

On positive Rockland operators

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On positive Rockland operators

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Abstract

Let G be a homogeneous Lie group with a left Haar measure dg and L the action of G as left translations on $L_p(G; dg)$. Further let $H = dL(C)$ denote a homogeneous operator associated with L . If H is positive and hypoelliptic on L_2 we prove that it is closed on each of the L_p -spaces, $p \in \langle 1, \infty \rangle$, and that it generates a semigroup S with a smooth kernel K which, with its derivatives, satisfies Gaussian bounds. The semigroup is holomorphic in the open right half-plane on all the L_p -spaces, $p \in [1, \infty]$.

Further extensions of these results to non-homogeneous operators and general representations are also given.

1 Introduction

A differential operator H on a homogeneous group G is defined to be a Rockland operator if it is right-invariant, homogeneous and injective in each nontrivial irreducible unitary representation. If H is positive on $L_2(G; dg)$ then it generates a holomorphic semigroup with a kernel and recently Dziubański [Dzi] proved that the kernel, together with its derivatives, decreases exponentially on the right half-plane. Dziubański also raised the question whether one could establish stronger decrease properties than exponential. Stronger bounds have been derived by Hebisch [Heb] for Rockland operators which are sums of even powers and in fact Hebisch showed that the kernel, and its derivatives, satisfy ‘Gaussian’ bounds on the positive real line. The aim of this paper is to show that the kernel of the semigroup generated by a general positive Rockland operator has ‘Gaussian’ bounds on the right half-plane. Similar bounds are also established for the derivatives.

The theory of Rockland operators began with Rockland’s analysis of differential operators on the Heisenberg group [Roc]. Helffer and Nourrigat [HeN1] proved that a Rockland operator on a graded group is hypoelliptic and in addition they derived several inequalities between the norm on the C^n spaces and the operator norm. Then Miller [Mil] showed that one can replace a graded group by a homogeneous group in the Helffer–Nourrigat theorem. Subsequently, Folland and Stein [FoS] used the proof of an earlier theorem of Nelson and Stinespring [NeS] to deduce that a positive Rockland operator is essentially self-adjoint on the space $C_c^\infty(G)$. Moreover, they established that the closure generates a continuous semigroup with a kernel which is in the Schwartz space over the group. Dziubański [Dzi] then extended their result and showed that on the right half-plane one has an exponential decay for the kernel and all its derivatives. Our results give more precise ‘Gaussian’ bounds, more detailed analyticity properties of the semigroup and stronger regularity properties of the Rockland operators. Consequently, we are able to extend many of the previous results to non-homogeneous operators and to general representations of the group.

Throughout the sequel we adopt the general notation of [Rob]. Let G be a (connected, simply connected) homogeneous group with Lie algebra \mathfrak{g} and let (\mathcal{X}, G, U) denote a strongly continuous, or weakly*, continuous representation of G on the Banach space \mathcal{X} by bounded operators $g \mapsto U(g)$. If $a_i \in \mathfrak{g}$ then $A_i (= dU(a_i))$ will denote the generator of the one-parameter subgroup $t \mapsto U(e^{-ta_i})$ of the representation. Let $(\gamma_t)_{t>0}$ be a family of **dilations** on \mathfrak{g} , i.e., a one-parameter group of automorphisms of the form

$$\gamma_t(a_i) = t^{w_i} a_i$$

for some basis a_1, \dots, a_d of \mathfrak{g} and some positive numbers w_1, \dots, w_d , which we call **weights**. We always assume that the smallest weight is at least one. Let $D = \sum_{i=1}^d w_i$ be the **homogeneous dimension** of G . For the multi-indices we introduce the following notation. If $n \in \mathbb{N}_0$ let

$$J_n(d) = \bigoplus_{k=0}^n \{1, \dots, d\}^k$$

and set

$$J(d) = \bigcup_{n=0}^{\infty} J_n(d) \quad .$$

Then if $\alpha = (i_1, \dots, i_n) \in J(d)$ we denote the **Euclidean length** n of α by $|\alpha|$ and the **weighted length** by

$$\|\alpha\| = \sum_{k=1}^n w_{i_k} .$$

If $n \in \mathbf{N}$ we define $\mathcal{X}_n = \mathcal{X}_n(U) = \bigcap_{\alpha \in J_n(d)} D(A^\alpha)$ and

$$\|x\|_n = \max_{\substack{\alpha \in J(d) \\ |\alpha| \leq n}} \|A^\alpha x\| ,$$

where $A^\alpha = A_{i_1} \dots A_{i_n}$ if $\alpha = (i_1, \dots, i_n)$. Similarly we define the weighted spaces

$$\mathcal{X}'_n = \mathcal{X}'_n(U) = \bigcap_{\substack{\alpha \in J(d) \\ \|\alpha\| \leq n}} D(A^\alpha)$$

for all $n \in \mathbf{R}$ with $n > 0$. Now, however, it can happen for a given n that there are no multi-indices α such that $\|\alpha\| = n$. Therefore the corresponding norms and seminorms are given by

$$\|x\|'_n = \begin{cases} \max_{\substack{\alpha \in J(d) \\ \|\alpha\| \leq n}} \|A^\alpha x\| & \text{if there exist } \alpha \in J(d) \text{ with } \|\alpha\| = n , \\ 0 & \text{otherwise} , \end{cases}$$

$$N'_n(x) = \begin{cases} \max_{\substack{\alpha \in J(d) \\ \|\alpha\| = n}} \|A^\alpha x\| & \text{if there exist } \alpha \in J(d) \text{ with } \|\alpha\| = n , \\ 0 & \text{otherwise} , \end{cases} .$$

Moreover, let $\mathcal{X}_\infty = \mathcal{X}_\infty(U) = \bigcap_{n=1}^\infty \mathcal{X}_n$. It follows by a line by line extension of Lemma 2.4 of [EIR2] that the Gårding space, and in particular the space \mathcal{X}_∞ , is dense in \mathcal{X}'_n for all $n > 0$. The density is with respect to the weak, or weak*, topology. If U is the left regular representation on $L_p(G; dg)$ then we denote the corresponding spaces by $L_{p;n}, L'_{p;n}, L_{p;\infty}$ and the norms and seminorms by $\|\cdot\|_{p;n}$ etc. Further we let L denote the left regular representation on L_2 .

Let $m \in \langle 0, \infty \rangle$ and let $C: J(d) \rightarrow \mathbf{C}$ be such that $C(\alpha) = 0$ if $\|\alpha\| > m$ and there exists at least one $\alpha \in J(d)$ with $\|\alpha\| = m$ and $C(\alpha) \neq 0$. We call C a **form** of order m . We write $c_\alpha = C(\alpha)$. The **principal part** P of C is the form

$$P(\alpha) = \begin{cases} C(\alpha) & \text{if } \|\alpha\| = m , \\ 0 & \text{if } \|\alpha\| < m . \end{cases}$$

The **formal adjoint** C^\dagger of C is the function $C^\dagger: J(d) \rightarrow \mathbf{C}$ defined by

$$C^\dagger(\alpha) = (-1)^{|\alpha|} \overline{C(\alpha_*)} ,$$

where $\alpha_* = (i_n, \dots, i_1)$ if $\alpha = (i_1, \dots, i_n)$. We consider the operators

$$dU(C) = \sum_{\alpha \in J(d)} c_\alpha A^\alpha$$

with domain $D(dU(C)) = \mathcal{X}'_m$. If (\mathcal{F}, G, U_*) is the dual representation of (\mathcal{X}, G, U) then $dU_*(C^\dagger)$ is called the dual operator and denoted by H^\dagger .

If P is the principal part of a form C we call P a **Rockland form** if the operator $dU(P)$ is injective on the space $\mathcal{X}_\infty(U)$ for every nontrivial irreducible unitary representation U of G . It follows then from the Helffer–Nourrigat theorem [HeN1] that $dL(P)|_{C^\infty}$ is a hypoelliptic operator. In fact the Helffer–Nourrigat theorem is formulated for graded groups. But it follows from Propositions 1.3 and 1.4 of [Mil] that the existence of a Rockland form ensures that the order m of P is an integer multiple of the smallest weight and all weights are rational multiples of this smallest weight. Therefore G is a graded group if one rescales the original weights by a large enough constant.

