Low-frequency diffraction by a slit in a conducting plane

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Memorandum 1982-15

August 1982

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Abstract. The diffraction of a normally incident plane wave by a slit in a conducting plane screen is treated by the low-frequency approach of Hurd and Hayashi [1]. Supplementary to [1], analytical and numerical results are presented for the transmission coefficient in the two cases of E- and H-polarization.

1. Introduction

In a recent paper [1], Hurd and Hayashi presented a new low-freqency approach to the problems of diffraction of an H- or E-polarized plane wave by a narrow slit in a perfectly conducting screen. The approach starts from a formulation of the diffraction problems in terms of an integral equation (in the case of H-polarization) or a differential-integral equation (in the case of E-polarization), both with a Hankel function difference kernel. This Hankel function is then replaced by the first three terms of its series expansion whereupon the resulting approximating integral equations can be solved exactly. The approximate solutions obtained are claimed to be superior to those derived by the traditional approach of solving a succession of integral equations, each with a static kernel. The accuracy of Hurd and Hayashi's approach is briefly discussed in [1, Sec. 6] where it is indicated that numerical results for the transmission coefficient in the E-polarized case agree quite well with the exact results of Skavlem [2] up to $kd = 1.6$; here, k is the wave number, and $2d$ is the slit width.

It is the aim of this note to present a more detailed evaluation of the transmission coefficient, supplementary to [1, Sec. 6]. Our analysis is restricted to diffraction of a normally incident plane wave, either E-polarized or H-polarized. Section 2 contains a statement of the diffraction problems and a summary of the solutions by Hurd and Hayashi [1]. In section 3 we derive analytical expressions for the transmission coefficient in the case of normal incidence, both for E- and H-polarization. Numerical results based on these analytical expressions, are presented in section 4, and are compared with exact results for the transmission coefficient due to Skavlem [2] and Van de Scheur [3]; in addition, a comparison is made with numerical results obtained from Millar's [4] low-frequency expansion for the transmission coefficient. Some mathematical details are deferred to Appendices A and B.

2. Statement of the diffraction problems, and summary of the solution by Hurd and Hayashi [1]

We consider the diffraction of a plane normally incident wave by a slit in a perfectly conducting screen. In terms of rectangular coordinates x,y,z, the screen coincides with the plane $y = 0$, and the slit is described by $-d < x < d$, $y = 0$, $-\infty < z < \infty$; see fig. 1.

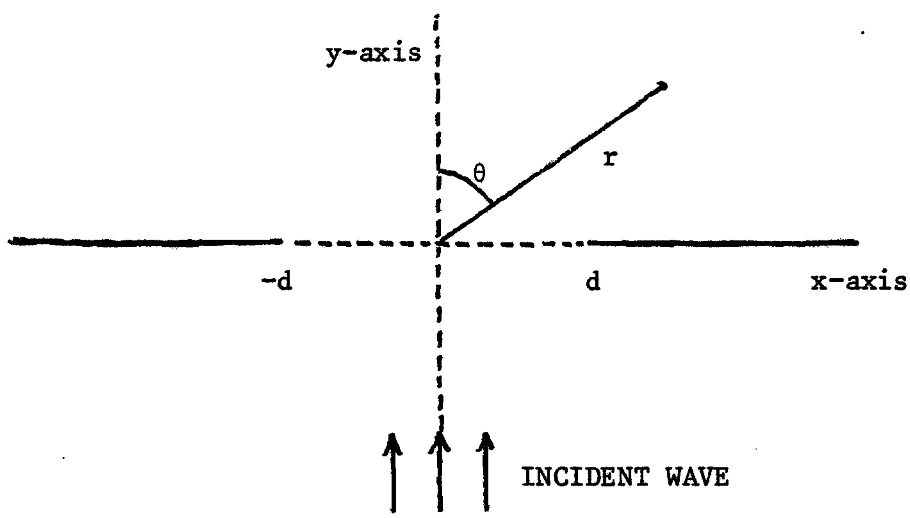

Fig. 1. Geometry of the diffraction problem.

The incident wave will be independent of z, hence, the diffraction problem is scalar and two-dimensional. Two polarizations are to be distinguished:

(i) E-polarization. Then the electromagnetic field has non-zero components E_z, H_x, H_y, expressible in terms of $E_z(x,y)$. On the perfectly conducting screen one has the Dirichlet boundary condition $E_z = 0$. The electromagnetic diffraction problem is equivalent to the acoustic diffraction problem for a slit in a soft screen.

