,A\otimes I}, I \otimes B)$ comprises all continuous linear operators from $T_{X,A}$ into $T_{Y,B}$.

IV. Consider the special case $Y = X$ and $B = A$. The space $T(S_{X\otimes X,A\otimes I}, I \otimes A)$ comprises all continuous linear operators from $T_{X,A}$ into itself iff for each $t > 0$ the operator e^{-tA} is HS.

Proof.

I. We use Theorem 4.6.III. For each $t > 0$, $e^{-tB}\Phi \in S_{X\otimes Y,A\otimes I}$. Then according to case a, $e^{-tB}\Phi$ is a continuous linear map from $T_{X,A}$ into $S_{Y,B}$.

II. For α and δ sufficiently small and positive

$$\langle\!\langle \Phi, F \otimes g\rangle\!\rangle_B = \langle \Phi(\alpha), (I \otimes e^{\alpha B})(F \otimes g)\rangle_{X\otimes Y} =$$

$$= \langle \Phi(\alpha), F \otimes e^{\alpha B} g\rangle_{X\otimes Y} = (e^{\delta B}\,\Phi(\alpha) e^{\delta A}, F(\delta) \otimes e^{(\alpha-\delta)B} g)_{X\otimes Y} =$$

$$= (e^{\delta B}\,\Phi(\alpha) e^{\delta A}\,F(\delta), e^{(\alpha-\delta)B} g)_Y = (\Phi(\alpha) e^{\delta A}\,F(\delta), e^{\alpha B} g)_Y =$$

$$= \overline{\langle g, \Phi F\rangle_Y} .$$

III. Let $\Psi : T_{X,A} \to T_{Y,B}$ be continuous. According to Theorem 4.6.VI for each $t > 0$ there exists $\beta(t) > 0$ such that $e^{-tB}\,\Psi_r e^{\beta(t)A}$ is a densely defined and bounded operaror from X into Y. If one of the operators $e^{-\alpha A}$, $e^{-\alpha B}$ is HS for arbitrary small positive α it follows that $e^{-tB}\Psi_r$ is HS for $t > 0$, because

$$e^{-tB}\Psi_r = e^{-tB}\,\overline{\Psi_r e^{\beta(t)A}}\; e^{-\beta(t)A} .$$

Here $\overline{\Psi_r e^{\beta(t)A}}$ denotes the extension of $\Psi_r e^{\beta(t)A}$ to the whole of X.
Since

$$e^{-tB}\Psi_r = e^{-\frac{1}{2}tB}(e^{-\frac{1}{2}tB}\,\overline{\Psi_r e^{\beta(\frac{1}{2}t)A}})e^{-\beta(\frac{1}{2}t)A}$$

it follows that $e^{-tB}\Psi_r \in S_{X\otimes Y,A\otimes I}$. Hence $e^{-tB}\Psi_r$ as a function of t, belongs to $T(S_{X\otimes Y,A\otimes I}, I \otimes B)$. By recipe (d) the operator Ψ is reproduced.

IV. The if-part is a special case of III. For the only-if-part consider the identity map I. In order that e^{-tA}, as a function of t, can be considered as an element in $T(S_{X\otimes X,A\otimes I}, I \otimes A)$ the operator e^{-tA} should be HS for all $t > 0$. □

Remark. In [ETh] a kernel theorem for extendable continuous linear mappings has been stated and proved.

REFERENCES

See part A.

ACKNOWLEDGEMENT

The author thanks Dr. S.J.L. van Eyndhoven for contributing to the revision of the manuscript.