A Rockland form P is called a **positive Rockland form** if $dL(P)$ is symmetric and $(\varphi, dL(P)\varphi) \geq 0$ for all φ in the Schwartz space on G (see [FoS] page 129). Throughout this paper we assume that C is a form of order m and that the principal part P of C is a positive Rockland form. We call $dL(P)$ a **positive Rockland operator**.

An important feature of the class of Rockland operators that we will use repeatedly is that it is closed under the operation of taking powers. If H is a Rockland operator then H^n is a Rockland operator for all $n \in \mathbb{N}$. Moreover, if H is a positive Rockland operator then all the powers, H^n , are also positive Rockland operators.

The group G can be equipped with two distances. The first is the Euclidean modulus $|\cdot|$, as defined in [Rob], page 256. In addition it has a homogeneous modulus $|\cdot|'$, which also induces a right invariant metric on G by $d'(g; h) = |gh^{-1}|'$ (see [HeS]). The balls with radius s are denoted by B_s and B'_s .

The main theorem of this paper is the following.

Theorem 1.1 *Let P be a positive Rockland form, (\mathcal{X}, G, U) a continuous representation of G and $H = dU(P)$ the associated operator. Then one has the following.*

- I. *The closure \overline{H} of H generates a continuous semigroup S .*
- II. *The semigroup S is holomorphic in the open right half-plane.*
- III. *The semigroup S has a representation independent kernel K in $L_{1;\infty}(G; dg) \cap C_{0;\infty}(G)$ such that*

$$A^\alpha S_t x = \int_G dh (A^\alpha K_t)(h) U(h)x$$

for all $\alpha \in J(d)$, $x \in \mathcal{X}$ and $g \in G$.

- IV. *For each $\varepsilon \in \langle 0, \pi/2 \rangle$ and all $\alpha \in J(d)$ there exist $a, b > 0$ such that*

$$|(A^\alpha K_z)(g)| \leq a|z|^{-(D+\|\alpha\|)/m} e^{-b((|g|')^m |z|^{-1})^{1/(m-1)}}$$

for all $g \in G$ and $z \in \Lambda(\pi/2 - \varepsilon) = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \pi/2 - \varepsilon\}$.

The outline of this paper is as follows. First we prove the theorem on L_2 and then we lift the result to a general representation by a transference argument. The L_2 proof is based on the examination of high powers $(\lambda I + H)^{-n}$ of the resolvent of H . Using Davies' method we obtain bounds on the kernels of these operators and then convert these to Gaussian bounds on the kernel of the semigroup by Cauchy integral techniques. The key point is that powers of the resolvent have improved regularity and boundedness properties.

The technique of using powers of the resolvent to improve regularity was implicit in the iterative argument of Nirenberg [Nir] and was used explicitly by Nelson and Stinespring [NeS]. It also occurs in a slightly different context in Agmon's work, [Agm] pages 250–260.

2 Positive Rockland operators on L_2

Let P be a positive Rockland form of order m and $H = dL(P)$ the associated positive Rockland operator on L_2 . In this section we prepare for the proof of Theorem 1.1 by establishing analogous results for the left regular representation L on $L_2(G; dg)$.

Proposition 2.1 *Let H be a positive Rockland operator which is homogeneous of degree m .*

I. *The operator H is self-adjoint.*

II. *For all $n \in \mathbf{N}$ one has $D(H^n) = L'_{2;nm}$, with equivalent norms. There exists a $c > 0$ such that*

$$cN'_{2;nm}(\varphi) \leq \|H^n \varphi\|_2$$

for all $\varphi \in D(H^n)$.

III. *The spaces $L_{2;\infty}$ and $C_c^\infty(G)$ are cores for H^n , for all $n \in \mathbf{N}$.*

IV. *If $n \in \mathbf{N}$ and $k \in \langle 0, nm \rangle$ then there exists a $c > 0$ such that*

$$N'_{2;k}(\varphi) \leq \varepsilon^{nm-k} N'_{2;nm}(\varphi) + c\varepsilon^{-k} \|\varphi\|_2$$

for all $\varepsilon > 0$ and $\varphi \in L'_{2;nm}$.

V. *If $n \in \mathbf{N}$ and $k \in \langle 0, nm \rangle$ then there exists a $c > 0$ such that*

$$\|\varphi\|'_{2;k} \leq \varepsilon^{nm-k} \|\varphi\|'_{2;nm} + c\varepsilon^{-k} \|\varphi\|_2$$

for all $\varepsilon > 0$ and $\varphi \in L'_{2;nm}$.

Proof The operator $H^n|_{C_c^\infty(G)}$ is essentially self-adjoint by the arguments of [NeS]. So

$$\overline{H^n} \subseteq (\overline{H^n})^* \subseteq \overline{H^n|_{C_c^\infty}}^* = \overline{H^n|_{C_c^\infty}} \subseteq \overline{H^n}$$

and $\overline{H^n|_{C_c^\infty}} = \overline{H^n}$. Therefore $\overline{H^n}$ is self-adjoint and $C_c^\infty(G)$ is a core for $\overline{H^n}$.

Next for all $\alpha \in J(d)$ and $n \in \mathbf{N}$ with $nm \geq \|\alpha\|$ there exists, by Helffer–Nourrigat [HeN1] Proposition 6.4, a $c > 0$ such that

$$\|A^\alpha \varphi\|_2 \leq c(\|H^n \varphi\|_2 + \|\varphi\|_2)$$

for all $\varphi \in C_c^\infty(G)$. Since $C_c^\infty(G)$ is dense in $D(\overline{H^n})$ and $L'_{2;k}$ is complete it follows that $D(\overline{H^n}) \subseteq L'_{2;k}$ if $k \leq nm$. But $L'_{2;nm} \subseteq D(H^n) \subseteq D(\overline{H^n})$, so $D(\overline{H^n}) = D(H^n) = L'_{2;nm}$, with equivalent norms. This proves Statements I, III and the first part of II.

It follows by scaling that there exists a $c > 0$ such that

$$\|A^\alpha \varphi\|_2 \leq \varepsilon^{nm-\|\alpha\|} \|H^n \varphi\|_2 + c\varepsilon^{-\|\alpha\|} \|\varphi\|_2$$

for all $\varphi \in C_c^\infty(G)$ and $\varepsilon > 0$, if $\|\alpha\| < nm$. In particular, for all $k < nm$ one has the estimates

$$N'_{2;k}(\varphi) \leq \varepsilon^{nm-k} \|H^n \varphi\|_2 + c \varepsilon^{-k} \|\varphi\|_2$$

and in addition

$$N'_{2;nm}(\varphi) \leq c \|H^n \varphi\|_2$$

for all $\varphi \in C_c^\infty(G)$. This proves Statements IV, V and the last part of II by closure and density. \square

Our aim is to bound the semigroup kernel associated with H . To this end we adopt the tactic of [AMT]. First, we obtain appropriate bounds on the kernel of a large power of the resolvent kernel where the power is large relative to the largest weight r . Secondly, we derive the semigroup estimates from the resolvent estimates by a Cauchy contour integration technique.

Note that with the choice of r one has the continuous embedding $L'_{2;kr} \subseteq L_{2;k}$ for each $k \in \mathbb{N}$.

Proposition 2.2 *If $n \in \mathbb{N}$, $k, \varepsilon > 0$ and $n \geq (dr + k)/m$ then $(\lambda I + H)^{-2n}$ has an integral kernel $R_\lambda^{(2n)} \in L'_{\infty;k}$ for all $\lambda \in \Lambda(\pi - \varepsilon)$. Moreover, one has bounds*

$$|(A^\alpha R_\lambda^{(2n)})(g)| \leq a |\lambda|^{-2n+(D+\|\alpha\|)/m} e^{-b|\lambda|^{1/m}|g|}$$

for all $g \in G$ and $\alpha \in J(d)$ with $\|\alpha\| \leq k$, where $a, b > 0$ are independent of λ .