(ii) H-polarization. Then the electromagnetic field has non-zero components H_z, E_x, E_y, expressible in terms of $H_z(x,y)$. On the perfectly conducting screen one has the Neumann boundary condition $\partial H_z/\partial y = 0$. The electromagnetic diffraction problem is equivalent to the acoustic diffraction problem for a slit in a rigid screen.

For the two polarizations the incident field is given by

$$E_z^i = H_z^i = e^{iky} , \tag{1}$$

at normal incidence from the region $y < 0$. Here k is the wave number, and a time dependence $e^{-i\omega t}$ is assumed and suppressed throughout. The diffraction problems for the two polarizations will be treated simultaneously, the solutions being distinguished by subscripts 1 and 2 corresponding to E- an H-polarization, respectively.

Following Bouwkamp [5, eqs. (2.1), (2.2)], the total fields E_z and H_z can be expressed as

$$E_z(x,y) = \begin{cases} e^{iky} - e^{-iky} + \phi_1(x,-y), & y \le 0 , \\ \phi_1(x,y), & y \ge 0 , \end{cases} \tag{2}$$

$$H_z(x,y) = \begin{cases} e^{iky} + e^{-iky} - \phi_2(x,-y), & y \le 0 , \\ \phi_2(x,y), & y \ge 0 . \end{cases} \tag{3}$$

Next, the fields $\phi_{1,2}$ are expressed in terms of the values of ϕ_1 and $\partial\phi_2/\partial y$ in the slit, by means of Rayleigh's integral representations

[5, eqs. (2.23), (2.24)]

$$\phi_1(x,y) = -\frac{i}{2}\frac{\partial}{\partial y}\int_{-d}^{d}\phi_1(x',0)\,H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right)dx' \,, \tag{4}$$

$$\phi_2(x,y) = -\frac{i}{2}\int_{-d}^{d}\left[\frac{\partial}{\partial y'}\phi_2(x',y')\right]_{y'=0} H_0^{(1)}\left(k\sqrt{(x-x')^2+y^2}\right)dx' \,, \tag{5}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. Finally, by requiring that $\partial E_z/\partial y$ and H_z be continuous through the slit, we are led to the differential-integral equation

$$\left(k^2+\frac{\partial^2}{\partial x^2}\right)\int_{-d}^{d}\phi_1(x',0)\,H_0^{(1)}(k|x-x'|)dx' = 2k, \quad -d < x < d \,, \tag{6}$$

in the case of E-polarization, and to the integral equation

$$\int_{-d}^{d}\left[\frac{\partial}{\partial y'}\phi_2(x',y')\right]_{y'=0} H_0^{(1)}(k|x-x'|)dx' = 2i, \quad -d < x < d \,, \tag{7}$$

in the case of H-polarization.

It is convenient to reduce eqs. (6), (7) to a dimensionless form. Thus we introduce

$$u = x'/d, \quad v = x/d, \quad \kappa = kd, \tag{8}$$

and we set

$$\phi_1(ud,0) = \tau_1(u), \quad d\left[\frac{\partial}{\partial y'}\phi_2(ud,y')\right]_{y'=0} = \tau_2(u) \,. \tag{9}$$

Then eqs. (6) and (7) reduce to

$$(10) \qquad \left(\kappa^2 + \frac{\partial^2}{\partial v^2}\right) \int_{-1}^{1} \tau_1(u)\, H_0^{(1)}\,(\kappa|u-v|)du = 2\kappa, \quad -1 < v < 1 ,$$

$$(11) \qquad \int_{-1}^{1} \tau_2(u)\, H_0^{(1)}\,(\kappa|u-v|)du = 2i, \quad -1 < v < 1 ,$$

in accordance with [1, eqs. (39), (5)] with $\beta = 0$, $\theta = \pi/2$ corresponding to normal incidence.

The integral equations (10) and (11) cannot be solved in a simple closed form and one has to resort to an approximate method of solution. Recently, Hurd and Hayashi [1] proposed a new low-frequency approach, the key step of which involves a replacement of the Hankel function kernel by the first three terms of its series expansion, viz.,

$$(12) \qquad H_0^{(1)}(\kappa|u-v|) = \frac{2i}{\pi}\,[\psi_0 + \psi_1\kappa^2 + \psi_2\kappa^4 + O(\kappa^6)]$$

where

$$(13) \qquad \psi_0 = p + \log(2|u-v|) ,$$

$$(14) \qquad \psi_n = \frac{(-1)^n}{2^{2n}(n!)^2}\,(u-v)^{2n}\,[p - 1 - \tfrac{1}{2} \ldots - \frac{1}{n} + \log(2|u-v|)] ,$$