Proof By Proposition 2.1 there exist $c_1, c_2 > 0$ such that $N'_{nm}(\varphi) \leq c_1 \|H^n \varphi\|_2$ and

$$\|H^{n-j} \varphi\|_2 \leq \delta^j \|H^n \varphi\|_2 + c_2 \delta^{-n+j} \|\varphi\|_2$$

for all $\varphi \in L'_{2;nm}$, $j \in \{1, \dots, n-1\}$ and $\delta > 0$. Let $\lambda \in \Lambda(\pi - \varepsilon)$. Then

$$\begin{aligned} \|H^n \varphi\| &\leq \|(\lambda I + H)^n \varphi\|_2 + \sum_{j=1}^n \binom{n}{j} |\lambda|^j \|H^{n-j} \varphi\|_2 \\ &\leq \|(\lambda I + H)^n \varphi\|_2 + \sum_{j=1}^n \binom{n}{j} |\lambda|^j \delta^j \|H^n \varphi\|_2 + c_2 |\lambda|^j \delta^{-n+j} \|\varphi\|_2 \\ &= \|(\lambda I + H)^n \varphi\|_2 + \left((1 + |\lambda| \delta)^n - 1 \right) \|H^n \varphi\|_2 + \delta^{-n} c_2 \left((1 + |\lambda| \delta)^n - 1 \right) \|\varphi\|_2 . \end{aligned}$$

By the spectral theorem one has $\|\varphi\|_2 \leq |\lambda|^{-n} \|(\lambda I + H)^n \varphi\|_2$. Taking $\delta > 0$ such that $(1 + |\lambda| \delta)^n - 1 = 1/2$ one establishes that

$$N'_{2;nm}(\varphi) \leq c_1 \|H^n \varphi\|_2 \leq c_3 \|(\lambda I + H)^n \varphi\|_2$$

where $c_3 = 2c_1 + 2c_1 c_2 ((3/2)^{1/n} - 1)^{-n}$. Next, using Proposition 2.1.IV one deduces that there exists a $c_4 > 0$ such that

$$\|\varphi\|'_{2;nm} \leq 2^{-1} c_4 (1 + |\lambda|^{-nm}) \|(\lambda I + H)^n \varphi\|_2 .$$

So the operators $(\lambda I + H)^{-n}$ map the Banach space L_2 continuously into the Banach space $L'_{2;nm}$ and

$$\left(\|(\lambda I + H)^{-n}\|_{L_2 \rightarrow L'_{2;nm}} \right)^{-1} \geq c_4^{-1}$$

uniformly for all $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$.

Fix a real $\psi \in C_{b;\infty}(G)$. Let

$$n_1(\psi) = \sup\{|(A_i\psi)(g)| : g \in G, i \in \{1, \dots, d\}\}$$

and for $j \in \mathbf{N}$ define recursively

$$n_{j+1}(\psi) = \sup_{i \in \{1, \dots, d\}} n_j(A_i\psi) \vee n_j(\psi) \quad .$$

If $j \leq t < j + 1$, with $j \in \mathbf{N}$, define $n_t(\psi) = n_j(\psi)$. Next for $\rho \in \mathbf{R}$ define $U_\rho: L_2 \rightarrow L_2$ by

$$(U_\rho\varphi)(g) = e^{-\rho\psi(g)}\varphi(g) \quad .$$

Then U_ρ maps $L_{2;l}$ continuously into $L_{2;l}$ for all $l \in \mathbf{N}_0$. Let

$$H^{(\rho)} = U_\rho H U_\rho^{-1} \quad .$$

The starting point is the observation that

$$U_\rho A_i U_\rho^{-1} = A_i + \rho M_{\psi_i}$$

for all $i \in \{1, \dots, d\}$, where $\psi_i = A_i\psi$ and M_{ψ_i} is the multiplication operator with ψ_i . Therefore $U_\rho A_i U_\rho^{-1}$ is a perturbation of A_i .

The operator $(\lambda I + H^{(\rho)})^n$ can be expressed in the form

$$(\lambda I + H^{(\rho)})^n = (\lambda I + H)^n + \sum_{\substack{\beta \in J(d) \\ \|\beta\| \leq nm-1}} M_\beta(\rho, \lambda, \psi) A^\beta$$

where $M_\beta(\rho, \lambda, \psi)$ are bounded multiplication operators satisfying bounds

$$\|M_\beta(\rho, \lambda, \psi)\| \leq c_5 |\rho| (1 + |\rho|^{nm-1}) (1 + n_{nm}(\psi))^{nm}$$

uniformly for all $\rho \in \mathbf{R}$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$, $\beta \in J(d)$ with $\|\beta\| \leq nm - 1$ and real $\psi \in C_{b;\infty}(G)$. Therefore

$$\|(\lambda I + H^{(\rho)})^n - (\lambda I + H)^n\|_{L'_{2;nm} \rightarrow L_2} \leq nm d^{nm-1} c_5 |\rho| (1 + |\rho|^{nm-1}) (1 + n_{nm}(\psi))^{nm} \quad .$$

So if $|\rho| \leq 1$, $2^{nm+1} c_5 d^{nm-1} nm |\rho| < (2c_4)^{-1}$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$ and real $\psi \in C_{b;\infty}(G)$ with $n_{nm}(\psi) \leq 1$ then $(\lambda I + H^{(\rho)})^n$ is invertible as an operator from $L'_{2;nm}$ from L_2 and

$$\|(\lambda I + H^{(\rho)})^{-n}\varphi\|'_{2;nm} \leq 2c_4 \|\varphi\|_2$$

for all $\varphi \in L_2$. Hence there exist $c_6, c_7 > 0$ such that

$$\|U_\rho A^\alpha U_\rho^{-1} (\lambda I + H^{(\rho)})^{-n} \varphi\|'_{2;nm-k} \leq c_6 \|\varphi\|_2$$

for all $\rho \in [-c_7, c_7]$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$, $\alpha \in J(d)$ with $\|\alpha\| \leq k$, real $\psi \in C_{b;\infty}(G)$ with $n_{nm}(\psi) \leq 1$ and $\varphi \in L_2$. Thus the operator $U_\rho A^\alpha (\lambda I + H)^{-n} U_\rho^{-1}$ maps L_2 continuously into $L'_{2;nm-k}$. But $L'_{2;nm-k}$ is continuously embedded into $L_{2;d}$ by the condition on n . Moreover, by the Sobolev embedding theorem for Lie groups, [Rob] B2.2, the space $L_{2;d}$ is continuously embedded into C_0 . Therefore there exists $R_{\alpha,\lambda}^{(n,\rho)} \in L_2$ such that

$$(U_\rho A^\alpha U_\rho^{-1} (\lambda I + H^{(\rho)})^{-n} \check{\varphi})(e) = \int_G dh R_{\alpha,\lambda}^{(n,\rho)}(h) \varphi(h)$$

for all $\varphi \in L_2$, where $\check{\varphi}(g) = \varphi(g^{-1})$. Moreover, $\|R_{\alpha,\lambda}^{(n,\rho)}\|_2 \leq c_8$ for some constant $c_8 > 0$ uniformly for all $|\rho| \leq c_7$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$, $\alpha \in J(d)$ with $\|\alpha\| \leq k$ and real $\psi \in C_{b;\infty}(G)$ with $n_{nm}(\psi) \leq 1$. Then by right invariance it follows that

$$(A^\alpha (\lambda I + H)^{-n} \varphi)(g) = \int_G dh R_{\alpha,\lambda}^{(n,0)}(h) \varphi(h^{-1}g)$$

for all $\varphi \in L_2$ and $g \in G$. Let $R_{\alpha,\lambda}^{(n)} = R_{\alpha,\lambda}^{(n,0)}$ and define $R_\lambda^{(n)} = R_{\beta,\lambda}^{(n,0)}$ if $\|\beta\| = 0$.

Since $(\bar{\lambda}I + H^{(-\rho)})^{-n}$ is bounded from L_2 into L_∞ it follows by duality that $(\lambda I + H^{(\rho)})^{-n}$ is bounded from L_1 into L_2 . Therefore one concludes, by composition, that

$$R_{\alpha,\lambda}^{(2n,\rho)} = R_{\alpha,\lambda}^{(n,\rho)} * R_{(\cdot),\lambda}^{(n,\rho)} \in C_0(G) .$$

Using the fact that $H^{(\rho)} = U_\rho H U_\rho^{-1}$, it then follows that

$$R_{\alpha,\lambda}^{(2n,\rho)}(g) = R_{\alpha,\lambda}^{(2n)}(g) e^{\rho(\psi(g^{-1}) - \psi(e))}$$

for all $g \in G$, $|\rho| \leq c_7$, $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$, $\alpha \in J(d)$ with $\|\alpha\| \leq k$ and real $\psi \in C_{b;\infty}(G)$ with $n_{nm}(\psi) \leq 1$. By [Rob], Chapter III, Equation (4.31), there exists a $c_9 > 0$ such that $|\psi(g^{-1}) - \psi(e)| \geq c_9|g|$ uniformly for all $g \in G$ and real $\psi \in C_{b;\infty}(G)$ with $n_{nm}(\psi) \leq 1$. So

$$|R_{\alpha,\lambda}^{(2n)}(g)| \leq \|R_{\alpha,\lambda}^{(2n,\rho)}\|_\infty e^{-c_7 c_9 |g|} \leq c_8^2 e^{-c_7 c_9 |g|}$$

uniformly for all $g \in G$. By [VSC], Proposition III.4.2, there exists a $c_{10} > 0$ such that $|g| \geq c_{10}|g'|$ for all $g \in G$ with $|g'| \geq 1$. So there exist $a, b > 0$ such that

$$|R_{\alpha,\lambda}^{(2n)}(g)| \leq a e^{-b|g'|}$$

uniformly for all $g \in G$ and $\lambda \in \Lambda(\pi - \varepsilon)$ with $|\lambda| = 1$.

Next we use a scaling argument to remove the restriction on λ . Let $\lambda \in \Lambda(\pi - \varepsilon)$. Since $H(\varphi \circ \gamma_u) = u^m(H\varphi) \circ \gamma_u$ and $(A^\alpha \varphi) \gamma_u^{-1} = u^{|\alpha|} A^\alpha(\varphi \circ \gamma_u^{-1})$ for all $u > 0$ one readily deduces that

$$(A^\alpha (\lambda I + H)^{-2n} \varphi) \circ \gamma_u^{-1} = u^{-2nm + |\alpha|} A^\alpha (\lambda u^{-m} I + H)^{-2n} (\varphi \circ \gamma_u^{-1}) .$$

Choose $u = |\lambda|^{1/m}$. Then

$$(A^\alpha (\lambda I + H)^{-2n} \varphi)(g) = |\lambda|^{-2n + (D + |\alpha|)/m} \int_G dh R_{\alpha,\lambda}^{(2n)}(\gamma_u(h)) \varphi(h^{-1}g) .$$

So $A^\alpha (\lambda I + H)^{-2n}$ has a kernel $R_{\alpha,\lambda}^{(2n)}$ and $R_{\alpha,\lambda}^{(2n)}(h) = |\lambda|^{-2n + (D + |\alpha|)/m} R_{\alpha,\lambda}^{(2n)}(\gamma_u(h))$ for all $h \in G$. Then

$$|R_{\alpha,\lambda}^{(2n)}(h)| \leq a |\lambda|^{-2n + (D + |\alpha|)/m} e^{-b|\lambda|^{1/m}|h'|}$$

for all $h \in G$.

Finally we prove that $R_\lambda^{(2n)}$ is k -times differentiable and that $A^\alpha R_\lambda^{(2n)} = R_{\alpha,\lambda}^{(2n)}$ for all $\alpha \in J(d)$ with $\|\alpha\| \leq k$. It suffices to consider the case $|\lambda| = 1$. Then

$$R_{\alpha,\lambda}^{(2n)} = R_{\alpha,\lambda}^{(n)} * R_\lambda^{(n)} = A^\alpha(\lambda I + H)^{-n} R_\lambda^{(n)} = A^\alpha((\lambda I + H)^{-n} R_\lambda^{(n)}) = A^\alpha R_\lambda^{(2n)} .$$

This completes the proof of the proposition. \square

Corollary 2.3 *The holomorphic self-adjoint semigroup S generated by H has a kernel $K \in L_{1,\infty}(G; dg) \cap C_{0,\infty}(G)$ such that*

$$(A^\alpha S_z \varphi)(g) = \int_G dh (A^\alpha K_z)(h) \varphi(h^{-1}g)$$

for all $\alpha \in J(d)$, $z \in \Lambda(\pi/2)$, $\varphi \in L_2$ and $g \in G$. Moreover, the function $z \mapsto K_z(g)$ is analytic on $\Lambda(\pi/2)$, uniformly for $g \in G$, and for each $\alpha \in J(d)$ and $\varepsilon \in \langle 0, \pi/2 \rangle$ there exist $a, b > 0$ such that

$$|(A^\alpha K_z)(g)| \leq a|z|^{-(D+\|\alpha\|)/m} e^{-b((|g|)^m|z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\pi/2 - \varepsilon)$ and all $g \in G$.

Proof The semigroup S can be constructed from the resolvent $(\lambda I + H)^{-1}$ using the standard Cauchy integral representation of the exponential. Alternatively, one can use the Cauchy representation for the derivatives of the exponential. Let $n \in \mathbf{N}$ and $k > 0$ and suppose that $n \geq (dr + k)/m$. Assume that $z \in \Lambda(\pi/2 - \varepsilon)$ then

$$S_z = (2\pi i)^{-1} (2n - 1)! z^{-(2n-1)} \int_{\Gamma_R} d\lambda e^{\lambda z} (\lambda I + H)^{-2n}$$

where Γ_R is the contour in the complex plane formed by connecting the two line segments $L_{R,\pm} = \{\lambda \in \mathbf{C} : \arg \lambda = \pm(\pi - \varepsilon/2), |\lambda| \geq R|z|^{-1}\}$ and the arc $A_R = \{\lambda \in \mathbf{C} : \arg \lambda \in [-\pi + \varepsilon/2, \pi - \varepsilon/2], |\lambda| = R|z|^{-1}\}$. We will deduce below, from the bounds of Proposition 2.2, that the semigroup kernel K exists,

$$K_z = (2\pi i)^{-1} (2n - 1)! z^{-(2n-1)} \int_{\Gamma_R} d\lambda e^{\lambda z} R_\lambda^{(2n)}$$

and $z \mapsto K_z(g)$ is analytic for all $g \in G$.

First, assume $R \geq 1$. If $\lambda \in L_{R,\pm}$ one has $\operatorname{Re}(\lambda z) < -|\lambda||z|\delta$, where $\delta = \sin(\varepsilon/2)$. Therefore the foregoing integral exists pointwise and one has bounds

$$|K_z(g)| \leq a(2\pi)^{-1} (2n - 1)! |z|^{-(2n-1)} \int_{\Gamma_R} d\lambda e^{-|\lambda||z|\delta} |\lambda|^{-(2n-D/m)} \leq a'|z|^{-D/m}$$

for some $a' > 0$ and for all $g \in G$. Moreover, a similar estimate bounds the derivative of $z \mapsto K_z(g)$, uniformly for $g \in G$, and this establishes uniform analyticity of the function.

Secondly, $|K_z(g)|$ can be estimated more accurately as follows. Assume $|z| = 1$. The integral over the arc A_R gives a contribution to $K_z(g)$ which is bounded by

$$\begin{aligned} B_z^{(A)}(g) &= a(2\pi)^{-1} (2n - 1)! \int_{-\pi+\varepsilon/2}^{\pi-\varepsilon/2} d\theta R^{-(2n-1-D/m)} e^R e^{-bR^{1/m}|g|'} \\ &\leq a' e^{R-bR^{1/m}|g|'} \end{aligned}$$

where we have used $2n - 1 \geq D/m$ and $R \geq 1$. But the exponential is minimized by choosing $R = (b|g|'m^{-1})^{m/(m-1)}$ and hence if $b|g|' \geq m$ one has bounds

$$B_z^{(A)}(g) \leq a'' e^{-b'(|g|')^{m/(m-1)}} .$$

Thirdly, the line segments $L_{R,\pm}$ in Γ_R both give contributions which can be bounded by

$$B_z^{(L)}(g) = a(2\pi)^{-1}(2n-1)! \int_R^\infty d\mu \mu^{-(2n-D/m)} e^{-\mu\delta - b\mu^{1/m}|g|'} .$$

Since $R = (b|g|'m^{-1})^{m/(m-1)} \geq 1$ one again has bounds of the form

$$B_z^{(L)}(g) \leq a'' e^{-b'(|g|')^{m/(m-1)}} .$$

Therefore one concludes that

$$|K_z(g)| \leq a e^{-b(|g|')^{m/(m-1)}}$$

for some $a, b > 0$, uniformly for all $z \in \Lambda(\pi/2 - \varepsilon)$ with $|z| = 1$ and all $g \in G$ with $|g|'$ sufficiently large. But we have already established that K_z is uniformly bounded if $|z| = 1$. Therefore the foregoing bounds extend to all $g \in G$, and $z \in \Lambda(\pi/2 - \varepsilon)$ with $|z| = 1$, by increasing the value of a .