$$(15) \qquad p = \log(\kappa/4) + \gamma - \pi i/2, \quad \gamma = 0.57721\ldots \text{ (Euler's constant)} ,$$

cf. [1, eqs. (6) - (8)]. On inserting (12) into (10) and (11), the integral equations take the approximate form [1, eqs. (40), (9)]

$$(16) \qquad \left(\kappa^2 + \frac{\partial^2}{\partial v^2}\right) \int_{-1}^{1} \tau_1(u)\,[p + \bar{q}(u-v)^2 + r(u-v)^4$$

$$+ \{1 + a(u-v)^2 + b(u-v)^4\}\log(2|u-v|)]du = -i\pi\kappa ,$$

$$\int_{-1}^{1} \tau_2(u)\,[p + q(u-v)^2 + r(u-v)^4 + \{1 + a(u-v)^2 + b(u-v)^4\}\log(2|u-v|)]du = \pi \tag{17}$$

where

$$q = -(p-1)\kappa^2/4, \quad r = (p-3/2)\kappa^4/64, \quad a = -\kappa^2/4, \quad b = \kappa^4/64\ . \tag{18}$$

The present approximate integral equations can be solved in closed form by means of a function-theoretic technique. Referring to [1] for the details of the calculations, we just quote the solutions from [1, eqs. (47), (23)], specialized to normal incidence,

$$\tau_i(u) = \frac{1}{\pi^2(1-u^2)^{\frac{1}{2}}} \int_{-1}^{1} \frac{(1-v^2)^{\frac{1}{2}} g_i(v)}{v-u}\, dv + \frac{D_i}{\pi^2(1-u^2)^{\frac{1}{2}}}\ , \tag{19}$$

$$g_i(v) = \sum_{j=1}^{3} C_{ij} \sinh(\alpha_j v), \tag{20}$$

where $i = 1,2$. Here the parameters α_j are given by

$$\alpha_{\substack{1\\2}} = \left[-a \pm \sqrt{a^2 - 24b}\right]^{\frac{1}{2}} = \frac{\kappa}{2}\,[1 \pm i\sqrt{5}]^{\frac{1}{2}}, \quad \alpha_3 = i\kappa\ , \tag{21}$$

cf. [1, eq. (17)]. In the case of E-polarization ($i = 1$) the coefficients C_{1j}, D_1 are identical to C_j, D in [1, eqs. (46), (47)]; C_{1j}, D_1 are to be determined from the system of four linear equations shown in [1, eqs. (50), (52)]. In the case of H-polarization ($i = 2$), $C_{23} = 0$ and the coefficients C_{21}, C_{22}, D_2 are identical to C_1^c, C_2^c, D in [1, eqs. (22), (23)]; C_{2j}, D_2 are to be determined from the system of three linear equations shown in [1, eq. (33)], specialized to normal incidence, i.e., $\beta = 0$. Note that [1, eq. (22)] contains two further coefficients C_1^s, C_2^s, which however vanish in the case of normal incidence.

For later use it is convenient to further reduce the approximate solution for $\tau_i(u)$. By combining (19) and (20) we are led to integrals of the form

$$T(\alpha) = \frac{1}{\pi} \int_{-1}^{1} \frac{(1-v^2)^{\frac{1}{2}} e^{\alpha v}}{v-u}\, dv \tag{22}$$

where $\alpha = \pm\, \alpha_j$. In Appendix A it is shown that $T(\alpha)$ can be reduced to

$$T(\alpha) = e^{\alpha u} \int_0^{\alpha} \frac{I_1(t)}{t}\, e^{-tu} dt - u e^{\alpha u} \tag{23}$$

where I_1 denotes the modified Bessel function of order 1. By use of (23), the approximate solution (19), (20) for $\tau_i(u)$ is now re-expressed as

$$\tau_i(u) = \frac{1}{2\pi(1-u^2)^{\frac{1}{2}}} \sum_{j=1}^{3} C_{ij} \left[\int_0^{\alpha_j} \frac{I_1(t)}{t} \left\{ e^{(\alpha_j - t)u} + e^{-(\alpha_j - t)u} \right\} dt - u\left(e^{\alpha_j u} - e^{-\alpha_j u} \right) \right] + \frac{D_i}{\pi^2 (1-u^2)^{\frac{1}{2}}} . \tag{24}$$

As it will be seen in the next section, the present representation of $\tau_i(u)$ provides a suitable starting point for the evaluation of the transmission coefficient.