Finally, since H is homogeneous one obtains the bounds

$$|K_z(g)| \leq a|z|^{-D/m} e^{-b((|g|')^m|z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\pi/2 - \varepsilon)$ and all $g \in G$ by scaling.

It now follows straightforwardly that K is indeed the kernel of S .

If $\alpha \in J(d)$ with $\|\alpha\| \leq k$ then one can define similarly the function $K_{\alpha,z}: G \rightarrow \mathbf{C}$ by

$$K_{\alpha,z}(g) = (2\pi i)^{-1}(2n-1)! z^{-(2n-1)} \int_{\Gamma_R} d\lambda e^{\lambda z} (A^\alpha R_\lambda^{(2n)})(g)$$

It follows as above that one has bounds

$$|K_{\alpha,z}(g)| \leq a|z|^{-(D+\|\alpha\|)/m} e^{-b((|g|')^m|z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\pi/2 - \varepsilon)$ and all $g \in G$. It then follows that K_z is pointwise differentiable and that $A^\alpha K_z = K_{\alpha,z}$ for all α with $\|\alpha\| \leq k$. Hence K_z is infinitely often differentiable in both the L_1 and the L_∞ sense. \square

Remark The kernel K is usually not positive. It follows from [Rob], Chapter III, Section 5, that K is positive if and only if H is a second-order operator, in the unweighted sense, with real coefficients and with the principal coefficients satisfying an ellipticity condition.

3 Miscellani

In this section we extend the foregoing results to general positive Rockland operators as defined in the introduction and to general continuous representations. In particular we complete the proof of Theorem 1.1. Subsequently, we examine properties of Rockland operators which are special to unitary representations or to translations on the L_p -spaces.

3.1 Lower order terms

In the previous section we derived Gaussian bounds for the kernel of the semigroup generated by H on L_2 under the assumption that H is homogeneous. Next we use perturbation theory to remove this homogeneity hypothesis.

Proposition 3.1 *Let $H = dL(C)$ where C is a form of order m whose principal part is a positive Rockland form.*

I. *For all $n \in \mathbf{N}$ one has $D(H^n) = L'_{2;nm}$, with equivalent norms. Moreover, there exists a $c > 0$ such that*

$$N'_{2;nm}(\varphi) \leq c(\|H^n\varphi\|_2 + \|\varphi\|_2)$$

for all $\varphi \in D(H^n)$.

II. *The spaces $L_{2;\infty}$ and $C_c^\infty(G)$ are cores for H^n , for all $n \in \mathbf{N}$.*

Proof Let $H_0 = dL(P)$ denote the principal part of H .

Suppose $n = 1$. Then $D(H_0) = L'_{2;m}$ by Proposition 2.1. Moreover, since $H_1 = H - H_0$ is of degree at most $m - 1$ one has

$$\|H_1\varphi\|_2 \leq \varepsilon\|H_0\varphi\|_2 + c\varepsilon^{-m+1}\|\varphi\|_2$$

for some $c > 0$ and all $\varepsilon \in \langle 0, 1 \rangle$ by Statements I and II of Proposition 2.1 applied to H_0 . Thus H_1 is a relatively bounded perturbation of H_0 with relative bound zero. Hence $D(H) = D(H_0) = L'_{2;m}$ with equivalent norms. The rest of the proof then follows the proof of Proposition 2.1.

Next suppose $n > 1$. Then the principal part H_0^n of the operator H^n is a positive Rockland operator. Therefore we can repeat the foregoing argument with H_0 replaced by H_0^n , H_1 replaced by $H^n - H_0^n$ and the degree m replaced by nm . \square

The semigroup results now follow by use of perturbation theory.

Theorem 3.2 *Let $H = dL(C)$ where C is a form of order m whose principal part is a positive Rockland form. Then H generates a continuous semigroup S with a kernel $K \in L_{1;\infty}(G; dg) \cap C_{0;\infty}(G)$. Moreover, for each $\alpha \in J(d)$ and $\varepsilon \in \langle 0, \pi/2 \rangle$ there exist $a, b > 0$ and an $\omega \geq 0$ such that*

$$|(A^\alpha K_z)(g)| \leq a|z|^{-(D+\|\alpha\|)/m} e^{\omega|z|} e^{-b(|g|^m|z|^{-1})^{1/(m-1)}}$$

for all $z \in \Lambda(\pi/2 - \varepsilon)$ and all $g \in G$.

If K^\dagger is the kernel of the semigroup generated by the dual operator H^\dagger then $K_t^\dagger(g) = \overline{K_t(g^{-1})}$ for all $g \in G$.

Proof Again we use H_0 to denote the principal part of H and H_1 the lower order terms. Since H_0 is positive self-adjoint and H_1 is relatively bounded by H_0 with relative bound zero it follows from standard perturbation theory that H generates a continuous semigroup S on L_2 . Moreover, it follows from the perturbation theory of holomorphic semigroups that S is holomorphic in the open right half-plane.

Next, we exploit the perturbation arguments developed in the appendix of [BrR2] and subsequently used in [ElR1], [ElR2] and [ElR4] to construct the kernel K of S from the kernel $K^{(0)}$ of the semigroup $S^{(0)}$ generated by the principal part H_0 of H . Since the proof is very similar to the previous applications we only sketch the outline of the argument.

Define $K_t^{(n)}$ by the recursion relation $K_t^{(n)} = -(K^{(n-1)} \hat{\star} H_1 K^{(0)})_t$ where the convolution product $\hat{\star}$ is defined on $\mathbf{R} \times G$ by

$$\begin{aligned} (\varphi \hat{\star} \psi)_t(g) &= \int_{\mathbf{R}} ds \int_G dh \varphi_s(h) \psi_{t-s}(h^{-1}g) \\ &= \int_{\mathbf{R}} ds \int_G dh \varphi_{t-s}(h) \psi_s(h^{-1}g) . \end{aligned}$$

Then one proves that the perturbation series

$$K_t = \sum_{n \geq 0} K_t^{(n)} ,$$

is L_p -convergent, for all $p \in [1, \infty]$ and all $t > 0$, and identifies K_t as the kernel of S . In fact the series is convergent on exponentially weighted L_p -spaces and this allows one to extend the Gaussian bounds from $K^{(0)}$ to K .

Let $\rho \geq 0$ and define L_1^ρ to be the L_1 -space with respect to the measure $dg e^{\rho|g|'}$ and denote the norm by

$$\|\varphi\|_1^\rho = \int_G dg e^{\rho|g|'} |\varphi(g)| .$$

Similarly, let L_∞^ρ be the space of measurable functions φ for which $g \mapsto e^{\rho|g|'} |\varphi(g)|$ is essentially bounded with norm

$$\|\varphi\|_\infty^\rho = \text{ess sup}_{g \in G} e^{\rho|g|'} |\varphi(g)| .$$

Since $|\cdot|'$ satisfies the triangle inequality it follows from the recursion relation for $K^{(n)}$ that one has the coupled integral inequalities

$$\|K_t^{(n)}\|_\infty^\rho \leq \int_0^t ds \left(\|K_{t-s}^{(n-1)}\|_\infty^\rho \|H_1 K_s^{(0)}\|_1^\rho \right) \wedge \left(\|K_{t-s}^{(n-1)}\|_1^\rho \|H_1 K_s^{(0)}\|_\infty^\rho \right) \quad (1)$$

and

$$\|K_t^{(n)}\|_1^\rho \leq \int_0^t ds \|K_{t-s}^{(n-1)}\|_1^\rho \|H_1 K_s^{(0)}\|_1^\rho . \quad (2)$$

But it follows from the Gaussian bounds of Corollary 2.3 applied to $K^{(0)}$ that with a suitable choice of $a > 0$ and $\omega \geq 0$ one has the set of bounds

$$\begin{aligned} \|K_t^{(0)}\|_1^\rho &\leq a e^{\omega(1+\rho^m)t} & , & \quad \|K_t^{(0)}\|_\infty^\rho \leq a t^{-D/m} e^{\omega(1+\rho^m)t} \\ \|H_1 K_t^{(0)}\|_1^\rho &\leq a t^{-(m-1)/m} e^{\omega(1+\rho^m)t} & , & \quad \|H_1 K_t^{(0)}\|_\infty^\rho \leq a t^{-(D+m-1)/m} e^{\omega(1+\rho^m)t} \end{aligned}$$

for all $\rho \geq 0$. Now these estimates allow one to solve the integral inequalities (1) and (2). One obtains bounds

$$\|K_t^{(n)}\|_1^\rho \leq a (b^n t^n / n!)^{1/m} e^{\omega(1+\rho^m)t} , \quad \|K_t^{(n)}\|_\infty^\rho \leq a t^{-D/m} (b^n t^n / n!)^{1/m} e^{\omega(1+\rho^m)t} .$$

Then estimating the sum K of the series $K^{(n)}$ with the Hölder inequality one obtains the bounds

$$\|K_t\|_\infty^\rho \leq a't^{-D/m} e^{\omega(1+\rho^m)t} .$$

But the latter bound gives

$$e^{\rho|g|'} |K_t(g)| \leq a't^{-D/m} e^{\omega(1+\rho^m)t}$$

and minimizing over ρ one finds

$$|K_t(g)| \leq at^{-D/m} e^{\omega t} e^{-b((|g|')^{m-1})^{1/(m-1)}}$$

for some $a, b > 0$ and $\omega \geq 0$.

The function K defined by the perturbation expansion is now a function of the Gaussian type and it follows from the standard arguments of ‘time-dependent’ perturbation theory that it is indeed equal to the semigroup kernel of the perturbed semigroup S . But applying similar arguments to the semigroup $S_t^\theta = S_{e^{i\theta}t}$ with $\theta \in \Lambda(\pi/2)$ one establishes that K_z is defined in the open right half-plane and satisfies the desired Gaussian bounds in $\Lambda(\pi/2 - \varepsilon)$. It remains to bound the left derivatives of K .

The estimation of the derivatives $A^\alpha K_t$ is almost identical to the estimation of K_t if $\alpha \in J(d)$ and $\|\alpha\| < m$. The starting point is now the recursion relation

$$(A^\alpha K_t^{(n)})(h) = - \int_0^t ds \int_G dk (A^\alpha K_{t-s}^{(n-1)})(k) (H_1 K_s^{(0)})(k^{-1}h)$$

which gives a set of coupled integral inequalities on the weighted spaces which can then be solved with the aid of the above bounds together with the estimates

$$\|A^\alpha K_t^{(0)}\|_1^\rho \leq at^{-\|\alpha\|/m} e^{\omega(1+\rho^m)t} , \quad \|A^\alpha K_t^{(0)}\|_\infty^\rho \leq at^{-(D+\|\alpha\|)/m} e^{\omega(1+\rho^m)t}$$

which again follow from the bounds of Corollary 2.3 applied to $K^{(0)}$. Now the derivatives only introduce an extra factor $(t-s)^{-\|\alpha\|/m}$ in the bound on $K_{t-s}^{(n-1)}$ in the integral inequalities and since this factor is integrable at $s=t$ this presents no problem. It merely introduces an additional factor $t^{-\|\alpha\|/m}$ in the overall estimate. If, however, $\|\alpha\| \geq m$ a new problem arises because the additional factor $(t-s)^{-\|\alpha\|/m}$ is not integrable and the recursion relation is ill-defined. But this can be dealt with by the methods of [ElR1].

The $n=0$ estimates are valid for all α and by splitting the convolution integral into two parts one obtains an alternative recursion relation which is well defined and can be used to estimate the higher derivatives. This second relation has the form

$$\begin{aligned} (A^\alpha K_t^{(n)})(h) = & - \int_0^{t/2} ds \int_G dk (A^\alpha K_{t-s}^{(n-1)})(k) (H_1 K_s^{(0)})(k^{-1}h) \\ & - \int_{t/2}^t ds \int_G dk K_{t-s}^{(n-1)}(k) (A^\alpha L(k) K_s^{(0)})(h) . \end{aligned}$$

In the first integral the extra singularity $(t-s)^{-\|\alpha\|/m}$ introduced by the derivatives plays no role because $s \leq t/2$. In the second integral the term $A^\alpha L(k) K_s^{(0)}$ will have a non-integrable singularity at $s=0$ but this will not cause any difficulty because it is excluded from the region of integration. Nevertheless, it is important for the form of the final bounds

that the singularity of this second term is of the correct order. But this follows because $L(k^{-1})A^\alpha L(k)$ is close to A^α in a suitable sense. The appropriate comparison is given by Lemma 4.3 of [ElR4] which is a weighted version of Lemma 7.3 of [ElR1]. If $w \in \langle 0, \infty \rangle$ is such that $w/w_i \in \mathbf{N}$ for all i one has to use the equivalent modulus $|\cdot|''$ defined by

$$|\exp(\sum_{i=1}^d \xi_i a_i)|'' = \left(\sum_{i=1}^d |\xi_i|^{2w/w_i} \right)^{1/(2w)}$$

in the proof of Lemma 4.3 in [ElR4]. The detailed estimation of the derivatives is then given by repetition of the arguments of [ElR1]. We omit further details.

The proof of the last statement of the theorem, concerning the dual kernel, is straightforward. \square

3.2 General representations

In the previous subsection we proved that if the principal part P of the form C is a positive Rockland form then the operator $dL(C)$ associated with the left regular representation L of G in L_2 generates a holomorphic semigroup with a kernel which, together with all its derivatives, satisfies Gaussian bounds. Now let (\mathcal{X}, G, U) be a strongly, or weakly*, continuous representation of G . Let $H = dU(C)$ and let K be the kernel of the semigroup generated by $dL(C)$. We shall show that \overline{H} generates a continuous semigroup S which is holomorphic in the open right half-plane and which has K as kernel. The idea of the proof is to use the kernel K to define a semigroup on \mathcal{X} and then show that the generator is \overline{H} .

First we show that $(K_t)_{t>0}$ is a bounded approximation of the identity.

Lemma 3.3

$$\lim_{t \rightarrow 0} \int_G dg K_t(g) = 1 \quad .$$

Proof Let $\varphi \in C_c^\infty(G)$, $\varphi \geq 0$, $\int dg \varphi(g) = 1$. Let $s > 0$ be such that $\text{supp } \varphi \subseteq B_s$. Further let $\psi \in C_c^\infty(G)$, $\psi \geq 0$ be such that $\psi(g) = 1$ for all $g \in B_{s+1}$ and set $\tau = \varphi * \psi$. Then $\tau \in C_c^\infty(G)$ and $\tau(g) = 1$ for all $g \in B_1$. Consider the decomposition

$$\int_G dg K_t(g) = \int_G dg K_t(g) \tau(g) + \int_G dg K_t(g) (1 - \tau(g)) \quad .$$

It follows from the Gaussian bounds of Theorem 3.2 that $\lim_{t \rightarrow 0} \int dg K_t(g) (1 - \tau(g)) = 0$, and hence it remains to prove that $\lim_{t \rightarrow 0} \int dg K_t(g) \tau(g) = 1$. Now

$$\begin{aligned} \int_G dg K_t(g) \tau(g) &= \int_G dg \int_G dh K_t(g) \varphi(h) \psi(h^{-1}g) \\ &= \int_G dg \int_G dh K_t(g) \varphi(h) \check{\psi}(g^{-1}h) \\ &= \int_G dh (T_t \check{\psi})(h) \varphi(h) \quad , \end{aligned}$$

where $\check{\psi}(g) = \psi(g^{-1})$ and T is the semigroup generated by $dL(C)$. Because $\lim_{t \rightarrow 0} T_t \check{\psi} = \check{\psi}$ in L_2 we obtain

$$\lim_{t \rightarrow 0} \int_G dg K_t(g) \tau(g) = \int_G dh \check{\psi}(h) \varphi(h) = \tau(e) = 1$$

and the lemma is proved. \square

We now prove a result which, together with Corollary 2.3, completes the proof of Theorem 1.1. Moreover, in combination with Theorem 3.2, it extends Theorem 1.1 to non-homogeneous operators.