3. Evaluation of the transmission coefficient

The transmission coefficient is defined as the ratio of the energy transmitted through the slit to the incident energy with the slit's area as basis. The transmission coefficient is denoted by $t_1(t_2)$ in the case of E-polarization (H-polarization). In this section analytical expressions will be derived for t_1, t_2.

As a preliminary we first determine the transmitted field ϕ_i at a large distance behind the slit. Let r, θ be polar coordinates defined by $x = r\sin\theta$, $y = r\cos\theta$, $-\pi/2 \le \theta \le \pi/2$; see fig. 1. Then for large r one has

$$(25) \qquad \sqrt{(x-x')^2+y^2} = r - x'\sin\theta + O(\tfrac{1}{r}) \ ,$$

$$(26) \qquad H_0^{(1)}(k\sqrt{(x-x')+y^2}) \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{ikr-\pi i/4}\, e^{-ikx'\sin\theta} \ ,$$

by means of the well-known asymptotic expansion of the Hankel function. By inserting (26) into the integral representations (4) and (5), we find that the transmitted far-field is an outgoing cylindrical wave, viz.,

$$(27) \qquad \phi_i \sim \frac{e^{ikr+\pi i/4}}{(2\pi kr)^{\frac{1}{2}}} \ A_i(\theta), \quad r \to \infty, \quad i = 1,2,$$

with amplitudes $A_i(\theta)$ given by

$$(28) \qquad A_1(\theta) = -ik\cos\theta \int_{-d}^{d} \phi_1(x',0)e^{-ikx'\sin\theta}dx'$$

$$= -i\kappa\cos\theta \int_{-1}^{1} \tau_1(u)e^{-i\kappa u\sin\theta}\,du \ ,$$

$$(29) \qquad A_2(\theta) = -\int_{-d}^{d} \left[\frac{\partial}{\partial y'}\,\phi_2(x',y')\right]_{y'=0} e^{-ikx'\sin\theta}\,dx'$$

$$= -\int_{-1}^{1} \tau_2(u)e^{-i\kappa u\sin\theta}\,du \ ,$$

by means of (8) and (9).

For a scalar wave ϕ the time average energy flux per unit area is proportional to

$$(30) \qquad \frac{1}{2i}\,(\phi^*\nabla\phi - \phi\nabla\phi^*) = \mathrm{Im}(\phi^*\nabla\phi)$$

where the asterisk denotes that the complex conjugate value should be taken. Thus for $\phi = \exp(iky)$, the energy incident on the slit is proportional to

$$(31) \qquad \mathrm{Im}\int_{-d}^{d} ik\,dx = 2\kappa\ .$$

Consequently the transmission coefficient t_i $(i = 1,2)$ is given by

$$(32) \qquad t_i = \frac{1}{2\kappa}\,\mathrm{Im}\left[\int_L \phi_i^*\,\frac{\partial\phi_i}{\partial n}\,ds\right]\ .$$

Here the integration is over an arbitrary curve L in the region $y \geq 0$, that connects the two parts of the screen, $x \geq d$, $y = 0$, and $x \leq -d$, $y = 0$; n is the unit normal to L, directed away from the slit. It can easily be shown that t_i is independent of the choice for L. We first take L to be the semi-circle of radius $r \to \infty$, $-\pi/2 \leq \theta \leq \pi/2$, then by means of (27) it is readily found that

$$(33) \qquad t_i = \frac{1}{4\pi\kappa}\int_{-\pi/2}^{\pi/2} |A_i(\theta)|^2\,d\theta\ , \qquad i = 1,2\ .$$

As a second choice, let L coincide with the slit $-d \leq x \leq d$, $y = 0$. In the slit one has

$$(34) \qquad \frac{\partial\phi_1}{\partial y} = ik, \quad \phi_2^* = 1\ ,$$

as it follows from (2) and (3) by requiring that $\partial E_z/\partial y$ and H_z be continuous through the slit. By using (34) and (9) in (32), we find

$$(35) \qquad t_1 = \tfrac{1}{2}\,\mathrm{Im}\left[i \int_{-1}^{1} \tau_1^*(u)\,du \right], \quad t_2 = \frac{1}{2\kappa}\,\mathrm{Im}\left[\int_{-1}^{1} \tau_2(u)\,du \right].$$

By means of (28) and (29), both integrals in (35) can be expressed in terms of $A_i(0)$, that is, the far-field amplitude in the direction of incidence. For both polarizations we obtain

$$(36) \qquad t_i = -\frac{1}{2\kappa}\,\mathrm{Im}\,A_i(0), \quad i = 1,2 .$$

The equivalence of the two results (33) and (36) for t_i is precisely Levine and Schwinger's [6] cross-section theorem. Notice that the equivalence only holds for the exact solution of the diffraction problems. Substitution of approximate values of $A_i(\theta)$ into (33) and (36), based on the approximate solution (24) for $\tau_i(u)$, will yield different results for t_i.