Theorem 3.4 *Let (\mathcal{X}, G, U) be a continuous representation of G and C a form of order m whose principal part is a positive Rockland form. Let $H = dU(C)$ and $H^\dagger = dU_*(C)$ be the dual operator in the dual representation (\mathcal{F}, G, U_*) . Then \overline{H} generates a semigroup S which is holomorphic in the right half-plane. Moreover, $\overline{H} = H^{\dagger*}$ and the kernel of the semigroup generated by $dL(C)$ on L_2 is the kernel of S .*

Proof Let K be the kernel of the semigroup generated by $H_0 = dL(C)$ on L_2 . For $t > 0$ define the operator S_t by $S_t x = \int_G dg K_t(g) U(g)x$. Then it follows from Lemma 3.3 that S is a continuous semigroup. But the same argument can be applied to the kernels $t \mapsto K_{e^{i\theta}t}$ with $\theta \in \langle -\pi/2, \pi/2 \rangle$. Since all constants involved are locally uniform in θ it follows from Kato [Kat], Theorem XI.1.23, that S is a holomorphic semigroup with holomorphy sector the open right half-plane. Let \widetilde{H} be its generator and for $\lambda \in \mathbf{R}$ large enough let $R_\lambda = (\lambda I + \widetilde{H})^{-1}$ be the resolvent. Then R_λ is a continuous operator, so if one replaces \mathcal{X} by \mathcal{X}_n it follows that R_λ maps \mathcal{X}_n into \mathcal{X}_n if (\mathcal{X}, G, U) is a strongly continuous representation. Hence R_λ maps \mathcal{X}_∞ into \mathcal{X}_∞ for any representation.

In order to identify the generator of S , we first assume that U is the left regular representation in $L_1^\rho = L_1(G; e^{\rho|g|} dg)$, where $\rho \geq 0$ is fixed.

Note that $R_\lambda \varphi = r_\lambda * \varphi$, where r_λ is the resolvent kernel associated with the kernel K . Since $R_\lambda \varphi = (\lambda I + H_0)^{-1} \varphi$ for all $\varphi \in C_c^\infty(G)$ and $C_c^\infty(G) \subseteq L_{1,\infty}^\rho$ one deduces that

$$(\psi, (\lambda I + H)R_\lambda \varphi) = ((\lambda I + H^\dagger)\psi, R_\lambda \varphi) = (\psi, \varphi)$$

for all $\psi, \varphi \in C_c^\infty(G)$. So $(\lambda I + \overline{H})R_\lambda \varphi = \varphi$ for all $\varphi \in C_c^\infty(G)$. Since R_λ is a continuous operator and $C_c^\infty(G)$ is dense in L_1^ρ it follows that $\widetilde{H} \subseteq \overline{H}$. Next, using the dual kernel K^\dagger one can define similarly a semigroup S^\dagger on \mathcal{F} with generator \widetilde{H}^\dagger . Then it follows from the above that $\widetilde{H}^\dagger \subseteq \overline{H}^\dagger$. But $S^\dagger = S^*$, so $\widetilde{H}^{\dagger*} = \widetilde{H}$. Hence

$$\overline{H} \subseteq H^{\dagger*} = \overline{H}^{\dagger*} \subseteq \widetilde{H}^{\dagger*} = \widetilde{H} \subseteq \overline{H}$$

So $\overline{H} = H^{\dagger*} = \widetilde{H}$ is the generator of S .

Finally we deduce that the generator \widetilde{H} of the semigroup S is the closure \overline{H} of $dU(C)$ for a general representation U . Let $\rho > 0$ be so large that $\|U(g)\| \leq M e^{\rho|g|}$ uniformly for all $g \in G$, for some $M > 0$. Let H^L and S^L be the operator and semigroup corresponding to the left regular representation in L_1^ρ . Let $\varphi \in C_c^\infty(G)$ and $x \in \mathcal{X}$. Then for all $t > 0$ one has

$$S_t U(\varphi)x = U(K_t * \varphi)x = U(S_t^L \varphi)x .$$

Hence by the Duhamel formula

$$\begin{aligned} S_t U(\varphi)x - U(\varphi)x &= U(S_t^L \varphi - \varphi)x \\ &= -U\left(\int_0^t ds H^L S_s^L \varphi\right)x = -\int_0^t ds \left(U(S_s^L H^L \varphi)x\right) . \end{aligned}$$

Therefore

$$\begin{aligned} \left\| t^{-1} \left(S_t U(\varphi)x - U(\varphi)x \right) - U(H^L \varphi)x \right\| &= \left\| t^{-1} \int_0^t ds U(S_s^L H^L \varphi - H^L \varphi)x \right\| \\ &\leq \sup_{0 < s \leq t} \| S_s^L H^L \varphi - H^L \varphi \|_1^p \|x\| \quad . \end{aligned}$$

Since S^L is a continuous semigroup, it follows that $U(\varphi)x$ is in the domain of the generator \widetilde{H} of S and

$$\widetilde{H}U(\varphi)x = U(H^L \varphi)x = HU(\varphi)x \quad .$$

Let $\mathcal{D} = \text{span} \{U(\varphi)x : \varphi \in C_c^\infty(G), x \in \mathcal{X}\}$ be the Gårding space. Then $\widetilde{H} \supseteq H|_{\mathcal{D}}$. But \mathcal{D} is dense in \mathcal{X}'_m . So $\widetilde{H} \supseteq H$ and hence $\widetilde{H} \supseteq \overline{H}$. Since $R_\lambda x \in \mathcal{X}_\infty$ for all $x \in \mathcal{X}_\infty$ it follows that $(\lambda I + H)R_\lambda x = x$ for all $x \in \mathcal{X}_\infty$. One can then deduce as above for the left regular representation in L^p_1 by using duality that $\widetilde{H} = \overline{H} = H^*$. This proves the theorem. \square

Corollary 3.5 *The kernel $z \mapsto K_z(g)$ is analytic in the open right half-plane uniformly for all $g \in G$.*

Proof If S is the holomorphic semigroup generated by $dU(C)$ on L_∞ , where U is the left regular representation, then $S_{z_1}K_{z_2} = S_{z_1+z_2}$ for all z_1, z_2 in the right half-plane. Hence the map $z \mapsto K_z$ from $\Lambda(\pi/2)$ into L_∞ is holomorphic. Fix $z_0 \in \Lambda(\pi/2)$. Then there exists $K'_{z_0} \in L_\infty$ such that

$$\lim_{z \rightarrow z_0} \left\| \frac{K_z - K_{z_0}}{z - z_0} - K'_{z_0} \right\|_\infty = 0 \quad .$$

But $g \mapsto (K_z - K_{z_0})(g)/(z - z_0)$ is a continuous function, so

$$\lim_{z \rightarrow z_0} \frac{K_z(g) - K_{z_0}(g)}{z - z_0} = K'_{z_0}(g)$$

uniformly for all $g \in G$. Therefore $z \mapsto K_z(g)$ is analytic. \square

3.3 Regularity

Finally we consider unitary representations and prove optimal regularity results.

Theorem 3.6 *Let C be a form of order m whose principal part P is a positive Rockland form. Suppose (\mathcal{X}, G, U) is a unitary representation and let $H = dU(C)$. Then one has the following.*

- I. *The operator H is closed.*
- II. *For all $n \in \mathbf{N}$ one has $D(H^n) = \mathcal{X}'_{nm}$ with equivalent norms.*

Proof We only have to prove Statement I since the principal part of C^n is a positive Rockland form. The equivalence of the norms follows from the closed graph theorem.

Consider the form C_1 defined such that

$$dV(C_1) = \sum_{\substack{\alpha \in J(d) \\ \|\alpha\| \leq m}} (-1)^{|\alpha|} A^{(\alpha^*, \alpha)}$$

in any representation (\mathcal{Y}, G, V) , where $\langle \alpha_*, \alpha \rangle$ denotes the multi-index formed by composition of α and the index α_* obtained by reversing the order of α . Then C_1 is a form of order $2m$ whose principal part P_1 is a positive Rockland form. Indeed, if V is a nontrivial irreducible unitary representation, $x \in \mathcal{Y}_\infty(V)$ and $dV(P_1)x = 0$, then $(x, dV(P_1)x) = 0$, so $A^\alpha x = 0$ in the representation V for all $\alpha \in J(d)$ with $\|\alpha\| = m$. Hence $dV(P)x = 0$ and $x = 0$ since P is a Rockland form.

By Proposition 2.1 there exists $\mu_1 > 0$ such that $\|dL(P)\varphi\|_2 \geq \mu_1 N'_{2;m}(\varphi)$ for all $\varphi \in L'_{2;m}$. Moreover, by Theorem 2.1 in Chapter I of [HeN2], and the remark immediately after it, there exists $\mu_2 > 0$ such that $\|dV(P)x\| \geq \mu_2 N'_m(x)$ uniformly for all unitary irreducible representations (\mathcal{Y}, G, V) and $x \in \mathcal{Y}'_m(V)$. Let $\mu = 2^{-1} \min(\mu_1, \mu_2)$. Then

$$(x, dV(P^2 - \mu P_1)x) \geq \mu (x, dV(P_1)x)$$

if V is an irreducible unitary representation or if V is the left regular representation on L_2 and x is a C^∞ -vector. Therefore $P^2 - \mu P_1$ is a positive Rockland form. So $C^\dagger C - \mu C_1$ is a form of order $2m$ whose principal part is a positive Rockland form.

Then the closure of $dU(C^\dagger C - \mu C_1)$ is a generator of a semigroup, so it is self-adjoint and by spectral theory it is lower bounded. Let $-\rho$ be a lower bound. Then

$$\|dU(C)x\|^2 = \mu (dU(C_1)x, x) + (dU(C^\dagger C - \mu C_1)x, x) \geq \mu (\|x\|'_m)^2 - \rho \|x\|^2$$

for all $x \in \mathcal{X}'_{2m}(U)$. Since $\mathcal{X}_\infty(U)$ is a core for $\overline{dU(C)}$, by [BrR1] Corollary 3.1.7, and \mathcal{X}'_m is complete it follows that $D(\overline{dU(C)}) \subseteq \mathcal{X}'_m$ and hence $dU(C)$ is closed. \square

It is also possible to obtain regularity results for the left regular representation on the L_p -spaces with respect to left Haar measure if $p \in \langle 1, \infty \rangle$. These are basically a result of the good kernel bounds and the regularity on L_2 .

Theorem 3.7 *Let U be the left regular representation on L_p , where $p \in \langle 1, \infty \rangle$ and let $H = dU(C)$. Then*

- I. *The operator H is closed.*
- II. *For all $n \in \mathbf{N}$ one has $D(H^n) = L'_{p;nm}$ with equivalent norms.*

Proof The proof is precisely the same as that for subcoercive operators in [BER]. It is based on a Lie group version of the usual weak L_1 -estimates of singular integration theory combined with the regularity properties already obtained for L_2 together with duality and interpolation arguments. \square

Theorem 3.8 *Let U be the left regular representation on L_p , where $p \in \langle 1, \infty \rangle$, and let $H = dU(C)$. If $\theta \in \langle 0, \pi/2 \rangle$ then there is a $\nu_0 \geq 0$, independent of p , such that the operators $\nu I + H$, $\nu > \nu_0$, have a bounded functional analysis over the functions which are bounded and holomorphic in the sector $\Lambda(\varphi)$ with $\varphi \in \langle \pi/2 - \theta, \pi \rangle$.*

Proof The proof is precisely the same as in [ElR3]. It is again based on a Lie group version of arguments of singular integration theory. \square

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References

- [Agm] AGMON, S., *Lectures on elliptic boundary value problems*. Van Nostrand Mathematical Studies 2. D. van Nostrand Company, Inc., Princeton etc., 1965.
- [AMT] AUSCHER, P., MCINTOSH, A. and TCHAMITCHIAN, PH. Noyau de la chaleur d'opérateurs elliptiques complexes. To appear.
- [BrR1] BRATTELI, O., and ROBINSON, D.W., *Operator algebras and quantum statistical mechanics*, vol. 1. Second edition. Springer-Verlag, New York etc., 1987.
- [BrR2] ———, Subelliptic operators on Lie groups: variable coefficients. Research Report CMA-MR3-92, The Australian National University, Canberra, Australia, 1992.
- [BER] BURNS, R.J., ELST, A.F.M. TER, and ROBINSON, D.W., L_p -regularity of subelliptic operators on Lie groups. *J. Operator Theory* **30** (1993). To appear.
- [Dzi] DZIUBAŃSKI, J., On semigroups generated by subelliptic operators on homogeneous groups. *Coll. Math.* **64** (1993), 215–231.
- [ElR1] ELST, A.F.M. TER, and ROBINSON, D.W., Subcoercivity and subelliptic operators on Lie groups. Research Report CMA-MR4A-92, The Australian National University, Canberra, Australia, 1992.
- [ElR2] ———, Subcoercive and subelliptic operators on Lie groups: variable coefficients. *Publ. RIMS, Kyoto Univ.* (1993). To appear.
- [ElR3] ———, Functional analysis of subelliptic operators on Lie groups. *J. Operator Theory* **30** (1993). To appear.
- [ElR4] ———, Weighted strongly elliptic operators on Lie groups. Research Report CMA-MR14-93, The Australian National University, Canberra, Australia, 1993.
- [FoS] FOLLAND, G.B., and STEIN, E.M., *Hardy spaces on homogeneous groups*. Mathematical Notes 28. Princeton University Press, Princeton, 1982.
- [Heb] HEBISCH, W., Sharp pointwise estimate for the kernels of the semigroup generated by sums of even powers of vector fields on homogeneous groups. *Stud. Math.* **95** (1989), 93–106.
- [HeS] HEBISH, W., and SIKORA, A., A smooth subadditive homogeneous norm on a homogeneous group. *Stud. Math.* **96** (1990), 231–236.

- [HeN1] HELFFER, B., and NOURRIGAT, J., Caractérisation des opérateurs hypoelliptiques homogènes invariants à gauche sur un groupe de Lie nilpotent gradué. *Comm. Part. Diff. Eq.* **4** (1979), 899–958.
- [HeN2] ———, *Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs*. Progress in Mathematics 58. Birkhäuser, Boston etc., 1985.
- [Kat] KATO, T., *Perturbation theory for linear operators*. Second edition, Grundlehren der mathematischen Wissenschaften 132. Springer-Verlag, Berlin etc., 1984.
- [Mil] MILLER, K.G., Parametrices for hypoelliptic operators on step two nilpotent Lie groups. *Comm. Part. Diff. Eq.* **5** (1980), 1153–1184.
- [NeS] NELSON, E., and STINESPRING, W.F., Representation of elliptic operators in an enveloping algebra. *Amer. J. Math.* **81** (1959), 547–560.
- [Nir] NIRENBERG, L., Remarks on strongly elliptic partial differential operators. *Comm. Pure Appl. Math.* **8** (1955), 649–675.
- [Rob] ROBINSON, D.W., *Elliptic operators and Lie groups*. Oxford Mathematical Monographs. Oxford University Press, Oxford etc., 1991.
- [Roc] ROCKLAND, C., Hypoellipticity for the Heisenberg group. *Trans. Amer. Math. Soc.* **240** (1978), 1–52.
- [VSC] VAROPOULOS, N.T., SALOFF-COSTE, L., and COULHON, T., *Analysis and geometry on groups*. Cambridge Tracts in Mathematics 100. Cambridge University Press, Cambridge, 1992.