We shall now further evaluate the far-field amplitudes $A_i(\theta)$, $i = 1,2$, given by (28) and (29). Setting $\sin\theta = s$, we consider the integral

$$(37) \qquad F_i(s) = \int_{-1}^{1} \tau_i(u)\,e^{-i\kappa s u}\,du ,$$

where $\tau_i(u)$ is given by (24). To evaluate $F_i(s)$, we establish the following auxiliary results by means of Poisson's integral for Bessel functions [7, eq. 3.3(4)]:

$$(38) \qquad \frac{1}{\pi} \int_{-1}^{1} \frac{e^{-i\kappa s u}}{(1-u^2)^{\frac{1}{2}}}\,du = J_0(\kappa s) ;$$

$$(39)\qquad \frac{1}{\pi}\int_{-1}^{1}\frac{ue^{\pm\alpha u}e^{-i\kappa su}}{(1-u^2)^{\frac{1}{2}}}du = \frac{i}{\kappa}\frac{d}{ds}\left[\frac{1}{\pi}\int_{-1}^{1}\frac{e^{-i(\kappa s\pm i\alpha)u}}{(1-u^2)^{\frac{1}{2}}}du\right]$$

$$= \frac{i}{\kappa}\frac{d}{ds}J_0(\kappa s\pm i\alpha) = -i\,J_1(\kappa s\pm i\alpha)\ ;$$

$$(40)\qquad \frac{1}{\pi}\int_{-1}^{1}\frac{e^{-i\kappa su}}{(1-u^2)^{\frac{1}{2}}}du\int_{0}^{\alpha}\frac{I_1(t)}{t}e^{\pm(\alpha-t)u}du$$

$$= \frac{1}{\pi}\int_{0}^{\alpha}\frac{I_1(t)}{t}dt\int_{-1}^{1}\frac{\exp[-i(\kappa s\pm i\alpha\mp it)u]}{(1-u^2)^{\frac{1}{2}}}du$$

$$= \int_{0}^{\alpha}\frac{I_1(t)}{t}J_0(\kappa s\pm i\alpha\mp it)dt\ .$$

Here J_ν and I_ν are the standard notations [7] for the Bessel function and the modified Bessel function of order ν.

Furthermore, it is shown in Appendix B that

$$(41)\qquad i[J_1(\kappa s+i\alpha) - J_1(\kappa s-i\alpha)] = -2\sum_{m=0}^{\infty}\varepsilon_m(-1)^m J_{2m}(\kappa s)\,I'_{2m}(\alpha)\ ,$$

$$(42)\qquad \int_{0}^{\alpha}\frac{I_1(t)}{t}[J_0(\kappa s+i\alpha-it) + J_0(\kappa s-i\alpha+it)]dt$$

$$= 2\sum_{m=0}^{\infty}\varepsilon_m(-1)^m J_{2m}(\kappa s)\,I_{2m+1}(\alpha)$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ for $m = 1,2,3,\ldots$ (Neumann's factor); see (B6), (B5).

Finally, we need the recurrence relation [7, eq. 3.71(4)]

$$(43)\qquad I_{2m+1}(\alpha) - I'_{2m}(\alpha) = -\frac{2m}{\alpha}I_{2m}(\alpha)\ .$$

By means of these auxiliary results we find for the integral (37),

$$(44) \qquad F_i(s) = \frac{D_i}{\pi} J_0(\kappa s) - 4 \sum_{j=1}^{3} C_{ij} \sum_{m=1}^{\infty} (-1)^m m \frac{I_{2m}(\alpha_j)}{\alpha_j} J_{2m}(\kappa s)$$

where $i = 1,2$. From (44), the amplitudes $A_1(\theta)$ and $A_2(\theta)$ can be determined by multiplication by $-i\kappa \cos\theta$ and -1, respectively. In particular, we find for $\theta = 0$,

$$(45) \qquad A_1(0) = -\frac{i\kappa}{\pi} D_1, \quad A_2(0) = -\frac{D_2}{\pi} .$$

From (36) and (45) we then obtain the following simple expressions for the transmission coefficients,

$$(46) \qquad t_1 = \frac{1}{2\pi} \operatorname{Re} D_1, \quad t_2 = \frac{1}{2\pi\kappa} \operatorname{Im} D_2 .$$

Notice that this expression for t_i only depends on the coefficient D_i. Next we turn to the evaluation of expression (33) for the transmission coefficient. We expand $F_i(s)$, as given by (44), in a power-series in powers of $s = \sin\theta$. Thus, by replacing $J_{2m}(\kappa s)$ by its expansion

$$(47) \qquad J_{2m}(\kappa s) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}\kappa s)^{2m+2n}}{n!(2m+n)!}$$

and after some re-arrangement of series, we obtain

$$(48) \qquad F_i(s) = \sum_{n=0}^{\infty} (-1)^n (\tfrac{1}{2}\kappa s)^{2n} B_n^{(i)}, \quad i = 1,2 ,$$

where the coefficients $B_n^{(i)}$ are given by

$$(49) \qquad B_n^{(i)} = \frac{D_i}{\pi (n!)^2} - 4 \sum_{j=1}^{3} C_{ij} \sum_{m=1}^{n} \frac{m}{(n-m)!(n+m)!} \frac{I_{2m}(\alpha_j)}{\alpha_j} .$$

Corresponding expansions for the amplitudes $A_1(\theta)$ and $A_2(\theta)$ are immediately found from (28) and (29). The latter expansions are inserted into (33) and the integral is evaluated through term-by-term integration. Here we need the integrals

$$\text{(50)} \qquad \int_{-\pi/2}^{\pi/2} \sin^{2n}\theta \, d\theta = \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n+1)} = \frac{\pi(2n)!}{2^{2n}(n!)^2} ,$$

$$\text{(51)} \qquad \int_{-\pi/2}^{\pi/2} \sin^{2n}\theta \cos^2\theta \, d\theta = \frac{\Gamma(n+\frac{1}{2})\Gamma(3/2)}{\Gamma(n+2)} = \frac{\pi(2n)!}{2^{2n+1} n!(n+1)!} .$$

As a result we obtain the following low-frequency series-representations for the transmission coefficients t_1, t_2,

$$\text{(52)} \qquad t_1 = \frac{\kappa}{8} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{n!(n+1)!} \left(\frac{\kappa}{4}\right)^{2n} \sum_{m=0}^{n} B_m^{(1)} B_{n-m}^{(1)*} ,$$

$$\text{(53)} \qquad t_2 = \frac{1}{4\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{n!n!} \left(\frac{\kappa}{4}\right)^{2n} \sum_{m=0}^{n} B_m^{(2)} B_{n-m}^{(2)*} .$$

The present series-representations are most suitable for numerical purposes, especially for small κ where the series converge rapidly. The numerical evaluation of the coefficients $B_n^{(i)}$ requires the summation of a finite number of modified Bessel functions, see (49). These functions are readily determined by the well-known Miller's algorithm based on the backward recurrence relation for modified Bessel functions.
Summarizing, we derived two analytical expressions for the transmission coefficient t_i $(i = 1,2)$: (i) the Levine-Schwinger-type expressions (46); (ii) the expressions (52), (53) in terms of convergent low-frequency series expansions. As it will be seen in the next section, the two analytical expressions give rise to different numerical results.

4. Numerical results for the transmission coefficient

Numerical results for the transmission coefficients t_1 and t_2 (corresponding to the cases of E- and H-polarization, respectively) are presented in Tables 1 and 2. The parameter $\kappa = kd$ ranges from 0.1 to 2. The first column of the tables shows the numerical values of t_1, t_2, calculated from the series (52), (53), truncated at the first term less than 10^{-5}. It was found that at most 9 terms of the series should be taken into account to meet the accuracy criterion. The second column contains the values of t_1, t_2, calculated from the Levine-Schwinger-type expressions (46). In the third column we present exact results for the transmission coefficients, quoted from Skavlem [2] and Van de Scheur [3]. The numerical values in the fourth column have been obtained from Millar's [4] low-frequency expansion for the transmission coefficient, to order κ^6 consistent with (12). For the sake of convenience we present Millar's expansion in detail, viz.,

$$t_1 = \frac{\pi^2\kappa^3}{32}\left[1 + \frac{5}{16}\kappa^2\left(1 - \frac{8}{5}\delta\right) + \frac{1}{1536}\kappa^4(109 - 336\delta + 288\delta^2 - 24\pi^2) + O(\kappa^6)\right], \tag{54}$$

$$t_2 = \frac{\pi^2}{\kappa(\pi^2 + 4\delta^2)}\left[1 + \frac{1}{4}\kappa^2 + \frac{3}{256}\kappa^4\left(1 + \frac{16}{3}\frac{\delta}{\pi^2 + 4\delta^2}\right) + O(\kappa^6)\right], \tag{55}$$

where

$$\delta = \log(\kappa/4) + \gamma, \quad \gamma = 0.5772\text{-}1566\ldots \quad \text{(Euler's constant)}. \tag{56}$$

Table 1. Transmission coefficient for E-polarization

κ	t_1, eq. (52)	t_1, eq. (46)	t_1 exact	t_1, eq. (54)
0.1	0.000314	0.000314		0.000314
0.2	0.002624	0.002624	0.00262	0.002624
0.3	0.009396	0.009397		0.009391
0.4	0.02392	0.02393	0.02392	0.02388
0.5	0.05057	0.05062		0.05042
0.6	0.09490	0.09511	0.09484	0.09454
0.7	0.16326	0.16390		0.16304
0.8	0.2613	0.2631	0.26059	0.2638
0.9	0.3912	0.3955		0.4054
1.0	0.5477	0.5568	0.54540	0.5963
1.1	0.7151	0.7327	0.71431	0.8442
1.2	0.8701	0.9006	0.87693	1.1542
1.3	0.9893	1.0374	1.01482	1.5273
1.4	1.0587	1.1285	1.11719	1.9583
1.5	1.0766	1.1710	1.18271	2.4330
1.6	1.0510	1.1716	1.21669	2.9256
1.7	0.9936	1.1407	1.22701	3.3948
1.8	0.9158	1.0887	1.22129	3.7803
1.9	0.8274	1.0241	1.20559	3.9985
2.0	0.7356	0.9533	1.18426	3.9379

Table 2. Transmission coefficient for H-polarization

κ	t_2, eq. (53)	t_2, eq. (46)	t_2 exact	t_2, eq. (55)
0.1	2.0359	2.0359		2.0359
0.2	1.4983	1.4983		1.4983
0.24	1.3965	1.3965	1.39651	1.3965
0.3	1.2899	1.2899		1.2900
0.4	1.1785	1.1785		1.1785
0.48	1.1215	1.1216	1.12162	1.1217
0.5	1.1101	1.1102		1.1103
0.6	1.0651	1.0653		1.0657
0.7	1.0343	1.0346		1.0356
0.8	1.0124	1.0133	1.01431	1.0153
0.9	0.9963	0.9981		1.0019
1.0	0.9835	0.9870	0.99085	0.9940
1.1	0.9723	0.9783	0.98510	0.9902
1.2	0.9606	0.9708	0.98202	0.9900
1.3	0.9468	0.9631	0.98092	0.9929
1.4	0.9293	0.9542	0.98126	0.9985
1.5	0.9066	0.9429	0.98262	1.0067
1.6	0.8776	0.9282	0.98465	1.0172
1.7	0.8418	0.9093	0.98708	1.0300
1.8	0.7997	0.8857	0.98969	1.0451
1.9	0.7526	0.8573	0.99229	1.0625
2.0	0.7029	0.8245	0.99478	1.0820

Finally, we present some conclusions on the accuracy of the various low-freqency approximations as compared to the exact values of the transmission coefficients. Our criterion for good agreement will be that the relative error is less than some 5%. Thus it is found from Table 1 that the approximate values for t_1 obtained from (52), are satisfactory up to $\kappa = 1.4$, where the relative error is 5.2%. The Levine-Schwinger-type approximation (46) for t_1 is usable just beyond $\kappa = 1.6$, the relative errors being 3.7% and 7% at $\kappa = 1.6$ and $\kappa = 1.7$, respectively. Notice also that for $\kappa \leq 1.2$ the approximation (52) is more accurate than the approximation (46), whereas for $\kappa \geq 1.3$ the second (Levine-Schwinger-type) approximation is superior.

Similar conclusions can be drawn from Table 2. The approximate values for t_2 obtained from (53), are satisfactory up to $\kappa = 1.4$, where the relative error is 5.3%. The Levine-Schwinger-type approximation (46) for t_2 is more accurate, and is usable up to $\kappa = 1.6$, where the relative error is 5.7%.

The low-frequency expansion (54) for t_1 provides satisfactory numerical results up to $\kappa \approx 0.9$; at $\kappa = 1$ the relative error is already over 9%. Thus, in the case of E-polarization, the approximate results (52) and (46) obtained by the low-frequency approach of Hurd and Hayashi [1], are indeed superior to the low-frequency expansion (54) obtained by the traditional approach of Millar [4] and others. However, such a superior accuracy is not found in the case of H-polarization; on the contrary, the numerical values in the fourth column of Table 2 are seen to be more accurate than those in the first and second columns. The low-frequency expansion (55) can be used up to $\kappa = 1.8$, where the relative error is 5.6%.

Appendix A. Evaluation of the integral $T(\alpha)$

The integral $T(\alpha)$, introduced in (22), is defined by

$$\text{(A1)} \qquad T(\alpha) = \frac{1}{\pi} \int_{-1}^{1} \frac{(1-v^2)^{\frac{1}{2}} e^{\alpha v}}{v-u} \, dv ,$$

to be understood as a Cauchy principal value. Multiply $T(\alpha)$ by $e^{-\alpha u}$ and differentiate with respect to α, then we have by means of Watson [7, eq. 3.71(9)],

$$\text{(A2)} \qquad \frac{d}{d\alpha}\left[e^{-\alpha u} T(\alpha) \right] = \frac{1}{\pi} \int_{-1}^{1} (1-v^2)^{\frac{1}{2}} e^{\alpha(v-u)} \, dv = \frac{I_1(\alpha)}{\alpha} e^{-\alpha u} .$$

The initial value $T(0)$ is determined from [8, eq. 15.2(19)], viz.,

$$\text{(A3)} \qquad T(0) = \frac{1}{\pi} \int_{-1}^{1} \frac{(1-v^2)^{\frac{1}{2}}}{v-u} \, dv = -u, \quad -1 < u < 1 .$$

Thus by integration of (A2), taking into account (A3), we obtain

$$\text{(A4)} \qquad T(\alpha) = e^{\alpha u} \int_{0}^{\alpha} \frac{I_1(t)}{t} e^{-tu} \, dt - u e^{\alpha u} ,$$

valid for $-1 < u < 1$ and complex α.

Appendix B. Some Bessel-function series expansions

From Neumann's addition theorem [7, eq. 11.2(1)] for Bessel functions we deduce

$$\text{(B1)} \qquad J_0(u \pm v) = \sum_{m=0}^{\infty} \varepsilon_m (\mp 1)^m J_m(u) \, J_m(v) ,$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ for $m = 1,2,3,\ldots$ (Neumann's factor). By substituting $u = \kappa s$, $v = i\alpha - it$ in (B1), we find

$$\text{(B2)} \qquad \begin{aligned} J_0(\kappa s + i\alpha - it) + J_0(\kappa s - i\alpha + it) &= \sum_{m=0}^{\infty} \varepsilon_m[(-1)^m + 1]\, J_m(\kappa s)\, J_m(i\alpha - it) \\ &= 2\sum_{m=0}^{\infty} \varepsilon_m (-1)^m J_{2m}(\kappa s)\, I_{2m}(\alpha - t) \ . \end{aligned}$$

Here J_ν and I_ν are the standard notations [7] for the Bessel function and the modified Bessel function of order ν.

Next we evaluate the convolution integral

$$\text{(B3)} \qquad \int_0^\alpha \frac{I_1(t)}{t} I_\nu(\alpha - t)dt = I_{\nu+1}(\alpha), \quad \nu > -1 \ ,$$

derived by use of the Laplace transforms [8, eqs. 4.16(1),(3)]

$$\text{(B4)} \qquad \begin{aligned} L\left\{\frac{I_1(t)}{t}\right\} &= \int_0^\infty \frac{I_1(t)}{t} e^{-st}dt = \left(s + \sqrt{s^2 - 1}\right)^{-1} ; \\ L\{I_\nu(t)\} &= (s^2 - 1)^{-\frac{1}{2}}\left(s + \sqrt{s^2 - 1}\right)^{-\nu}, \quad \nu > -1 \ . \end{aligned}$$

By combining (B2) and (B3) we readily find

$$\text{(B5)} \qquad \begin{aligned} &\int_0^\alpha \frac{I_1(t)}{t} [J_0(\kappa s + i\alpha - it) + J_0(\kappa s - i\alpha + it)]dt \\ &\qquad = 2\sum_{m=0}^{\infty} \varepsilon_m (-1)^m J_{2m}(\kappa s)\, I_{2m+1}(\alpha) \ . \end{aligned}$$

Finally, by differentiation of (B2) with respect to t and setting $t = 0$, we obtain the series expansion

$$\text{(B6)} \qquad i[J_1(\kappa s + i\alpha) - J_1(\kappa s - i\alpha)] = -2\sum_{m=0}^{\infty} \varepsilon_m (-1)^m J_{2m}(\kappa s)\, I'_{2m}(\alpha) \ .$$

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